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CYCLOTOMIC NUMERICAL SEMIGROUPS

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ABSTRACT. Given a numerical semigroup S, we let $\mathrm{P}_S(x) = (1-x)\sum_{s\in S} x^s$ be its *semigroup polynomial*. We study *cyclotomic numerical semigroups*; these are numerical semigroups S such that $\mathrm{P}_S(x)$ has all its roots in the unit disc. We conjecture that S is a cyclotomic numerical semigroup if and only if S is a complete intersection numerical semigroup and present some evidence for it.

Aside from the notion of cyclotomic numerical semigroup we introduce the notion of *cyclotomic exponents* and *polynomially related semigroups*. We derive some properties and give some applications of these new concepts.

1. INTRODUCTION

A *numerical semigroup* S is a submonoid of $\mathbb{N}$ (the set of nonnegative integers) under addition, with finite complement in $\mathbb{N}$. The nonnegative integers not in S are its *gaps*, and the largest integer not in S is its *Frobenius number*, $\mathrm{F}(S)$. A numerical semigroup admits a unique minimal generating system; its cardinality is called its *embedding dimension* $\mathrm{e}(S)$, and its elements *minimal generators*. The smallest positive integer in S is called the *multiplicity* of S, and it is denoted by $\mathrm{m}(S)$ (see for instance [24] for an introduction to numerical semigroups).

To a numerical semigroup S, we can associate $\mathrm{H}_S(x) := \sum_{s\in S} x^s$, its *Hilbert series*, and $\mathrm{P}_S(x) = (1-x)\sum_{s\in S} x^s$, its *semigroup polynomial*. Since all elements larger than $\mathrm{F}(S)$ are in S, $\mathrm{H}_S(x)$ is not a polynomial, but $\mathrm{P}_S(x)$ is. On noting that $\mathrm{H}_S(x) = (1-x)^{-1} - \sum_{s\notin S} x^s$, we see that

$$\mathrm{P}_S(x) = 1 + (x-1)\sum_{s\notin S} x^s, \tag{1}$$

and hence the degree of $\mathrm{P}_S(x)$ equals $\mathrm{F}(S)+1$. (With $s \notin S$ we mean the sum over the numbers in $\mathbb{N}\backslash S$.) Note that $\mathrm{P}_S(x)$ is a monic polynomial.

The following result is a generalization of [21, Corollary 2].

Lemma 1. *Let S be a numerical semigroup and assume that $\mathrm{P}_S(x) = a_0 + a_1x + \cdots + a_kx^k$. Then for $s \in \{0,\ldots,k\}$,*

$$a_s = \begin{cases} 1 & \text{if } s \in S \text{ and } s-1 \notin S,\\ -1 & \text{if } s \notin S \text{ and } s-1 \in S,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. The proof easily follows from the fact that $\mathrm{P}_S(x) = (1-x)\mathrm{H}_S(x)$ and that a coefficient of $x\mathrm{H}_S(x)$ is one if and only if its degree minus one belongs to S. □

Corollary 1. *The nonzero coefficients of $\mathrm{P}_S(x)$ alternate between 1 and -1.*

Recall (see, for instance, Damianou [6]) that a *Kronecker polynomial* is a monic polynomial with integer coefficients having all its roots in the unit disc.

We say that a numerical semigroup is *cyclotomic* if its semigroup polynomial is a Kronecker polynomial.

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Lemma 2 (Kronecker, 1857, cf. [6])**.** *If f is a Kronecker polynomial with $f(0)\neq 0$, then all roots of f are actually on the unit circle and f factorizes as a product of cyclotomic polynomials.*

As usual we will denote the nth cyclotomic polynomial by Φ_n. The cyclotomic polynomials Φ_n are monic polynomials of degree $\varphi(n)$ (where φ denotes Euler's totient function) and are irreducible over $\mathbb{Q}$ (see, e.g., Weintraub [28]). Over the rational numbers x^m-1 factorizes into irreducibles as

$$x^m-1=\prod_{d|m}\Phi_d(x). \tag{2}$$

By Möbius inversion it follows from (2) that

$$\Phi_n(x)=\prod_{d|n}(x^d-1)^{\mu(n/d)}, \tag{3}$$

where $\mu(n)$ denotes the Möbius function. It follows for example from (3) that if p and q are distinct primes, then

$$\Phi_{pq}(x)=\frac{(x^{pq}-1)(x-1)}{(x^p-1)(x^q-1)}. \tag{4}$$

On using that $\sum_{d|n}\mu(d)=0$ for $n>1$, we infer from (3) that

$$\Phi_n(x)=\prod_{d|n}(1-x^d)^{\mu(n/d)}, \tag{5}$$

and on using that $\sum_{d|n} d\mu(n/d)=\varphi(n)$, we deduce that

$$x^{\varphi(n)}\Phi_n\left(\frac{1}{x}\right)=\Phi_n(x), \tag{6}$$

and hence Φ_n is selfreciprocal for $n>1$. Note that $\Phi_1(x)=x-1$ is not selfreciprocal.

Throughout, the letters p and q are used to denote primes.

In the sequel we will use often that if $p|n$, then $\Phi_{pn}(x)=\Phi_n(x^p)$ (this is easily derived from (3)).

Lemma 3. *Suppose that S is a cyclotomic numerical semigroup. Then we have*

$$\mathrm{P}_S(x)=\prod_{d\in\mathscr{D}}\Phi_d(x)^{e_d}, \tag{7}$$

where $\mathscr{D}=\{d_1,d_2,\ldots,d_s\}$ is a set of positive integers and the e_d are positive integers too. Furthermore, all elements of $\mathscr{D}$ are composite.

Proof. The first assertion is a consequence of Lemma 2 and the fact that, by (1), $\mathrm{P}_S(0)\neq 0$.

It is a well-known fact, see, e.g., Lang [18, p. 74], that

$$\Phi_n(1)=\begin{cases}0 & \text{if } n=1;\\ p & \text{if } n=p^m;\\ 1 & \text{otherwise.}\end{cases}$$

By (1) and (7) we have $1=\mathrm{P}_S(1)=\prod_{d\in\mathscr{D}}\Phi_d(1)^{e_d}$, and hence by the above fact the proof is completed. □

Corollary 2. *If S is cyclotomic, then P_S is selfreciprocal.*

Proof. This follows on using that Φ_n is selfreciprocal for $n>1$, i.e. that (6) holds. □

We can now formulate the main problem we like to address:

Problem 1. *Find an intrisic characterization of the numerical semigroups S for which S is cyclotomic, that is a characterization that does not involve P_S or its roots in any way.*

Recall that a numerical semigroup S is said to be *symmetric* if $S \cup (\mathrm{F}(S) - S) = \mathbb{Z}$, thus symmetry is an example of an intrinsic characterization of S.

Theorem 1. *If S is cyclotomic, then it must be symmetric.*

Proof. Using (1) it is not difficult to conclude (see Moree [21]) that S is symmetric if and only if P_S is selfreciprocal. By Corollary 2 P_S is selfreciprocal. □

If $\mathrm{e}(S) = 2$, then S is cyclotomic if and only if S is symmetric. One thus might be tempted to think that this also holds if $\mathrm{e}(S) \geq 3$. The following observation is not deep, see Section 3.

Lemma 4. *Every complete intersection numerical semigroup is cyclotomic.*

As every symmetric numerical semigroup with embedding dimension $\mathrm{e}(S) \leq 3$ is a complete intersection numerical semigroup ([13]), we get a positive answer for $\mathrm{e}(S) \leq 3$. We thus have the following result.

Lemma 5. *If $\mathrm{e}(S) \leq 3$, then S is cyclotomic if and only if S is symmetric.*

It is not difficult to show that the condition $\mathrm{e}(S) \leq 3$ is necessary. The example with smallest Frobenius number of a symmetric numerical semigroup S that is not cyclotomic is $S = \langle 5, 6, 7, 8 \rangle$, where we have $\mathrm{P}_S(x) = x^{10} - x^9 + x^5 - x + 1$ and $\mathrm{F}(S) = 9$ (for $A \subseteq \mathbb{N}$, we use $\langle A \rangle$ to denote set of integers of the form $\sum_{a \in A} \lambda_a a$ with $a \in A$, $\lambda_a \in \mathbb{N}$ and all but finitely many of them equal to zero). For Frobenius number 11, we have two symmetric numerical semigroups that are not cyclotomic: $\langle 5, 7, 8, 9 \rangle$ and $\langle 6, 7, 8, 9, 10 \rangle$. These examples were found with an implementation using [8] of the Graeffe method, [4]; the procedures used in this paper are all included in the development version of `numericalsgps` (see `https://bitbucket.org/mdelgado/numericalsgps`), and will be added to the next official release.

On the basis of Lemma 4 and computer evidence we make the following conjecture.

Conjecture 1. *Every cyclotomic numerical semigroup is a complete intersection numerical semigroup.*

The notion of complete intersection numerical semigroup does not involve P_S or its roots in anyway and thus this gives together with Lemma 4, conjecturally, the answer to Problem 1, namely we conjecture that S is cyclotomic if and only if S is a complete intersection numerical semigroup.

1.1. **Cyclotomic numerical semigroups of prescribed height and depth.** It follows from Lemma 3 and the identity (2) that if S is a cyclotomic numerical semigroup, then $\mathrm{P}_S(x) | (x^m - 1)^e$ for some integers m and e.

We say that a numerical semigroup S is cyclotomic of *depth d* and *height h* if $\mathrm{P}_S(x) | (x^d - 1)^h$, where both d and h are chosen minimally, that is, $\mathrm{P}_S(x)$ does not divide $(x^n - 1)^{h-1}$ for any n and it does not divide $(x^{d_1} - 1)^h$ for any divisor $d_1 < d$ of d.

On noting that $\Phi_m(x) | (x^n - 1)$ if and only if $m | n$ one arrives at the following conclusion.

Lemma 6. *Suppose that S is a cyclotomic numerical semigroup with P_S factorizing as in (7), then S is of depth $\mathrm{lcm}(d_1, d_2, \ldots, d_s)$ and of height $\max\{e_1, \ldots, e_s\}$.*

Problem 2. *Classify all cyclotomic numerical semigroups of a prescribed depth and height.*

As in this problem we focus on a subclass of the cyclotomic numerical semigroups, it might be less challenging to resolve it than to resolve Problem 1. In the rest of the paper we present contributions towards resolving Problem 1 and the more restricted Problem 2.

For a more pedestrian introduction to numerical semigroups and cyclotomic polynomials the reader is referred to Moree [21].

2. APÉRY SETS AND SEMIGROUP POLYNOMIALS

The *Apéry set* of S with respect to a nonzero $m \in S$ is defined as

$$\mathrm{Ap}(S; m) = \{s \in S \mid s - m \notin S\}.$$

Note that

$$S = \mathrm{Ap}(S;m) + m\mathbb{N} \tag{8}$$

and that $\mathrm{Ap}(S;m)$ consists of a complete set of residues modulo m. Thus we have

$$\mathrm{H}_S(x) = \sum_{w\in \mathrm{Ap}(S;m)} x^w \sum_{i=0}^{\infty} x^{mi} = \frac{1}{1-x^m} \sum_{w\in \mathrm{Ap}(S;m)} x^w, \tag{9}$$

cf. [23, (4)].

Apéry sets can also be defined in a natural way for integers m not in the semigroup (see for instance [12] or [7]), but in this case $\#\mathrm{Ap}(S;m) \neq m$.

Proposition 1. *Let S be a numerical semigroup and m be a positive integer. Then $\#\mathrm{Ap}(S;m) = m$ if and only if $m \in S$.*

Proof. For $i \in \{0,\dots,m-1\}$, set $w_i = \min\{s \in S \mid s \equiv i \bmod m\}$. By definition, $w_0 = 0$ and $\{w_0,\dots,w_{m-1}\} \subseteq \mathrm{Ap}(S;m)$. Hence $\#\mathrm{Ap}(S;m) \geq m$, and equality holds if and only if $\{0 = w_0, w_1,\dots,w_{m-1}\} = \mathrm{Ap}(S;m)$.

If $m \in S$, [24, Lemma 2.4] asserts that $\{0 = w_0, w_1,\dots,w_{m-1}\} = \mathrm{Ap}(S;m)$.

Now assume that $\{0 = w_0, w_1,\dots,w_{m-1}\} = \mathrm{Ap}(S;m)$. Then for every $i \in \{0,\dots,m-1\}$ and every $k \in \mathbb{N}$, $w_i + km \in S$. In particular $w_0 + m = m \in S$. □

Example 1. Let S be a numerical semigroup minimally generated by $\{a,b\}$. Assume that u, v are integers with $0 \leq u < b$ and $1 = ua + vb$. By Lemma 1, the number of ones in P_S equals $\#\mathrm{Ap}(S;1)$ and in view of [7, Theorem 14], we have that $\#\mathrm{Ap}(S;1) = u(a+v)$ (compare with [21, Corollary 1]). Given $0 < \gamma < 1/2$, let $\mathrm{C}_\gamma(x)$ denote the number of numerical semigroups $S = \langle p,q\rangle$ with p,q primes and $m = pq \leq x$ such that $\#\mathrm{Ap}(\langle p,q\rangle;1) \leq m^{1/2+\gamma}$. Bzdęga [5] was the first to obtain sharp upper and lower bounds for this quantity. Fouvry [11], using deep methods from analytic number theory, even obtained an asymptotic for $\mathrm{C}_\gamma(x)$ in the range $\gamma \in (\frac{12}{25}, \frac{1}{2})$.

Example 2. Let m and q be positive integers such that $m \geq 2q+3$ and let

$$S = \langle m, m+1, qm+2q+2,\dots,qm+(m-1)\rangle.$$

Then by [24, Lemma 4.22] S is symmetric with multiplicity m and embedding dimension $m-2q$. It is easy to deduce that $\mathrm{Ap}(S;m) = \{0, m+1, 2m+2,\dots,qm+q, qm+2q+2,\dots,qm+(m-1),(q+1)(m+1),\dots,(2q+1)(m+1)\}$. On invoking (9) we have $\mathrm{P}_S(x) = \frac{1-x}{1-x^m}\sum_{w\in\mathrm{Ap}(S;m)} x^w$ and computations give

$$\mathrm{P}_S(x) = \frac{1-x}{1-x^m}\left(\frac{1-x^{2(m+1)(q+1)}}{1-x^{m+1}} + x^{qm+2q+2}\frac{1-x^{m-2q-2}}{1-x}\right).$$

For instace, taking $m=6$ and $q=1$ leads to $S = \langle 6,7,10,11\rangle$, with $e(S) = 4$ and $\mathrm{P}_S(x) = x^{16} - x^{15} + x^{10} - x^8 + x^6 - x + 1$, which is not Kronecker. Hence S is not cyclotomic. We have done an exhaustive search in this family of numerical semigroups up to multiplicity 30 with our `GAP` functions, and only those with embedding dimension three were cyclotomic.

Example 3. Let m and q be nonnegative integers such that $m \geq 2q+4$ and let

$$S = \langle m, m+1, (q+1)m+q+2,\dots,(q+1)m+m-q-2\rangle.$$

Then by [24, Lemma 4.23] S is symmetric with multiplicity m and embedding dimension $m-2q-1$. By appealing to the Apéry set again we compute

$$\mathrm{P}_S(x) = \frac{1-x}{1-x^m}\left(\sum_{k=0}^{q+1}(x^{m+1})^k + x^{(q+1)m}\sum_{h=q+2}^{m-q-2} x^h + x^{(q+3)m-q-1}\sum_{l=0}^{q}(x^{m+1})^l\right).$$

For instance, taking $m=8$ and $q=1$ leads to $S = \langle 8,9,19,20,21\rangle$, with e($S$)=5 and $\mathrm{P}_S(x) = x^{32} - x^{31} + x^{24} - x^{22} + x^{16} - x^{10} + x^8 - x + 1$, which is not Kronecker. Hence S is not cyclotomic.

The following problem could turn out to be difficult.

Problem 3. *Prove that the numerical semigroups given in Examples 2 and 3 are not cyclotomic or find a counterexample.*

3. Gluings of numerical semigroups

Let T, T_1 and T_2 be submonoids of $\mathbb{N}$. We say that T is the gluing of T_1 and T_2 if

(1) $T = T_1 + T_2$,
(2) $\mathrm{lcm}(d_1, d_2) \in T_1 \cap T_2$, with $d_i = \gcd(T_i)$ for $i \in \{1,2\}$.

We will denote this fact by $T = T_1 +_d T_2$, with $d = \mathrm{lcm}(d_1, d_2)$.

Every submonoid T of $\mathbb{N}$ is isomorphic as a monoid to $T/\gcd(T)$, which is a numerical semigroup. Hence, in the above definition if $T = S$ is a numerical semigroup, and $S = T_1 +_d T_2$, then $T_i = d_i S_i$, with $S_i = T_i/d_i$, and $\gcd(d_1, d_2) = \gcd(S) = 1$. Hence $\mathrm{lcm}(d_1, d_2) = d_1 d_2$, which leads to $d_i \in S_j$ for $\{i, j\} = \{1,2\}$.

In [1] it is shown that

$$\mathrm{H}_{a_1 S_1 +_{a_1 a_2} a_2 S_2}(x) = (1 - x^{a_1 a_2})\mathrm{H}_{S_1}(x^{a_1})\mathrm{H}_{S_2}(x^{a_2}). \tag{10}$$

For the particular case $S = \langle a_1, a_2\rangle = a_1\mathbb{N} +_{a_1 a_2} a_2\mathbb{N}$, we obtain (see also [21])

$$\mathrm{H}_{\langle a_1, a_2\rangle}(x) = \frac{1 - x^{a_1 a_2}}{(1 - x^{a_1})(1 - x^{a_2})}, \tag{11}$$

and by using (2),

$$\mathrm{P}_{\langle a_1, a_2\rangle}(x) = \frac{(1 - x)(1 - x^{a_1 a_2})}{(1 - x^{a_1})(1 - x^{a_2})} = \prod_{d \mid a_1 a_2,\ d\nmid a_1,\ d\nmid a_2} \Phi_d(x). \tag{12}$$

Hence

$$\mathrm{P}_{a_1 S_1 +_{a_1 a_2} a_2 S_2}(x) = \frac{(1 - x)(1 - x^{a_1 a_2})}{(1 - x^{a_1})(1 - x^{a_2})}\mathrm{P}_{S_1}(x^{a_1})\mathrm{P}_{S_2}(x^{a_2}) = \mathrm{P}_{\langle a_1, a_2\rangle}(x)\mathrm{P}_{S_1}(x^{a_1})\mathrm{P}_{S_2}(x^{a_2}). \tag{13}$$

Delorme in [10] proved (but with a different notation) that a numerical semigroup S is a complete intersection if and only if either S is $\mathbb{N}$ or it is the gluing of two complete intersection numerical semigroups. If we proceed recursively and $A = \{a_1, \ldots, a_t\}$ is a minimal generating system of S, we will find positive integers $g_1, \ldots, g_{t-1}$ such that

$$S = a_1\mathbb{N} +_{g_1} \cdots +_{g_{t-1}} a_t\mathbb{N}.$$

By using [1, Theorem 20], we obtain

$$\mathrm{P}_S(x) = (1 - x)\prod_{i=1}^{t-1}(1 - x^{g_i})\prod_{i=1}^{t}(1 - x^{a_i})^{-1}, \tag{14}$$

and we deduce the following consequence.

Corollary 3. *Every complete intersection numerical semigroup is cyclotomic.*

This proves one of the directions of Conjecture 1. There is some computer evidence that this conjecture might hold. As we have seen above, cyclotomic implies symmetric. We took all symmetric numerical semigroups with Frobenius number less than or equal to 70, and saw that in this set cyclotomic equals complete intersection. The strategy we used to try to prove this conjecture, without success so far, was the following. For $S = \langle n_1, \ldots, n_e\rangle$, according to [26, (1)], the only nonzero terms of $Q(x) = \mathrm{H}_S(x)\prod_{i=1}^{e}(1 - x^{n_i})$ are those of degrees $n \in S$ such that the *Euler characteristic* of the *shaded set* of n, $\Delta_n = \{L \subset \{n_1, \ldots, n_e\} \mid n - \sum_{s\in L} s \in S\}$, is not zero, that is, $\chi_S(n) := \sum_{L\in\Delta_n}(-1)^{\#L} \neq 0$. We have been trying to determine when $Q(x)$ factors as $\prod_{b\in\mathrm{Betti}(S)}(1 - x^b)^{m_b}$, where Betti($S$) is the set of elements for which the underlying graph of Δ_n is not connected (the graph whose vertices are the elements $n_i \in \{n_1, \ldots, n_e\}$ such that $n - n_i \in S$, and $n_i n_j$ is an edge whenever $i, j \in \{1, \ldots, e\}$, $i \neq j$ and $n - (n_i + n_j) \in S$; see [24, §7.3]) and $m_b \in \mathbb{N}$. This is what actually happens in (14).

3.1. **Free semigroups.** Let S be a numerical semigroup generated by $\{n_1,\dots,n_t\}$. We say that S is *free* if either $S=\mathbb{N}$ or it is the gluing of the free semigroup $\langle n_1,\dots,n_{t-1}\rangle$ and $\langle n_t\rangle$ (see [3]). The way we enumerate the generators is relevant. For instance S is free for the arrangement $\{n_1=4,n_2=6,n_3=9\}$ and it is not for $\{n_1=4,n_2=9,n_3=6\}$.

Example 4. Let S be an embedding dimension three symmetric numerical semigroup. Then S is free and it has a system of generators of the form $\langle am_1, am_2, bm_1+cm_2\rangle$, with $a,b,c\in\mathbb{N}$ such that $a\ge 2$, $b+c\ge 2$ and $\gcd(a,bm_1+cm_2)=1$ ([24, Theorem 10.6]). It follows that S can be expressed as $S=a\langle m_1,m_2\rangle +_{a(bm_1+cm_2)}(bm_1+cm_2)\mathbb{N}$. From (14),

$$\mathrm{P}_S(x)=\frac{(1-x)(1-x^{a(bm_1+cm_2)})(1-x^{am_1m_2})}{(1-x^{am_1})(1-x^{am_2})(1-x^{bm_1+cm_2})}.$$

If $S=\langle n_1,n_2,n_3\rangle$ is nonsymmetric with embedding dimension three, then it can be deduced from [26] and [2] (see also [23, Theorem 4]) that

$$\mathrm{P}_S(x)=\frac{(1-x)(1-x^{c_1n_1}-x^{c_2n_2}-x^{c_3n_3}+x^{f_1+n_1+n_2+n_3}+x^{f_2+n_1+n_2+n_3})}{(1-x^{n_1})(1-x^{n_2})(1-x^{n_3})},$$

where

- $c_i=\min\{m\in\mathbb{N}\setminus\{0\}\mid mn_i\in\langle n_j,n_k\rangle\}$ for all $\{i,j,k\}=\{1,2,3\}$,
- $f_1=\mathrm{F}(S)$ and $f_2\neq f_1$ is such that $f_2+S\setminus\{0\}\subset S$ (f_1 and f_2 are the *pseudo-Frobenius numbers* of S; their expression can be found for instance in [2, Corollary 11]).

Formulas for symmetric and pseudo-symmetric embedding dimension four can be derived from [2, Section 4], and the number of nonzero coefficients of $\mathrm{H}_S(x)\prod_{i=1}^4(1-x_i^n)$ is 12 and 14, respectively (recall that S is pseudo-symmetric if $\mathrm{F}(S)$ is even and for every $x\in\mathbb{Z}\setminus S$, either $x=\mathrm{F}(S)/2$ or $\mathrm{F}(S)-x\in S$). From [26] it is derived that the number of nonzero coefficients is not bounded for embedding dimension four.

Let $n\ge 2$ and $(a_1,a_2,\dots,a_n)$ be a sequence of relatively prime positive integers. For every $k\in\{1,\dots,n\}$, let $d_k=\gcd(a_1,\dots,a_k)$. For $k\in\{2,\dots,n\}$, let $c_k=d_{k-1}/d_k$. Let S_k be the semigroup generated by $a_1,\dots,a_k$. We say that the sequence $(a_1,a_2,\dots,a_n)$ is *smooth* if $c_ka_k\in S_{k-1}$ for every $k=2,\dots,n$.

Observe that a numerical semigroup S is generated by a smooth sequence if and only if S is free. Also $c_ka_k\in S_{k-1}$ is equivalent to $\frac{a_k}{d_k}\in\frac{1}{d_{k-1}}S_{k-1}$ (and $\frac{1}{d_{k-1}}S_{k-1}$ is a numerical semigroup). Notice that $S_k=S_{k-1}+a_k\mathbb{N}$. With the notation of gluing, we have $\frac{1}{d_k}S_k=c_k\left(\frac{1}{d_{k-1}}S_{k-1}\right)+_{c_k\frac{a_k}{d_k}}\frac{a_k}{d_k}\mathbb{N}$. By using (14), we recover the following result.

Lemma 7. (Leher [19, Corollary 8].) *Let $n\ge 2$ and $(a_1,a_2,\dots,a_n)$ be a smooth sequence. Let S be the semigroup generated by $a_1,\dots,a_n$. We have*

$$\mathrm{P}_S(x)=(1-x)\prod_{i=2}^{n}(1-x^{c_ia_i})\prod_{i=1}^{n}(1-x^{a_i})^{-1},$$

which factorizes as

$$\mathrm{P}_S=\Phi_1\prod_{d\mid a_1}\Phi_d^{-1}\prod_{i=2}^{n}\prod_{d\mid c_ia_i,\ d\nmid a_i}\Phi_d. \tag{15}$$

From this lemma we recover the following well known facts.

Corollary 4.

a) $\mathrm{F}(S)=\sum_{i=2}^n c_ia_i-\sum_{i=1}^n a_i$ *(this formula can also be derived from* [14] *or* [10]*).*
b) *S is symmetric.*
c) *S is cyclotomic.*

Special families of free numerical semigroups are the telescopic ones (free with respect to the arrangement $n_1 < n_2 < \cdots < n_t$, [16]) and numerical semigroups associated to irreducible plane curve singularities ([29]).

Example 5 (Binomial semigroups). Consider $B_m(a,b) := \langle a^m, ba^{m-1}, \ldots, b^{m-1}a, b^m\rangle$, where $a, b > 1$ are relatively prime. Putting $a_k = a^{m-k}b^k$, $k \in \{0, \ldots, m\}$, we see that the sequence $(a_0, \ldots, a_m)$ is smooth (with $c_k = a$ for $k \in \{1, \ldots, m\}$ and $c_k a_k = ba_{k-1} \in \langle a_0, \ldots, a_{k-1}\rangle$). By Corollary 4 it follows that

$$F(B_m(a,b)) = \sum_{k=1}^{m} a^{m+1-k}b^k - \sum_{k=0}^{m} a^{m-k}b^k.$$

Further, we have

$$\mathrm{P}_{B_n(a,b)}(x) = (1-x)\prod_{k=1}^{m}(1-x^{a^{m+1-k}b^k})\prod_{k=0}^{m}(1-x^{a^{m-k}b^k})^{-1}.$$

In particular, let $B = B_n(p,q)$ be a binomial semigroup with p and q two different primes. From (15) we infer that

$$\begin{aligned} \mathrm{P}_B &= \Phi_1(\Phi_1\Phi_p\cdots\Phi_{p^n})^{-1}\prod_{k=1}^{n}\prod_{j=0}^{k}\Phi_{p^{n+1-k}q^j}; \\ &= \prod_{k=1}^{n}\prod_{j=1}^{k}\Phi_{p^{n+1-k}q^j} = \prod_{l=2}^{n+1}\prod_{\substack{i+j=l\\1\le i,j\le l}}\Phi_{p^iq^j}. \end{aligned}$$

By Lemma 6 we see that P_B is of depth $d = p^{n+1}q^{n+1}$ and of height $h = 1$.

4. Cyclotomic exponents

The reader might wonder whether the expression in the right hand side of (14) is unique. It is easy to see the answer is yes and indeed a little more can be shown, see Moree [20, Lemma 1].

Lemma 8. *Let $f(x) = 1 + a_1x + \cdots + a_dx^d \in \mathbb{Z}[x]$ be a polynomial of degree d (hence $a_d \neq 0$). Let $\alpha_1, \ldots, \alpha_d$ be its roots. Put $s_f(k) = \alpha_1^{-k} + \cdots + \alpha_d^{-k}$. Then the numbers $s_f(k)$ are integers and satisfy the recursion*

$$s_f(k) + a_1s_f(k-1) + \cdots + a_{k-1}s_f(1) + ka_k = 0,$$

with $a_k = 0$ for every $k > d$. Put

$$b_f(k) = \frac{1}{k}\sum_{d|k} s_f(d)\mu\left(\frac{k}{d}\right).$$

Then $b_f(k)$ is an integer. Moreover, we have the formal identity

$$1 + a_1x + \cdots + a_dx^d = \prod_{j=1}^{\infty}(1-x^j)^{b_f(j)}.$$

It is a consequence of this lemma that given a numerical semigroup S, there are unique integers $e_1, e_2, \ldots$ such that

$$\mathrm{P}_S(x) = \prod_{j=1}^{\infty}(1-x^j)^{e_j}.$$

The sequence $\mathbf{e} = \{e_1, e_2, \ldots\}$ we call *the cyclotomic exponent sequence of S.*

Problem 4. *Relate the properties of S to its cyclotomic exponent sequence.*

By Lemma 10 we have $e_1 = 0$ if $S = \langle 1\rangle$ and $e_1 = 1$ otherwise.

Lemma 9. *A numerical semigroup S has a cyclotomic exponent sequence with finitely many nonzero terms if and only if S is a cyclotomic numerical semigroup.*

Proof. *Necessity.* We can write $\mathrm{P}_S(x)=\prod_{j=1}^{k}(1-x^j)^{e_j}$ for some k and hence $\mathrm{P}_S(x)$ has only roots of unity as zeros and so S is a cyclotomic numerical semigroup.
Sufficiency. Use Lemma 5 and formula (3). □

Write $S=\langle n_1,n_2,\dots,n_e\rangle$, with $e=e(S)$ and $0<n_1<\cdots<n_e$. Note that

$$(1-x^{n_1})(1-x^{n_2})\cdots(1-x^{n_e})=\sum_{j_1=0}^{1}\sum_{j_2=0}^{1}\cdots\sum_{j_e=0}^{1}(-1)^{j_1+j_2+\cdots+j_e}x^{j_1n_1+j_2n_2+\cdots+j_en_e}.$$

We can thus write

$$\mathrm{P}_S(x)=\frac{1-x}{(1-x^{n_1})\cdots(1-x^{n_e})}\left(\sum_{j_1=0}^{1}\sum_{j_2=0}^{1}\cdots\sum_{j_e=0}^{1}(-1)^{j_1+j_2+\cdots+j_e}x^{j_1n_1+j_2a_2+\cdots+j_en_e+S}\right),\tag{16}$$

where $m+S:=\{m+s\mid s\in S\}$. We can rewrite (16) as

$$\mathrm{P}_S(x)=\frac{1-x}{(1-x^{n_1})\cdots(1-x^{n_e})}\sum_n\chi_S(n)x^n.\tag{17}$$

Note that $\sum_n\chi_S(n)x^n$ is a (finite) polynomial since, for every $n>\mathrm{F}(S)+n_1+\cdots+n_e$, n can be written as $\sum u_ia_i$ with $u_i\geq 1$ for $1\leq i\leq e(S)$ and hence $\chi_S(n)=0$; this recovers the formula given in [26]. Alternatively, this can be seen by noting that $\sum_n\chi_S(n)x^n$ is the product of the polynomials $P_S(x)$ and $(1-x^{n_1})\cdots(1-x^{n_e})/(1-x)$.

Remark 1. As a first step in proving our conjecture, the following can be shown. Let μ be the minimum of $\{n>1\mid\chi_S(n)\neq 0\}$ and let $\mathfrak{d}(n)$ be the *denumerant* of n, that is, the number of different ways in which n can be written as sum of the generators n_i. If $\mu>n_e$ and if there is $s\in S$, $s\leq n_e$ with $\mathfrak{d}(s)\geq 2$, then there exist $1<d_1<d_2<\cdots<d_k$ and $e_i\geq 1$, $i=1,\dots,k$ (with $\sum_{i=1}^k e_i=e-1$) such that

$$\mathrm{H}_S(x)=\frac{(1-x^{d_1})^{e_1}\cdots(1-x^{d_k})^{e_k}}{(1-x^{n_1})\cdots(1-x^{n_e})}.\tag{18}$$

Note that the above conditions are rather restrictive. However, solely from the factorization (18), it is easy to prove that $d_i\in S$, $\mathfrak{d}(d_i)\geq 2$ for all $i=1,\dots,k$ and $d_1=\min\{s\mid s\in\mathrm{Betti}(S)\}$.

5. Semigroup polynomial divisors of x^n-1

Various authors studied the coefficients of divisors of x^n-1 [9, 15, 22, 25, 27]. Note that if a divisor $f(x)$ of x^n-1 is of the form $\mathrm{P}_S(x)$ we immediately know that its nonzero coefficients alternate between 1 and -1, hence it is of some interest to find divisors of x^n-1 that are semigroup polynomials.

We start with considering Problem 2 for height $h=1$. We will need the following trivial observation.

Lemma 10. *If* $S\neq\langle 1\rangle$, *then* $\mathrm{P}_S(x)\equiv 1-x \pmod{x^2}$.

Proof. If $S\neq\langle 1\rangle$, then $0\in S$ and $1\notin S$ and hence $\sum_{s\in S}x^s\equiv 1\pmod{x^2}$. □

Theorem 2. *Let p,q and r be distinct primes. Suppose S is cyclotomic of depth $d=pqr$ and height $h=1$. Then $S=\langle pr,q\rangle$ or one of its cyclic permutations.*

Proof. Suppose that $\mathrm{P}_S(x)|x^{pqr}-1$ for some S. Then by (2) and Lemma 3 we have $\mathrm{P}_S=\Phi_{pq}^{k_1}\Phi_{qr}^{k_2}\Phi_{pr}^{k_3}\Phi_{pqr}^{k_4}$ with $0\leq k_i\leq 1$. Since the problem is symmetric in p,q and r, we may assume without loss of generality that $k_1\geq k_2\geq k_3$. Note that, modulo x^2, $f(x)=1+(k_4-k_1-k_2-k_3)x$. On invoking Lemma 10 we now deduce that $(k_1,k_2,k_3,k_4)\in\{(1,0,0,0),(1,1,0,1)\}$. The first case we can exclude, as this leads to a depth $d=pq$. By (12) we have $\Phi_{pq}\Phi_{qr}\Phi_{pqr}=\mathrm{P}_{\langle pr,q\rangle}$. □

Theorem 3. *Suppose T is a cyclotomic numerical semigroup of depth $d=p^nq$ and height $h=1$. Then $T=\langle p^n,q\rangle$.*

The proof makes use of the following lemma.

Lemma 11. *Let $k \geq 1$ be an integer, $0 \leq e_i \leq 1$ ($i \in \{1,\ldots,k-1\}$) arbitrary and $e_k = 1$. Suppose that*

$$\Phi_{pq}^{e_1}\Phi_{p^2q}^{e_2}\cdots\Phi_{p^kq}^{e_k} = \mathrm{P}_T, \tag{19}$$

with T a numerical semigroup. Then $e_i = 1$ for $1 \leq i \leq k$ and $T = S(p^k, q)$.

Proof. In case $e_i = 1$ for $1 \leq i \leq k$ the identity (19) holds with $S = \langle p^k, q\rangle$ by (12) with $a_1 = p^k$ and $a_2 = q$. Since, modulo x^2, $\Phi_{p^mq} = 1$ for $m \geq 2$ and $\Phi_{pq} = 1 - x$, we infer that $e_1 = 1$. Suppose now we are not in the case where $e_i = 1$ for $1 \leq i \leq k$, hence the largest integer j_1 with $e_{j_1} = 1$ satisfies $1 \leq j_1 < k$. We let j_2 be the smallest integer such that $j_2 > j_1$ and $e_{j_2} = 1$. Since $e_k = 1$, j_2 exists. We now rewrite the left hand side of (19) as

$$\mathrm{P}_{\langle p^{j_1},q\rangle}(x)\Phi_{pq}(x^{p^{j_2}})^{e_{j_2}}\cdots\Phi_{pq}(x^{p^k})^{e_k},$$

which by (4) equals, modulo $x^{p^{j_2}+1}$,

$$\mathrm{P}_{\langle p^{j_1},q\rangle}(x)(1 - x^{p^{j_2}}).$$

From this and (19) we infer that

$$\sum_{s\in S(p^{j_1},q)} x^s(1 - x^{p^{j_2}}) \equiv \mathrm{H}_T(x) \pmod{x^{p^{j_2}+1}}.$$

It follows that $p^{j_1} \in T$ and $p^{j_2} \notin T$ and hence T is not a numerical semigroup, contradicting our assumption. □

Proof of Theorem 3. By (2) with $m = p^nq$ and Lemma 3 we deduce that

$$\mathrm{P}_T = \Phi_{pq}^{e_1}\Phi_{p^2q}^{e_2}\cdots\Phi_{p^nq}^{e_n}, \tag{20}$$

with $0 \leq e_i \leq 1$. Since, modulo x^2, $\Phi_{p^iq} = 1$ for $i \geq 2$ and $\Phi_{pq} = 1 - x$, we infer that $e_1 = 1$. Note that $e_n = 1$, for otherwise $d | p^{n-1}q$. The proof is concluded with the help of Lemma 11. □

6. Polynomially related Hilbert series

We say that two numerical semigroups S and T are *polynomially related* if there exists $f(x) \in \mathbb{Z}[x]$ and an integer $w \geq 1$ such that

$$\mathrm{H}_S(x^w)f(x) = \mathrm{H}_T(x). \tag{21}$$

From (21) we infer that

$$\mathrm{P}_S(x^w)f(x) = \mathrm{P}_T(x)(1 + x + \cdots + x^{w-1}). \tag{22}$$

Note that (21) and (22) are equivalent formulations of S and T being polynomially related.

Example 6. Put $S_1 = \langle p, q\rangle$ and $S_3 = \langle p^3, q\rangle$. By (12) we have $\Phi_{pq}\Phi_{p^2q}\Phi_{p^3q} = \mathrm{P}_{S_3}$. Recall that $\mathrm{P}_{S_1} = \Phi_{pq}$. We have

$$\mathrm{P}_{S_1}\Phi_{p^2q}\Phi_{p^3q} = \mathrm{P}_{S_3},\ \mathrm{P}_{S_1}(x^p)\Phi_{pq}\Phi_{p^3q} = \mathrm{P}_{S_3},\ \mathrm{P}_{S_1}(x^{p^2})\Phi_{pq}\Phi_{p^2q} = \mathrm{P}_{S_3},$$

giving three different polynomial relations between S_1 and S_3.

Problem 5. *Find necessary and sufficient conditions for S and T to be polynomially related.*

In proving the following result we make repeatedly use of the fact that $P_S(1) = 1$ and $P_S'(1) = \mathrm{g}(S)$, where $\mathrm{g}(S)$ denotes the number of gaps in S, the *genus* of S (this follows from (1)).

Lemma 12. *Suppose that* (21) *holds with S, T numerical semigroups. Then*

a) $f(0) = 1$.
b) $f(1) = w$.
c) $f'(1) = w(\mathrm{g}(T) - w\mathrm{g}(S) + (w-1)/2)$.
d) $\mathrm{F}(T) = w\mathrm{F}(S) + \deg(f)$.

Proof.
a) We have $\mathrm{P}_S(0)=\mathrm{P}_T(0)=1$.
b) On substituting $x=1$ in the identity (22) and noting that $\mathrm{P}_S(1)=\mathrm{P}_T(1)=1$, we obtain $f(1)=w$.
c) The identity (22) yields (on differentiating both sides) that

$$\mathrm{P}_S'(x^w)wx^{w-1}f(x)+P_S(x^w)f'(x)=\mathrm{P}_T'(x)(1+x+\cdots+x^{w-1})+\mathrm{P}_T(x)\sum_{j=0}^{w-2}(j+1)x^j.$$

The claim now easily follows on setting $x=1$ and invoking part b.
d) Use that $\deg(\mathrm{P}_S)=\mathrm{F}(S)+1$. □

Lemma 13. *Being polynomially related defines a partial order on the numerical semigroups.*

Proof. Obviously a numerical semigroup is polynomially related with itself. Further being polynomially related is clearly transitive. Using part d of Lemma 12 we see that $\mathrm{F}(S)<\mathrm{F}(T)$ unless $S=T$. This implies that being polynomially related defines an antisymmetric binary relation on the numerical semigroups. □

Lemma 14. *Suppose that S and T are numerical semigroups. Then $\mathrm{H}_S(x^w)f(x)=\mathrm{H}_T(x)$ for some integer $w\geq 1$ and $f\in\mathbb{N}[x]$ if and only if there are $0=e_1<e_2<\cdots<e_w$ such that $f(x)=\sum_{i=1}^w x^{e_w}$ and every $t\in T$ can be written in a unique way as*

$$t=e_i+s\cdot w,\ 1\leq i\leq w,\ s\in S.$$

Proof. Necessity. If f were to have a coefficient greater than 1, this would lead to a coefficient greater than 1 in H_T, which is not possible. By Lemma 12 we have $f(0)=0$ and $f(1)=w$, and hence it follows that $f(x)=\sum_{i=1}^w x^{e_w}$ with $0=e_1<\cdots<e_w$. The identity $\sum_{i=1}^w x^{e_w}\sum_{s\in S}x^{sw}=\mathrm{H}_T(x)$, yields that every element $t\in T$ can be written as $t=e_i+s\cdot w$, with $1\leq i\leq w$ and $s\in S$. Since every nonzero coefficient of H_T is 1, this writing way of t must be unique.

Sufficiency. Obvious. □

Remark 2. By Lemma 12 we have $\sum_{i=1}^w e_i=w(\mathrm{g}(T)-w\mathrm{g}(S)+(w-1)/2)$.

Let us write $S\leq_P T$ if S and T are polynomially related.

Corollary 5.

a) We have $\langle p^a,q^b\rangle\leq_P\langle p^m,q^n\rangle$ if $1\leq a\leq m$ and $1\leq b\leq n$.
b) We have $\langle p^a,q^b\rangle\leq_P B_n(p,q)$ if $a,b\geq 1$ and $2\leq a+b\leq n+1$.
c) Let V be a numerical semigroup generated by $\{n_1,\ldots,n_k\}$. Let $d=\gcd(n_1,\ldots,n_{k-1})$ and set $U=S(n_1/d,\ldots,n_{k-1}/d,n_k)$. The numerical semigroups U and V are polynomially related.

Proof.
a) This is a consequence of the identity

$$\mathrm{P}_{S(p^m,q^n)}(x)=\prod_{\substack{1\leq\alpha\leq m\\1\leq\beta\leq n}}\Phi_{p^\alpha q^\beta}(x),\tag{23}$$

which is a consequence of (12).
b) Results on comparing (23) with the factorization of P_B given in Example 5.
c) It is easy to see (cf. [24, Lemma 2.16]) that $\mathrm{Ap}(V;n_k)=d\,\mathrm{Ap}(U;n_k)$. By using this identity and (9) we derive

$$\mathrm{H}_U(x^d)\left(\frac{1-x^{n_kd}}{1-x^{n_k}}\right)=\mathrm{H}_V(x).$$ □

6.1. **An application.** We will use our insights in polynomially related Hilbert series to establish the following result.

Theorem 4. *Let $p \neq q$ be primes and m, n positive integers. The quotient*

$$Q(x) := \mathrm{P}_{\langle p^m, q^n\rangle}(x)/\Phi_{p^m q^n}(x)$$

is monic, is in $\mathbb{Z}[x]$ and has constant coefficient 1. Its nonzero coefficients alternate between 1 and -1.

Proof. On using that $\mathrm{P}_S(x) = (1-x)\mathrm{H}_S(x)$ and the identity (12), we infer that

$$\mathrm{H}_{\langle p^m, q^n\rangle}(x) = \mathrm{H}_{\langle p,q\rangle}(x^{p^{m-1}q^{n-1}}) \sum_{j=0}^{q^{n-1}-1} x^{jp^m} \sum_{k=0}^{p^{m-1}-1} x^{kq^n}. \tag{24}$$

The identity (23) yields that $Q(x)$ is a polynomial in $\mathbb{Z}[x]$. On noticing that

$$\mathrm{P}_{\langle p,q\rangle}(x^{p^{m-1}q^{n-1}}) = \Phi_{p^m q^n}(x)$$

we obtain from (24) that

$$Q(x) = \frac{1-x}{1-x^{p^{m-1}q^{n-1}}} \sum_{j=0}^{q^{n-1}-1} x^{jp^m} \sum_{k=0}^{p^{m-1}-1} x^{kq^n}.$$

The set

$$\{\alpha p^m + \beta q^n \mid 0 \le \alpha \le q^{n-1}-1,\ 0 \le \beta \le p^{m-1}-1\}$$

forms a complete residue system modulo $p^{m-1}q^{n-1}$ and it follows that around $x=0$ we can write $Q(x) = (1-x)\sum_{s\in S'} x^s$ for some set S' containing zero and all large enough integers. From this it follows that $Q(x)$ is a monic polynomial and that the nonzero coefficients of $Q(x)$ alternate between 1 and -1. □

Remark 3. An alternative, much more conceptual proof of the identity (24) is obtained on using the following lemma; one notes that on writing down the Hilbert series for both sides of (25), we obtain the identity (24).

Lemma 15. *Let $T = \langle p^m, q^n\rangle$ and $S = \langle p, q\rangle$. Every element of T can be uniquely written as*

$$t = \alpha p^m + \beta q^n + s p^{m-1} q^{n-1}, 0 \le \alpha \le q^{n-1}-1, 0 \le \beta \le p^{m-1}-1, s \in S. \tag{25}$$

Proof. Suppose that $t \in T$. Then

$$t = ap^m + bq^n = (q^{n-1}a_1 + \alpha)p^m + (p^{m-1}b_1 + \beta)q^n, \tag{26}$$

with $0 \le \alpha \le q^{n-1}-1$ and $0 \le \beta \le p^{m-1}-1$. Put $s = a_1 p + b_1 q$. Clearly $s \in S$. From (26) we then infer that $t = \alpha p^m + \beta q^n + s p^{m-1} q^{n-1}$, as required. The congruence class of t modulo $p^{m-1}q^{n-1}$ determines α and β uniquely. Since α and β are determined uniquely, so is s. □

Theorem 4 can be alternatively proven on invoking the following more general result together with Lemma 15.

Theorem 5. *Suppose that S and T are numerical semigroups with $\mathrm{H}_S(x^w)f(x) = \mathrm{H}_T(x)$ for some $w \ge 1$ and $f \in \mathbb{N}[x]$. Put $Q(x) = \mathrm{P}_T(x)/\mathrm{P}_S(x^w)$. Then $Q(0) = 1$, $Q(x)$ is a monic polynomial and its nonzero coefficients alternate between 1 and -1.*

Proof. By Lemma 14 we can write $f(x) = \sum_{i=1}^{w} x^{e_i}$. Since T contains all integers sufficiently large, it follows that $e_1, \ldots, e_w$ forms a complete residue system modulo w. By (22) we see that

$$Q(x) = \frac{f(x)}{1-x^w}(1-x).$$

Around $x = 0$ we have $f(x)/(1-x^w) = \sum_{z\in Z} x^z$ for some infinite set of integers Z. Since $e_1, \ldots, e_w$ forms a complete residue system modulo w, it follows that all integers large enough are in Z. From this we then infer that $Q(x)$ is a monic polynomial. Note that $Q(0) = f(0) = 1$ by Lemma 12 and so $0 \in Z$. For any set $Z' \subseteq \mathbb{N}$ containing zero the nonzero coefficients in $(1-x)\sum_{z\in Z'} x^z$ alternate between 1 and -1. □

6.2. **Gluings and polynomially related numerical semigroups.** Assume that S is a gluing of two numerical semigroups, say $S = a_1S_1 +_{a_1a_2} a_2S_2$. The following lemma is inspired by [12, Lemma 8.(2)].

Lemma 16. *Every element s in $S = a_1S_1 +_{a_1a_2} a_2S_2$ can be written uniquely as $s = a_1w_1 + a_2s_2$, with $w_1 \in \mathrm{Ap}(S_1; a_2)$ and $s_2 \in S_2$.*

Proof. First proof. Using (10) and (9) we obtain

$$\mathrm{H}_S(x) = \mathrm{H}_{S_2}(x^{a_2}) \sum_{w \in \mathrm{Ap}(S_1;a_2)} x^{a_1 w}. \tag{27}$$

This shows that each $s \in S$ can be written in the indicated way. Since the nonzero coefficients of all monomials in $\mathrm{H}_S(x)$ are 1, the representation is unique.

Second proof. Let $s \in S$. Then there exists $t_1 \in S_1$ and $t_2 \in S_2$ such that $s = a_1t_1 + a_2t_2$. As $t_1 \in S_1$ and $a_2 \in S_1$, $t_1 = w_1 + ka_2$ for some $w_1 \in \mathrm{Ap}(S_1; a_2)$ and $k \in \mathbb{N}$ (this is easy to deduce, and a proof can be found in [24, Lemma 2.6]). Hence $s = a_1w_1 + a_2(t_2 + ka_1)$. Observe that $a_1 \in S_2$, and consequently $t_2 + ka_1 \in S_2$.

Now assume that $a_1w_1 + a_2s_2 = a_1w_1' + a_2s_2'$ with $w_1, w_1' \in \mathrm{Ap}(S_1; a_2)$ and $s_2, s_2' \in S_2$. Assume without loss of generality that $w_1 \geq w_1'$. Then $a_1(w_1 - w_1') = a_2(s_2' - s_2)$. Since $\gcd(a_1, a_2) = 1$, this forces $a_2 | w_1 - w_1'$. But then $w_1 = w_1' + ka_2$ for some $k \in \mathbb{N}$. The fact $w_1, w_1' \in \mathrm{Ap}(S_1; a_2)$ yields $k = 0$, and hence $w_1 = w_1'$ and $s_2 = s_2'$. □

The following lemma is an immediate consequence of (27). It can also be proved (without using Hilbert series) on invoking Lemma 16 and Lemma 14. (Observe that $a_1 \mathrm{Ap}(S_1; a_2)$ is a complete system modulo a_2 since $\gcd(a_1, a_2) = 1$. Thus $w = a_2$ (or a_1 by symmetry) and $\{e_1, \ldots, e_w\}$ in Lemma 14 is $a_1 \mathrm{Ap}(S_1; a_2)$.)

Theorem 6. *Assume that S is a gluing of two numerical semigroups, say $S = a_1S_1 +_{a_1a_2} a_2S_2$, then $S_1 \leq_P S$ and $S_2 \leq_P S$ with a relating polynomial $f \in \mathbb{N}[x]$.*

Notice that Theorem 6 implies that part d of Lemma 12 is a generalization of Delorme's formula for the conductor of a gluing [10, 10. Proposition (i)] (the conductor is just the Frobenius number plus one). As a consequence of the Theorems 5 and 6 we obtain the following.

Corollary 6. *Assume that S is a gluing of two numerical semigroups, say $S = a_1S_1 +_{a_1a_2} a_2S_2$. Put $Q_i(x) = \mathrm{P}_S(x)/\mathrm{P}_{S_i}(x^{a_i})$ for $i = 1, 2$. Then $Q_i(0) = 1$, $Q_i(x)$ is a monic polynomial and its nonzero coefficients alternate between 1 and -1.*

Lemma 17. *Let S and T be numerical semigroups with $wS \subseteq T$ for some positive integer w. Assume that there are nonnegative integers $e_1, \ldots, e_w$ such that $T = \{e_1, \ldots, e_w\} + wS$ and for every $t \in T$, $t = e_i + ws$ for some $i \in \{1, \ldots, w\}$ and $s \in S$. Let $u = \gcd(e_1, \ldots, e_w)$ and $U = \langle w, e_1/u, \ldots, e_w/u \rangle$. If $u \in S$, then $T = uU +_{uv} wS$.*

Proof. It suffices to prove that $uU \subseteq T$. But $uw \in wS \subseteq T$ and $ue_i/u = e_i \in \{e_1, \ldots, e_w\} + wS = T$. □

Theorem 7. *Let S and T be two numerical semigroups. Assume that $S \leq_P T$, with $\mathrm{H}_S(x^w)f(x) = \mathrm{H}_T(x)$ for some $w \in \mathbb{N}$ and some polynomial $f \in \mathbb{N}[x]$. Let u be the greatest common divisor of the exponents of f. If u is in S, then there exists a numerical semigroup U such that*

$$T = uU +_{uw} wS.$$

Proof. The proof follows from Lemmas 14 and 17. □

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