APPROXIMATIONS OF THE MAASS FORMS BY MEANS OF ANALYTIC MODULAR FORMS

Alexei B. Venkov

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
53225 Bonn

Germany

MPI / 94-33

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ALEXEI B. VENKOV

To Ralph Phillips on the occasion of his 80th birthday

1. INTRODUCTION

Let H be the upper half plane, Γ be the modular group $\Gamma = PSL(2,\mathbb{Z})$ and $\mathcal{H} = L_2(\Gamma\backslash H)$ be the standard Hilbert space of automorphic functions. The goal of this paper is to prove that any even function $f \in \mathcal{H}$ is represented by some special series

$$f(z,\overline{z}) = \sum_{n=2}^{\infty} a_n(k,m) W_n(z,\overline{z};k)\overline{W_n(z,\overline{z};m)} + a_0$$

Here a_n are some constants and

$$W_n(z,\overline{z};k) = y^n R^{k_1}(z) Q^{k_2}(z) S^{k_3}(z,\overline{z})$$
$$R(z) = E_6(z), Q(z) = E_4(z), S(z,\overline{z}) = E_2(z) - \frac{3}{\pi y}, \qquad y = \operatorname{Im} z$$

$E_l(z)$ are the analytic Eisenstein series $l = 2,4,6;\ k = (k_1,k_2,k_3)$, $m = (m_1,m_2,m_3)$ $\in \mathbb{Z}^3_+$, $n = 6k_1 + 4k_2 + 2k_3 = 6m_1 + 4m_2 + 2m_3$.

The dash means the complex conjugation. For the precise assertion see the Main Theorem. This theorem is utmost improvement of the main result of the paper [1].

The important ingredients of the proof are the spectral theory of the Schroedinger operator with automorphic potentials, the Ramanujan's formulae for the derivatives of the Eisenstein series and Phillips-Sarnak approach to investigate a disappearance of the Maass cusp forms under perturbations.

2. THE MAIN PART

Let H be the hyperbolic plane with Poincaré metric ds^2 and the corresponding Laplacian L, Γ be the modular group $\Gamma = PSL(2,\mathbb{Z})$.

We consider H as the upper half plane

$$H = \{z \in \mathbb{C}, z = x + iy \mid y > 0\}$$

and

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$$L = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

The differential operator $-L$ generates the nonnegative selfadjoint operator $A(\Gamma) = A$ in the Hilbert space $\mathcal{H} = L_2(F; d\mu)$ in the natural way. A is called the automorphic Laplacian.

We introduce some notation. F is a fundamental domain of the group Γ on H. $d\mu$ is the Riemann measure which generated by the metric ds^2. The norm $\|f\|$ of a function $f \in \mathcal{H}$ is defined by the integral

$$\|f\| = \int_F |f(z, \overline{z})|^2 d\mu(z, \overline{z})$$

f is a function of the two variables x, y or $z, \overline{z}$ where $z = x + iy, \overline{z} = x - iy$. For analytic function f of the variable z we write $f(z)$ and omit $\overline{z}$.

The following spectral decomposition of the Hilbert space $\mathcal{H}$ is very well known

(1) $$\mathcal{H} = \mathcal{H}_0 \otimes \mathbb{C} \otimes \Theta$$

$\mathcal{H}_0$ is the space of cusp forms (functions, for to be more precise). In the same time $\mathcal{H}_0$ is the closed subspace of $\mathcal{H}$ spanned by all of the eigenfunctions of the discrete spectrum λ_j of the operator A and $\lambda_j \in [1/4, \infty)$.

Θ is the space of the continuous spectrum of A.

For any function $f \in \mathcal{H}$ we have related to (1) the expansion in eigenfunctions of the operator A

(2)
$$f(z, \overline{z}) = \sum_{n=1}^{\infty} (f, v_n) v_n(z, \overline{z}) + \frac{1}{|F|} \int_F f(z, \overline{z}) d\mu(z, \overline{z}) + \frac{1}{4\pi i} \int_{\mathrm{Re}s=1/2} (f, E_s) E_s(z, \overline{z}) ds$$

Here $(.,.)$ means the standard scalar product in $\mathcal{H}$ and $\{v_n\}_{n=1}^{\infty}$ is a basis of eigenfunction in $\mathcal{H}_o$

$$(v_j, v_k) = \delta_{jk}, A v_k = \lambda_k v_k$$

$1/4 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots |F|$ is the $d\mu$ volume of F. For $PSL(2, \mathbb{Z}) |F| = \pi/3. E_s = E(z, \overline{z}; s)$ is the Eisenstein-Maass series or the non-analytic Eisenstein series.

For $\mathrm{Re}s > 1$ we define the Eisenstein-Maass series by the absolutely convergent Pointcaré series

$$E(z, \overline{z}; s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}^s(\gamma z)$$

where Im means the imaginary part of a complex number. $\Gamma_\infty \subset \Gamma$ is the subgroup generated by the transformation of $H : z \to z + 1$.

The main property of a continuous function $f \in \mathcal{H}_0$ is the vanishing of the zero coefficient of the Fourier series expansion which is defined by the action of the group Γ_∞ in H

$$\int_0^1 f(z,\overline{z})dx = 0$$

for all $y > 0, z = x + iy$.

The following decomposition is important also

$$\mathcal{H} = \mathcal{H}^{(1)} \oplus \mathcal{H}^{(2)}$$

By definition

$$\mathcal{H}^{(1)} = \{f \in \mathcal{H} | -f(z,\overline{z}) = f(-\overline{z},-z)\}$$
$$\mathcal{H}^{(2)} = \{f \in \mathcal{H} | f(z,\overline{z}) = f(-\overline{z},-z)\}$$

are the subspaces of odd and even functions correspondingly. We remark that the map $z \to -\overline{z}, H \to H$, commutes with Laplacian L and transfer an automorphic (modular)function to automorphic (modular) one. It is not hard to see that

$$\Theta \subset \mathcal{H}^{(2)}$$

because the Eisenstein-Maass series is the even function. This observation follows from the Fourier decomposition of $E(z,\overline{z};s)$. All these results related to the spectral theory of the operator A are contained in the papers [2], [3], [4], [5].

Let $\mathcal{H}_0^{(i)} = \mathcal{H}_0 \cap \mathcal{H}^{(i)}$ be the spaces of cusp forms $j = 1,2$. We recall now some properties of analytic modular forms. A function $f : H \to \mathbb{C}$ is said to be analytic modular form of weight k for Γ if 1) f is analytic on H, 2) $f(\gamma z) = (cz+d)^k f(z)$ for any

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$$

and any $z \in H$, 3) f is analytic at infinity, i.e. it has the Fourier series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n \exp 2\pi \mathrm{i} n z \tag{3}$$

Let R, Q, P be the analytic Eisenstein series of weights $6,4,2$ correspondingly. We have well known Fourier series expansions

$$R(z) = E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) \exp 2\pi \mathrm{i} n z$$
$$Q(z) = E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) \exp 2\pi \mathrm{i} n z$$
$$P(z) = E_2(z) = 1 - \ 24 \sum_{n=1}^{\infty} \sigma_1(n) \exp 2\pi \mathrm{i} n z$$

where $\sigma_k(n)$ is the sum of k degrees of the devisors of n.

The functions R, Q generate the algebra of all analytic modular forms of positive even weights. Any modular form of the weight k is represented as a complex linear combination of the monomials (see [6])

$$R^{k_1}Q^{k_2}, 6k_1 + 4_{k_2} = k$$

We come back to the subspaces $\mathcal{H}^{(j)}$ now and we consider one well known way of constructing some elements of the subspace $\mathcal{H}_0^{(1)}$ using analytic modular forms (see §3,7 [7] for the similar idea). Let g_1, g_2 be two analytic modular forms of the weight k with real Fourier coefficients (see (3)), then it is simple to see that

$$f(z,\overline{z}) = y^k(g_1(z)\overline{g_2(z)} - \overline{g_1(z)}g_2(z)) \in \mathcal{H}_0^{(1)}, \tag{4}$$

$y = \mathrm{Im}z$, dash means the complex conjugation, of course.

The interesting question is the following one. What are the elements of $\mathcal{H}_0^{(1)}$ which are approximated by linear combinations of the functions (4) when g_j run through the whole algebra B? We can not answer this question now but we answer the similar question for the space $\mathcal{H}^{(2)}$ and it is just the subject of the Main Theorem.

Let $S(z,\overline{z}) = P(z) - 3\pi y, y = \mathrm{Im}z$, and let $\mathcal{M}_\infty$ be the $\mathbb{C}$-linear space generated by the system of all functions e_k, where k runs all nonnegative even integers. By definition

$$e_k = y^k R^{k_1}Q^{k_2}S^{k_3}\overline{R^{m_1}Q^{m_2}S^{m_3}}, e_0 = 1$$
$$k = 6_{k_1} + 4_{k_2} + 2_{k_3} = 6_{m_1} + 4_{m_2} + 2_{m_3},$$

$$k_j, m_j \in \mathbb{Z}, k_j \geqslant 0, m_j \geqslant 0, j = 1,2,3.$$

The dash is the complex conjugation. One remark is important here. We add mentally that e_k depends on the numbers k_j, m_j mentioned above but we not write it in the denotation of e_k.

We define now the linear space $M_\infty^{\mathcal{H}}$ as the intersection

$$M_\infty^{\mathcal{H}} = M_\infty \cap \mathcal{D}(A),$$

where $\mathcal{D}(A)$ is the domain of definition of the selfadjoint unbounded operator A in $\mathcal{H}$. We recall that $\mathcal{D}(A)$ is the dense set in $\mathcal{H}$. We define after all the linear space $\hat{M}_\infty^{\mathcal{H}}$ as the closure of $M_\infty^{\mathcal{H}}$ in $\mathcal{H}$.

We formulate now the main theorem of this paper

Theorem (Main Theorem). *The following equality holds*

$$\hat{M}_\infty^{\mathcal{H}} = \mathcal{H}^{(2)}$$

We devide the proof of this theorem for several parts.

Firstly we prove

Lemma 1.

$$LM_\infty \subset M_\infty$$

Proof. We remark that any function e_k is an automorphic one, i.e. for all $\gamma \in \Gamma, z \in H$ the equality holds

$$e_k(\gamma z, \gamma \overline{z}) = e_k(z, \overline{z})$$

It is well known that the function $P(z)$ is not a modular form. For example, we have

$$P(-1/z) = z^2 P(z) + \frac{6z}{\pi i}$$

One can proves that $S(z, \overline{z})$ is the modular form of the weight 2 but not analytic form. Therefore the monomial

$$R^{k_1} Q^{k_2} S^{k_3}$$

is the modular form of the weight k if $6k_1 + 4k_2 + 2k_3 = k$ and the function $e_k(z, \overline{z})$ is the modular one.

We want to compute now the action of the Laplacian L on the function $e_k(z, \overline{z})$. It is well known (see [6]) that albebra B is not closed relative to a differentiation but the algebra generated by R, Q, P has this property and this result is proved by means of remarkable Ramanujan's formulae

$$P_z = \frac{\pi i}{6}(P^2 - Q) \qquad Q_z = \frac{2\pi i}{3}(PQ - R) \qquad R_z = \pi i(PR - Q^2),$$

where $f_z = \frac{\partial f}{\partial z}$. The modified Ramanujan's formulae for R, Q, S take the form

$$(5) \quad S_z = \frac{\pi i}{6}(S^2 + \frac{6}{\pi y} S - Q), \qquad S_{\overline{z}} = \frac{3i}{2\pi y^2}$$
$$Q_z = \frac{2\pi i}{3}(SQ + \frac{3}{\pi y} Q - R), \qquad R_z = \pi i(SR + \frac{3}{\pi y} R - Q^2)$$

Using (5) we have

$$\begin{aligned}
&(z - \overline{z}^2) \frac{\partial^2 e_k}{\partial z \partial \overline{z}} = \\
&\quad e_k \left\{ k + (k - k_3)k_3 + (k - m_3)m_3 - k_3 - m_3 + k_3 \frac{Q}{S^2} + m_3 \frac{\overline{Q}}{\overline{S}^2} - 6k_1 k_3 \frac{Q^2}{RS} \right. \\
&\quad - 4k_2 k_3 \frac{R}{QS} - k_3^2 \frac{Q}{S^2} - 6m_1 m_3 \frac{\overline{Q}^2}{\overline{RS}} - 4m_2 m_3 \frac{\overline{R}}{\overline{QS}} - m_3^2 \frac{\overline{Q}}{\overline{S}^2} \\
&\quad + \frac{36 k_3 m_3}{\pi^2 (z - \overline{z})^2 |S|^2} - \pi^2 (z - \overline{z})^2 \left(\frac{k - k_3}{6} S - k_1 \frac{Q^2}{R} - \frac{2k_2}{3} \cdot \frac{R}{Q} - \frac{k_3}{6} \frac{Q}{S} \right) \\
&\quad \left. \left(m_1 \frac{\overline{Q}^2}{\overline{R}} + \frac{2m_2}{3} \frac{\overline{R}}{\overline{Q}} + \frac{m_3}{6} \frac{\overline{Q}}{\overline{S}} - \frac{k - m_3}{6} \overline{S} \right) \right\}
\end{aligned}$$

It is not hard to see from this equality that Le_k is the linear combinations of the functions e_{k-2}, e_k, e_{k+2}. It is clear when $k_3 \geq 2, m_3 \geq 2, k_1 \geq 1, m_1 \geq 1, k_2 \geq 1, m_2 \geq 1$. If $k_3 = 1$ (or $m_3 = 1$) then the dangerous terms to diminish. When some of k_j (or m_j) is equal to zero then the corresponding dangerous term is equal to zero too, $1 \leq j \leq 3$. We name dangerous all terms with nontrivial denominators. The proof of Lemma 1 is complete

Using Lemma 1 the proof of the following lemma is standard in the theory of selfadjoint operators in Hilbert spaces and we shall not give this proof here.

Lemma 2.

1) *Let $h(A)$ be any continuous bounded function of the automorphic Laplacian A. Then we have*

$$h(A)\hat{M}_\infty^{\mathcal{H}} \subset \hat{M}_\infty^{\mathcal{H}}$$

2) *Let P_θ, P_M be ortogonal projectors in $\mathcal{H}$ on the subspaces $\Theta, \hat{M}_\infty^{\mathcal{H}}$ correspondingly, then the equality holds*

$$P_\theta P_M = P_M P_\theta$$

Lemma 3. *There is the inclusion*

$$\Theta \subset \hat{M}_\infty^{\mathcal{H}}$$

Proof. Let $f_\Delta(z, \overline{z})$ be the function

$$f_\Delta(z, \overline{z}) = y^{12}|\Delta(z)|^2, \qquad \Delta(z) = c_1(Q^3(z) - R^2(z))$$
$$\Delta(z) = c_2 q \prod_{n=1}^{\infty}(1 - q^n)^{24}, \qquad q = \exp 2\pi i z$$

c_1, c_2 are some constants. It is clear that $f_\Delta \in \hat{M}_\infty^{\mathcal{H}}$, because $\Delta(z)$ is the analytic modular cusp-form of weight 12.

Let P_θ be the orthogonal projector in $\mathcal{H}$ to Θ like in Lemma 2. We consider standard Hilbert space $L_2(\mathbb{R}_+)$ of functions $\eta : \mathbb{R}_+ \longrightarrow \mathbb{C}$ with the scalar product

$$< \xi, \eta >= \frac{1}{2\pi}\int_0^\infty \xi(t)\overline{\eta(t)}dt$$

As we know from the theorem of eigenfunctions expansion for the automorphic Laplacian (see [7]) the map

$$U : \Theta \longrightarrow L_2(\mathbb{R}_+)$$
$$(Uf)(t) = \int_F E(z, \overline{z}; 1/2 - it) f(z, \overline{z}) d\mu(z, \overline{z})$$

is isometric. The operator $Uh(A)U^*$ is the operator of multiplication by the function h. We recall that h is an bounded and continuous function. The star means the conjugation of an operator in $\mathcal{H}$.

We consider now the function $(UP_\theta f_\Delta)(t)$. It is continuous, and even analytic function for $t \in (0, \infty)$ as the Rankin-Selberg convolution. It has no poles and it has only a discrete set of zeroes. Therefore the set of functions $Uh(A)P_\theta f_\Delta$ fill in a dense subset in $L_2(\mathbb{R}_+)$ when h run through the set of all continuous and bounded functions. Thus the space containing all functions $h(A)P_\theta f_\Delta$ after the closing in $\mathcal{H}$ coincides to Θ. Lemma 3 follows from Lemma 2. The proof is complete.

Lemma 4. *The following inclusion holds*

$$\mathbb{C} \subset \hat{M}_\infty^{\mathcal{H}}$$

Proof. Because $e_0 = 1$.

The idea of the proof of the Main Theorem now is the following. Together with the spectral problem $Af = \lambda f$ we consider the spectral problems for the automorphic Schroedinger operators A_q in $\mathcal{H}, A_q f = \lambda f$.

The corresponding differential equation is

$$-Lf + qf = \lambda f, \qquad q \in \hat{M}_\infty^{\mathcal{H}} \tag{6}$$

We will prove that there exists a potential q_∞ and the operator A_{q_∞} which satisfy

1) A_{q_∞} is the selfadjoint nonnegative operator in $\mathcal{H}$
2) A_{q_∞} is well defined on the dense set $\mathcal{D}(A_{q_\infty})$ in $\mathcal{H}$.

   In $\mathcal{D}(A_{q_\infty}) \cap \mathcal{H}^{(2)}$ it has one dimensional absolutely continuous spectrum $\lambda \in [1/4, \infty)$ and the discrete spectrum which is outside of the continuous one $\lambda \in [0, 1/4)$ as a finite set of eigenvalues of a finite multiplicity, and it has no other spectrum.
3) There is inclusion

$$h(A_{q_\infty})\hat{M}_\infty^{\mathcal{H}} \subset \hat{M}_\infty^{\mathcal{H}}$$

   where the function h is defined in Lemma 2.

After that we will prove that $\hat{M}_\infty^{\mathcal{H}}$ coincides to $\mathcal{H}^{(2)}$ up to the finite dimensional subspace which is in the space of the discrete spectrum of the operator A_{q_∞} in $\mathcal{H}^{(2)}$. In the last step of the proof of the Main Theorem we will prove that this finite dimensional subspace is trivial.

Lemma 5.

There exists a set of potentials $\mathcal{N} \subset \Theta \oplus \mathbb{C}$ which satisfy

1) *Any $q \in \mathcal{N}$ is a continuous nonnegative modular function, i.e. $q \in C(H)$, $q \geq 0$, $q(\gamma z, \gamma \bar{z}) = q(z, \bar{z})$ for all $\gamma \in \Gamma$, $z \in H$.*
2) *$\mathbb{C}$-linear space generated by all elements of $\mathcal{N}$ is a dense set in $\Theta \oplus \mathbb{C}$*

Proof. It is well known that the space $\Theta \oplus \mathbb{C}$ fill in by incomplete theta series (see [4], [5], for example)

$$\theta_\varphi(z,\overline{z}) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \varphi(\mathrm{Im}\gamma z)$$

where φ run through the set of all continuous functions $\varphi : (0,\infty) \longrightarrow \mathbb{C}$ with compact supports. For the proof of Lemma 5 it is enough to choose $\mathcal{N}$ as the set of all θ_φ with nonnegative φ. The proof is complete.

Remark. We further suppose that $\mathcal{N}$ is the set defined in the proof of Lemma 5.

The following lemma comes from the spectral theory of the Schroedinger operator with automorphic potentials (see §2 [9] and [10] for more general theory). We formulate now slightly simplified version, more advanced one will be important in the proof of Lemma 8.

We introduce some notation relating to this spectral theory. Let $\tau_q(z,\overline{z};z'\overline{z}';s)$ be the kernel of the resolvent $(A_q-\lambda)^{-1}1$ of the operator A_q in $\mathcal{H}, \lambda = s(1-s), \mathrm{Re}s > 1$. Let $\psi_q(z,\overline{z};s)$ be the generalized eigenfunction of the continous spectrum of A_q with "eigenvalue" $\lambda = s(1-s), \mathrm{Re}s = 1/2$, and $S_q(s)$ be the corresponding one dimensional scattering matrix (see §2 [9]).

Lemma 6. *Let $q \in \mathcal{N}$ be a potential from the Remark to Lemma 5 then we have the following assertions*

1) *The left hand side of equality (6) defines the selfadjoint nonnegative operator A_q in $\mathcal{H}$ (the Friedrichs extension)*
2) *A_q has one dimensional absolutely continuous spectrum for $\lambda \in [1/4,\infty)$ and a discrete one of eigenvalues of a finite multiplicity $\lambda_j \in [0,\infty)$ and it has no other spectrum.*
3) *The functions $\tau(.,.,.,.,s), \psi_q(.,.,.,s), S_q(s)$ are meromorphic for $s \in \mathbb{C}$ and they satisfy the functional equations*

   a)

$$\tau_q(z,\overline{z};z',\overline{z}';s) - \tau_q(z,\overline{z};z',\overline{z}';1-s) = \frac{1}{2s-1}\psi_q(z,\overline{z};s)\psi_q(z',\overline{z}';1-s)$$

   b)

$$\psi_q(z,\overline{z};s) = \psi_q(z,\overline{z};1-s)S_q(s)$$

   c)

$$S_q(s)S_q(1-s) = 1$$

The proof of this lemma is given in above mentioned papers. The method of this proof is essentially the same as Faddeev's one for the theorem on eigenfunctions expansion for the automorphic Laplacian (see [8]). It is important for this theorem that a potential is sufficiently decreasing in the cusp of the fundamental domain.

We are interested now in a special potential $q_\infty \in \hat{M}^{\mathcal{H}}_\infty$ or, more precisely, in a family of such potentials which satify

1) For A_{q_∞} Lemma 6 holds
2) A_{q_∞} in $\mathcal{H}^{(2)}$ has only a finite spectrum besides a continuous one.

We know from the paper of Colin de Verdiere (see [11] and [10]) that a generic potential q has this property. But the question is in the important restriction $q \in \hat{M}^{\mathcal{H}}_{\infty}$ and we have to prove the existence of such potentials. Here we use some idea from the paper [12] on a condition to destroy cusp forms. We modify the Phillips-Sarnak method for the automorphic Schroedinger operator.

We fix some potential q from the Remark to Lemma 5 and we consider the Schroedinger operator $A\varepsilon q$ where ε is a small positive parameter.

Let $\lambda_n > 0$ be an arbitrary fixed eigenvalue of the operator A for the corresponding even eigenfunction $v_n, Av_n = \lambda_n v_n$. We know that λ_n is embedded in the continuous spectrum of $A, 1/4 < \lambda_n = s_n(1-s_n), \mathrm{Re} s_n = 1/2$. Using the same arguments as in §2 [12] we obtain that the pair s_n, v_n is included in the analytic family of the solutions of the differential equation

$$-Lv_n(z,\overline{z},;\varepsilon) + \varepsilon q(z,\overline{z})v_n(z,\overline{z};\varepsilon) = \lambda_n(\varepsilon)v_n(z,\overline{z};\varepsilon) \tag{7}$$

in the neighbourhood of the $\varepsilon = 0$. Here $v_n(z,\overline{z};0) = v_n(z,\overline{z}), \lambda_n(0) = \lambda_n, \lambda_n(\varepsilon) = s_n(\varepsilon)(1-s_n(\varepsilon))$. Besides we have

$$v_n(z,\overline{z};\varepsilon) = c_1(\varepsilon)y^{s_n(\varepsilon)} + c_2(\varepsilon)y^{1-s_n(\varepsilon)} + \underset{y\to\infty}{\mathcal{O}}(exp - ay)$$

for some $a > 0$. For the analytic functions $c_j(\varepsilon)$ there is the condition $c_1(0) = c_2(0) = 0$. For the analytic functions $s_n(\varepsilon)$ there is the alternative 1) To have $\mathrm{Re}s_n(\varepsilon) = 1/2$ for all $\varepsilon \geq 0$ small enough, or 2) To have $\mathrm{Re}s_n(\varepsilon) < 1/2$ for all $\varepsilon > 0$ small enough.

In the case 1) we have $c_1(\varepsilon) = 0, c_2(\varepsilon) = 0$. The function $v_n(z,\overline{z};\varepsilon)$ and the value $\lambda_n(\varepsilon)$ are correspondingly the eigenfunction and the eigenvalue of the discrete spectrum of $A_{\varepsilon q}$.

In the case 2) the eigenvalue λ_n of the discrete spectrum of A disappears under the deformation of εq.

We find now the sufficient condition for the case 2). It is the similar condition, of course, as Phillips-Sarnak's one for the deformation in the Theichmueller space (see [12]). We have

$$v_n(z,\overline{z};\varepsilon) = v_n(z,\overline{z}) + \varepsilon w_n(z,\overline{z}) + \underset{\varepsilon\to 0}{\mathcal{O}}(\varepsilon^2)$$
$$\lambda_n(\varepsilon) = \lambda_n + \varepsilon\nu_n + \underset{\varepsilon\to 0}{\mathcal{O}}(\varepsilon^2)$$

From the equation (7) it follows

$$-Lw_n(z,\overline{z}) + q(z,\overline{z})v_n(z,\overline{z}) = \nu_n(z,\overline{z}) + \lambda_n w_n(z,\overline{z}) \tag{8}$$

We multiply (8) by the Eisenstein series and we integrate it over the fundamental domain F. More precisely we have

$$(9)\quad \int_F q(z,\overline{z})v_n(z,\overline{z})E(z,\overline{z};s_n)d\mu(z,\overline{z}) =$$
$$\lim_{Y\to\infty}\left(\int_{F_Y}[\lambda_n w_n(z,\overline{z})E(z,\overline{z};s_n)+(Lw_n(z,\overline{z}))E(z,\overline{z};s_n)]d\mu(z,\overline{z})\right)$$

Here F_y is some compact part of the fundamental domain F

$$F \doteq \{z = x+iy \in H \| x| < 1/2, |z| > 1\}$$

$F_y = \{z \in F | y \leqslant Y\}$, Y is fixed and $Y > 0$ (big enough), $\doteq$ means the equality up to the points on the boundary F. We remember also in (9) that

$(v_n, E_{s_n}) = 0$.

Let $f(z,\overline{z}), g(z,\overline{z})$ be arbitrary smooth automorphic functions then the following Green formula is well known (see [2]. [5])

$$(10)\qquad \int_{F_Y}(f\Delta g - g\Delta f)dx\wedge dy = \int_0^1 (f\cdot\frac{\partial g}{\partial y} - g\frac{\partial f}{\partial y})dx|_{y=Y}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. The integral in the right hand side of (9) is equal to

$$\int_{F_Y}(E_{s_n}\Delta w_n - w_n\Delta E_{s_n})dx\wedge dy$$

Then we have

$$w_n(z,\overline{z}) = c_1'(0)y^{s_n} + c_2'(0)y^{1-s_n} + \underset{y\to\infty}{O}(\exp -ay)$$
$$E(z,\overline{z};s_n) = y^{s_n} + \varphi(s_n)y^{1-s_n} + \underset{y\to\infty}{O}(\exp -ay)$$

for some $a > 0$. We obtain from (10)

$$\int_{F_Y}(E_{s_n}\Delta w_n - w_n\Delta E_{s_n})dx\wedge dy = (1-2s)(c_2'(0) - \varphi(s_n)c_1'(0)) + \underset{Y\to\infty}{o}(1)$$

Using (9) we obtain at last

$$\int_F q(z,\overline{z})v_n(z,\overline{z})E(z,\overline{z};s_n)d\mu(z,\overline{z}) =$$
$$(1-2s_n)(c_2'(0) - \varphi(s_n)c_1'(0))$$

Therefore if the inequality holds

$$(11)\qquad \int_F q(z,\overline{z})v_n(z,\overline{z})E(z,\overline{z};s_n)d\mu(z,\overline{z}) \neq 0$$

then one of $c_1'(0), c_2'(0)$ at least is not equal to the zero, and we have the case 2), i.e. the eigenvalue λ_n disappears under the deformation εq.

Lemma 7. *For any eigenvalue $\lambda_n \neq 0$ for the even eigenfunction $v_n(z,\overline{z})$ of the operator A there exists a potential $q(z,\overline{z}) = q_n(z,\overline{z}), q_n \in \mathcal{N}$ (see Remark to Lemma 5) such that (11) holds.*

Proof. We assume the contrary. Then for some fixed s_n the following equality holds

$$(12) \qquad \int_F q(z,\overline{z})v_n(z,\overline{z})E(z,\overline{z};s_n)d\mu(z,\overline{z}) = 0$$

for all $q \in \mathcal{N}$ (see Remark to Lemma 5). Therefore the function $v_n E_{s_n}$ is orthogonal to the whole subspace Θ in $\mathcal{H}$ and as a result we have

$$(13) \qquad \int_0^1 v_n(z,\overline{z})E(z,\overline{z};s_n)dx = 0$$

for all $y > 0$. We multiply (13) by y^s, Res > 1 and we obtain

$$0 = \int_0^\infty y^s \frac{dy}{y^2} \int_0^1 v_n(z,\overline{z})E(z,\overline{z};s_n)dx =$$

$$\int_F E(z,\overline{z};s_n)v_n(z,\overline{z})E(z,\overline{z};s_n)d\mu(z,\overline{z})$$

The last integral is not equal to zero identicaly, because it is some special Rankin-Selberg convolution (see [13], for example). We have the contradiction to (12) and it proves Lemma 7.

Lemma 8. *There exists a potential (a set of potentials) q_∞ with properties 1) For the corresponding operator A_{q_∞} the assertions of Lemma 6 are valid. 2) The discrete spectrum of A_{q_∞} of even eigenfunctions is a finite set of eigenvalues of a finite multiplicity $\lambda_n \in [0, 1/4)$.*

Proof. We consider the following series

$$(14) \qquad q(z,\overline{z}) = \sum_{n=1}^{\infty} d_n q_n(z,\overline{z}),$$

where q_n run through the set of all potentials from Lemma 7 for all eigenvalues λ_n of A in the space $\mathcal{H}_0^{(2)}$ $d_n \geqslant 0$ are some constants which we define now. We remark that the potential q_n from Lemma 7 is defined up to the multiplication by a positive constant at least. Therefore we can suppose that

$$\max_{z \in F} \quad q_n(z,\overline{z})$$

decreases as fast as we want. We write here "max" instead of "sup" because q_n has a compact support. Thus for the absolute convergence of the series (14) it is enough to consider bounded sequences $\{d_n\}$.

We prove now that there exists a bounded sequences $\{d_n\}$, $d_n \geqslant 0$, such that for the corresponding potential $q(z,\overline{z})$ from (14) the inequality (11) holds for all n from Lemma 7.

We assume the contrary. That means there exists an eigenvalue λ_n and an even eigenfunction v_n of A with the equality (12) which holds for any potential (14) constructed for all bounded sequences $\{d_k\}$. But there is the contradition. Namely, for the sequences

$$d_k = \begin{cases} 1 & k = n \\ 0 & \text{otherwise} \end{cases}$$

we have $q = q_n$ and the inequality (11) from the definition of q_n. Therefore we proved the existence of a desired bounded sequences d_n. We fix it and we define the corresponding potential q. We remark that from the decreasing property of

$$\max_F \quad q_n(z,\overline{z})$$

mentioned above and $|d_n| \leqslant c$, for some $c > 0$ and for all n, if follows that we can suppose the estimate

$$q(z,\overline{z}) = O(e^{-\alpha y}), z \in F, y = \mathrm{Im} z$$

$\alpha > 0$ is some constant.

Therefore the potential q satisfies the conditions of Theorem 4 from §2 [9] and for the corresponding operator A_q the assertions of Lemma 6 of this paper are valid. The same result is true for any potential εq where $\varepsilon > 0$ is some constant.

We consider now ε from $0 \leqslant \varepsilon \leqslant \varepsilon_0$ where $\varepsilon_0 > 0$ is some small constant and we investigate the eigenvalues $\lambda_n(\varepsilon)$ of $A_{\varepsilon q}$ for even eigenfunctions, $\lambda_n(\varepsilon) \geqslant 1/4$. We watch these eigenvalues when ε varies from 0 to ε_0.

Let $s_n(\varepsilon)$ be the new variable $s_n(\varepsilon)(1 - s_n(\varepsilon)) = \lambda_n(\varepsilon)$. We suppose $\mathrm{Im} s_n(\varepsilon) \geqslant 1/2$. From the principle of analiticity it follows that $s_n(\varepsilon)$ comes or from $s_n^{(1)}(0)$ or from $s_n^{(2)}(0)$. Here $\lambda_n = s_n^{(i)}(0)(1 - s_n^{(i)}(0))$ is the eigenvalue of the operator A and $s_n^{(2)}(0) = s_n^{(2)}$ is the pole of the automorphic scattering matrix $\varphi(s)$ (one dimensional)

$$E(z,s) = y_s + \varphi(s) y^{1-s} + \underset{y \to \infty}{\mathcal{O}(1)}$$

In both cases for any fixed n there exists only a finite number of values ε, say e_j we name them as "bad" values, for which the equality $\mathrm{Re} s_n(\varepsilon_j) = 1/2$ holds. We define the set

$$\Omega = \bigcup_n \bigcup_j \varepsilon_j(n)$$

of all "bad" values for all $\lambda_n(\varepsilon)$. It is clear Ω is the countable set. Therefore there exists a sequences of "good" values $\{\delta_k\}, \delta_k \in \{(0,\varepsilon_0) - \Omega\}$ such that $\delta_k \to 0$ when $k \to \infty$ and for any $\delta_k q$ the corresponding $A_{\delta_k q}$ has no discrete spectrum embedded in continuous one (in the space $\mathcal{H}^{(2)}$). The proof of Lemma 8 is complete.

Proof of the Main Theorem *Using Lemmas 1-8 the proof is almost complete. We consider the set of operators $A_{\delta_k}q$ from Lemma 8 in the space $\mathcal{H}^{(2)}$. Let $\Theta(\delta_k q)$ be the subspace of the continuous spectrum of $A_{\delta_k}q$. By analogy we can see*

$$\Theta(\delta_k q) \subset \hat{M}_\infty^{\mathcal{H}}$$

for all k. For any k we have the spectral decomposition

$$\mathcal{H}^{(2)} = \Theta(\delta_k q) \oplus \mathcal{D}_k$$

where $\mathcal{D}_k$ is a finite dimensional subspace of the discrete spectrum of $A_{\delta_k q}$. We remark

$$\mathcal{H}^{(2)} = \hat{M}_\infty^{\mathcal{H}} \oplus \mathcal{D}$$
$$\mathcal{D} = \bigcap_k \mathcal{D}_k$$

We know also that in the limit $\delta_k \to 0$ the operator A has no any discrete spectrum in the interval $\lambda \in (0, 1/4]$. Therefore $\mathcal{D}$ is a trivial linear space $\mathcal{D} = \{0\}$ and the proof of the Main Theorem is complete.

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S.-PETERSBOURG DEPARTMENT OF THE STEKLOV MATHEMATICAL INSTITUTE OF THE RUSSIAN ACADEMY OF SCIENCES

Current address: Max-Planck-Institut für Mathematik Gottfried-Claren-Str. 26 53225 Bonn Germany