Explicit equivariant quantization on coadjoint orbits of $GL(n,\mathbb{C})$

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Abstract

We present an explicit $\mathcal{U}_h\big(gl(n,\mathbb{C})\big)$-equivariant quantization on coadjoint orbits of $GL(n,\mathbb{C})$. It forms a two-parameter family quantizing the Poisson pair of the reflection equation and Kirillov-Kostant-Souriau brackets.

1 Introduction

Let G be a reductive complex connected Lie group with Lie algebra $\mathfrak{g}$. Let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and $\mathcal{U}_h(\mathfrak{g})$ the corresponding quantum group, [Dr]. We consider the problem of $\mathcal{U}_h(\mathfrak{g})$-equivariant quantization on $\mathfrak{g}^*$ and on coadjoint orbits of G.

By a $\mathcal{U}_h(\mathfrak{g})$-equivariant quantization on a G-manifold M we mean a two-parameter family $\mathcal{A}_{t,h}$ of $\mathcal{U}_h(\mathfrak{g})$-module algebras such that $\mathcal{A}_{0,0} = \mathcal{A}(M)$, the function algebra on M, and the $\mathcal{U}_h(\mathfrak{g})$-action on $\mathcal{A}_{t,h}$ is an extension of the $\mathcal{U}(\mathfrak{g})$-action on $\mathcal{A}(M)$. Briefly, we refer to such a quantization as (t,h)-quantization while to its $(0,h)$- and $(t,0)$-subfamilies as h- and t-quantizations. This definition implies that the family $\mathcal{A}_{t,0}$ is G-equivariant.

At the infinitesimal level, a (t,h)-quantization gives rise to a pair of compatible Poisson brackets. The one along the parameter h is represented as a difference of bivector fields, $f - r_M$, [D2], where f is a G-invariant one and r_M is induced on M via the group action by the r-matrix $r \in \wedge^2\mathfrak{g}$ corresponding to $\mathcal{U}_h(\mathfrak{g})$. The Poisson bracket along the parameter t is G-invariant.

In the paper, as G-manifolds we consider $\mathfrak{g}^*$ and coadjoint orbits, as a G-invariant Poisson bracket along t we take the Poisson-Lie bracket on $\mathfrak{g}^*$ and its restriction to an orbit; the latter is known as the Kirillov-Kostant-Souriau (KKS) bracket. As was shown in [D1], a (t,h)-quantization on $\mathfrak{g}^*$ exists only for $\mathfrak{g}$ with simple components being $sl(n,\mathbb{C})$. So we assume $\mathfrak{g} = gl(n,\mathbb{C})$ in the paper.

Firstly, we show that the quantization on $\mathfrak{g}^*$ is given by the so-called extended reflection equation (ERE) algebra, $\mathcal{L}_{t,h}$. At $(t,h) = (0,0)$, it is the polynomial algebra on $\mathfrak{g}^*$ (or symmetric algebra of $\mathfrak{g}$). The ordinary or quadratic reflection equation (RE) algebra $\mathcal{L}_h$ was studied in [KSkl, KS]. The ERE algebra naturally appears in non-commutative differential calculus related to quantum groups, see e.g. [IP].

Secondly, using the generalized Verma module over $\mathcal{U}(\mathfrak{g})$, we show that the ERE algebra can be restricted to any semisimple orbit in $\mathfrak{g}^*$. By the restriction we mean the following. Let $\{f\} \subset \mathcal{A}(\mathfrak{g}^*)$ be a set of functions generating the ideal of the orbit O. Then, there exist their extensions $\{f_{t,h}\} \subset \mathcal{L}_{t,h}$ generating an ideal in $\mathcal{L}_{t,h}$; the quotient of $\mathcal{L}_{t,h}$ by that ideal gives a flat deformation over $\mathbb{C}[[t,h]]$.

Thirdly, we give explicit formulas for the generators $\{f_{t,h}\}$.

Let us describe our approach in more detail. In the classical case, an orbit of rank $k-1$ is specified by a matrix polynomial equation,

$$(X-\mu_1)\dots(X-\mu_k) = 0, \tag{1}$$

and the conditions on traces

$$\mathrm{Tr}(X^m) = \sum_{i=1}^{k} n_i \mu_i^m, \quad m = 1, \dots, k-1,$$

where n_i are non-negative integers such that $\sum_{i=1}^{k} n_i = n$. This orbit consists of matrices with eigenvalues μ_i of multiplicities n_i, $i = 1, \dots, k$. The ideal of the orbit is generated by the $n \times n$ entries of the matrix polynomial and $k-1$ functions involving traces. Matrix polynomial equation (1) defines a G-invariant subvariety M_μ^k, which is a finite collection of rank $< k$ orbits at generic μ, namely, when $\mu_i \neq \mu_j$ if $i \neq j$. The condition on traces defines a character of the subalgebra $\mathcal{S}$ of invariants restricted to M_μ^k. This restriction is a finite dimensional semisimple algebra and it has a finite spectrum whose points correspond to orbits.

We extend this picture to the quantum case. It turns out that the quotient of the algebra $\mathcal{L}_{t,h}$ by the relations

$$(L-\mu_1)\dots(L-\mu_k) = 0,$$

where L is the matrix whose entries are generators of $\mathcal{L}_{t,h}$, gives a (t,h)-quantization, $\mathcal{A}_{t,h}(M_\mu^k)$, on M_μ^k. Moreover, the quotient of $\mathcal{L}_{t,h}$ by the ideal generated by entries of *any* polynomial in L is a flat algebra over $\mathbb{C}[[t,h]]$. The first part of the paper is devoted to the proof of this statement.

With the matrix polynomial condition imposed, the equations on traces have to be deformed in a consistent way. The quantum version of these equations involves the quantum trace Tr_q, where $q = e^h$, that is the quantum analog of the ordinary trace, [FRT]. It is defined in such a way that the elements $\mathrm{Tr}_q(L^m)$, $m \in \mathbb{N}$, are $\mathcal{U}_h(\mathfrak{g})$-invariant. The first n traces $\mathrm{Tr}_q(L^m)$, $m = 1, \ldots, n$, generate the center of the algebra $\mathcal{L}_{t,h}$. The problem of restricting the quantization $\mathcal{A}_{t,h}(M_\mu^k)$ to particular orbits in M_μ^k reduces to the problem of computing the values of quantum traces on them. The second part of the paper is devoted to solution of this problem.

Let us illustrate our results on the simplest example of symmetric (rank one) orbits. The quantized orbit of matrices with eigenvalues μ_1, μ_2 of multiplicities n_1, n_2 is the quotient of the ERE algebra $\mathcal{L}_{t,h}$ by the relations

$$\begin{aligned} (L-\mu_1)(L-\mu_2) &= 0, \\ \mathrm{Tr}_q(L) &= \hat{n}_1\mu_1 + \hat{n}_2\mu_2 + t\hat{n}_1\hat{n}_2. \end{aligned}$$

In the paper, we define $\hat{m}$ as $\frac{1-q^{-2m}}{1-q^{-2}}$ for $m \in \mathbb{Z}$ and normalize the quantum trace so that $\mathrm{Tr}_q(1) = \hat{n}$. Note that these particular formulas for symmetric orbits were derived in our paper [DM2] by different methods.

As a limit case of our quantization, we come, putting $h = 0$, to an explicit quantization of the KKS bracket. The problem of quantizing this Poisson structure was put forward in [BFFLS]. It is interesting that the G-invariant quantization of the KKS bracket is obtained with the substantial use of quantum groups.

Among other works relevant to our study, we would like to mention [Ast, Kar], where the Karabegov quantization with separation of variables is applied to building the quantum moment map. That construction establishes relations between G-equivariant quantizations and generalized Verma modules. A construction of the G-equivariant quantization on semisimple orbits by the generalized Verma modules is presented in [DGS].

None of the mentioned methods gives explicit formulas for quantized orbits. In [DGK], explicit formulas was given for quantizing $\mathbb{C}P^n$ type orbits. In our papers [DM1, DM2], we developed a method of quantization on orbits of small ranks and built an explicit quantization on symmetric and bisymmetric orbits of $gl(n, \mathbb{C})$. In the present paper, we generalize those formulas to all semisimple orbits.

Let us remark that our quantized orbits are relevant to the so-called fuzzy spaces considered in the physics literature (see, for example, [DolJ, PawSt] and references therein).

The paper is organized as follows. Section 2 deals with the t-quantization of the KKS bracket. There, we prove that the natural G-equivariant quantization on $\mathfrak{g}^*$ can be restricted to semisimple orbits. In Section 3 we generalize that approach to the (t,h)-quantization. Therein, we prove that the quotient of $\mathcal{L}_{t,h}$ by any matrix polynomial equation is a flat deformation over $\mathbb{C}[[t,h]]$. In Section 4, we study the subalgebra of invariants in $\mathcal{L}_{t,h}$ restricted to M_μ^k. We prove that it is a finite dimensional semisimple commutative algebra and compute its characters. The explicit formulas for the equivariant quantization on orbits are given in Section 5.

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2 G-equivariant quantization on orbits

2.1 Quantization on $\mathfrak{g}^*$

Let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of a complex Lie algebra $\mathfrak{g}$ with the Lie bracket $[\,.\,,.\,]$. Let G be a connected Lie group corresponding to $\mathfrak{g}$. We consider $\mathfrak{g}[t]$ as a Lie algebra over $\mathbb{C}[t]$ with respect to the bracket $[x,y]_t = t[x,y]$. Let $\mathcal{L}_t$ denote the universal enveloping algebra of $\mathfrak{g}[t]$. By definition, it is a quotient of the tensor algebra $\mathrm{T}(\mathfrak{g})[t]$ by the ideal generated by relations $x\otimes y - y\otimes x - t[x,y]$, $x,y\in\mathfrak{g}$. The algebra $\mathcal{L}_t$ is a G-equivariant quantization of the symmetric algebra of $\mathfrak{g}$ (considered as the polynomial algebra on $\mathfrak{g}^*$) with the Poisson-Lie bracket induced by $[\,.\,,.\,]$. The assignment $x\mapsto tx$, $x\in\mathfrak{g}$, defines a $\mathbb{C}[t]$-algebra morphism,

$$\phi_t\colon \mathcal{L}_t \to U(\mathfrak{g})[t], \tag{2}$$

which is obviously an embedding of free algebras over $\mathbb{C}[t]$.

2.2 Levi and parabolic subalgebras

Let $\mathfrak{g}$ be a complex reductive Lie algebra with the Cartan subalgebra $\mathfrak{h}$ and $\mathfrak{g}=\mathfrak{n}^-\oplus\mathfrak{h}\oplus\mathfrak{n}^+$ its polarization with respect to $\mathfrak{h}$.

We fix a Levi subalgebra $\mathfrak{l}$, which is, by definition, the centralizer of an element in $\mathfrak{h}$. The algebra $\mathfrak{l}$ is reductive, so it is decomposed into the direct sum of its center and the semisimple part, $\mathfrak{l}=\mathfrak{c}\oplus[\mathfrak{l},\mathfrak{l}]$. Also, there exists a decomposition

$$\mathfrak{g}=\mathfrak{n}_{\mathfrak{l}}^-\oplus\mathfrak{l}\oplus\mathfrak{n}_{\mathfrak{l}}^+, \tag{3}$$

where $\mathfrak{n}_{\mathfrak{l}}^\pm$ are subalgebras in $\mathfrak{n}^\pm$.

It is clear that $\mathfrak{c} \subset \mathfrak{h}$ and $\mathfrak{h} = \mathfrak{c} \oplus \mathfrak{h}_{\mathfrak{l}}$, where $\mathfrak{h}_{\mathfrak{l}} = \mathfrak{h} \cap [\mathfrak{l}, \mathfrak{l}]$. So, we have a projection $\mathfrak{h} \to \mathfrak{c}$. On the other hand, the Cartan decomposition defines a projection

$$\pi \colon \mathfrak{g} \to \mathfrak{h}. \tag{4}$$

Composition of these maps gives the natural projection $\mathfrak{g} \to \mathfrak{c}$ which defines an embedding $\mathfrak{c}^* \to \mathfrak{g}^*$.

Taking an element $\mu \in \mathfrak{c}^*$, we consider the coadjoint orbit of G in $\mathfrak{g}^*$ passing through μ. For generic μ, this orbit is semisimple with $\mathfrak{l}$ being the Lie algebra of the stabilizer at the point μ. Denote by O^k, $k = \dim \mathfrak{c}$, the closure in $\mathfrak{g}^* \times \mathfrak{c}^*$ of this family of semisimple orbits and by O^k_μ the fiber over μ. It is known that the family O^k is flat over $\mathfrak{c}^*$. For generic $\mu \in \mathfrak{c}^*$ the variety O^k_μ is a semisimple orbit in $\mathfrak{g}^*$.

Let $\mathfrak{p}$ denote the parabolic subalgebra $\mathfrak{l} \oplus \mathfrak{n}_{\mathfrak{l}}^+$. Decomposition (3) turns into the decomposition

$$\mathfrak{g} = \mathfrak{n}_{\mathfrak{l}}^- \oplus \mathfrak{p}. \tag{5}$$

It is clear that characters (one-dimensional representations) of $\mathfrak{p}$ (as well as of $\mathfrak{l}$) are generated by linear forms from $\mathfrak{c}^*$.

2.3 Generalized Verma modules

For any $\mu \in \mathfrak{c}^*$, projection (4) defines a representation π_μ of $\mathfrak{p}$ on $\mathbb{C}$: $\pi_\mu(x) = \mu\big(\pi(x)\big)$, $x \in \mathfrak{p}$. It extends to a representation of $\mathcal{U}(\mathfrak{p})$, which we still denote by π_μ.

The generalized Verma $\mathcal{U}(\mathfrak{g})$-module corresponding to $\mu \in \mathfrak{c}^*$ is the left module $\mathcal{V}_\mu = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} \mathbb{C}$. In the tensor product, $\mathbb{C}$ is a $\mathcal{U}(\mathfrak{p})$-module with respect to π_μ, while $\mathcal{U}(\mathfrak{g})$ is considered as a right $\mathcal{U}(\mathfrak{p})$-module. The element $v_0 = 1 \otimes 1 \in \mathcal{V}_\mu$ is the highest weight vector. In particular, $xv_0 = \pi_\mu(x)v_0$ for $x \in \mathcal{U}(\mathfrak{p})$. Moreover, the map $\mathcal{U}(\mathfrak{n}_{\mathfrak{l}}^-) \to \mathcal{V}_\mu$, $x \mapsto xv_0$, is an isomorphism of vector spaces. So, all $\mathcal{V}_\mu$ are canonically isomorphic to $\mathcal{V} = \mathcal{U}(\mathfrak{n}_{\mathfrak{l}}^-)$ as vector spaces. The representation $\mathcal{V}_\mu$ is given by a homomorphism of algebras

$$\varphi_\mu \colon \mathcal{U}(\mathfrak{g}) \to \mathrm{End}(\mathcal{V}). \tag{6}$$

We treat $\mathcal{V}_\mu$ as belonging to the family of Verma modules parameterized by $\mathfrak{c}^*$. Namely, we consider trivial bundles over $\mathfrak{c}^*$ with the fibers $\mathcal{U}(\mathfrak{g})$ and $\mathrm{End}(\mathcal{V})$; then φ_μ is a map of their polynomial sections, which we denote by $\mathcal{U}(\mathfrak{g})[\mu]$ and $\mathrm{End}(\mathcal{V})[\mu]$. One can think of μ as a collection of formal coordinates on $\mathfrak{c}^*$, $\mu = (\mu_1, ..., \mu_k)$. Then, $\mathcal{U}(\mathfrak{g})[\mu]$ and $\mathrm{End}(\mathcal{V})[\mu]$ are polynomials in μ with values in $\mathcal{U}(\mathfrak{g})$ and $\mathrm{End}(\mathcal{V})$, respectively.

It is easy to check that the map $\varphi_{\mu/t}\circ\phi_t\colon \mathcal{L}_t \to \mathrm{End}(\mathcal{V})[[t]]$, where ϕ_t is given by (2), defines a map of polynomial sections over $\mathfrak{c}^*$,

$$\Phi_{t,\mu}\colon \mathcal{L}_t[\mu] \to \mathrm{End}(\mathcal{V})[[t]][\mu]. \tag{7}$$

Theorem 2.1 ([Ast, DGS]). *The image $\mathcal{A}_{t,\mu}$ of $\mathcal{L}_{t,\mu}$ in $\mathrm{End}(\mathcal{V})[[t]][\mu]$ with respect to $\Phi_{t,\mu}$ is a free module over $\mathbb{C}[[t]][\mu]$. For all $\mu\in\mathfrak{c}^*$, the algebra $\mathcal{A}_{t,\mu}$ specified to the fiber O^k_μ gives a G-equivariant quantization of the KKS bracket on it.*

In particular, this theorem gives G-equivariant quantizations on all semisimple orbits.

3 $U_h(\mathfrak{g})$-equivariant quantization on orbits

3.1 Extended reflection equation algebra and (t,h) quantization on $gl(n)^*$

We specialize our further considerations to the case $\mathfrak{g}=gl(n,\mathbb{C})$. The G-module $\mathfrak{g}^*$ is identified with $\mathrm{End}(\mathbb{C}^n)$ using the trace pairing. The multiplication on matrix units $\{e^i_j\}^n_{i,j=1}\subset \mathrm{End}(\mathbb{C}^n)$ is given by $e^i_j e^l_k=\delta^l_j e^i_k$, where δ^l_j is the Kronecker symbol. Let $S\in\mathrm{End}^{\otimes 2}(\mathbb{C}^n)$ be the Hecke symmetry related to the representation of $\mathcal{U}_h(\mathfrak{g})$ on $\mathbb{C}^n$. It satisfies the relations

$$S_{12}S_{23}S_{12} = S_{23}S_{12}S_{23}, \tag{8}$$

$$S^2-(q-q^{-1})S = 1\otimes 1,\quad q=e^h. \tag{9}$$

The symmetry S is expressed through the image R of the universal R-matrix of $\mathcal{U}_h(\mathfrak{g})$ and the flip operator P on $\mathbb{C}^n\otimes\mathbb{C}^n$, $S=PR$. Let $\{L^i_j\}\subset\mathrm{End}^*(\mathbb{C}^n)$ be the dual basis to $\{e^j_i\}$. Consider an associative algebra, $\mathcal{L}_{t,h}$, over $\mathbb{C}[[t,h]]$ generated by L^i_j subject to the quadratic-linear relations

$$SL_2SL_2-L_2SL_2S=qt\bigl(L_2S-SL_2\bigr). \tag{10}$$

Here, L is the matrix $\sum_{i,j}L^i_j e^j_i$ and $L_2=1\otimes L$. Note that at $h=0$ these relations give the algebra $\mathcal{L}_t$ defined in Subsection 2.1.

Let $\mathcal{R}$ be the universal R-matrix of the quantum group $\mathcal{U}_h(\mathfrak{g})$. Consider the element $\bar{\mathcal{Q}}=qt/(q-q^{-1})(\mathcal{Q}-1\otimes 1)$, where $\mathcal{Q}=\mathcal{R}_{21}\mathcal{R}$. It is easy to check that the map $\mathcal{L}_{t,h}\to\mathcal{U}_h(\mathfrak{g})[[t]]$, $L^i_j\mapsto L^i_j(\bar{\mathcal{Q}}_1)\bar{\mathcal{Q}}_2$ respects relations (10). So we have the homomorphism

$$\phi_{t,h}\colon\mathcal{L}_{t,h}\to\mathcal{U}_h(\mathfrak{g})[[t]]. \tag{11}$$

Proposition 3.1. *The map $\phi_{t,h}$ is an embedding of free $\mathbb{C}[[t,h]]$-modules.*

Proof. At the point $h=0$, the quotient of the tensor algebra $\mathrm{T}\big(\mathrm{End}^*(\mathbb{C}^n)\big)$ by relations (10) coincides with $\mathcal{L}_t$ and the map $\phi_{t,0}$ coincides with (2), so the proposition is a corollary of the following lemma. □

Lemma 3.2. *Let $\mathcal{E}$ and $\mathcal{F}$ be vector spaces over $\mathbb{C}$ and λ a set of formal parameters. Let $\psi_\lambda\colon \mathcal{E}[[\lambda]] \to \mathcal{F}[[\lambda]]$, $\psi_\lambda = \psi_0 \bmod \lambda$, be a map of free $\mathbb{C}[[\lambda]]$-modules and $\mathcal{K}_\lambda$ a submodule in $\mathcal{E}[[\lambda]]$ satisfying the conditions: a) $\mathcal{K}_\lambda \subset \ker\psi_\lambda$, b) $\mathcal{K}_0 = \ker\psi_0$, where $\mathcal{K}_0 = \{a \in \mathcal{E} | \exists f = a + \lambda b \in \mathcal{K}_\lambda\}$. Then, the $\mathbb{C}[[\lambda]]$-module $\mathcal{E}[[\lambda]]/\mathcal{K}_\lambda$ is free and the map $\mathcal{E}[[\lambda]]/\mathcal{K}_\lambda \to \mathcal{F}[[\lambda]]$ is an embedding.*

Proof. Clear. □

Corollary 3.3. *The algebra $\mathcal{L}_{t,h}$ is a $\mathcal{U}_h(\mathfrak{g})$-equivariant quantization on $gl^*(n,\mathbb{C})$.*

3.2 Double quantization on orbits in $gl^*(n,\mathbb{C})$

Let $\mathfrak{g} = \mathrm{End}(\mathbb{C}^n)$. As a Cartan subalgebra in $\mathfrak{g}$ we take diagonal matrices. As a Levi subalgebra $\mathfrak{l}$ we take the subspace of matrices commuting with a diagonal matrix with eigenvalues $\mu_1, ..., \mu_k$. Then $\mathfrak{c}$ consists of matrices commuting with $\mathfrak{l}$. Abusing notations, we will consider $(\mu_1, ..., \mu_k)$ as coordinates in $\mathfrak{c}^* \simeq \mathbb{C}^k$.

Since $\mathcal{U}_h(\mathfrak{g}) \simeq \mathcal{U}(\mathfrak{g})[[h]]$ as associative algebras, there is an extension,

$$\varphi_{h,\mu}\colon \mathcal{U}_h(\mathfrak{g}) \to \mathrm{End}(\mathcal{V})[[h]],$$

of map (6). This algebra homomorphism is automatically equivariant with respect to the adjoint actions of the Hopf algebra $\mathcal{U}_h(\mathfrak{g})$. It is easy to check that the map $\varphi_{h,\mu/t}\circ\phi_{t,h}\colon \mathcal{L}_{t,h} \to \mathrm{End}(\mathcal{V})[[t,h]]$, where $\phi_{t,h}$ is given by (11), defines a map of polynomial sections in μ,

$$\tilde{\Phi}_{t,h,\mu}\colon \mathcal{L}_{t,h}[\mu] \to \mathrm{End}(\mathcal{V})[[t,h]][\mu]. \tag{12}$$

Theorem 3.4. *The image $\tilde{\mathcal{A}}_{t,h,\mu}$ of $\mathcal{L}_{t,h}[\mu]$ in $\mathrm{End}(\mathcal{V})[[t,h]][\mu]$ with respect to $\tilde{\Phi}_{t,h,\mu}$ is flat over $\mathbb{C}[[t,h]][\mu]$. For a fixed $\mu \in \mathfrak{c}^*$, $\tilde{\mathcal{A}}_{t,h,\mu}$ gives a (t,h)-quantization on O^k_μ.*

Proof. The statement follows from Theorem 2.1 and Lemma 3.2. □

In the classical case, elements of O^k_μ satisfy the matrix equation

$$(X-\mu_1)...(X-\mu_k) = X^k + \xi_{k-1}(\mu)X^{k-1} + ... + \xi_0(\mu) = 0, \tag{13}$$

where coefficients ξ_i are symmetric functions of μ. Entries of the matrix X^m, $m \in \mathbb{N}$, generate a $\mathcal{U}(\mathfrak{g})$-submodule in $\tilde{\mathcal{A}}_{0,0,\mu}$ of $\mathrm{End}^*(\mathbb{C}^n)$-type, which contains a one-dimensional trivial module and a submodule of $sl(n,\mathbb{C})$-type. Moreover, invariants form a one dimensional submodule in $\tilde{\mathcal{A}}_{0,0,\mu}$, while the $sl(n,\mathbb{C})$-isotypic component has multiplicity $k-1$ and is generated by $\bar{X}^1, \ldots, \bar{X}^{k-1}$, where $\bar{X}^m$ is the $sl(n,\mathbb{C})$-type component in X^m.

Since $\tilde{\mathcal{A}}_{t,h,\mu}$ is a $\mathcal{U}_h(\mathfrak{g})$-equivariant $\mathbb{C}[[t,h]][\mu]$-flat quotient of $\mathcal{L}_{t,h}[\mu]$, we conclude that $\tilde{\mathcal{A}}_{t,h,\mu}$ has the same decomposition into irreducible components as $\tilde{\mathcal{A}}_{0,0,\mu}$, the classical algebra of polynomial functions on O^k_μ, and (13) extends to a matrix polynomial equation,

$$L^k + \xi_{k-1}(t,h,\mu)L^{k-1} + \ldots + \xi_0(t,h,\mu) = 0. \tag{14}$$

Let us fix μ such that O^k_μ is a semisimple orbit. Let $U \subset \mathbb{C}^k$ be a neighborhood of μ such that $O^k_{\mu'}$ is a semisimple orbit for $\mu' \in U$. Then, we can think of elementary symmetric functions $\xi_i(\mu')$ of μ' as local coordinates on U. Let $\mathcal{A}(U)$ denote the algebra of analytic functions on U. Coefficients of the polynomial in the left hand side of (14) define an algebra map $\psi\colon \mathcal{A}(U)[[t,h]] \to \mathcal{A}(U)[[t,h]]$, $(\xi_{k-1}(\mu'), \ldots, \xi_0(\mu'), t, h) \mapsto \big(\xi_{k-1}(t,h,\mu'), \ldots, \xi_0(t,h,\mu'), t, h\big)$. Since ψ is identical at $(t,h) = (0,0)$, it is an algebra automorphism. Therefore, ψ defines a transformation of parameters (t,h,μ') in the neighborhood U of μ. These arguments lead to the following

Proposition 3.5. *For every semisimple orbit O^k_μ, there exists a (t,h)-quantization, $\mathcal{A}_{t,h,\mu}$, on O^k_μ and an epimorphism*

$$\Phi_{t,h,\mu}\colon \mathcal{L}_{t,h} \to \mathcal{A}_{t,h,\mu} \tag{15}$$

factored through the ideal $\mathcal{J}_{t,h,\mu} \subset \mathcal{L}_{t,h}$ generated by the relations

$$(L-\mu_1)\ldots(L-\mu_k) = L^k + \xi_{k-1}(\mu)L^{k-1} + \ldots + \xi_0(\mu) = 0. \tag{16}$$

Proof. Let $\tilde{\mathfrak{I}}_{t,h,\mu'}$ be the kernel of the map $\tilde{\Phi}_{t,h,\mu'}$ from (12), where μ' are considered as local coordinates on the neighborhood U of the point μ. Let $\mathfrak{I}_{t,h,\mu} = \tilde{\mathfrak{I}}_{\psi^{-1}(t,h,\mu)}$ and $\mathcal{A}_{t,h,\mu} = \mathcal{L}_{t,h}/\mathfrak{I}_{t,h,\mu}$. Since ψ^{-1} transforms the coefficients of polynomial (14) to those of polynomial (16), the natural map $\mathcal{L}_{t,h} \to \mathcal{A}_{t,h,\mu}$ satisfies the proposition. $\square$

Remark 3.6. Polynomial (16) is minimal for the semisimple orbit O^k_μ. Proposition 3.5 obviously holds for any polynomial divided by (16).

In the next subsection, we show that the quotient of the algebra $\mathcal{L}_{t,h}$ by the ideal generated by entries of *any* matrix polynomial is a flat deformation.

3.3 Flatness of matrix polynomial relations

Let $a = (a_{m-1}, ..., a_0)$ be coordinates in $\mathbb{C}^m$. Let us consider the subvariety M^m in $\mathfrak{g}^* \times \mathbb{C}^m$ generated by the matrix equation

$$X^m + a_{m-1}X^{m-1} + ... + a_0 = 0. \tag{17}$$

It is clear that if the polynomial $f_a(x) = x^m + a_{m-1}x^{m-1} + ... + a_0$ has no multiple roots, the fiber M_a^m is the union of semisimple orbits passing through diagonal matrices with eigenvalues from the set $\{\lambda_i\}$, $i = 1, ..., m$, of simple roots of $f(x)$. It is easy to see that the variety M^m is flat over $\mathbb{C}^m$.

Theorem 3.7. *Let $\mathcal{J}_a$ be the ideal in the ERE algebra $\mathcal{L}_{t,h}$ generated by entries of a matrix polynomial relation,*

$$f_a(L) = L^m + a_{m-1}L^{m-1} + ... + a_0 = 0. \tag{18}$$

Then the quotient algebra $\mathcal{L}_{t,h}/\mathcal{J}_a$ is flat over $\mathbb{C}[[t, h]][a]$ and for any $a \in \mathbb{C}^m$ gives a (t, h)-quantization on M_a^m.

Proof. First, we suppose that $f_a(x)$ has only simple roots $(\lambda_1, \ldots, \lambda_m)$. Let the fiber M_a^k be the union of semisimple orbits O_ℓ, $\ell = 1, ..., K$. As pointed out in Remark 3.6, for every ℓ there exists an epimorphism $\Phi^\ell_{t,h,} \colon \mathcal{L}_{t,h}/\mathcal{J}_a \to \mathcal{A}^\ell_{t,h}$, where $\mathcal{A}^\ell_{t,h}$ is a (t, h)-quantization on O_ℓ. Consider the algebra homomorphism

$$\oplus_{\ell=1}^K \Phi^\ell_{t,h} \colon \mathcal{L}_{t,h}/\mathcal{J}_a \to \oplus_{\ell=1}^K \mathcal{A}^\ell_{t,h}. \tag{19}$$

This map is an isomorphism at the classical point $(t, h) = (0, 0)$. Therefore, as follows from Lemma 3.2, it is an isomorphism in the quantum case, too. This proves flatness of $\mathcal{L}_{t,h}/\mathcal{J}_a$ over neighborhoods of all $a \in \mathbb{C}^m$ corresponding to polynomials with simple roots.

Let $f_a(x)$ be an arbitrary polynomial. Then, it is a limit of polynomials with simple roots. Now, the flatness of $\mathcal{L}_{t,h}/\mathcal{J}_a$ over $\mathbb{C}[[t, h]][a]$ follows from the above part of the proof and from the fact that the quotient algebra $\mathcal{L}_{0,0}/\mathcal{J}_a$ is flat over $\mathbb{C}[a]$. □

The orbits in M_a^m correspond to points of the spectrum of the subalgebra of invariants in the function algebra on M_a^m. In order to pass to the quantization on a particular orbit, we should study the subalgebra of invariants in $\mathcal{L}_{t,h}$ restricted to M_a^m. That is done in the next section.

4 Algebra of invariants on M^k

4.1 Center of the ERE algebra

The subalgebra $\mathcal{S}_{t,h} \subset \mathcal{L}_{t,h}$ of $\mathcal{U}_h\big(gl(n,\mathbb{C})\big)$-invariants coincides with the center of $\mathcal{L}_{t,h}$. It is generated by elements $s_1, \ldots, s_n$, where $s_m = \mathrm{Tr}_q(L^m)$, $m \in \mathbb{N}$. The quantum trace Tr_q of a matrix A is defined as, [FRT],

$$\mathrm{Tr}_q(A) = \mathrm{Tr}(DA), \quad q = e^h, \tag{20}$$

with the weight matrix D normalized by the condition $(\mathrm{id} \otimes \mathrm{Tr}_q)(S) = q\,1$. In this normalization, $\mathrm{Tr}_q(1) = \hat{n}$. We use the notation $\hat{m}$ for quantum integers, $\hat{m} = \frac{1-q^{-2m}}{1-q^{-2}}$, $m \in \mathbb{Z}$.

The algebra $\mathcal{S}_{t,h}$ is isomorphic to the polynomial algebra $\mathbb{C}[[h]][\sigma_1, \ldots, \sigma_n]$, where the elements σ_i are coefficients of the characteristic polynomial in the quantum Cayley-Hamilton identity

$$L^n - \sigma_1 L^{n-1} + \ldots + (-1)^n \sigma_n = 0 \tag{21}$$

which follows from the same representation theory arguments as in the proof of (14).

We substantially rely on relations between generators $\{\sigma_i\}$ and $\{s_i\}$ calculated by Pyatov and Saponov, [PS], for the algebra $\mathcal{S}_h = \mathcal{S}_{0,h}$. In our normalization of the quantum trace, their result is formulated as follows.

Theorem 4.1 ([PS]). *Elements $\{s_i\}_{i=1}^{\infty}$ and $\{\sigma_i\}_{i=1}^{\infty}$, where $\sigma_i = 0$ for $i > n$, of the algebra $\mathcal{S}_h$ satisfy the quantum Newton identities*

$$\hat{m}\sigma_m - s_1\sigma_{m-1} + \ldots + (-1)^m s_m = 0, \quad m = 1, 2, \ldots. \tag{22}$$

Given a sequence $\vec{x} = \{x_m\}_{m=0}^{\infty}$ with values in a $\mathbb{C}[\mu]$-module, where $\mu = (\mu_1, ..., \mu_k) \in \mathbb{C}^k$, $k \leq n$, let us define the sequence $\{r_m(\vec{x})\}_{m=k}^{\infty}$,

$$r_m(\vec{x}) = x_m - \sigma_1(\mu)x_{m-1} + \ldots + (-1)^k \sigma_k(\mu) x_{m-k}, \tag{23}$$

where $\sigma_i(\mu)$ are the elementary symmetric functions in μ, $\sigma_i(\mu) = \sum_{1 \leq j_1 < \ldots < j_i \leq k} \mu_{j_1} \cdots \mu_{j_i}$. The following proposition describes the intersection of $\mathcal{S}_{t,h}$ with the ideal $\mathcal{J}_{t,h,\mu} \subset \mathcal{L}_{t,h}$ generated by relations (16).

Proposition 4.2. *Let $\vec{s} = \{s_m\}_{m=0}^{\infty}$ be the sequence of q-traces, $s_m = \mathrm{Tr}_q(L^m)$. Then, the set $\{r_m(\vec{s})\}_{m=k}^{\infty}$ generates the ideal $\mathcal{J}^{\mathcal{S}}_{t,h,\mu} = \mathcal{S}_{t,h} \cap \mathcal{J}_{t,h,\mu}$ in $\mathcal{S}_{t,h}$.*

Proof. Denote by p the polynomial on the left-hand side of (16) whose entries p^i_j generate the ideal $\mathcal{J}_{t,h,\mu}$. Clearly $r_m(\vec{s}) = \mathrm{Tr}_q\big(L^{m-k}p\big)$, so the elements $r_m(\vec{s})$ belong to $\mathcal{J}^{\mathcal{S}}_{t,h,\mu}$ for all $m \geq k$. Let us prove that they generate $\mathcal{J}^{\mathcal{S}}_{t,h,\mu}$. Suppose $y \in \mathcal{J}_{t,h,\mu}$; then it is representable as $y = \sum_{i,j=1}^{n} y^i_j p^j_i$, where $\{y^i_j\} \subset \mathcal{L}_{t,h}$. If $y \in \mathcal{S}_{t,h}$, the elements y^i_j generate a module of $\mathrm{End}(\mathbb{C}^n)$-type. But the $\mathrm{End}(\mathbb{C}^n)$-type component in $\mathcal{L}_{t,h}$ is an $\mathcal{S}_{t,h}$-module spanned by entries of the matrix powers L^l, $l = 0, \ldots, n-1$. Therefore y^i_j are entries of a polynomial $\sum_{l=0}^{n-1} y_l\, L^l$ with coefficients $y_l \in \mathcal{S}_{t,h}$, so $y = \sum_{l=0}^{n-1} y_l\, r_{l+k}(\vec{s})$. ☐

4.2 Special polynomials

Characters of the algebra $\mathcal{S}_{t,h}$ restricted to M^k are described by means of special polynomials, which are introduced in this subsection.

Let us define the set $\{n:k\} \subset \mathbb{Z}^k$ of k-tuples $\mathbf{n} = (n_1, \ldots, n_k)$ such that $0 \leq n_i$ and $n_1 + \ldots + n_k = n$; the subset $\{n:k\}_+ \subset \{n:k\}$ consists of $\mathbf{n}$ with all $n_i > 0$. We denote by $|n:k|$ the number of elements in $\{n:k\}$.

Given $\mathbf{n} \in \{n:k\}$ let us introduce polynomials in μ of degree $m = 0, 1, \ldots$ setting

$$\vartheta_m(\mathbf{n}, q^{-2}, \mu) \;=\; \sum_{\ell=1}^{k} (1-q^{-2})^{\ell-1} \sum_{\mathbf{d}\in\{m:\ell\}_+} \; \sum_{1\leq i_1<\ldots<i_\ell\leq k} \hat{n}_{i_1} \ldots \hat{n}_{i_\ell}\, \mu_{i_1}^{d_1} \ldots \mu_{i_\ell}^{d_\ell} \tag{24}$$

for $m > 0$ and $\vartheta_0(\mathbf{n}, q^{-2}, \mu) = \hat{n}$. At the classical point $q = 1$, the polynomial $\vartheta_m(\mathbf{n}, q^{-2}, \mu)$ is equal to $\sum_{i=1}^{k} n_i \mu_i^m$, i.e., the trace $\mathrm{Tr}(A^m)$ of a matrix A with eigenvalues $(\mu_1, \ldots, \mu_k)$ of multiplicities $(n_1, \ldots, n_k)$.

The following proposition will be important for our consideration.

Proposition 4.3. *The polynomials $\vartheta_m(\mathbf{n}, q^{-2}, \mu)$, $m \geq 0$, can be represented as*

$$\vartheta_m(\mathbf{n}, q^{-2}, \mu) \;=\; \sum_{j=1}^{k} C_j(\mathbf{n}, q^{-2}, \mu)\mu_j^m, \tag{25}$$

$$\textit{where}\quad C_j(\mathbf{n}, q^{-2}, \mu) \;=\; \hat{n}_j + \hat{n}_j \sum_{\ell=1}^{k-1} (1-q^{-2})^{\ell} \sum_{\substack{1\leq i_1<\ldots<i_\ell\leq k\\ i_1,\ldots,i_\ell\neq j}} \frac{\hat{n}_{i_1}\mu_{i_1}}{\mu_j-\mu_{i_1}} \cdots \frac{\hat{n}_{i_\ell}\mu_{i_\ell}}{\mu_j-\mu_{i_\ell}}. \tag{26}$$

Proof. It follows from definition (24) that the polynomials $\vartheta_m(\mathbf{n}, q^{-2}, \mu)$ satisfy the identity

$$\vartheta_m(\mathbf{n}, q^{-2}, \mu) = \vartheta_m(\mathbf{n}', q^{-2}, \mu') + \hat{n}_k \sum_{i=1}^{m-1} \vartheta_{m-i}(\mathbf{n}', q^{-2}, \mu')\mu_k^i + \hat{n}_k \mu_k^m, \tag{27}$$

where $\mu' = (\mu_1, \ldots, \mu_{k-1})$ and $\mathbf{n}' = (n_1, \ldots, n_{k-1})$. Equation (27) allows to apply induction on k. ☐

Using representation (25), let us introduce the functions

$$\vartheta_m(\mathbf{n}, q^{-2}, \mu, t) = \sum_{j=1}^{k} C_j\Bigl(\mathbf{n}, q^{-2}, \mu + \frac{t}{1-q^{-2}}\Bigr)\mu_j^m,$$

where we put $\mu+\frac{t}{1-q^{-2}} = (\mu_1+\frac{t}{1-q^{-2}}, \ldots, \mu_k+\frac{t}{1-q^{-2}})$. Although the coefficients $C_j(\mathbf{n}, q^{-2}, \mu)$ are rational, the functions $\vartheta_m(\mathbf{n}, q^{-2}, \mu, t)$ are, in fact, polynomials in t, q^{-2}, and μ. Evidently, $\vartheta_m(\mathbf{n}, q^{-2}, \mu) = \vartheta_m(\mathbf{n}, q^{-2}, \mu, t)|_{t=0}$.

4.3 Characters of the subalgebra of invariants in $\mathcal{L}_{0,h}$

Let us reserve the notation $\lambda = \mu$ for the case $k = n$ and consider polynomials $\vartheta_m(q^{-2}, \lambda)$; by definition,

$$\vartheta_m(q^{-2}, \lambda) = \vartheta_m(\mathbf{n}, q^{-2}, \lambda)|_{\mathbf{n}=(1,\ldots,1)} = \sum_{\ell=1}^{n}(1-q^{-2})^{\ell-1} \sum_{\mathbf{d}\in\{m:\ell\}_+} \sum_{1\leq j_1<\ldots<j_\ell\leq n} \lambda_{j_1}^{d_1}\ldots\lambda_{j_\ell}^{d_\ell}. \quad (28)$$

Lemma 4.4. *Substitution $\lambda_i = q^{-2(i-1)}$, $i = 1, \ldots, n$, to $\vartheta_m(q^{-2}, \lambda)$ returns $\hat{n}$.*

Proof. We apply induction on n. For $n = 1$ we obviously have $\vartheta_m(q^{-2}, \lambda) = 1$. Suppose the theorem holds for $n = l$. Using the induction assumption and identity (27) for the case $\mathbf{n} = (1, \ldots, 1)$ we find, for $n = l + 1$,

$$\begin{aligned}
\vartheta_m(q^{-2}, \lambda)|_{\lambda=(1,q^{-2},\ldots,q^{-2l})} &= \hat{l} + (q^{-2l})^m + (1-q^{-2})\hat{l}\sum_{d=1}^{m-1}(q^{-2l})^d \\
&= \hat{l} + (q^{-2l})^m + (1-q^{-2})\hat{l}(q^{-2l})\frac{1-(q^{-2l})^{m-1}}{1-q^{-2l}}.
\end{aligned}$$

After elementary transformations the last expression is brought to $\widehat{l+1}$. $\square$

Given $\mathbf{n} = (n_1, \ldots, n_k) \in \{n\!:\!k\}$ let us consider non-intersecting intervals $\mathrm{I}_i \subset \{1, \ldots, n\}$, $i = 1, \ldots, k$, some of them may be empty, satisfying the following requirements: 1) if $a \in I_i$ and $b \in I_j$, then $a < b$ whenever $i < j$, 2) $\#\mathrm{I}_i = n_i$. Clearly, $\mathrm{I}_1 \cup \ldots \cup \mathrm{I}_k = \{1, \ldots, n\}$. Let us introduce double indexing $\lambda_{j,i}$, $j = 1, \ldots, k$, $i = 1, \ldots, n_j$, of the indeterminates $\lambda_1, \ldots, \lambda_n$: if $m \in \mathrm{I}_j$ and i is the relative position of m within the interval I_j, then we identify $\lambda_{j,i} = \lambda_m$. Double indices are ordered by the lexicographic ordering, which coincides with that induced from $\mathbb{N}$. We shall consider the vectors $\bar{\lambda}_j = (\lambda_{j,1}, \ldots, \lambda_{j,n_j})$ and polynomials $\vartheta_m(q^{-2}, \bar{\lambda}_j)$, $j = 1, \ldots, k$.

Proposition 4.5. *Substitution $\lambda_{j,i} = \mu_j q^{-2(i-1)}$ to $\vartheta_m(q^{-2}, \lambda)$, $m \in \mathbb{N}$, gives $\vartheta_m(\mathbf{n}, q^{-2}, \mu)$.*

Proof. The case $k = 1$ follows from Lemma 4.4, because the polynomials $\vartheta_m(q^{-2}, \lambda)$, $m \in \mathbb{N}$, are homogeneous in λ of degree m. The general situation is reduced to the case $k = 1$ if one observes that, upon rearranging summation in (28), the polynomial $\vartheta_m(q, \lambda)$ may be rewritten as

$$\vartheta_m(q^{-2}, \lambda) = \sum_{\ell=1}^{k}(1 - q^{-2})^{\ell-1} \sum_{\mathbf{d}\in\{m:\ell\}_+} \sum_{1\leq j_1<\ldots<j_\ell\leq k} \vartheta_{d_1}(q^{-2}, \bar\lambda_{j_1}) \ldots \vartheta_{d_\ell}(q^{-2}, \bar\lambda_{j_\ell}), \qquad (29)$$

Now we apply Lemma 4.4 to each factor $\vartheta_{d_1}(q^{-2}, \bar\lambda_{j_1}), \ldots, \vartheta_{d_\ell}(q^{-2}, \bar\lambda_{j_\ell})$ on the right-hand side of (29). $\square$

Recall that the subalgebra of invariants $\mathcal{S}_h = \mathcal{S}_{0,h}$ of the quadratic RE algebra $\mathcal{L}_h = \mathcal{L}_{0,h}$ is a polynomial algebra in n variables σ_i, $i = 1\ldots, n$, which are the coefficients of the Cayley-Hamilton identity (21). Let us assign to $\lambda \in \mathbb{C}^n[[h]]$ a character $\chi^\lambda \colon \mathcal{S}_h \to \mathbb{C}[[h]]$ setting on the generators $\chi^\lambda(\sigma_m) = \sum_{i_1<\ldots<i_m} \lambda_{i_1} \ldots \lambda_{i_m} = \sigma_m(\lambda)$, the elementary symmetric polynomial of degree $m \in \{1, \ldots, n\}$. The correspondence $\lambda \to \chi^\lambda$ defines an epimorphism from $\mathbb{C}^n[[h]]$ to $\mathrm{ch}\, \mathcal{S}_h$, the set of characters of $\mathcal{S}_h$.

Proposition 4.6. *One has $\chi^\lambda(s_m) = \vartheta_m(q^{-2}, \lambda)$, where $s_m = \mathrm{Tr}_q(L^m)$, $m \in \mathbb{N}$.*

Proof. Assuming $\sigma_m = 0$ and $\sigma_m(\lambda) = 0$ for $m > n$ we must check that the substitution $\sigma_m \to \sigma_m(\lambda)$, $s_m \to \vartheta_m(q^{-2}, \lambda)$ satisfies Newton identities (22). Since representation (25) is valid for $\vartheta_m(q^{-2}, \lambda) = \vartheta_m(\mathbf{n}, q^{-2}, \lambda)|_{\mathbf{n}=(1,\ldots,1)}$, it is enough to check identities (22) for $m \leq n$ only. Let i and j be non-negative integer numbers and integers $d_1, \ldots, d_j > 1$ be such that $i + d_1 + \ldots + d_j = m \leq n$. Consider the coefficient before the monomial $p = p'p''$ in equality (22), where $p' = \lambda_1 \ldots \lambda_i$ and $p'' = \lambda_{i+1}^{d_1} \ldots \lambda_{i+j}^{d_j}$. Let us show that this coefficient is equal to zero. This will prove the statement because all the polynomials involved are symmetric in λ. Let us compute contributions coming, say, from the term $(-1)^l\sigma_l(\lambda)s_{m-l}(\lambda)$, where $l \geq 0$ (we assume $\sigma_0 = 1$). We should consider all the monomials $\frac{p}{r}$, where $r = \lambda_{\alpha_1} \ldots \lambda_{\alpha_l}$, $\alpha_1 < \ldots < \alpha_l \leq i + j$, enters $\sigma_l(\lambda)$ and take the coefficients before $\frac{p}{r}$ entering $s_{m-l}(\lambda)$, with the sign $(-1)^l$. We can represent r as the product $r = r'r''$, where r' contains only λ_l, $1 \leq l \leq i$, while r'' involves only λ_l, $i < l \leq i + j$; so r' and r'' divide p' and p'', respectively. Observe that the coefficients before $\frac{p}{r}$ in $s_m(\lambda)$ do not depend on r'' and are equal to $(1 - q^{-2})^{i+j-\deg(r')-1}$.

Consider separately two cases: $p'' = 1$ and $p'' \neq 1$. If $p'' = 1$, then the coefficient before the term $\lambda_1 \ldots \lambda_m$ entering (22) is

$$\hat m + \sum_{l=1}^{m}(-1)^l \binom{l}{m} (1 - q^{-2})^{l-1} = \hat m + \frac{\left(1 - (1 - q^{-2})\right)^m - 1}{(1 - q^{-2})} = 0. \qquad (30)$$

If $p'' \neq 1$, the first term in (22) gives no contribution to the coefficient, which we find to be

$$\sum_{r'}(1-q^{-2})^{i+j-\deg(r')-1}\sum_{r''}(-1)^{\deg(r)} = \sum_{r'}(1-q^{-2})^{i+j-1}\sum_{l=0}^{j}(-1)^{l}\binom{l}{j} = 0.$$

□

4.4 Algebra of invariants on M^k and its restriction to orbits

In this section, we study the image of the subalgebra of invariants $\mathcal{S}_{t,h} \subset \mathcal{L}_{t,h}$ in the quotient of $\mathcal{L}_{t,h}$ by the ideal $\mathcal{J}_{t,h,\mu}$ generated by relations (16). To analyze its structure and compute its characters, we use polynomials $\vartheta_m(\mathbf{n}, q^{-2}, \mu)$ and $\vartheta_m(\mathbf{n}, q^{-2}, \mu, t)$ introduced in Subsection 4.2.

Theorem 4.7. *The quotient of the algebra $\mathcal{S}_{t,h}$ by the ideal $\mathcal{J}^{\mathcal{S}}_{t,h,\mu} = \mathcal{S}_{t,h} \cap \mathcal{J}_{t,h,\mu}$ is generated by the images of the elements $\{s_1, \ldots, s_{k-1}\}$ in $\mathcal{S}_{t,h}/\mathcal{J}^{\mathcal{S}}_{t,h,\mu}$. For any $\mathbf{n} \in \{n\!:\!k\}$, the formula*

$$\chi_{t,h}(\mathbf{n})(s_m) = \vartheta_m(\mathbf{n}, q^{-2}, \mu, t), \quad m \in \mathbb{N}, \tag{31}$$

defines a character $\chi_{t,h}(\mathbf{n})$ of the algebra $\mathcal{S}_{t,h}/\mathcal{J}^{\mathcal{S}}_{t,h,\mu}$. The correspondence $\mathbf{n} \to \chi_{t,h}(\mathbf{n})$ is a bijection between $\{n\!:\!k\}$ and the set of characters $\operatorname{ch} \mathcal{S}_{t,h}/\mathcal{J}^{\mathcal{S}}_{t,h,\mu}$. The algebra $S_{t,h}/\mathcal{J}^{\mathcal{S}}_{t,h,\mu}$ is a free $\mathbb{C}[[t,h]]$-module of rank $|n\!:\!k|$. At generic μ, it is a direct sum of $|n\!:\!k|$ copies of $\mathbb{C}[[t,h]]$.

Proof. First of all, the theorem holds true in the classical case $(t,h) = (0,0)$. Clearly the images of the elements $\{s_1, \ldots, s_{k-1}\}$ generate $\mathcal{S}_{t,h}/\mathcal{J}^{\mathcal{S}}_{t,h,\mu}$ because other traces are related to them via recurrent relations, according to Proposition 4.2.

Let us describe characters of the algebra $\mathcal{S}_{t,h}/\mathcal{J}^{\mathcal{S}}_{t,h,\mu}$. Suppose first $t = 0$. An element χ from $\operatorname{ch} \mathcal{S}_{0,h}/\mathcal{J}^{\mathcal{S}}_{0,h,\mu}$ is also a character for $\mathcal{S}_{0,h}$, so we can assume $\chi = \chi^{\lambda}$, for some $\lambda \in \mathbb{C}^n[[h]]$ depending on μ. We have $\chi(s_m) = \vartheta_m(q^{-2}, \lambda)$ according to Proposition 4.6. For χ^{λ} to define a character of the quotient algebra $\mathcal{S}_{0,h}/\mathcal{J}^{\mathcal{S}}_{0,h,\mu}$, the sequence $\chi(\vec{s}) \subset \mathbb{C}[[h]]$, must satisfy the recurrent identities $r_m\big(\chi(\vec{s})\big) = 0$, $m \geq k$, by Proposition 4.2. By Proposition 4.5, for any $\mathbf{n} \in \{n\!:\!k\}$ the polynomials $\vartheta_m(\mathbf{n}, q^{-2}, \mu)$ are obtained by a substitution to $\vartheta_m(q^{-2}, \lambda)$. The sequence $\vec{a} = \{\vartheta_m(\mathbf{n}, q^{-2}, \mu)\}_{m=0}^{\infty}$ satisfies the recurrent relations $r_m(\vec{a}) = 0$, $m \geq k$, as follows from representation (25). Therefore, for any $\mathbf{n} \in \{n\!:\!k\}$ there exists an element $\chi_h(\mathbf{n}) \in \operatorname{ch} \mathcal{S}_{0,h}/\mathcal{J}^{\mathcal{S}}_{0,h,\mu}$ such that $\chi_h(\mathbf{n})(s_m) = \vartheta_m(\mathbf{n}, q^{-2}, \mu)$, for $m \geq 0$.

As follows from their definitions, the ERE and RE algebras are defined over $\mathbb{C}[q, q^{-1}]$, $q = e^h$, and all the $\mathbb{C}[[h]]$-algebras under consideration are, in fact, completions of the corresponding $\mathbb{C}[q, q^{-1}]$-algebras at the point $q = 1$. Since the functions $\vartheta_m(\mathbf{n}, q^{-2}, \mu)$ are polynomials in q^{-2}, the character $\chi_h(\mathbf{n})$ is the extension of a character $\chi_q(\mathbf{n}) \in \operatorname{ch}\mathcal{S}_{0,q}/\mathcal{J}^{\mathcal{S}}_{0,q,\mu}$.

Let us prove formula (31) for the case $t \neq 0$. Hecke condition (9) and relations (10) imply that the shift $L \mapsto L - \frac{t}{1-q^{-2}}$, $\mu \mapsto \mu - \frac{t}{1-q^{-2}}$ induces a homomorphism $\varrho_{t,q}\colon \mathcal{L}_{t,q} \to \mathcal{L}_{0,q}[t]$ (here we assume that the ring $\mathbb{C}[q,q^{-1}]$ is extended to the field $\mathbb{C}(q)$). Clearly $\varrho_{t,q}(\mathcal{J}_{t,q,\mu})$ coincides with $\mathcal{J}_{0,q,\mu}$ because equation (16) is invariant under this transformation. Therefore, the map $\varrho_{t,q}$ defines a homomorphism $\bar{\varrho}_{t,q}\colon \mathcal{S}_{t,q}/\mathcal{J}^{\mathcal{S}}_{t,q,\mu} \to \mathcal{S}_{0,q}/\mathcal{J}^{\mathcal{S}}_{0,q,\mu}$. Denote by $\chi_{t,q}(\mathbf{n})$ the composition $\chi_q(\mathbf{n}) \circ \bar{\varrho}_{t,q}$, which is a character of $\mathcal{S}_{t,q}/\mathcal{J}^{\mathcal{S}}_{t,q,\mu}$. It is easy to prove by induction on m, using representation (25), that $\chi_{q,t}(\mathbf{n})$ evaluated on s_m gives $\vartheta_m(\mathbf{n}, q^{-2}, \mu, t)$. Since the functions $\vartheta_m(\mathbf{n}, q^{-2}, \mu, t)$ are polynomials in q^{-2}, we can extend $\chi_{t,q}(\mathbf{n})$ to a character $\chi_{t,h}(\mathbf{n})$ of $\mathcal{S}_{t,h}/\mathcal{J}^{\mathcal{S}}_{t,h,\mu}$ over $\mathbb{C}[[h]]$.

Thus we constructed a deformation $\chi_{t,h}(\mathbf{n})$ for each classical character $\chi(\mathbf{n})$. We have the inequality

$$|n\!:\!k| = \#\mathrm{ch}\, \mathcal{S}/\mathcal{J}^{\mathcal{S}}_{\mu} \le \#\mathrm{ch}\, \mathcal{S}_{t,h}/\mathcal{J}^{\mathcal{S}}_{t,h,\mu} \le \mathrm{rk}\, \mathcal{S}_{t,h}/\mathcal{J}^{\mathcal{S}}_{t,h,\mu} \le N \le \mathrm{rk}\, \mathcal{S}/\mathcal{J}^{\mathcal{S}}_{\mu} = |n\!:\!k|,$$

where $\mathcal{S} = \mathcal{S}_{0,0}$ and $\mathcal{J}_{\mu} = \mathcal{J}^{\mathcal{S}}_{0,0,\mu}$ are the classical algebras and N the number of generators of the $\mathbb{C}[[t,h]]$-module $\mathcal{S}_{t,h}/\mathcal{J}^{\mathcal{S}}_{t,h,\mu}$. This inequality shows that $\mathcal{S}_{t,h}/\mathcal{J}^{\mathcal{S}}_{t,h,\mu}$ is a free module over $\mathbb{C}[[t,h]]$ of rank $|n\!:\!k|$.

Let us fix μ such that $\mu_i \neq \mu_j$ for $i \neq j$. Since at the classical point $(t,h) = (0,0)$ the algebra map

$$\oplus_{\mathbf{n}\in\{n:k\}}\chi_{t,h}(\mathbf{n})\colon \mathcal{S}_{t,h}/\mathcal{J}^{\mathcal{S}}_{t,h,\mu} \to \oplus_{\mathbf{n}\in\{n:k\}}\mathbb{C}[[t,h]]$$

is an isomorphism, it is an isomorphism of $\mathbb{C}[[t,h]]$-algebras. □

5 Quantum orbits

In this section, we present explicit formulas of the $\mathcal{U}_h\big(gl(n,\mathbb{C})\big)$-equivariant quantization on orbits using characters of the algebra $\mathcal{S}_{t,h}/\mathcal{J}^{\mathcal{S}}_{t,h,\mu}$ calculated in the previous section.

Theorem 5.1. *For any element $\mathbf{n} = (n_1, \ldots, n_k) \in \{n\!:\!k\}$, the quotient of the ERE algebra (10) by the relations*

$$(L-\mu_1)\ldots(L-\mu_k) = 0, \tag{32}$$

$$\mathrm{Tr}_q(L^m) = \vartheta_m(\mathbf{n}, q^{-2}, \mu, t), \quad m = 1, \ldots, k-1, \tag{33}$$

is a flat $\mathbb{C}[[t,h]]$-algebra. When $\mu_i \neq \mu_j$ for $i \neq j$, it is a quantization on the orbit of semisimple matrices with eigenvalues $(\mu_1, \ldots \mu_k)$ of multiplicities $(n_1, \ldots, n_k)$.

Proof. By Theorem 3.7, the quotient algebra $\mathcal{L}_{t,h}/\mathcal{J}_{t,h,\mu}$ defined by relations (32) is a quantization on M^k. The algebra of invariants on a semisimple orbit is one-dimensional thus corresponding to a character of $\mathcal{S}_{t,h}/\mathcal{J}^{\mathcal{S}}_{t,h,\mu}$. At the classical point, characters are $\chi(\mathbf{n})(s_m) = \sum_{i=1}^k n_i\mu_i^m$, for $\mathbf{n} \in \{n\!:\!k\}$. Their deformations are given by formula (31). So relations (33) should hold on the orbit of semisimple matrices with eigenvalues $(\mu_1, \ldots \mu_k)$ of multiplicities $(n_1, \ldots, n_k)$. Because relations (32) and (33) define this orbit at $(t, h) = (0, 0)$, they define a (t, h)-quantization on it. □

Since functions $\vartheta_m(\mathbf{n}, q^{-2}, \mu, t)$ are polynomials in q^{-2}, t, and μ, we obtain the following

Corollary 5.2. *At $q = 1$, formulas (32) and (33) give G-equivariant quantizations of the Kostant-Kirillov-Souriau brackets on semisimple orbits.*

Remark 5.3. One can consider the ERE algebra associated with arbitrary Hecke symmetry S and define "quantum orbits" by formulas (32) and (33). Equations (33) are forced by condition (32) and have the same form, since Newton identities (22) hold for any Hecke symmetry, [IOP]. The question is about the module structure (supply of "functions") of the orbits defined in this way. In our next paper we will prove that the supply of functions on an "orbit" for any even Hecke symmetry is rich enough.

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