Max-Planck-Institut für Mathematik Bonn

P-Schur positive P-Grothendieck polynomials

by

Graham Hawkes

Max-Planck-Institut für Mathematik
Preprint Series 2020 (8)

P-Schur positive P-Grothendieck polynomials

Graham Hawkes

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

MPIM 20-8

P-SCHUR POSITIVE P-GROTHENDIECK POLYNOMIALS

GRAHAM HAWKES

Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany

Abstract. The symmetric Grothendieck polynomials generalize Schur polynomials and are Schur-positive by degree. Combinatorially this is manifested as the generalization of semistandard Young tableaux by set-valued tableaux. We define a (weak) symmetric P-Grothendieck polynomial which generalizes P-Schur polynomials in the same way. Combinatorially this is manifested as the generalization of shifted semistandard Young tableau by a new type of tableaux which we call shifted multiset tableaux.

E-mail address: hawkes@math.ucdavis.edu.

Key words and phrases. Combinatorics [05], Grothendieck polynomials, P-Schur polynomials, multiset tableaux, RSK correspondence.

G. Hawkes is supported by the Max-Planck-Institut für Mathematik.

1. Introduction

Symmetric Grothendieck polynomials and their duals, weak symmetric Grothendieck polynomials, are families of nonhomogeneous symmetric polynomials indexed by Grassmaninian permutations, or equivalently, by partitions. The former are special cases of the Grothendieck polynomials of Lascoux and Schützenberger [LS82, LS83]. Moreover, the stable Grothendieck polynomials of Fomin and Kirillov [FK94, FK96] expand with positive integer coefficients in terms of symmetric Grothendieck polynomials [MPS18], and weak stable Grothendieck polynomials expand with positive integer coefficients in terms of weak symmetric Grothendieck polynomials [HS19]. For these reasons symmetric and weak symmetric Grothendieck polynomials are fundamental building blocks in the subject of nonhomogeneous symmetric functions of type A. Moreover, they are what we call *natural nonhomogeneous generalizations* of Schur polynomials by which we mean:

Definition 1.1. We say that a family $\mathscr{B}$ of polynomials indexed by partitions is a *natural nonhomogeneous generalization* of family of homogeneous polynomials $\mathscr{A}$ if:

- For any μ, there is an algebraically defined polynomial $\mathscr{C}_\mu(\mathbf{x}, \mathbf{t})$ such that $\mathscr{A}_\mu(\mathbf{x}) = \mathscr{C}_\mu(\mathbf{x}, \mathbf{0})$ and $\mathscr{B}_\mu(\mathbf{x}) = \mathscr{C}_\mu(\mathbf{x}, \mathbf{1})$.
- $\mathscr{B}_\mu(\mathbf{x}) = \sum c^\mu_\lambda \mathscr{A}_\lambda(\mathbf{x})$ for some nonnegative integer coefficients c^μ_λ.

The theory of Schubert polynomials of type C is also well developed [Lam95]: Whereas stable limits of Schubert polynomials of type A (Stanley symmetric functions [Sta84]) are known to expand in terms of Schur polynomials, stable limits of Schubert polynomials of type C are known to expand in terms of P-Schur polynomials [HPS17]. Our goal is to find a *natural nonhomogeneous generalization* of P-Schur polynomials, $\mathfrak{P}_\mu(\mathbf{x})$ to better understand the theory of nonhomogeneous symmetric functions of type C. These polynomials will play the role that weak symmetric Grothendieck polynomials, $\mathfrak{J}_\mu(\mathbf{x})$ play in type A. We also introduce a multiparameter $\mathbf{t} = t_1, \ldots, t_\ell$ deformation of both these polynomials, $\mathfrak{P}_\mu(\mathbf{x}, \mathbf{t})$ and $\mathfrak{J}_\mu(\mathbf{x}, \mathbf{t})$, respectively, which clarifies the relation between the algebraic and combinatorial definitions of these polynomials, makes the proofs easier to follow, and explains the definition of *natural nonhomogeneous generalization*.

We carry out a complete construction and analysis of both $\mathfrak{J}_\mu(\mathbf{x}, \mathbf{t})$ and $\mathfrak{P}_\mu(\mathbf{x}, \mathbf{t})$, even though the former are already well understood at $\mathbf{t} = (1 \ldots, 1)$ (e.g., [HS19]) for the following reasons: First, the arguments and constructions used in the $\mathfrak{P}_\mu$ are almost always generalizations or alterations of those used in the $\mathfrak{J}_\mu$ case and the former is much easier to comprehend once the latter (generally simpler) case is understood. Secondly, it is instructive to be able to compare the two situations side by side.

We give algebraic and combinatorial definitions of both $\mathfrak{J}_\mu(\mathbf{x}, \mathbf{t})$ and $\mathfrak{P}_\mu(\mathbf{x}, \mathbf{t})$. The main theorems of the paper are showing that they are equivalent. The underlying combinatorial objects in the first case are multiset tableaux and the underlying combinatorial objects in the second case a new type of tableaux which we call shifted multiset tableaux even though, as will be seen, they are a mix of the notions of multiset and set tableaux.

We note that similar nonhomogeneous, or K-theoretic, generalizations of P-Schur polynomials such as in [IN13] and [HKP$^+$17] have been made, but differ from ours in that they do not satisfy the second bullet point in our definition of *natural nonhomogeneous generalization*, i.e., are not themselves P-Schur positive.

2. Lemma

We begin with a basic lemma about how to multiply symmetric polynomials by a sequence of homogeneous symmetric polynomials in a weakly increasing number of variables. The case when $n = c_\ell = \cdots = c_1$ can be found in most books on symmetric functions, such as [Sta99].

Let $\mu = (\mu_1, \ldots, \mu_n)$ be a partition with distinct parts and fix integers $n \geq c_\ell \geq \cdots \geq c_1$. Then for any list of ℓ nonnegative integers, $\mathbf{T} = T_\ell, \ldots, T_1$ define a $\mathbf{T}$-extension of μ to be a sequence of compositions, $\lambda = \lambda^\ell \supseteq \cdots \supseteq \lambda^1 \supseteq \lambda^0 = \mu$ such that $|\lambda^h| - |\lambda^{h-1}| = T_h$ and $\lambda^h_k = \lambda^{h-1}_k$ for $k > c_i$ for all $1 \leq h \leq \ell$. A $\mathbf{T}$-extension of μ is called *good* if $\lambda^h_k < \lambda^{h-1}_{k-1}$ for $2 \leq k \leq c_h$ for all $1 \leq h \leq \ell$. A $\mathbf{T}$-extension which is not *good* is called *bad*. In particular, every composition in a *good* $\mathbf{T}$-extension is a partition.

Lemma 2.1.

$$\sum_{\sigma \in S_n} sgn(\sigma) h_{T_\ell}(x_{\sigma_1}, \ldots, x_{\sigma_{c_\ell}}) \cdots h_{T_1}(x_{\sigma_1}, \ldots, x_{\sigma_{c_1}}) x^{\mu_1}_{\sigma_1} \cdots x^{\mu_n}_{\sigma_n}$$

$$= \sum_{\lambda=\lambda^\ell \supseteq \cdots \supseteq \lambda^1 \supseteq \lambda^0 = \mu} \left(\sum_{\sigma \in S_n} sgn(\sigma) x^{\lambda_1}_{\sigma_1} \cdots x^{\lambda_n}_{\sigma_n} \right)$$

where the sum is over all good $\mathbf{T}$*-extensions.*

Proof. It suffices to show that

$$\sum_{\lambda=\lambda^\ell \supseteq \cdots \supseteq \lambda^1 \supseteq \lambda^0 = \mu} \left(\sum_{\sigma \in S_n} sgn(\sigma) x^{\lambda_1}_{\sigma_1} \cdots x^{\lambda_n}_{\sigma_n} \right) = 0$$

where the sum is over all bad $\mathbf{T}$-extensions. It suffices to find a sign changing involution, ι on the set of pairs of the form (σ, Λ) where $\sigma \in S_n$, Λ is a bad $\mathbf{T}$-extension and the sign of the pair is the sign of the permutation σ, such that ι has the following property: If $\iota(\sigma, \Lambda) = (\bar{\sigma}, \bar{\Lambda})$ where λ is the largest composition of Λ and $\bar{\lambda}$ is the largest composition of $\bar{\Lambda}$ then $\bar{\lambda}_{(\bar{\sigma}^{-1}(p))} = \lambda_{(\sigma^{-1}(p))}$ for all $1 \leq p \leq n$.

Define $\iota(\sigma, \Lambda)$ as follows: Suppose Λ is the bad $\mathbf{T}$-extension $\lambda = \lambda^\ell \supseteq \cdots \supseteq \lambda^1 \supseteq \lambda^0 = \mu$. Choose i minimal such that there exists some $2 \leq k \leq c_i$ such that $\lambda^i_k \geq \lambda^{i-1}_{k-1}$. Choose the minimal such k, and then choose the minimal $1 \leq j < k$ such that $\lambda^i_k \geq \lambda^{i-1}_j$. Define $\bar{\sigma}(m) = \sigma(m)$ for $m \notin \{j, k\}$, $\bar{\sigma}(j) = \sigma(k)$, and $\bar{\sigma}(k) = \sigma(j)$. Next, for $h < i$ define $\bar{\lambda}^h = \lambda^h$. For $h \geq i$ define $\bar{\lambda}^h_m = \lambda^h_m$ for $m \notin \{j, k\}$, $\bar{\lambda}^h_j = \lambda^h_k$, and $\bar{\lambda}^h_k = \lambda^h_j$. Set $\iota(\sigma, \Lambda) = (\bar{\sigma}, \bar{\Lambda})$ where $\bar{\Lambda}$ is the bad $\mathbf{T}$-extension $\bar{\lambda} = \bar{\lambda}^\ell \supseteq \cdots \supseteq \bar{\lambda}^1 \supseteq \bar{\lambda}^0 = \mu$. (That the $\supseteq$ are correct, and that $\bar{\Lambda}$ is a bad $\mathbf{T}$-extension is proven below).

Note the following properties of ι.

(1) $\iota(\sigma, \Lambda)$ has the opposite sign as (σ, Λ).
(2) $\bar{\lambda}_{(\bar{\sigma}^{-1}(p))} = \lambda_{(\sigma^{-1}(p))}$ for all $1 \leq p \leq n$.
(3) $\bar{\Lambda}$ is a $\mathbf{T}$-extension.
   - That $|\bar{\lambda}^h| - |\bar{\lambda}^{h-1}| = T_h$ is immediate.

- Suppose that $m > c_h$, we wish to check that $\bar{\lambda}^h_m = \bar{\lambda}^{h-1}_m$. Now if $m \in \{j,k\}$ and $h \geq i$ we have $j,k \leq c_i \leq c_h$ so the condition $m > c_h$ is impossible to attain. Thus we may assume that $m \notin \{j,k\}$ or $h < i$ in which case we have $\lambda^h_m = \bar{\lambda}^h_m$ and $\lambda^{h-1}_m = \bar{\lambda}^{h-1}_m$ so that the equality $\lambda^h_m = \lambda^{h-1}_m$ implies the equality $\bar{\lambda}^h_m = \bar{\lambda}^{h-1}_m$.
- Next, it is clear that $\bar{\lambda}^h \supseteq \bar{\lambda}^{h-1}$ if $h \neq i$ and also that $\bar{\lambda}^h_m > \bar{\lambda}^{h-1}_m$ for $m \notin \{j,k\}$. We need only check that $\bar{\lambda}^i_j \geq \bar{\lambda}^{i-1}_j$ and $\bar{\lambda}^i_k \geq \bar{\lambda}^{i-1}_k$. The first is equivalent to saying that $\lambda^i_k \geq \lambda^{i-1}_j$ which is true by the choice of j and k. The second is equivalent to saying that $\lambda^i_j \geq \lambda^{i-1}_k$ but $\lambda^i_j \geq \lambda^{i-1}_j$ since $\lambda^i \supseteq \lambda^{i-1}$ and $\lambda^{i-1}_j \geq \lambda^{i-1}_k$ by minimality of i.

(4) $\bar{\Lambda}$ is a bad **T**-extension. Indeed, $\bar{\lambda}^i_k = \lambda^i_j \geq \lambda^{i-1}_j = \bar{\lambda}^{i-1}_j$.

(5) $\iota^2(\sigma,\Lambda) = (\sigma,\Lambda)$. It is clear from the definitions that this is true as long as the values of i,k,j chosen when applying ι to (σ,Λ) are the same as those (say $\bar{i},\bar{k},\bar{j}$) chosen when applying ι to $(\bar{\sigma},\bar{\Lambda})$. Clearly $\bar{i} \geq i$, and, by the step above, $\bar{i} \leq i$, so $\bar{i} = i$. If $j \neq \bar{k} < k$ then $\lambda^i_{\bar{k}} = \bar{\lambda}^i_{\bar{k}} \geq \bar{\lambda}^{i-1}_{\bar{k}-1} = \lambda^{i-1}_{\bar{k}-1}$, contradicting the minimality of k. If $\bar{k} = j$ then $\lambda^i_k = \bar{\lambda}^i_j \geq \bar{\lambda}^{i-1}_{j-1} = \lambda^{i-1}_{j-1}$, contradicting the minimality of j. Since the step above implies $\bar{k} \leq k$ this means $\bar{k} = k$. Finally, if $\bar{j} < j$ then $\lambda^i_j = \bar{\lambda}^i_k \geq \bar{\lambda}^{i-1}_{\bar{j}} = \lambda^{i-1}_{\bar{j}}$, contradicting the minimality of k. Again, the step above means $\bar{j} \leq j$ so together we get $\bar{j} = j$.

This shows that ι is a well defined sign changing involution with the desired property, proving the lemma. □

3. $\mathfrak{J}_\mu(\mathbf{x},\mathbf{t})$ and multiset tableaux

3.1. **Algebraic Definition of $\mathfrak{J}_\mu(\mathbf{x},\mathbf{t})$.** *We will always work in n variables and will set $V = \prod_{i<j}(x_i - x_j)$. In general we will define a symmetric polynomial f by defining the value of the skew-symmetric polynomial $V * f$.*

For a partition μ of n parts, the weak symmetric Grothendieck polynomial in n variables is defined by:

$$V * \mathfrak{J}_\mu(x_1,\ldots,x_n) = \sum_{\sigma\in S_n} sgn(\sigma) \prod_i \left(\left(\frac{x_{\sigma_i}}{1-x_{\sigma_i}} \right)^{\mu_i} x^{n-i}_{\sigma_i} \right)$$

We define a slight generalization of this polynomial. Suppose μ has longest part $\ell = \mu_1$. Let the weak symmetric Grothendieck polynomial in $n+\ell$ variables be defined by:

$$V * \mathfrak{J}_\mu(x_1,\ldots,x_n,t_1,\ldots,t_\ell) = \sum_{\sigma\in S_n} sgn(\sigma) \prod_i \left(\left(\frac{x_{\sigma_i}}{1-t_\ell x_{\sigma_i}} \right) \cdots \left(\frac{x_{\sigma_i}}{1-t_{\ell-\mu_i+1} x_{\sigma_i}} \right) x^{n-i}_{\sigma_i} \right)$$

Clearly $\mathfrak{J}_\mu(x_1,\ldots,x_n,1,\ldots,1) = \mathfrak{J}_\mu(x_1,\ldots,x_n)$ whereas $\mathfrak{J}_\mu(x_1,\ldots,x_n,0,\ldots,0) = s_\mu(x_1,\ldots,x_n)$. Note that the coefficient of $t_1^{T_1}\cdots t_\ell^{T_\ell}$ in $V * \mathfrak{J}_\mu(x_1,\ldots,x_n,t_1,\ldots,t_\ell)$ is given by:

$$\sum_{\sigma \in S_n} sgn(\sigma) h_{T_\ell}(x_{\sigma_1}, \ldots, x_{\sigma_{c_\ell}}) \cdots h_{T_1}(x_{\sigma_1}, \ldots, x_{\sigma_{c_1}}) x_{\sigma_1}^{\mu_1+n-1} \cdots x_{\sigma_n}^{\mu_n+0}$$

where $(c_\ell, \ldots, c_1) = \mu'$. Since $n \geq c_\ell \geq \cdots \geq c_1$ Lemma 2.1 implies that this coefficient is:

$$\sum_{\lambda=\lambda^\ell \supseteq \cdots \supseteq \lambda^1 \supseteq \lambda^0 = \mu+\delta} \left(\sum_{\sigma \in S_n} sgn(\sigma) x_{\sigma_1}^{\lambda_1} \cdots x_{\sigma_n}^{\lambda_n} \right)$$

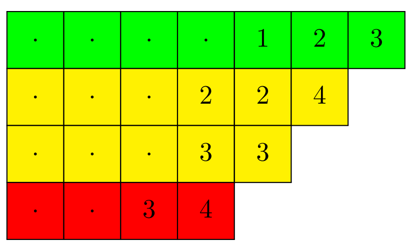
where the sum is over all good $\mathbf{T}$-extensions and $\delta = (n-1, \ldots, 0)$. Interpreting each $\lambda_i \setminus \lambda_{i-1}$ as a strip filled with is and then shifting the result to the left by δ one can deduce that this coefficient is the same as:

$$V * \sum_{\lambda \supseteq \mu} (M^{\mathbf{T}}_{\lambda/\mu}) s_\lambda$$

where $M^{\mathbf{T}}_{\lambda/\mu}$ is the number of semistandard Young tableaux of shape λ/μ and weight $T_1, \ldots, T_\ell$ such that every entry i occurs on or above row c_i.

Definition 3.1. Let μ be a partition with n parts and conjugate $\mu' = (c_\ell, \ldots, c_1)$. We define a ***restricted tableau*** of shape λ/μ, or element of $RT(\lambda/\mu)$, to be a semistandard Young tableau of shape λ/μ in the alphabet $\{1, \ldots, \ell\}$ such that each entry i occurs on or above row c_i. If $R \in RT(\lambda/\mu)$ then the weight of R, denoted $wt(R)$ is the vector $(w_1, \ldots, w_\ell)$ where w_i is the number of is which appear in R.

Example 3.2. Let $\lambda = (7, 6, 5, 4)$ and $\mu = (4, 3, 3, 2)$ so that $c_4 = 4$, $c_3 = 4$, $c_2 = 3$, $c_1 = 1$.



Since all 1s lie in the green all 2s lie in the green or yellow and all 3s and all 4s lie in the red, yellow, or green, this is an element of $RT(\lambda/\mu)$. It has weight $(1, 3, 4, 2)$.

With this definition, the computation before the definition shows:

Theorem 3.3. *Let* $\mathbf{t} = t_1, \ldots, t_\ell$, $\mathbf{x} = (x_1, \ldots, x_n)$ *then*

$$\mathfrak{J}_\lambda(x_1, \ldots, x_n, t_1, \ldots, t_\ell) = \sum_{\lambda \supseteq \mu} \sum_{R \in RT(\lambda/\mu)} \mathbf{t}^{wt(R)} s_\lambda(\mathbf{x})$$

3.2. **Straight-shape multiset tableaux.**

Definition 3.4 ([LP07])**.** Given a partition μ, with conjugate $(c_\ell, \ldots, c_1) = \mu'$ a ***multiset tableau*** of shape μ, or an element of $MT(\mu)$ is a collection of boxes with μ_i boxes in each row and the rows left-justified, along with a filling of said boxes with the following properties.

(1) Each box contains a nonempty multiset of the numbers $\{1, 2, \ldots\}$.
(2) The maximum value of each box is strictly less than the minimum value of the box below it (if it exists) and weakly less than the minimum value of the box to its right (if it exists).

The weight, denoted wt, of a multiset tableaux is the vector $(w_1, w_2, \ldots)$ where w_i is the total number of is appearing in the tableau. We label the columns from left to right by $\ell, \ell-1, \ldots, 1$. That is, by box b_{ij} we refer to the box which is in the i^{th} row from the top row and the $\ell-j+1^{st}$ column from the leftmost column. Define the column weight of a multiset tableau, cw, to be the vector $(T_1, \ldots, T_\ell)$ where T_i is the difference between the number of entries in column i and the height of that column (c_i). By $|b_{ij}|$ we simply mean the total number of entries in box b_{ij} and $|b_{ij}(x)|$ refers more specifically to box the number of entries in box b_{ij} in tableau x. By the nonemptiness property $|b_{ij}| \geq 1$ if box b_{ij} exists and, by convention is 0 otherwise.

Example 3.5. Let $\mu = (3, 3, 2)$. Then

11	12	333
2	3	445
34	4	

is an element $P \in MT(\mu)$ with $wt(T) = (3, 2, 5, 4, 1)$ and $cw(P) = (4, 1, 2)$.

Definition 3.6. A ***maximal multiset tableau*** of shape μ, or element of $\overline{MT}(\mu)$, is a multiset tableau of shape μ with the following properties:

(1) Each box b_{ij} may only contain is.
(2) For each $i \geq 1$ and $k \geq 0$ we have $\sum_{1 \leq j \leq k} |b_{(i+1)j}| - |b_{i(j-1)}| \leq 1$

where by convention $|b_{i0}| = 0$.

Example 3.7. Let $\mu = (4, 3, 3, 1)$. Then

1	11	11	11
22	2	222	
3	333	3	
44	44		

is an element $P \in \overline{MT}(\mu)$ with $wt(T) = (7, 6, 5, 4)$ and $cw(P) = (1, 3, 4, 2)$.

Proposition 3.8. *There is a bijection from the subset of $\overline{MT}(\mu)$ with weight λ and column weight $\mathbf{T}$ to the subset of $RT(\lambda/\mu)$ with weight $\mathbf{T}$.*

Proof. Let X be the subset of $MT(\mu)$ with weight λ and column weight $\mathbf{T}$ that satisfy property (1) above. Let Y be the set of weakly increasing by row fillings of shape λ/μ and weight $\mathbf{T}$ such that every entry i occurs on or above row c_i (equivalently: row i only contains entries greater than $\ell - \mu_i$). The map $x \to y$ where y is defined by the property that for each (i, j), row i of y contains exactly

$|b_{ij}(x)|-1$ copies of j is a bijection from X to Y. Moreover if $x \to y$ then x satisfies property (2) above if and only if the columns of y are strictly decreasing down rows: Indeed, if there is some i and some k such that $\sum_{1\le j\le k} b_{(i+1)j} - b_{i(j-1)} > 1$ then for the minimal such k, row $i+1$ of y will have an entry k that lies above an entry k' of row i with $k' \geq k$. On the other hand, if row $i+1$ of y contains a k which lies above some k' in row i with $k' \geq k$ then we are guaranteed to have $\sum_{1\le j\le k} b_{(i+1)j} - b_{i(j-1)} > 1$. Since the elements of Y that are strictly decreasing down columns are exactly the elements of $RT(\lambda/\mu)$ with weight $\mathbf{T}$, the map restricted to the elements of X that satisfy property (2) gives the desired bijection. □

Example 3.9. The tableaux of examples 3.2 and 3.7 correspond under this bijection.

Corollary 3.10. *Set* $\mathbf{t} = t_1, \ldots, t_\ell$, $\mathbf{x} = (x_1, \ldots, x_n)$ *then*

$$\mathfrak{J}_\mu(x_1,\ldots,x_n,t_1,\ldots,t_\ell) = \sum_{P\in \overline{MT}^\lambda_\mu} \mathbf{t}^{cw(P)} s_{wt(P)}(\mathbf{x})$$

3.3. **Combinatorial Definition of $\mathfrak{J}_\mu(\mathbf{x},\mathbf{t})$.** In this section we will give an equivalent combinatorial definition of $\mathfrak{J}_\mu$. We will need to use the *dual RSK* column insertion algorithm (see, for instance [Sta99]). We refer to dual RSK insertion of an element into a column, and the reverse insertion of an element under dual RSK as *insert* and *reverse insert*. These maps are reviewed below.

Let K be a valid column (each box of K contains exactly one number and the numbers strictly decrease from top to bottom). One *inserts* a into K, denoted $a \to K$ as follows: Let $\hat{a}$ denote the uppermost entry in K such that $a \leq \hat{a}$. If $\hat{a}$ exists, replace $\hat{a}$ with a and bump out $\hat{a}$. Otherwise, append a to the bottom of K. The result is recorded as the pair $(K', \hat{a})$ if the second of this pair exists and just K' otherwise. On the other hand if $z \leq a$ for some $a \in K$ then we define *reverse insertion* of z into K or $K \leftarrow a$ as follows: Let $\hat{z}$ denote the bottommost entry in K such that $z \geq \hat{z}$. Replace $\hat{z}$ with z and bump out $\hat{z}$. The result is recorded as the pair $(\hat{z}, K')$.

Notice the basic properties:

(1) If $a \to K = K'$ then K' is a valid column.
(2) if $a \to K = (K', \hat{a})$ then K' is a valid column.
(3) If $K \leftarrow z = (\hat{z}, K')$ then K' is a valid column.
(4) If $a \leq z$ then either
   - $z \to K = K'$ and $a \to K' = (K'', \hat{a})$ for some $\hat{a}$.
   - $z \to K = (K', \hat{z})$ and $a \to K' = (K'', \hat{a})$ where $\hat{a} \leq \hat{z}$.
(5) If $a \leq z$ and $K \leftarrow a = (\hat{a}, K')$ and $K' \leftarrow z = (\hat{z}, K'')$ then $\hat{a} \leq \hat{z}$.

Fix μ a partition with conjugate $\mu' = (c_\ell, \ldots, c_1)$.

Proposition 3.11. *There is a bijection* $\Psi : MT(\mu) \to \bigcup_{\lambda\supseteq\mu} SSYT(\lambda) \times RT(\lambda/\mu)$, *such that if* $P \to (Q, R)$ *then:*

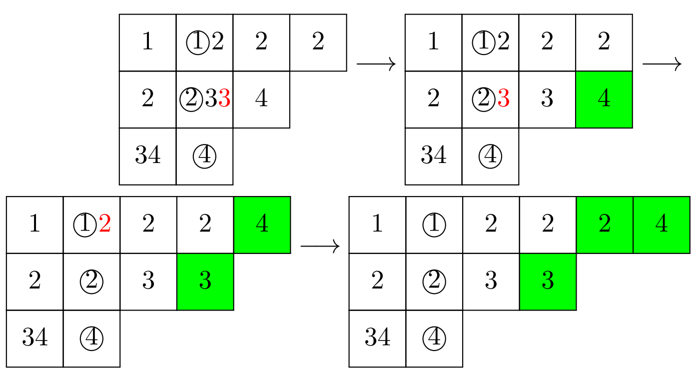
(1) $wt(P) = wt(Q)$.
(2) $cw(P) = wt(R)$.

First some reductions. Define the set $MT_k(\lambda)$ to be the subset of $MT(\lambda)$ which have only single entries in columns $k-1, \ldots, 1, 0, -1, \ldots$. Define the set $RT_k(\lambda/\mu)$ to be the subset of $RT(\lambda/\mu)$ which have only entries from $\{1, 2, \ldots, k-1\}$. Given a pair $(Q, R) \in MT_k(\lambda) \times RT_k(\lambda/\mu)$ define the weight and column weight of this pair as $wt(Q, R) = wt(Q)$ and $cw(Q, R) = cw(Q) + wt(R)$. To achieve our goal it suffices to find a weight and column weight preserving bijection for each k (and then compose: $\Psi = \Psi_\ell \circ \cdots \circ \Psi_1$) from $\bigcup_{\lambda \supseteq \mu} MT_k(\lambda) \times RT_k(\lambda/\mu)$ to $\bigcup_{\lambda \supseteq \mu} MT_{k+1}(\lambda) \times RT_{k+1}(\lambda/\mu)$. To do the latter, it is enough to find a weight preserving bijection $\Psi_k : MT_k(\lambda) \to \bigcup_{\nu \supseteq \lambda} MT_{k+1}(\nu)$ where the union is over all ν such that ν/λ is a horizontal strip with no box below row c_k (equivalently, below the lowest box in column k of λ: in the previous map the only $\lambda \supseteq \mu$ appearing have length of column k equal to c_k (equal to the length of column k of μ)).

Ψ_k will be defined by repetitively applying the following map: Let $T \in MT_k(\lambda)$. Define $out(T)$ as follows: First, in each box of column k circle (one of) the minimum entry(s) from that box. Now find (one of) the largest noncircled entry(s) in column k and remove it and *insert* it into the column to the right of the column from which it was removed. After this, each time an element is bumped, *insert* it into the next column to the right until some entry is eventually appended to a (possibly empty) column. ① Note the following properties of *out*.

(1) The path of positions where an element is bumped/appended moves weakly down as we move to the right.
(2) The result of *out* is a multiset tableau.
(3) If $out(T)$ and $out(out(T))$ are both defined then the box which *out* appends to $out(T)$ lies strictly to the right of the box that *out* appends to T.

Example 3.12. Suppose that $k = 2$. Each $\longrightarrow$ represents an application of *out*.



Uncircled numbers being removed are shown in red, and the boxes being added appear in green.

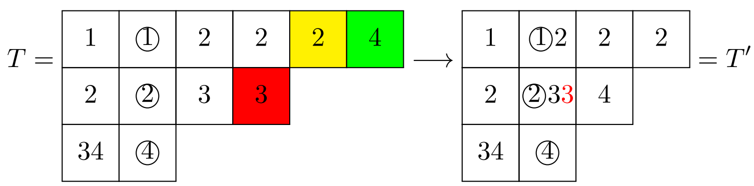
We will also need a map called in_b. Let $T \in MT_k(\nu)$ for some ν such that ν/λ is a horizontal strip with no box below row c_k and suppose b is some corner box of this strip. First, in each box of column k circle (one of) the minimum entry(s) from that box. Define $in_b(T)$ as follows: Remove the entry from box b and *reverse insert* it into the column to the left. After this, each time an element is bumped *reverse insert* it into the column to the left until an element is removed from column $k-1$. Then add this element to the lowest box of column k such that the resulting column satisfies the column strict requirement in (2) of the definition of multiset tableau. Note the following properties of $\mathbf{in}_b$.

(1) The path of positions where an element is bumped/added moves weakly up as we move to the left.
(2) The result of in_b is a multiset tableau.
(3) If b' lies to the left of b and if $in_b(T)$ and $in_{b'}(in_b(T))$ are both defined then the element that $in_{b'}$ adds to column k of $in_b(T)$ is greater than or equal to the element in_b adds to column k of T.

Moreover, *out* and in_b are related as follows:

(1) If *out* appends box b when applied to T, then $in_b(out(T)) = T$.
(2) If the element that in_b adds to column k when applied to T is the largest or tied for the largest uncircled element on column k then $out(in_b(T)) = T$.

Example 3.13. Let $k = 2$. Then $in_{red}(in_{yellow}(in_{green}(T))) = T'$ where:



Note that T is the last tableau in example 3.13 and T' is the first tableau in example 3.13.

Proof. We prove there exists a bijection $\Psi_k : MT_k(\lambda) \to \bigcup_{\nu \supseteq \lambda} MT_{k+1}(\nu)$. If $T \in MT_k(\lambda)$ we define $\Psi_k(T)$ simply by applying *out* until column k only contains single entries. This is an element of $\bigcup_{\nu \supseteq \lambda} MT_{k+1}(\nu)$ because of the properties (1), (2), and (3) of *out*. If $T \in \bigcup_{\nu \supseteq \lambda} MT_{k+1}(\nu)$ we define $\Psi_k^{-1}(T)$ by successively applying in_b to the rightmost box b which lies outside of the shape of λ, until the result has shape λ. This is an element of $MT_k(\lambda)$ because of the property (2) of in_b. If $T \in MT_k(\lambda)$ then $\Psi_k^{-1}(\Psi_k(T)) = T$ because of property (3) of *out* and property (1) of how *out* and in_b are related. If $T \in \bigcup_{\nu \supseteq \lambda} MT_{k+1}(\nu)$ then $\Psi_k(\Psi_k^{-1}(T)) = T$ by property (3) of in_b and property (2) of how *out* and in_b are related. ☐

Theorem 3.14. *Set* $\mathbf{t} = t_1, \ldots, t_\ell$, $\mathbf{x} = (x_1, \ldots, x_n)$

$$\mathfrak{J}_\mu(x_1, \ldots, x_n, t_1, \ldots, t_\ell) = \sum_{P \in MT(\mu)} \mathbf{t}^{cw(P)} \mathbf{x}^{wt(P)}$$

Proof.

$$
\begin{aligned}
& \mathfrak{J}_\mu(x_1,\dots,x_n,t_1,\dots,t_\ell) \\
= & \sum_{\lambda\supseteq\mu}\sum_{R\in RT(\lambda/\mu)} \mathbf{t}^{wt(R)} s_\lambda(\mathbf{x}) && \text{Theorem 3.3} \\
= & \sum_{\lambda\supseteq\mu}\sum_{R\in RT(\lambda/\mu)}\sum_{Q\in SSYT(\lambda)} \mathbf{t}^{wt(R)}\mathbf{x}^{wt(Q)} && \text{Def. of } s_\lambda \\
= & \sum_{P\in MT(\mu)} \mathbf{t}^{cw(P)}\mathbf{x}^{wt(P)} && \text{Prop. 3.11}
\end{aligned}
$$

□

Remark 3.15. There is a natural crystal structure on the set of semistandard Young tableaux [BS17]. Moreover, it is not difficult to see that the bijection Ψ has the property that whenever $\Psi(P) = (Q,R)$ then $P \in \overline{MT}(\mu)$ if and only if Q is highest weight. Thus Ψ^{-1} induces a natural crystal structure on $MT(\mu)$ where the highest weight elements are precisely those that lie in $\overline{MT}(\mu)$. This crystal structure is interpreted algebraically by comparing Corollary 3.10 (where the sum is over highest weight elements) with Theorem 3.14 (where the sum is over all elements). This crystal structure coincides with that given in [HS19].

4. $\mathfrak{J}_\mu(\mathbf{x},\mathbf{t})$ and shifted multiset tableaux

4.1. **Algebraic Definition of $\mathfrak{J}_\mu(\mathbf{x},\mathbf{t})$.** For a strict partition μ of m nonzero parts, we define the weak symmetric P-Grothendieck polynomial in $n \geq m$ variables by:

$$
V * \mathfrak{P}_\mu(x_1,\dots,x_n) = \\
\sum_{\sigma\in S_n/S_{n-m}} sgn(\sigma)\left(\prod_i \left(\frac{x_{\sigma_i}}{1-x_{\sigma_i}}\right)^{\mu_i}\right)\left(\prod_{i<j,i\leq m} x_{\sigma_i}+x_{\sigma_j}\right)\left(\prod_{m<i<j} x_{\sigma_i}-x_{\sigma_j}\right)
$$

where S_n/S_{n-m} refers to the set of permutations of n with no descents after position m. We define a slight generalization of this polynomial. Suppose μ has longest part $\ell = \mu_1$. Let the weak symmetric P-Grothendieck polynomial in $n+\ell$ variables be defined by:

$$
V * \mathfrak{P}_\mu(x_1,\dots,x_n,t_1,\dots,t_\ell) = \\
\sum_{\sigma\in S_n/S_{n-m}} sgn(\sigma)\left(\prod_i \left(\frac{x_{\sigma_i}}{1-t_\ell x_{\sigma_i}}\right)\cdots\left(\frac{x_{\sigma_i}}{1-t_{\ell-\mu_i+1}x_{\sigma_i}}\right)\right)\left(\prod_{i<j,i\leq m} x_{\sigma_i}+x_{\sigma_j}\right)\left(\prod_{m<i<j} x_{\sigma_i}-x_{\sigma_j}\right)
$$

Clearly $\mathfrak{P}_\mu(x_1,\dots,x_n,1,\dots,1) = \mathfrak{P}_\mu(x_1,\dots,x_n)$ whereas $\mathfrak{P}_\mu(x_1,\dots,x_n,0,\dots,0) = P_\mu(x_1,\dots,x_n)$, the P-Schur polynomial. Note that the coefficient of $t_1^{T_1}\cdots t_\ell^{T_\ell}$ in $V * \mathfrak{P}_\mu(x_1,\dots,x_n,t_1,\dots,t_\ell)$ is given by:

$$\sum_{\sigma\in S_n/S_{n-m}} sgn(\sigma)h_{T_\ell}(x_{\sigma_1},\ldots,x_{\sigma_{c_\ell}})\cdots h_{T_1}(x_{\sigma_1},\ldots,x_{\sigma_{c_1}})$$
$$*x_{\sigma_1}^{\mu_1}\cdots x_{\sigma_m}^{\mu_m}\left(\prod_{i<j,i\le m} x_{\sigma_i}+x_{\sigma_j}\right)\left(\prod_{m<i<j} x_{\sigma_i}-x_{\sigma_j}\right)$$

where $(c_\ell,\ldots,c_1)=\mu'$. We can create each permutation in S_n/S_{n-m} by first selecting m variables and then permuting them. This yields:

$$\sum_{\tau\in S_n/(S_{n-m}\times S_m)} sgn(\tau)\sum_{\sigma\in S_m(\tau_1,\ldots,\tau_m)} sgn(\sigma)h_{T_\ell}(x_{\sigma(\tau_1)},\ldots,x_{\sigma(\tau_{c_\ell})})\cdots h_{T_1}(x_{\sigma(\tau_1)},\ldots,x_{\sigma(\tau_{c_1})})$$
$$*x_{\sigma(\tau_1)}^{\mu_1}\cdots x_{\sigma(\tau_m)}^{\mu_m}\left(\prod_{i<j\le m} x_{\sigma(\tau_i)}+x_{\sigma(\tau_j)}\right)\left(\prod_{i\le m,j>m} x_{\sigma(\tau_i)}+x_{\tau_j}\right)\left(\prod_{m<i<j} x_{\tau_i}-x_{\tau_j}\right)$$

The last three products are constant over the choice of σ so we may apply Lemma 2.1 since again $n\ge c_\ell\ge\cdots\ge c_1$. We are left with:

$$\sum_{\tau\in S_n/(S_{n-m}\times S_m)} sgn(\tau)\sum_{\lambda=\lambda^\ell\supseteq\cdots\supseteq\lambda^1\supseteq\lambda^0=\mu}\ \sum_{\sigma\in S_m(\tau_1,\ldots,\tau_m)} sgn(\sigma)x_{\sigma(\tau_1)}^{\lambda_1}\cdots x_{\sigma(\tau_m)}^{\lambda_m}$$
$$*\left(\prod_{i<j\le m} x_{\sigma(\tau_i)}+x_{\sigma(\tau_j)}\right)\left(\prod_{i\le m,j>m} x_{\sigma(\tau_i)}+x_{\tau_j}\right)\left(\prod_{m<i<j} x_{\tau_i}-x_{\tau_j}\right)$$

where the sum is over all good **T**-extensions. Reverting to a sum over a single set of permutations this becomes:

$$\sum_{\lambda=\lambda^\ell\supseteq\cdots\supseteq\lambda^1\supseteq\lambda^0=\mu}\ \sum_{\sigma\in S_n/S_{n-m}} sgn(\sigma)x_{\sigma_1}^{\lambda_1}\cdots x_{\sigma_m}^{\lambda_m}\left(\prod_{i<j,i\le m} x_{\sigma_i}+x_{\sigma_j}\right)\left(\prod_{m<i<j} x_{\sigma_i}-x_{\sigma_j}\right)$$

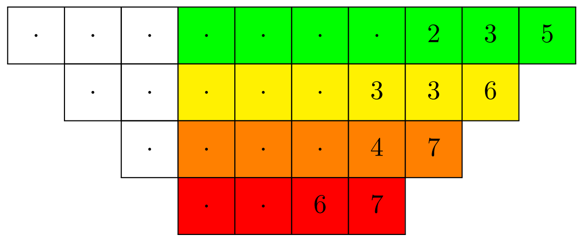
Interpreting each $\lambda_i\ \setminus\lambda_{i-1}$ as a strip filled with is and then shifting the result to the left by $\delta=(m-1,\ldots,0)$ one can deduce that this coefficient is the same as:

$$\sum_{\lambda\supseteq(\mu-\delta)}(N_{\lambda/\mu}^{\mathbf{T}})P_{\lambda+\delta}(x_1,\ldots,x_n)$$

where $N_{\lambda/\mu}^{\mathbf{T}}$ is the number of semistandard Young tableaux of shape $\lambda/(\mu-\delta)$ and weight $T_1,\ldots,T_\ell$ such that every entry i occurs on or above row c_i.

Definition 4.1. Let μ be a partition with m distinct, nonzero parts and conjugate $\mu'=(c_\ell,\ldots,c_1)$ and set $\delta=(m-1,\ldots,0)$. If $\lambda\supseteq\mu$ is a partition of m distinct parts then a ***shifted restricted tableau*** of shape $(\lambda-\delta)/(\mu-\delta)$ is a semistandard Young tableau of this shape using entries in the alphabet $\{1,\ldots,\ell\}$ such that each entry i occurs on or above row c_i. We denote the set of all such tableaux by $SRT(\lambda/\mu)$. If $R\in SRT(\lambda/\mu)$ then the weight of R, denoted $wt(R)$ is the vector $(w_1,\ldots,w_\ell)$ where w_i is the number of is which appear in R.

Example 4.2. Let $\lambda = (10, 8, 6, 4)$ and $\mu = (7, 5, 4, 2)$ so that $c_7 = 4$, $c_6 = 4$, $c_5 = 3$, $c_4 = 3$, $c_3 = 2$, $c_2 = 1$, $c_1 = 1$.



Since all 1s and 2s lie in the green all 3s lie in the green or yellow, all 4s and all 5s lie in the orange, yellow, or green, and all 6s and 7s lie in the red, orange, yellow, or green, this is an element of $SRT(\lambda/\mu)$. It has weight $(0, 1, 3, 1, 1, 2, 2)$.

The statement before the definition now becomes:

Theorem 4.3. *Set* $\mathbf{t} = t_1, \ldots, t_\ell$*,* $\mathbf{x} = (x_1, \ldots, x_n)$ *then:*

$$\mathfrak{P}_\mu(x_1, \ldots, x_n, t_1, \ldots, t_\ell) = \sum_{\lambda \supseteq (\mu - \delta)} \sum_{R \in SRT((\lambda+\delta)/\mu)} \mathbf{t}^{wt(R)} P_{\lambda+\delta}(\mathbf{x})$$

Remark 4.4. Note that $RT_{\lambda/\mu}$ is *not* the same as $SRT_{(\lambda+\delta)/(\mu+\delta)}$ since in the first case we use the constants $(c_\ell, \ldots, c_1) = \mu'$ and the alphabet $\{1, \ldots, \ell\}$ and in the second we would use the constants $(d_{\ell+m-1}, \ldots, d_1) = (\mu + \delta)'$ and the alphabet $\{1, \ldots, \ell + m - 1\}$.

4.2. **Shifted shape multiset tableaux.** In this section we will use the following ordered entries to fill tableaux: $S' = \{1' < 1 < 2' < 2 < 3' < \cdots\}$. We use the following notation. Let $a, z \in S'$

- $a <_u z$ means $a < z$ or else $a = z$ and they are unprimed.
- $a <_p z$ means $a < z$ or else $a = z$ and they are primed.
- $a >_u z$ means $a > z$ or else $a = z$ and they are unprimed.
- $a >_p z$ means $a > z$ or else $a = z$ and they are primed.

Definition 4.5. Given a partition with distinct parts, $\mu = (\mu_1, \ldots, \mu_\ell)$, a ***signed shifted multiset tableau*** of shape μ, or element of $SMT^{\pm}(\mu)$, is an arrangement of boxes with μ_i adjacent boxes in row i for each i and where the rows are situated such that the leftmost box of row i lies one column to the left of the leftmost box of row $i + 1$, along with a filling of said boxes with the following properties.

(1) Each box contains a nonempty multiset of the numbers $\{1', 1, 2', 2, 3', \ldots\}$ such that the multiplicity of each primed number is 0 or 1.
(2) Suppose entry z lies in a box directly to the right of box b. Then for *all* $a \in b$ we have $a <_u z$.
(3) Suppose entry z lies in a box directly below box b. Then for *some* $a \in b$ we have $a <_p z$.

If, in addition the smallest entry in each row is not primed we call such a tableau simply a ***shifted multiset tableau*** of shape μ or an element of $SMT(\mu)$.[1]

[1]Compare to the definitions of *weak set-valued shifted tableaux* in [HKP+17] and *set-valued shifted tableaux* in [IN13].

The weight of a (signed) shifted multiset tableau is the vector $(w_1, w_2, \dots)$ where w_i is the total number of is or i's appearing in the tableau. We label the $\backslash$ direction diagonals from left to right by $\{\ell, \ell-1, \dots, 2, 1\}$ where $\ell = \mu_1$. By box d_{ij} we refer to the box that is in the i^{th} row (from top to bottom) of diagonal j. Define the diagonal weight of a shifted multiset tableau, dw, to be the vector $(T_1, \dots, T_\ell)$ where T_j is the difference between the number of entries in diagonal j and the number of boxes in diagonal j. Let, $|d_{ij}|$ mean the total number of entries in box d_{ij} and $|d_{ij}(x)|$ refer, more specifically, to the number of entries in box d_{ij} in tableau x. The convention is $|d_{ij}| = 0$ if d_{ij} describes a position not in the tableau.

Example 4.6. Let $\mu = (5, 4, 2)$. Then

1	1113	3	$4'45$	$7'7$
	22	$4'4$	$5'6'$	$7'$
		$45'$	55	

is an element $P \in SMT(\mu)$ with $wt(T) = (4, 2, 2, 5, 5, 1, 3)$ and $dw(P) = (1, 2, 1, 5, 2)$.

Definition 4.7. An element of $SMT^{\pm}(\mu)$ with diagonal weight $(0, \dots, 0)$ is called a ***signed shifted semistandard tableau*** of shape μ, or element of $SST^{\pm}(\mu)$. An element of $SMT(\mu)$ with diagonal weight $(0, \dots, 0)$ is called a ***shifted semistandard tableau*** of shape μ, or element of $SST(\mu)$.

Remark 4.8. Note that $SST(\mu)$, which is the subset of $SST^{\pm}(\mu)$ with no primes in the leftmost $\backslash$ direction diagonal, agrees with the classical definition of shifted semistandard tableau (e.g., [Ser09]) and is therefore the generating set for the P-Schur function P_μ. Moreover, if m is the number of parts of μ, it is not difficult to see that $SST^{\pm}(\mu)$ differs from $SST(\mu)$ and $SMT^{\pm}(\mu)$ differs from $SMT(\mu)$ only by a power of 2^m.

Definition 4.9. A ***maximal shifted multiset tableau*** of shape μ, or element of $\overline{SMT}(\mu)$ is an element of $SMT(\mu)$ with the following properties:

(1) Each box d_{ij} may only contain is.
(2) For each $i \geq 1$ and $k \geq 0$ we have $\sum_{1 \leq j \leq k} |d_{(i+1)j}| - |d_{i(j-1)}| \leq 0$

Example 4.10. Let $\mu = (4, 3, 3, 1)$. Then

1	1	11	1	11	11	1
	2	22	2	2	222	
		33	3	3	33	
			44	44		

is an element $P \in \overline{MT}(\mu)$ with $wt(P) = (7, 6, 5, 4)$ and $cw(P) = (1, 3, 4, 2)$.

Proposition 4.11. *There is a bijection from the subset of $\overline{SMT}(\mu)$ with weight λ and diagonal weight $\mathbf{T}$ to the subset of $SRT(\lambda/\mu)$ with weight $\mathbf{T}$.*

Proof. Let X be the subset of $SMT(\mu)$ with weight λ and diagonal weight $\mathbf{T}$ that satisfy property (1) above. Let Y be the set of weakly increasing by row fillings of shape $(\lambda-\delta)/(\mu-\delta)$ and weight $\mathbf{T}$ such that every entry i occurs on or above row c_i (equivalently: row i only contains entries greater than $\ell-\mu_i$). The map $x \to y$ where y is defined by the property that for each (i,j), row i of y contains exactly $|d_{ij}(x)|-1$ copies of j is a bijection from X to Y. Moreover if $x \to y$ then x satisfies property (2) above if and only if the columns of y are strictly decreasing down rows: Indeed, if there is some i and some k such that $\sum_{1\le j\le k} d_{(i+1)j} - d_{i(j-1)} > 0$ then for the minimal such k, row $i+1$ of y will have an entry k that lies above an entry k' of row i with $k' \ge k$. On the other hand, if row $i+1$ of y contains a k which lies above some k' in row i with $k' \ge k$ then we are guaranteed to have $\sum_{1\le j\le k} d_{(i+1)j} - d_{i(j-1)} > 0$. Since the elements of Y that are strictly decreasing down columns are exactly the elements of $SRT(\lambda/\mu)$ with weight $\mathbf{T}$, the map restricted to the elements of X that satisfy property (2) gives the desired bijection. □

Example 4.12. The tableaux of examples 4.2 and 4.10 correspond under this bijection.

Corollary 4.13. *Let μ be a partition with m distinct, nonzero parts and set $\mathbf{t} = t_1,\dots,t_\ell$, $\mathbf{x} = (x_1,\dots,x_n)$ then*

$$\mathfrak{P}_\mu(x_1,\dots,x_n,t_1,\dots,t_\ell) = \sum_{Q\in\overline{SMT}(\mu)} \mathbf{t}^{dw(Q)} P_{wt(Q)}(\mathbf{x})$$

4.3. **Combinatorial Definition of $\mathfrak{P}_\mu(\mathbf{x},\mathbf{t})$.** In this section we will give an equivalent combinatorial definition of $\mathfrak{P}_\mu$. We will need a certain column insertion algorithm. In the below, we describe how to **insert** and **reverse insert** an element into a column.

Let K be a valid column (each box of K contains exactly one element from S' and whenever a lies above z in K we have $a <_p z$). Now let $a \in S'$. We **insert** a into K, denoted $a \hookrightarrow K$ as follows: Let $\hat{a}$ denote the uppermost entry in K such that $a <_u \hat{a}$. If $\hat{a}$ exists, replace $\hat{a}$ with a and bump out $\hat{a}$. Otherwise, append a to the bottom of K. The result is recorded as the pair $(K',\hat{a})$ if the second of this pair exists and just K' otherwise. On the other hand if $z \in S'$ is any element such that $z >_u a$ for some $a \in K$ then we define **reverse insertion** of z into K as follows: Let $\hat{z}$ denote the bottommost entry in K such that $z >_u \hat{z}$. Replace $\hat{z}$ with z and bump out $\hat{z}$. The result is recorded as the pair $(\hat{z}, K')$.

Notice the basic properties:

(1) If $a \hookrightarrow K = K'$ then K' is a valid column.
(2) if $a \hookrightarrow K = (K',\hat{a})$ then K' is a valid column.
(3) If $K \hookleftarrow z = (\hat{z}, K')$ then K' is a valid column.
(4) If $a <_u z$ then either
  - $z \hookrightarrow K = K'$ and $a \hookrightarrow K' = (K'',\hat{a})$.
  - $z \hookrightarrow K = (K',\hat{z})$ and $a \hookrightarrow K' = (K'',\hat{a})$ where $\hat{a} <_u \hat{z}$.
(5) If $a <_u z$ and $K \hookleftarrow a = (\hat{a}, K')$ and $K' \hookleftarrow z = (\hat{z}, K'')$ then $\hat{a} <_u \hat{z}$.

Now, fix a partition μ with m distinct nontrivial parts and with conjugate $\mu' = (c_\ell,\dots,c_1)$. We will refer to both columns and diagonals. Both are labeled in decreasing order from left to right starting on ℓ.

Proposition 4.14. *There is a bijection* $SMT^{\pm}(\mu) \to \bigcup_{\lambda \supseteq \mu} SST^{\pm}(\lambda) \times SRT(\lambda/\mu)$, *such that if* $P \to (Q, R)$ *then:*

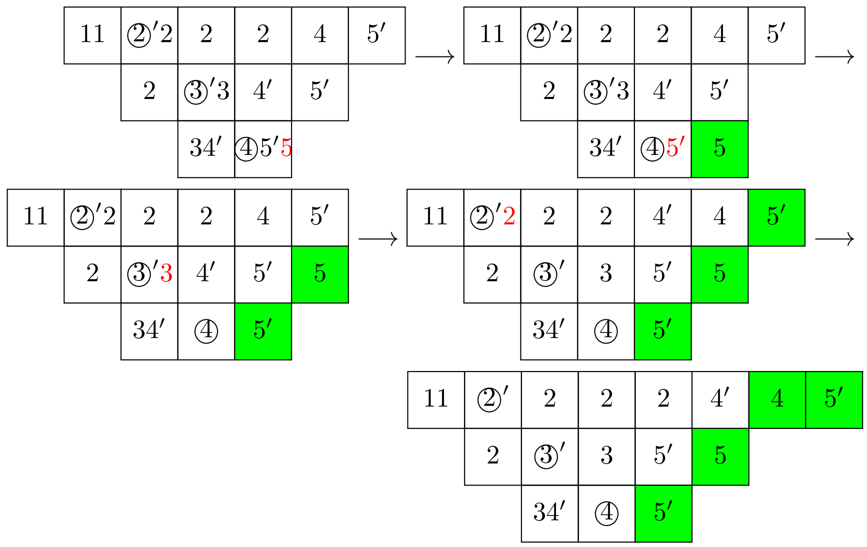
(1) $wt(P) = wt(Q)$.
(2) $dw(P) = wt(R)$.

First some reductions. Define the set $SMT_k(\lambda)$ to be the subset of $SMT^{\pm}(\lambda)$ which have only single entries in diagonals $k-1, \dots, 1, 0, -1, \dots$. Define the set $SRT_k(\lambda/\mu)$ to be the subset of $SRT(\lambda/\mu)$ which have only entries from $\{1, 2, \dots, k-1\}$. Given a pair $(Q, R) \in SMT_k(\lambda) \times SRT_k(\lambda/\mu)$ define the weight and diagonal weight of this pair as $wt(Q, R) = wt(Q)$ and $dw(Q, R) = dw(Q) + wt(R)$. To achieve oar goal it suffices to find a weight and diagonal weight preserving bijection for each k (and then compose) from $\bigcup_{\lambda \supseteq \mu} SMT_k(\lambda) \times SRT_k(\lambda/\mu)$ to $\bigcup_{\lambda \supseteq \mu} SMT_{k+1}(\lambda) \times SRT_{k+1}(\lambda/\mu)$. To do the latter, it is enough to find a weight preserving bijection $\Phi_k : SMT_k(\lambda) \to \bigcup_{\nu \supseteq \lambda} SMT_{k+1}(\nu)$ where the union is over all ν such ν/λ is a horizontal strip with no box below row c_k. (equivalently, below the lowest box of diagonal k of λ: in the previous map the only $\lambda \supseteq \mu$ appearing have length of diagonal k equal to c_k (equal to the length of diagonal k of μ)).

Φ_k will be defined by repetitively applying the following map: Let $T \in SMT_k(\lambda)$. Define $\mathbf{out}(T)$ as follows: First, in each box of diagonal k circle (one of) the minimum entry(s) from that box. Now find (one of) the largest noncircled entry(s) in diagonal k and remove it and **insert** it into the undercolumn to the right of the column from which it was removed (where the *undercolumn* denotes the part of the column that lies below a circled entry, or, if there is no circled entry in the column, the entire column). After this, each time an element is bumped, **insert** it into the next undercolumn to the right until some entry is eventually appended to an undercolumn. Note the following properties of **out**.

(1) The path of positions where an element is bumped/appended moves weakly down as we move to the right.
(2) Properties (1), (2), and (3) in the definition of shifted multiset tableaux are preserved under **out**.
(3) If $\mathbf{out}(T)$ and $\mathbf{out}(\mathbf{out}(T))$ are both defined then the box which **out** appends to $\mathbf{out}(T)$ lies strictly to the right of the box that **out** appends to T.

Example 4.15. Suppose that $k = 2$. Each $\longrightarrow$ represents an application of **out**.



Uncircled numbers being removed are shown in red, and the boxes being added appear in green.

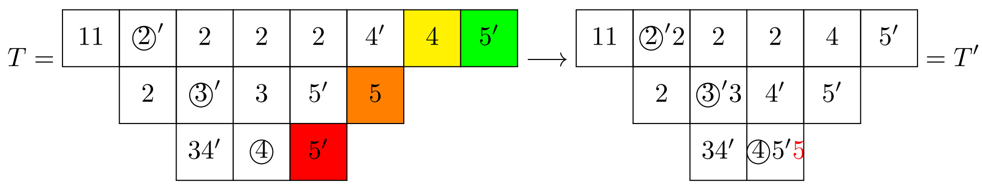
We will also need a map called $\mathbf{in}_b$. Let $T \in SMT_k(\nu)$ for some ν such that ν/λ is a horizontal strip with no box below row c_k and suppose b is some corner box of T that lies on or above row c_k. Define $\mathbf{in}_b(T)$ as follows: First, in each box of diagonal k circle (one of) the minimum entry(s) from that box. Now remove the entry from box b. If this entry less than the circled entry in the column to the left or both are equal and primed, **reverse insert** it into the undercolumn of the column to the left. After this, each time an element is bumped that is less than the circled entry in the column to its left or equal to it and primed, **reverse insert** it into the undercolumn of the column to the left. When an element is bumped that is greater than the circled entry in the column to its left or equal to it and unprimed, add it to the box containing this circled element. Note the following properties of $\mathbf{in}_b$.

(1) The path of positions where an element is bumped/added moves weakly up as we move to the left.
(2) Properties (2), and (3) in the definition of shifted multiset tableaux are preserved under $\mathbf{in}_b$. Property (1) is satisfied unless $\mathbf{in}_b$ adds a primed entry to a box already containing a the same noncircled primed entry.
(3) If b' lies to the left of b and if $\mathbf{in}_b(T)$ and $\mathbf{in}_{b'}(\mathbf{in}_b(T))$ are both defined then the element that $\mathbf{in}_{b'}$ adds to diagonal k of $\mathbf{in}_b(T)$ is greater than, or equal to and unprimed, the element $\mathbf{in}_b$ adds to diagonal k of T.

Moreover, **out** and $\mathbf{in}_b$ are related as follows:

(1) If **out** appends box b when applied to T, then $\mathbf{in}_b(\mathbf{out}(T)) = T$.
(2) If the element that $\mathbf{in}_b$ adds to diagonal k when applied to T is the largest, or tied for the largest and unprimed, uncircled element on diagonal k then $\mathbf{in}_b(T)$ satisfies property (1) in the definition of shifted multiset tableaux (and hence is a shifted multiset tableau), and $\mathbf{out}(\mathbf{in}_b(T)) = T$.

Example 4.16. Set $k = 2$. Then $\mathbf{in}_{red}(\mathbf{in}_{orange}(\mathbf{in}_{yellow}(\mathbf{in}_{green}(T)))) = T'$ where:



Note that T is the last tableau of example 4.15 and T' is the first tableau of 4.15.

Proof. We define Φ_k simply by applying **out** until diagonal k only contains single entries.

(1) Φ_k is well defined. For any tableau T *denote the shape of* T *by* T^s. If $T \in SMT_k(\lambda)$ then Property (3) of **out** implies $\Phi_k(T)^s/T^s$ is a horizontal strip and Property (1) of **out** implies all of its boxes lie on or above row c_k. On the other hand Property (2) of **out** implies that $\Phi_k(T)$ is a valid shifted multiset tableau, and, by construction $\Phi_k(T)$ has only single entries in diagonals $k, k-1, \ldots, 0, -1, \ldots$.
(2) Φ_k is injective. Suppose $T \neq T' \in SMT_k(\lambda)$ with $\Phi_k(T) = \Phi_k(T')$ then by Property (3) of **out** and construction of Φ_k there is some ν and some $S \neq S' \in SMT_k(\nu)$ with $\mathbf{out}(S) = \mathbf{out}(S')$. But then if b is the box that **out** adds to S or equivalently to S', property (1) of how **out** and $\mathbf{in}_b$ are related says $S = \mathbf{in}_b(\mathbf{out}(S)) = \mathbf{in}_b(\mathbf{out}(S')) = S'$).
(3) Φ_k is surjective. Let $T \in \bigcup_{\nu \supseteq \lambda} SMT_{k+1}(\nu)$ where the union is over all ν such ν/λ is a horizontal strip with no box below row c_k. Let $b_1, \ldots, b_r$ denote the boxes labeled from left to right of T^s/λ. Set $S = \mathbf{in}_{b_1}(\cdots(\mathbf{in}_{b_r}(T)\cdots)$. Property (3) of $\mathbf{in}_b$ implies that for each i we have that $\mathbf{in}_{b_i}$ adds a an element to diagonal k when applied to $\mathbf{in}_{b_{i+1}}(\cdots(\mathbf{in}_{b_r}(T)\cdots)$ that is the largest, or tied for largest and unprimed, noncircled element in diagonal k. This along with property (2) of $\mathbf{in}_b$ implies $\mathbf{in}_{b_i}(\cdots(\mathbf{in}_{b_r}(T)\cdots)$ is a valid shifted multiset tableau. Moreover, property (2) of how **out** and $\mathbf{in}_b$ are related says that in this case $\mathbf{out}(\mathbf{in}_{b_i}(\cdots(\mathbf{in}_{b_r}(T)\cdots)) = \mathbf{in}_{b_{i+1}}(\cdots(\mathbf{in}_{b_r}(T)\cdots)$. All together, this implies that S is a valid shifted multiset tableau and that $\Phi_k(S) = T$. By construction, S has shape λ and has only single entries in diagonals $k-1, \ldots, 0, -1, \ldots$, i.e., $S \in SMT_k(\lambda)$.

□

Theorem 4.17. *Let* $\mathbf{t} = t_1, \ldots, t_\ell$, $\mathbf{x} = (x_1, \ldots, x_n)$ *then:*

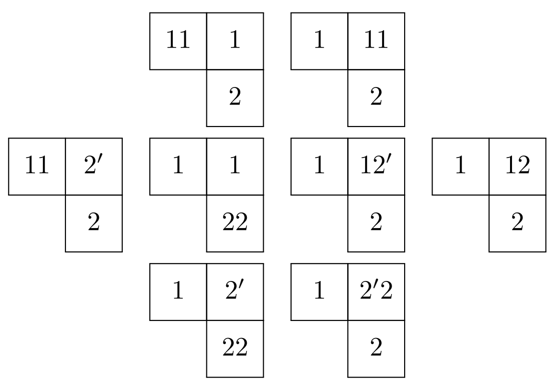
$$\mathfrak{P}_\mu(x_1, \ldots, x_n, t_1, \ldots, t_\ell) = \sum_{P \in SMT(\mu)} \mathbf{t}^{dw(P)} \mathbf{x}^{wt(P)}$$

Proof. Let m denote the number of parts of μ.

$$\begin{aligned}
&\mathfrak{P}_\mu(x_1,\dots,x_n,t_1,\dots,t_\ell) \\
=& \sum_{\lambda\supseteq(\mu-\delta)} \sum_{R\in SRT((\lambda+\delta)/\mu)} \mathbf{t}^{wt(R)} P_{\lambda+\delta}(\mathbf{x}) && \text{Theorem 4.3} \\
=& \sum_{\lambda\supseteq(\mu-\delta)} \sum_{R\in SRT((\lambda+\delta)/\mu)} \sum_{Q\in SST(\lambda+\delta)} \mathbf{t}^{wt(R)}\mathbf{x}^{wt(Q)} && \text{Def. of } P_{\lambda+\delta} \\
=& \sum_{\lambda\supseteq(\mu-\delta)} \sum_{R\in SRT((\lambda+\delta)/\mu)} \sum_{Q\in SST^{\pm}(\lambda+\delta)} (2^{-m})\mathbf{t}^{wt(R)}\mathbf{x}^{wt(Q)} && \text{Def. of } SST^{\pm} \\
=& \sum_{P\in SMT^{\pm}(\mu)} (2^{-m})\mathbf{t}^{dw(P)}\mathbf{x}^{wt(P)} && \text{Prop. 4.14} \\
=& \sum_{P\in SMT(\mu)} \mathbf{t}^{dw(P)}\mathbf{x}^{wt(P)} && \text{Def. of } SMT^{\pm}
\end{aligned}$$

□

Example 4.18. Let us consider $\mathfrak{P}_{2,1}(x_1,x_2,t_1,t_2)$. We will compute the degree 4 part in $\mathbf{x}$ (which is the degree 1 part in $\mathbf{t}$). We have the following tableaux:



Which yields $x_1^3x_2t_1 + x_1^3x_2t_2 + 2x_1^2x_2^2t_1 + 2x_1^2x_2^2t_2 + x_1x_2^3t_1 + x_1x_2^3t_2$, which can be expressed in terms of P-Schur polynomials as $t_1P_{3,1}(x_1,x_2) + t_2P_{3,1}(x_1,x_2)$. Compare with example 3.3 of [HKP+17].

Remark 4.19. There exists a $\mathfrak{q}$-crystal structure on the set of semistandard shifted tableaux [Hir18]. Under this structure, the highest weight elements are precisely those for which every entry on row i is an (unprimed) i. Moreover, the bijection Φ fixes the minimum entry on each row. Thus restricting Φ gives a bijection from $SMT(\mu) \to \bigcup_{\lambda\supseteq\mu} SST(\lambda)\times SRT(\lambda/\mu)$. Moreover, it is not difficult so see that this restriction of Φ has the property that whenever $\Phi(P) = (Q,R)$ then $P \in \overline{SMT}(\mu)$ if and only if Q is highest weight. Thus Φ^{-1} induces a queer crystal structure on $SMT(\mu)$ where the highest weight elements are precisely those that lie in $\overline{SMT}(\mu)$. This crystal structure is interpreted algebraically by comparing Corollary 4.13 (where the sum is over highest weight elements) with Theorem 4.17 (where the sum is over all elements).

References

[BS17] Daniel Bump and Anne Schilling. *Crystal bases*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017. Representations and combinatorics.

[FK94] Sergey Fomin and Anatol N. Kirillov. Grothendieck polynomials and the Yang-Baxter equation. In *Formal power series and algebraic combinatorics/Séries formelles et combinatoire algébrique*, pages 183–189. DIMACS, Piscataway, NJ, 1994.

[FK96] Sergey Fomin and Anatol N. Kirillov. The Yang-Baxter equation, symmetric functions, and Schubert polynomials. In *Proceedings of the 5th Conference on Formal Power Series and Algebraic Combinatorics (Florence, 1993)*, volume 153, pages 123–143, 1996.

[Hir18] Toya Hiroshima. q-crystal structure on primed tableaux and on signed unimodal factorizations of reduced words of type b. Preprint, arXiv:1803.05775, 2018.

[HKP+17] Zachary Hamaker, Adam Keilthy, Rebecca Patrias, Lillian Webster, Yinuo Zhang, and Shuqi Zhou. Shifted Hecke insertion and the K-theory of OG(n,2n + 1). *J. Combin. Theory Ser. A*, 151(1):207–240, 2017.

[HPS17] Graham Hawkes, Kirill Paramonov, and Anne Schilling. Crystal analysis of type C Stanley symmetric functions. *Electron. J. Combin.*, 24(3):Paper P3.51, 2017.

[HS19] Graham Hawkes and Travis Scrimshaw. Crystal structures for canonical Grothendieck functions. Preprint, arXiv:1907.11415, 2019.

[IN13] Takeshi Ikeda and Hiroshi Naruse. K-theoretic analogues of factorial Schur P- and Q-functions. *Adv. Math.*, 243:22–66, 2013.

[Lam95] Tao Kai Lam. *B and D analogues of stable Schubert polynomials and related insertion algorithms*. ProQuest LLC, Ann Arbor, MI, 1995. Thesis (Ph.D.)–Massachusetts Institute of Technology.

[LP07] Cristian Lenart and Alexander Postnikov. Affine Weyl groups in K-theory and representation theory. *Int. Math. Res. Not. IMRN*, (12):Art. ID rnm038, 65, 2007.

[LS82] Alain Lascoux and Marcel-Paul Schützenberger. Structure de Hopf de l'anneau de cohomologie et de l'anneau de Grothendieck d'une variété de drapeaux. *C. R. Acad. Sci. Paris Sér. I Math.*, 295(11):629–633, 1982.

[LS83] Alain Lascoux and Marcel-Paul Schützenberger. Symmetry and flag manifolds. In *Invariant theory (Montecatini, 1982)*, volume 996 of *Lecture Notes in Math.*, pages 118–144. Springer, Berlin, 1983.

[MPS18] Cara Monical, Oliver Pechenik, and Travis Scrimshaw. Crystal structures for symmetric Grothendieck polynomials. Preprint, arXiv:1807.03294, 2018.

[Ser09] Luis Serrano. The shifted plactic monoid (extended abstract). In *21st International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2009)*, Discrete Math. Theor. Comput. Sci. Proc., AK, pages 757–768. Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2009.

[Sta84] Richard P. Stanley. On the number of reduced decompositions of elements of Coxeter groups. *European J. Combin.*, 5(4):359–372, 1984.

[Sta99] Richard Stanley. *Enumerative Combinatorics Volume 2*. Cambridge University Press, 1999.