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CATEGORIFICATION FOR PRINCIPAL COEFFICIENT CLUSTER ALGEBRAS

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Abstract. In earlier work, the author introduced a method for constructing a Frobenius categorification of a cluster algebra with frozen variables, requiring as input a suitable candidate for the endomorphism algebra of a cluster-tilting object in such a category. In this paper, we construct such candidates in the case of acyclic cluster algebras with 'polarised' principal coefficients, and study the resulting Frobenius categorifications. Since cluster algebras with principal coefficients are obtained from those with polarised principal coefficients by setting half of the frozen variables to 1, our categories also indirectly model cluster algebras with principal coefficients, for which no Frobenius categorification can exist. Many of the intermediate results remain valid without the acyclicity assumption, as we will indicate. Along the way, we establish a Frobenius version of Keller's result that the Ginzburg dg-algebra of a quiver with potential is bimodule 3-Calabi–Yau, and extend results of Buan–Iyama–Reiten–Smith to give conditions under which mutation of cluster-tilting objects is compatible with mutation of ice quivers with potential.

1. Introduction

Cluster algebras, introduced by Fomin–Zelevinsky [?fomincluster1], are combinatorially defined algebras with applications to many areas of mathematics, and currently the subject of intense study; see Keller [?kellercluster] for a survey of connections between cluster algebras and the representation theory of associative algebras, and the references therein for applications to other fields.

A key obstruction to studying cluster algebras is their recursive definition—one is given some initial data (a seed), and constructs the cluster algebra inductively via sequences of mutations of this seed. Typically one may obtain an infinite number of seeds in this way, and so the output is not easily controllable. To gain a better understanding of the cluster algebra, it has been fruitful to construct categorical models, which allow the combinatorics to be understood in a more global way. In such models $\mathcal{C}$, the clusters are replaced by cluster-tilting objects T, defined by the property that

$$\operatorname{add} T = \{X \in \mathcal{C} : \operatorname{Ext}^1_{\mathcal{C}}(T, X) = 0\} = \{X \in \mathcal{C} : \operatorname{Ext}^1_{\mathcal{C}}(X, T) = 0\}.$$

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When $\mathcal{C}$ is 2-Calabi–Yau in a suitable sense, these objects may be mutated via a process analogous to that of the mutation of seeds [?iyamamutation]. Categorification allows one to give clean, conceptual proofs of many key statements for any cluster algebra admitting such a categorical model, such as 'cluster determines seed' [?buanclusters], linear independence of cluster monomials [?cerullilinear], sign coherence of c-vectors [?speyeracyclic], and so on.

We consider skew-symmetric cluster algebras of geometric type, so that a seed is given by the data of a collection of cluster variables forming the vertices of some quiver. One passes to another seed by a process of mutation, which replaces a single cluster variable of the seed by a new cluster variable, and alters the quiver by Fomin–Zelevinsky mutation (see for example [?kellercluster, §3.2]). A subset of the variables may be frozen, indicating that they may not be mutated, and thus occur in every seed. We use the terminology 'ice quiver' to refer to a quiver with a specified 'frozen' subquiver; from the point of view of cluster algebras, the only relevant additional data is the set of frozen vertices, since arrows between these play no role in the cluster structure, but we will want to consider such arrows later.

For cluster algebras without frozen variables, categorical models, known as cluster categories, have been constructed in great generality, beginning with Buan–Marsh–Reineke–Reiten–Todorov [?buantilting] for the case of acyclic quivers, and later generalised by Amiot [?amiotcluster] to allow for the existence of cycles. Unfortunately, most cluster algebras appearing naturally in other contexts, such as the cluster structures on the coordinate rings of partial flag varieties [?geisspartial] and their double Bruhat cells [?berensteincluster3], do have frozen variables, which we would like to capture in a categorical model.

This can be achieved by replacing cluster categories, which are 2-Calabi–Yau triangulated categories, by stably 2-Calabi–Yau Frobenius categories. A Frobenius category is, by definition, an exact category with enough projective objects and enough injective objects, such that these two classes of objects coincide. It is the indecomposable projective-injective objects, which necessarily appear as summands of any cluster-tilting object, that will model the frozen variables. The fact that setting all frozen variables to 1 in a cluster algebra produces a new cluster algebra without frozen variables corresponds to the fact that the stable category of a Frobenius category, given by taking the quotient by the ideal of morphisms factoring through a projective object ('setting the projective objects to 0') produces a triangulated category [?happeltriangulated, §I.2], which we require to be 2-Calabi–Yau.

Such Frobenius models for cluster algebras with frozen variables have been constructed sporadically for families of examples, often geometric in nature. For example, Geiß–Leclerc–Schröer [?geisspartial] construct cluster structures on coordinate rings of open cells in partial flag varieties using a Frobenius model. They then combinatorially lift such structures to the homogeneous coordinate ring of the whole flag variety. Frobenius categorifications of the resulting structures have been obtained in the case of Grassmannians of planes by Demonet–Luo [?demoneticequivers1], all Grassmannians by Jensen–King–Su [?jensencategorification], and in general by Demonet–Iyama [?demonetlifting]. However, these constructions are all somewhat specialised to the case at hand, and depend to some extent on using the geometry of the partial flag varieties to gain some insight into the global structure of the cluster algebra, before constructing the categorification. Nájera Chávez [?najerachavez2calabiyau] has constructed Frobenius models for finite-type cluster algebras with universal coefficients, in this case using the finiteness to control the global structure.

By contrast, the constructions of cluster categories for cluster algebras without frozen variables by Buan–Marsh–Reineke–Reiten–Todorov and Amiot did not depend on such global information, but instead start from the data of a single seed, as in the original definition of a cluster algebra, possibly enhanced by some additional (but still local) data, such as a potential on the quiver in Amiot's case. In earlier work [?presslandinternally], the author introduced a similar framework for constructing Frobenius models of a cluster algebra, starting again from the data of an initial seed. Starting from the quiver Q of this seed (some of the vertices of which are frozen), one attempts to find a Noetherian algebra A such that the Gabriel quiver of A agrees with Q up to the addition of arrows between frozen vertices, the quotient of A by paths passing through these vertices is finite dimensional, and, most importantly, A is bimodule internally 3-Calabi–Yau with respect to these vertices [?presslandinternally, Defn. 2.4]. Such an algebra A then determines a candidate Frobenius model of the cluster algebra [?presslandinternally, Thm. 4.1, Thm. 4.10].

Passing from Q to A requires (mostly necessarily [?presslandinternally, Rem. 4.11]) the choice of a great deal of extra data, satisfying restrictive conditions. Thus it was not clear from the results of [?presslandinternally] how realistic it would be to apply this methodology in practice. In this work, we demonstrate that this approach is in fact workable, by using it to construct a Frobenius model for the cluster algebra with 'polarised principal coefficients' associated to any acyclic quiver. Fomin–Zelevinsky have shown that cluster algebras with principal coefficients play a 'universal' role in the theory, since their combinatorics can be used to control that of any other cluster algebra with the same principal part, meaning the cluster algebra obtained upon specialising all frozen variables to 1 [?fomincluster4]. Cluster algebras with polarised principal coefficients, which we define in Section 2, differ from those with principal coefficients only by the addition of frozen variables, and so also have this universality property. Our main theorem is the following.

Theorem 1. *Let Q be an acyclic quiver, and let $\mathscr{A}$ be the corresponding cluster algebra without frozen variables. Then there exists a Frobenius cluster category $\mathcal{E}$ such that*

(i) *the stable category $\underline{\mathcal{E}}$ is equivalent to the cluster category $\mathcal{C}_Q$, and*
(ii) *there is a bijection between cluster-tilting objects of $\mathcal{E}$ and seeds of the polarised principal coefficient cluster algebra with principal part $\mathscr{A}$, commuting with mutation, such that the ice quiver of the endomorphism algebra of each cluster-tilting object agrees, up to arrows between frozen vertices, with the ice quiver of the corresponding seed.*

We interpret the quivers of endomorphism algebras in (ii), and indeed throughout the paper, as ice quivers by declaring the frozen vertices to be those corresponding to indecomposable projective summands. The definition of a Frobenius cluster category is stated below (Definition 2.7); such categories always admit a weak cluster structure in the sense of Buan–Iyama–Reiten–Scott [?buancluster, §II.1], by [?buancluster, Thm. II.1.10]. By part (ii) of Theorem 1, this weak cluster structure on $\mathcal{E}$ is even a cluster structure, also defined in [?buancluster, §II.1].

As we will indicate, while the acyclicity assumption is needed for all of the ingredients in the proof of Theorem 1 to be available simultaneously, many of these intermediate results hold in much wider generality. In particular, we will see that the theorem remains true when Q is a 3-cycle (replacing $\mathcal{C}_Q$ in (i) by Amiot's cluster category $\mathcal{C}_{Q,W}$, where W is the potential on Q given by the 3-cycle).

It is then possible to use the category $\mathcal{E}$ from Theorem 1 to study the principal coefficient cluster algebra with principal part $\mathscr{A}$, essentially by 'ignoring' the extra projective-injective objects in the category, as we will illustrate in Section 8.

On the way to proving Theorem 1, we will obtain other results that are of wider interest, such as the following; here a positive grading of a quiver with potential (Q,W) is a $\mathbb{Z}$-grading of the Jacobian algebra $\mathcal{J}(Q,W)$ such that all arrows of Q have positive degree.

Theorem 2 (Corollary 4.12)**.** *Let (Q,W) be a quiver with potential admitting a positive grading, and let $\underline{A}$ be the corresponding Jacobian algebra. Then there is a bimodule internally 3-Calabi–Yau frozen Jacobian algebra A (constructed explicitly in Section 3) such that $\underline{A}=A/\langle e\rangle$ for e the frozen idempotent of A.*

We see this statement as analogous to a result of Keller [?kellerdeformed, Thm. 6.3, Thm. A.12], implying that any finite-dimensional Jacobian algebra may be realised as the 0-th homology of a bimodule 3-Calabi–Yau dg-algebra constructed by Ginzburg [?ginzburgcalabiyau], and based on this analogy conjecture that it remains true without the assumption on the grading.

In order to prove part (ii) of Theorem 1, we extend results of Buan–Iyama–Reiten–Smith [?buanmutation] to the Frobenius setting. The results of [?buanmutation] explain when, for a cluster-tilting object T of a 2-Calabi–Yau triangulated category $\mathcal{C}$ whose endomorphism algebra is isomorphic to the Jacobian algebra of a quiver with potential (Q,W), the endomorphism algebras of mutations of T are the Jacobian algebras of mutations of (Q,W), in the sense of Derksen–Weyman–Zelevinsky [?derksenquivers1]. Using similar methods, we give an analogous result (Theorem 6.9) when (Q,W) is replaced by an ice quiver with potential (Q,F,W), and the Jacobian algebra of (Q,W) by the frozen Jacobian algebra of (Q,F,W) (see Definition 2.1). This result can be applied to the Frobenius cluster categories constructed in this paper, to those defined from Weyl group elements by Buan–Iyama–Reiten–Smith [?buancluster], and to the Grassmannian cluster categories of Jensen–King–Su [?jensencategorification]. In particular, we are able to show that the quiver of the endomorphism algebra of a cluster-tilting object T in the Grassmannian cluster category coincides, up to arrows between frozen vertices, with the quiver of the corresponding seed of the Grassmannian cluster algebra (at least for T within a specific mutation class, so that it does in fact correspond to a seed!), a fact which had not been previously established.

The main results of the paper are contained in Sections 2–6. In Section 2 we describe the cluster algebras with polarised principal coefficients that we will categorify, and recall the results of [?presslandinternally], which we will use to produce the model. The algebra needed as input for this construction is defined in Section 3. In Section 4 we explain further results of [?presslandinternally] which allow one to check the bimodule internally 3-Calabi–Yau property for a frozen Jacobian algebra, and apply these to $A_{Q,W}$ under the assumption that (Q, W) admits a positive grading. This establishes Theorem 2. In Section 5, we show that the algebra A is finite-dimensional when we start from an acyclic quiver, and so it is in particular Noetherian. This allows us to conclude most of the statements of Theorem 1. The results on mutations are found in Section 6, where we use them to complete the proof of Theorem 1, as well as giving the promised applications to other Frobenius cluster categories, such as the Grassmannian cluster category.

The Frobenius cluster categories from Theorem 1 are, by definition, categories of Gorenstein projective modules over some Iwanaga–Gorenstein algebra B, which we describe explicitly via a quiver with relations in Section 7. In Section 8 we show that the Frobenius cluster categories from Theorem 1 may be graded, in the sense of [?grabowskigradedfrobenius], in a way that captures the grading of a principal coefficient cluster algebra by Fomin–Zelevinsky [?fomincluster4]. This allows us to recover an identity of Fomin–Zelevinsky, relating g-vectors and c-vectors. We close in Section 9 with some examples, in particular observing that Theorem 1 remains true when Q is a 3-cycle.

Throughout, algebras are assumed to be associative and unital. All modules are left modules, the composition of maps $f\colon X \to Y$ and $g\colon Y \to Z$ is denoted by gf, and we use the analogous convention for compositions of arrows in quivers. The Jacobsen radical of a module X is denoted by $\mathfrak{m}(X)$. If p is a path in a quiver, we denote its head by hp and its tail by tp.

2. Polarised principal coefficients

Let $\mathscr{A}$ be a cluster algebra of geometric type without frozen variables, and let s_0 be a seed of $\mathscr{A}$, with quiver Q and cluster variables $(x_1, \dots, x_n)$. By definition, the quiver Q has no loops or 2-cycles. The cluster algebra $\mathscr{A}_Q^\bullet$ with principal coefficients corresponding to this data is defined by an initial seed as follows. The mutable cluster variables are again $(x_1, \dots, x_n)$, and the frozen variables are $(y_1, \dots, y_n)$, where the indexing reveals a preferred bijection between the cluster variables and the frozen variables. The ice quiver $Q^\bullet$ of this seed contains Q as a full subquiver, with mutable vertices, and for each vertex $i \in Q_0$ (corresponding to the variable x_i), $Q^\bullet$ has a frozen vertex i^+ (corresponding to y_i) and an arrow $i \to i^+$. While $\mathscr{A}$ is isomorphic to the cluster algebra determined by any quiver mutation equivalent to Q, this is not true of $\mathscr{A}_Q^\bullet$.

By construction, setting all frozen variables of $\mathscr{A}_Q^\bullet$ to 1, to obtain a cluster algebra without frozen variables we call the *principal part* of $\mathscr{A}_Q^\bullet$, recovers $\mathscr{A}$, giving a bijection between the seeds of $\mathscr{A}_Q^\bullet$ and those of $\mathscr{A}$; we write $s^\bullet$ for the seed of $\mathscr{A}_Q^\bullet$ corresponding to a seed s of $\mathscr{A}$. Principal coefficients are important, since knowledge of the cluster algebra $\mathscr{A}_Q^\bullet$ gives strong information on every cluster algebra $\mathscr{A}'$ with principal part $\mathscr{A}$, via the theory of g-vectors and F-polynomials [?fomincluster4].

By choosing some extra data on Q, Amiot [?amiotcluster] is able to construct a categorical model of $\mathscr{A}$. The construction uses Jacobian algebras, so we recall some relevant definitions, which we will also need later in the paper.

Definition 2.1. An *ice quiver* (Q, F) consists of a finite quiver Q without loops and a (not necessarily full) subquiver F of Q. Denote by $\mathbb{K}Q$ the completion of the path algebra of Q over $\mathbb{K}$ with respect to the arrow ideal. A *potential* on Q is a linear combination W of cycles of Q, such that no two cycles with non-zero coefficient are cyclically equivalent.

A vertex or arrow of Q is called *frozen* if it is a vertex or arrow of F, and *mutable* or *unfrozen* otherwise. For brevity, we write $Q_0^{\mathrm{m}} = Q_0 \setminus F_0$ and $Q_1^{\mathrm{m}} = Q_1 \setminus F_1$ for the sets of mutable vertices and unfrozen arrows respectively. For $\alpha \in Q_1$ and $\alpha_n \cdots \alpha_1$ a cycle in Q, write

$$\partial_\alpha \alpha_n \cdots \alpha_1 = \sum_{\alpha_i = \alpha} \alpha_{i-1} \cdots \alpha_1 \alpha_n \cdots \alpha_{i+1}$$

and extend linearly. The ideal $\langle \partial_\alpha W : \alpha \in Q_1^{\mathrm{m}} \rangle$ of $\mathbb{K}Q$ is called the *Jacobian ideal*, and we may take its closure $\overline{\langle \partial_\alpha W : \alpha \in Q_1^{\mathrm{m}} \rangle}$ since $\mathbb{K}Q$ is a topological algebra. We define the *frozen Jacobian algebra* associated to (Q, F, W) by

$$\mathcal{J}(Q, F, W) = \mathbb{K}Q / \overline{\langle \partial_\alpha W : \alpha \in Q_1^{\mathrm{m}} \rangle}.$$

Write $A = \mathcal{J}(Q, F, W)$. The above presentation of A suggests a preferred idempotent $e = \sum_{v \in F_0} e_v$, which we call the *frozen idempotent*. We will call $B = eAe$ the *boundary algebra* of A. If $F = \varnothing$, then we refer to the pair (Q, W) as a quiver with potential, and call

$$\mathcal{J}(Q, W) = \mathcal{J}(Q, \varnothing, W)$$

the Jacobian algebra of (Q, W).

Remark 2.2. While $\mathbb{K}Q$ usually denotes the uncompleted path algebra of Q, we use it here as clean notation for the completed version since this is the version we will always take. Moreover, in much of the paper, we are interested in algebras $\mathbb{K}Q/\overline{\langle R \rangle}$ which turn out to be finite dimensional, which means that they are isomorphic to the quotient of the ordinary path algebra of Q by the ideal $\langle R \rangle$ and the distinction was irrelevant. We insist on taking complete path algebras in the general theory since it makes it more likely that the categories we construct will be Krull–Schmidt (cf. [?jensencategorification, Rem. 3.3]), and in order to apply results of Buan–Iyama–Reiten–Smith [?buanmutation] in Section 6.

Returning to the cluster algebra $\mathscr{A}$, choose a potential W on Q such that $\mathcal{J}(Q,W)$ is finite-dimensional. Then by work of Amiot [?amiotcluster], there is a 2-Calabi–Yau triangulated category $\mathcal{C} = \mathcal{C}_{(Q,W)}$ categorifying $\mathscr{A}$. In particular, the seeds of $\mathscr{A}$ correspond bijectively to additive equivalence classes of cluster-tilting objects of $\mathcal{C}$ related by a finite sequence of mutations from an initial cluster-tilting object T_0 with $\mathrm{End}_{\mathcal{C}}(T_0)^{\mathrm{op}} = \mathcal{J}(Q,W)$; we denote the seed corresponding to a cluster-tilting object T by s_T. A priori, we wish to find a Frobenius category $\mathcal{E}$ such that

(i) the stable category of $\mathcal{E}$ is triangle equivalent to $\mathcal{C}$ (so cluster-tilting objects of $\mathcal{E}$ are in bijection with those of $\mathcal{C}$), and
(ii) for any reachable cluster-tilting object $T \in \mathcal{E}$, the quiver of $\mathrm{End}_{\mathcal{E}}(T)^{\mathrm{op}}$ is, up to arrows between vertices corresponding to projective-injective summands of T, the quiver of the seed $s_T^\bullet$ of $\mathscr{A}_Q^\bullet$.

Considering the case that Q is a Dynkin quiver, so that $\mathcal{C}$ is representation-finite, one sees that there must be n indecomposable projective-injective objects of $\mathcal{E}$, whose coradicals are non-projective rigid objects $T_1,\dots,T_n$ corresponding to the cluster variables $x_1,\dots,x_n$ of the initial seed s_0. Unfortunately, one can then check that the radicals τT_i of the indecomposable projective-injectives have no projective cover, as follows. Since $\underline{\mathcal{E}}$ is 2-Calabi–Yau, τT_i is not projective, meaning any map $P \to \tau T_i$ with P projective is non-split, and so factors through the coradical T' of P, which lies in $\mathrm{add}(\bigoplus_{i=1}^n T_i)$. Moreover, if the map $P \to \tau T_i$ is a minimal projective cover, the induced map $T' \to \tau T_i$ must not factor through a projective object, so it is non-zero in the stable category $\underline{\mathcal{E}} = \mathcal{C}$. However, since $\bigoplus_{i=1}^n T_i$ is cluster-tilting in $\mathcal{C}$, we have

$$\mathrm{Hom}_{\mathcal{C}}(T',\tau T_i) = \mathrm{Ext}^1_{\mathcal{E}}(T',T_i) = 0.$$

It follows that the objects τT_i have no projective cover in $\mathcal{E}$, contradicting the assumption that $\mathcal{E}$ is Frobenius.

There are several possible ways of resolving this problem. One option, taken by Fu–Keller [?fucluster, §6], is to ask for a triangulated category modelling $\mathscr{A}_Q^\bullet$. Their construction yields a category with n 'too many' indecomposable objects, although one can think of these as corresponding to inverses of the frozen variables. More problematic, from our point of view, is that the objects corresponding to the frozen variables are not characterised intrinsically within the category.

The approach we will adopt here is similar—we simply add more frozen variables to $\mathscr{A}_Q^\bullet$ and then categorify the result using a Frobenius category $\mathcal{E}$. While, just as for Fu–Keller's construction, $\mathcal{E}$ will have too many indecomposable objects to be a 'strict' model for $\mathscr{A}_Q^\bullet$, the objects corresponding to frozen variables (of our extended cluster algebra) will be intrinsically characterised by the property of being indecomposable projective-injective. Since $\mathscr{A}_Q^\bullet$ can be obtained from the extended cluster algebra by specialising some frozen variables to 1, our category will still encode all of the combinatorial information about $\mathscr{A}_Q^\bullet$. By composing the usual cluster character on $\mathcal{E}$ [?fucluster] with the projection to $\mathscr{A}_Q^\bullet$, one can even write down a function which is morally a cluster character $\mathcal{E} \to \mathscr{A}_Q^\bullet$, but with the unusual feature that it 'has kernel' i.e. some non-zero objects of $\mathcal{E}$ have character 1.

Our chosen extension of $\mathscr{A}_Q^\bullet$ is the 'polarised principal coefficient' cluster algebra $\widetilde{\mathscr{A}_Q}$, which we now define. Starting from our seed s_0 of $\mathscr{A}$, with quiver Q and cluster variables $(x_1, \dots, x_n)$, we construct an initial seed $\widetilde{s}_0$ of $\widetilde{\mathscr{A}_Q}$ as follows. The mutable variables are $(x_1, \dots, x_n)$, and the frozen variables are $(y_1^+, \dots, y_n^+, y_1^-, \dots, y_n^-)$. The ice quiver $\widetilde{Q}$ contains Q as a full subquiver, with mutable vertices, and has two frozen vertices i^+ (corresponding to y_i^+) and i^- (corresponding to y_i^-) for each mutable vertex $i \in Q_0$, with arrows $i \to i^+$ and $i^- \to i$ for each i. In Section 3 we will also describe arrows between the frozen vertices of $\widetilde{Q}$, but since these play no role in the definition of the cluster algebra $\widetilde{\mathscr{A}_Q}$ we ignore them for now.

We adopt the word 'polarised', which refers to the partitioning of the frozen variables into two 'flavours', to differentiate this coefficient system from the system of 'double principal coefficients' studied by Rupel–Stella–Williams [?rupelgeneralized]. Since one encounters the same issues categorifying cluster algebras with double principal coefficients as one does in the case of ordinary principal coefficients, namely that the naïve categorification fails to have enough projective objects, our preference here is for the polarised version.

Remark 2.3. Like the double principal coefficient cluster algebras of [?rupelgeneralized], the cluster algebra $\widetilde{\mathscr{A}_Q}$ associated to a Dynkin quiver Q may (after inverting frozen variables) be realised as the coordinate ring of a double Bruhat cell, as we now briefly explain.

Assume Q is an orientation of a Dynkin diagram Δ with vertex set $\{1, \dots, n\}$. Quivers with underlying graph Δ are in bijection with Coxeter elements of the Weyl group W of Δ as follows. Let $s_1, \dots, s_n$ be the simple reflections generating W, and let $i_1, \dots, i_n$ be an ordering of $Q_0 = \{1, \dots, n\}$ such that $i_j < i_k$ whenever there is an arrow from j to k. Such an ordering is not unique, but any two determine the same Coxeter element

$$c = s_{i_1} \cdots s_{i_n}$$

of the Weyl group, and every Coxeter element arises in this way.

Let G be a simple connected Lie group of type Δ. After choosing a Borel subgroup and maximal torus, one may associate the *double Bruhat cell* $G^{u,v}$ to any pair $u, v \in W$, as in [?fomindouble]. Analogous to the classical Bruhat decomposition, G is then expressible as the disjoint union

$$G = \bigsqcup_{(u,v) \in W^2} G^{u,v}.$$

Berenstein–Fomin–Zelevinsky have shown that each coordinate ring $\mathbb{C}[G^{u,v}]$ has the structure of an upper cluster algebra [?berensteincluster3, Thm. 2.10] with invertible coefficients.

In the case of the double Bruhat cell $G^{c,c}$ associated to a Coxeter element c of W, the algebra $\mathbb{C}[G^{c,c}]$ is isomorphic to the cluster algebra (with invertible coefficients) from the ice quiver $\widetilde{Q}$, where Q is the orientation of Δ determined by c; i.e. it is the cluster algebra with polarised principal coefficients associated to Q (cf. [?rupelgeneralized, Thm. 2.13]). Note that we may drop the word 'upper' here, since Q is acyclic [?berensteincluster3, Cor. 2.17]. Our general results can thus be exploited to construct a Frobenius categorification of this cluster algebra.

As discussed in the introduction, we will construct a categorification $\mathcal{E}$ of $\widetilde{\mathscr{A}_Q}$ using methodology introduced by the author in [?presslandinternally]. We now recall the key definitions and results needed to explain this construction, using the notation $A^\varepsilon = A \otimes_{\mathbb{K}} A^{\mathrm{op}}$ for the enveloping algebra of A, modules of which are precisely A-bimodules.

Definition 2.4 ([?presslandinternally, Defn. 2.4])**.** An algebra A is *bimodule internally* 3*-Calabi–Yau* with respect to an idempotent $e \in A$ if

(i) $\mathrm{p.dim}_{A^\varepsilon} A \leq 3$,
(ii) $A \in \operatorname{per} A^\varepsilon$, and
(iii) there exists a triangle

$$A \xrightarrow{\psi} \Omega_A[3] \longrightarrow C \longrightarrow A[1]$$

in $\mathcal{D}A^\varepsilon$, such that

$$\mathbf{R}\mathrm{Hom}_A(C, M) = 0 = \mathbf{R}\mathrm{Hom}_{A^{\mathrm{op}}}(C, N)$$

for any $M \in \mathcal{D}_{\mathrm{fd},\underline{A}}(A)$ and $N \in \mathcal{D}_{\mathrm{fd},\underline{A}^{\mathrm{op}}}(A^{\mathrm{op}})$, where $\underline{A} = A/\langle e\rangle$.

Remark 2.5. Assume A is bimodule internally 3-Calabi–Yau with respect to e. Then $\mathrm{gl.dim}\, A \leq 3$, and there is a functorial duality

$$\mathrm{D}\,\mathrm{Ext}^i_A(M, N) = \mathrm{Ext}^{3-i}_A(N, M)$$

for finite-dimensional $M \in \operatorname{mod} \underline{A}$ and any $N \in \operatorname{Mod} A$ [?presslandinternally, Cor. 2.9]. Moreover, the same is true of A^{op} [?presslandinternally, Rem. 2.6]. In the language of [?presslandinternally, Defn. 2.1], we say that A and A^{op} are internally 3-Calabi–Yau with respect to e (without the word 'bimodule').

To construct a Frobenius category from our frozen Jacobian algebra A and its frozen idempotent e defined above, we will use the following theorem.

Theorem 2.6 ([?presslandinternally, Thm. 4.1, Thm. 4.10])**.** *Let A be an algebra, and $e \in A$ an idempotent. If A is Noetherian, $\underline{A}$ is finite-dimensional, and A is bimodule internally* 3*-Calabi–Yau with respect to e, then*

(i) *B is Iwanaga–Gorenstein of injective dimension at most* 3*, so*

$$\mathrm{GP}(B) = \{X \in \operatorname{mod} B : \mathrm{Ext}^i_B(X, B) = 0,\ i > 0\}$$

is a Frobenius category,

(ii) *$eA \in \mathrm{GP}(B)$ is cluster-tilting,*
(iii) *there are natural isomorphisms $\mathrm{End}_B(eA)^{\mathrm{op}} \xrightarrow{\sim} A$ and $\underline{\mathrm{End}}_B(eA)^{\mathrm{op}} \xrightarrow{\sim} \underline{A}$, and*
(iv) *the stable category $\underline{\mathrm{GP}}(B)$ is 2-Calabi–Yau.*

Ideally, we would like $\mathrm{GP}(B)$ as constructed above to be a Frobenius cluster category, in the sense of the following definition.

Definition 2.7 ([?presslandinternally, Defn. 3.3])**.** An exact category $\mathcal{E}$ is called a Frobenius cluster category if it is idempotent complete, stably 2-Calabi–Yau, and $\mathrm{gl.dim}\,\mathrm{End}_{\mathcal{E}}(T)^{\mathrm{op}} \leq 3$ for any cluster-tilting object $T \in \mathcal{E}$, of which there is at least one.

The category $\mathrm{GP}(B)$ constructed in Theorem 2.6 is Frobenius by (i), idempotent complete for arbitrary B, and stably 2-Calabi–Yau by (iv). Since $\mathrm{End}_B(eA)^{\mathrm{op}} \cong A$ is bimodule internally 3-Calabi–Yau, it has global dimension at most 3, but we do not know a priori that this is true for endomorphism algebras of other cluster-tilting objects. However, this does hold whenever such an algebra is Noetherian, by [?presslandinternally, Prop. 3.7]. In particular, if we additionally assume that the algebra A in Theorem 2.6 is finite-dimensional, then so is B, and then $\mathrm{GP}(B)$ is a Frobenius cluster category.

A key property of a Frobenius cluster category is the following mutation property for cluster-tilting objects, which follows immediately from work of Iyama–Yoshino [?iyamamutation] since the stable category is 2-Calabi–Yau. This will play a key role in Section 6.

Proposition 2.8 ([?iyamamutation])**.** *Let $\mathcal{E}$ be a Krull–Schmidt Frobenius cluster category, let $T \in \mathcal{E}$ be a cluster-tilting object, and choose an isomorphism $\Phi\colon \mathbb{K}Q/I \xrightarrow{\sim} \mathrm{End}_{\mathcal{E}}(T)^{\mathrm{op}}$ for some quiver Q and closed ideal $I \subseteq \mathfrak{m}(\mathbb{K}Q)$. Let k be a vertex such that $T_k = \Phi(e_k)(T)$ is non-projective; since $I \subseteq \mathfrak{m}(\mathbb{K}Q)$, the object T_k is indecomposable in $\mathcal{E}$. If Q has no 2-cycles incident with k, then there is a unique indecomposable object $T_k^* \in \mathcal{E}$, not isomorphic to T_k, such that $\mu_k T := T/T_k \oplus T_k^*$ is cluster-tilting. Such an object is determined by the short exact sequences*

$$0 \longrightarrow T_k \xrightarrow{\Phi b} \bigoplus_{\substack{b \in Q_1 \\ hb = k}} T_{tb} \longrightarrow T_k^* \longrightarrow 0,$$

$$0 \longrightarrow T_k^* \longrightarrow \bigoplus_{\substack{a \in Q_1 \\ ta = k}} T_{ha} \xrightarrow{\Phi a} T_k \longrightarrow 0.$$

Returning to the problem of categorifying the cluster algebra $\widetilde{\mathscr{A}}_Q$, our aim now is to construct an algebra A satisfying the conditions of Theorem 2.6, such that the Gabriel quiver of A agrees with the quiver $\widetilde{Q}$ up to arrows between frozen vertices.

3. An ice quiver with potential

Consider again our initial seed s_0 for $\mathscr{A}$, with quiver Q, and choose a potential W on Q. In this section, we will construct from (Q,W) an ice quiver with potential $(\widetilde{Q},\widetilde{F},\widetilde{W})$, and thus a frozen Jacobian algebra $A=\mathcal{J}(\widetilde{Q},\widetilde{F},\widetilde{W})$. It is this algebra A that we intend to use as the input for the construction of a Frobenius category by Theorem 2.6; for this we will require that $\mathcal{J}(Q,W)$ is finite-dimensional, but since the results of Section 4 do not require this assumption, we defer it for now.

Definition 3.1. Let (Q,W) be a quiver with potential. We define $\widetilde{Q}$ to be the quiver with vertex set given by

$$\widetilde{Q}_0=Q_0\sqcup Q_0^+\sqcup Q_0^-$$

where $Q_0^+=\{i^+:i\in Q_0\}$ is a set of formal symbols in natural bijection with Q_0, and similarly for $Q_0^-=\{i^-:i\in Q_0\}$. The set of arrows is given by

$$\widetilde{Q}_1=Q_1\sqcup\{\alpha_i:i\in Q_0\}\sqcup\{\beta_i:i\in Q_0\}\sqcup\{\delta_i:i\in Q_0\}\sqcup\{\delta_a:a\in Q_1\}.$$

The head and tail functions h and t on $\widetilde{Q}_1$ are extended from those on Q_1 by defining

$$\begin{aligned} t\alpha_i&=i, & h\alpha_i&=i^+,\\ t\beta_i&=i^-, & h\beta_i&=i,\\ t\delta_i&=i^+, & h\delta_i&=i^-,\\ t\delta_a&=(ha)^+, & h\delta_a&=(ta)^-. \end{aligned}$$

The frozen subquiver $\widetilde{F}$ is defined by

$$\begin{aligned} \widetilde{F}_0&=Q_0^+\sqcup Q_0^-,\\ \widetilde{F}_1&=\{\delta_i:i\in Q_0\}\sqcup\{\delta_a:a\in Q_1\}. \end{aligned}$$

Note that the head and tail of any arrow in $\widetilde{F}_1$ lies in $\widetilde{F}_0$, so these subsets describe a valid subquiver of $\widetilde{Q}$, that is in fact full. The quiver $\widetilde{F}$ is also bipartite, meaning that every vertex is either a source or a sink, and so it has no paths of length greater than 1; precisely, vertices of the form i^+ are sources, and those of the form i^- are sinks. When viewed as as a subquiver of $\widetilde{Q}$, each source i^+ of $\widetilde{F}$ is incident with a unique incoming arrow α_i and each sink i^- is incident with a unique outgoing arrow β_i.

Finally, we define a potential $\widetilde{W}$ on $\widetilde{Q}$ by

$$\widetilde{W}=W+\sum_{i\in Q_0}\beta_i\delta_i\alpha_i-\sum_{a\in Q_1}a\beta_{ta}\delta_a\alpha_{ha}, \tag{3.1}$$

and let

$$A_{Q,W}=\mathcal{J}(\widetilde{Q},\widetilde{F},\widetilde{W})$$

be the frozen Jacobian algebra determined by $(\widetilde{Q},\widetilde{F},\widetilde{W})$. We denote the boundary algebra of $A_{Q,W}$ by $B_{Q,W}=eA_{Q,W}e$, where $e=\sum_{i\in Q_0}(e_i^++e_i^-)$ is the frozen idempotent of $A_{Q,W}$.

Note. To aid legibility, if the vertices i^+ or i^- appear as subscripts, we will usually move the $+$ or $-$ sign into a superscript, so that, for example, X_{i^+} becomes X_i^+. When $W = 0$ is the zero potential, we will typically drop it from the notation; for example, we write $A_Q = A_{Q,0}$. The reader is warned that $\widetilde{0}$ is not the zero potential on $\widetilde{Q}$.

When Q is the quiver of the initial seed s_0 of $\mathscr{A}$ (and W is any potential on Q), the quiver $\widetilde{Q}$ 'is' by construction the ice quiver of the same name forming part of the data of our initial seed $\widetilde{s}_0$ of $\widetilde{\mathscr{A}_Q}$, but with the additional data of arrows between frozen vertices.

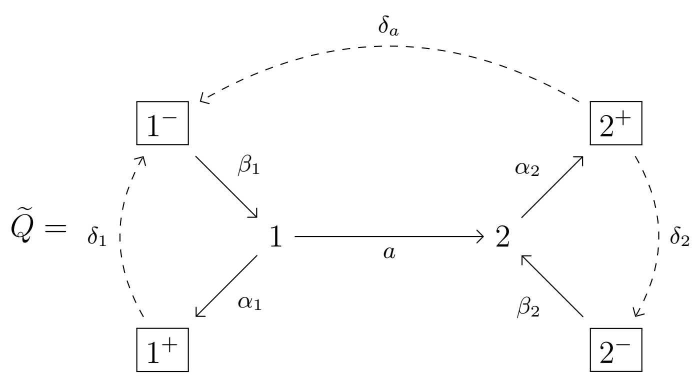
Since $\widetilde{W}$ has a straightforward combinatorial description in terms of W, so do the defining relations of $A = A_{Q,W}$; these form the set R consisting of

$$\begin{aligned} \partial_a\widetilde{W} &= \partial_a W - \beta_{ta}\delta_a\alpha_{ha}, \\ \partial_{\alpha_i}\widetilde{W} &= \beta_i\delta_i - \sum_{\substack{\gamma\in Q_1\\ h\gamma=i}} \gamma\beta_{t\gamma}\delta_\gamma, \\ \partial_{\beta_i}\widetilde{W} &= \delta_i\alpha_i - \sum_{\substack{\gamma\in Q_1\\ t\gamma=i}} \delta_\gamma\alpha_{h\gamma}\gamma, \end{aligned} \tag{3.2}$$

for $a \in Q_1$ and $i \in Q_0$. Having such an explicit generating set for the relations of A will prove to be extremely useful later in the paper.

To be able to apply Theorem 2.6, we wish to show that $A_{Q,W}$ is bimodule internally 3-Calabi–Yau with respect to its frozen idempotent e, in the sense of Definition 2.4. We will do this, under mild assumptions on (Q, W), in Section 4, but first give some examples.

Example 3.2. The quiver with potential $(Q, 0)$, for Q an A_2 quiver, provides the most basic example revealing all of the combinatorial features of the construction. In this case, we have



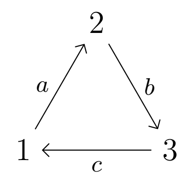
and $\widetilde{F}$ is indicated by the boxed vertices and dashed arrows. The potential on this ice quiver is given by

$$\widetilde{W} = \beta_1\delta_1\alpha_1 + \beta_2\delta_2\alpha_2 - a\beta_1\delta_a\alpha_2.$$

One can check that the frozen Jacobian algebra A_Q attached to this data is isomorphic to the endomorphism algebra of a cluster-tilting object in the Frobenius cluster category $\operatorname{Sub} Q_2$ of submodules of the injective module Q_2 for the preprojective algebra of type A_4 with socle isomorphic to the simple at one of the bivalent vertices of A_4. For a description of this category, and an explanation of why it is a Frobenius cluster category, see [?presslandinternally, Ex. 3.11].

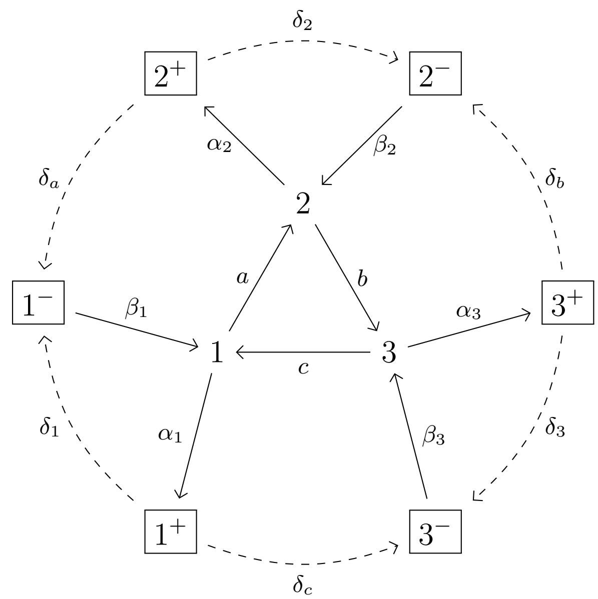
Now let (Q, W) be the quiver with potential in which

$Q =$



and $W = cba$. The Jacobian algebra of this quiver is a cluster-tilted algebra of type A_3, and has infinite global dimension (as indeed does any non-hereditary cluster-tilted algebra [?kellerclustertilted, Cor. 2.1]). In this case, we have

$\widetilde{Q} =$



with $\widetilde{F}$ again indicated by boxed vertices and dashed arrows. The potential is

$$\widetilde{W} = cba + \beta_1\delta_1\alpha_1 + \beta_2\delta_2\alpha_2 + \beta_3\delta_3\alpha_3 - a\beta_1\delta_a\alpha_2 - b\beta_2\delta_b\alpha_3 - c\beta_3\delta_c\alpha_1.$$

The associated frozen Jacobian algebra $A_{Q,W}$ also arises from a dimer model on a disk with six marked points on its boundary [?baurdimer], and is isomorphic to the endomorphism algebra of a cluster-tilting object in Jensen–King–Su's categorification of the cluster algebra structure on the Grassmannian G_2^6 [?jensencategorification]. This category is again a Frobenius cluster category [?presslandinternally, Ex. 3.12]. Unlike the first example, this algebra is infinite-dimensional. However, it is Noetherian, so Theorem 2.6 still applies.

Since in both of these cases the algebra $A_{Q,W}$ is the endomorphism algebra of a cluster-tilting object in a Frobenius cluster category, it is internally 3-Calabi–Yau with respect to its frozen idempotent by a result of Keller–Reiten [?kellerclustertilted, §5.4] (see also [?presslandinternally, Cor. 3.10]). This foreshadows Theorem 4.11 below, in which we prove that the stronger bimodule internal Calabi–Yau property holds.

Remark 3.3. Combinatorially, the algebra $A_{Q,W}$ seems to have a lot to do with the dg-algebra $\Gamma_{Q,W}$ associated to (Q,W) by Ginzburg [?ginzburgcalabiyau, §4.2]; the loops in cohomological degree -2 in $\Gamma_{Q,W}$ are replaced by the cycles $i\to i^+\to i^-\to i$, and the degree -1 arrows are replaced by the paths $ha\to ha^+\to ta^-\to ta$. Here we use Amiot's sign conventions [?amiotcluster, Defn. 3.1], which are opposite to Ginzburg's.

By a result of Keller [?kellerdeformed, Thm. 6.3], the dg-algebra $\Gamma_{Q,W}$ is always bimodule 3-Calabi–Yau, and we expect this to be related to the fact (at least under mild assumptions on (Q,W); see Theorem 4.11 below) that $A_{Q,W}$ is bimodule internally 3-Calabi–Yau with respect to the idempotent at the vertices not appearing in Ginzburg's construction. We will show in Theorem 5.3 that when Q is acyclic, there is an equivalence

$$\underline{\mathrm{GP}}(B_Q)\simeq\mathcal{C}_Q=\frac{\operatorname{per}\Gamma_Q}{\mathcal{D}^b\Gamma_Q}.$$

4. Calabi–Yau properties for frozen Jacobian algebras

We now recall from [?presslandinternally, §5] a sufficient condition on an ice quiver with potential (Q,F,W) for the associated frozen Jacobian algebra to be bimodule internally 3-Calabi–Yau with respect to the idempotent $e=\sum_{v\in F_0}e_v$. We will show, under mild assumptions on (Q,W), that the ice quiver with potential $(\widetilde{Q},\widetilde{F},\widetilde{W})$ from Definition 3.1 satisfies this condition, and so $A_{Q,W}$ has the necessary Calabi–Yau symmetry for us to be able to apply Theorem 2.6.

In [?presslandinternally, §5], it is explained how an ice quiver with potential (Q,F,W) determines a complex of projective bimodules for the associated frozen Jacobian algebra $A=\mathcal{J}(Q,F,W)$. We denote this complex by $\mathbf{P}(A)$, although strictly it depends on the presentation of A determined by (Q,F,W). We now recall its definition from [?presslandinternally, §5].

Recall that $Q_0^{\mathrm{m}}=Q_0\setminus F_0$ and $Q_1^{\mathrm{m}}=Q_1\setminus F_1$ denote the sets of mutable vertices and unfrozen arrows of Q respectively. For $v\in Q_0$, we write $\operatorname{out}(v)$ for the set of arrows of Q with tail v, and $\operatorname{in}(v)$ for the set of arrows of Q with head v. Denote the arrow ideal of A by $\mathfrak{m}(A)$, and let $S=A/\mathfrak{m}(A)$. Note that, as a left A-module, S is the direct sum of the vertex simple left A-modules. For the remainder of this section, we write $\otimes=\otimes_S$.

Introduce formal symbols ρ_α for each $\alpha \in Q_1$ and ω_v for each $v \in Q_0$, and define S-bimodule structures on the vector spaces

$$\begin{aligned}
\mathbb{K}Q_0 &= \bigoplus_{v\in Q_0} \mathbb{K}e_v, & \mathbb{K}Q_0^{\mathrm{m}} &= \bigoplus_{v\in Q_0^{\mathrm{m}}} \mathbb{K}e_v, & \mathbb{K}F_0 &= \bigoplus_{v\in F_0} \mathbb{K}e_v,\\
\mathbb{K}Q_1 &= \bigoplus_{\alpha\in Q_1} \mathbb{K}\alpha, & \mathbb{K}Q_1^{\mathrm{m}} &= \bigoplus_{\alpha\in Q_1^{\mathrm{m}}} \mathbb{K}\alpha, & \mathbb{K}F_1 &= \bigoplus_{\alpha\in F_1} \mathbb{K}\alpha,\\
\mathbb{K}Q_2 &= \bigoplus_{\alpha\in Q_1} \mathbb{K}\rho_\alpha, & \mathbb{K}Q_2^{\mathrm{m}} &= \bigoplus_{\alpha\in Q_1^{\mathrm{m}}} \mathbb{K}\rho_\alpha, & \mathbb{K}F_2 &= \bigoplus_{\alpha\in F_1} \mathbb{K}\rho_\alpha,\\
\mathbb{K}Q_3 &= \bigoplus_{v\in Q_0} \mathbb{K}\omega_v, & \mathbb{K}Q_3^{\mathrm{m}} &= \bigoplus_{v\in Q_0^{\mathrm{m}}} \mathbb{K}\omega_v, & \mathbb{K}F_3 &= \bigoplus_{v\in F_0} \mathbb{K}\omega_v,
\end{aligned}$$

via the formulae

$$\begin{aligned}
e_v \cdot e_v \cdot e_v &= e_v,\\
e_{h\alpha} \cdot \alpha \cdot e_{t\alpha} &= \alpha,\\
e_{t\alpha} \cdot \rho_\alpha \cdot e_{h\alpha} &= \rho_\alpha,\\
e_v \cdot \omega_v \cdot e_v &= \omega_v.
\end{aligned}$$

For each i, the S-bimodule $\mathbb{K}Q_i$ splits as the direct sum

$$\mathbb{K}Q_i = \mathbb{K}Q_i^{\mathrm{m}} \oplus \mathbb{K}F_i.$$

Since $\mathbb{K}Q_0 \cong S$, the A-bimodule $A \otimes \mathbb{K}Q_0 \otimes A$ is canonically isomorphic to $A \otimes A$, and we will use the two descriptions interchangeably.

We define maps $\bar{\mu}_i\colon A \otimes \mathbb{K}Q_i \otimes A \to A \otimes \mathbb{K}Q_{i-1} \otimes A$ for $1 \le i \le 3$. The map $\bar{\mu}_1$ is defined by

$$\bar{\mu}_1(x \otimes \alpha \otimes y) = x \otimes e_{h\alpha} \otimes \alpha y - x\alpha \otimes e_{t\alpha} \otimes y,$$

or, composing with the natural isomorphism $A \otimes \mathbb{K}Q_0 \otimes A \xrightarrow{\sim} A \otimes A$, by

$$\bar{\mu}_1(x \otimes \alpha \otimes y) = x \otimes \alpha y - x\alpha \otimes y.$$

For any path $p = \alpha_m \cdots \alpha_1$ of $\mathbb{K}Q$, we may define

$$\Delta_\alpha(p) = \sum_{\alpha_i=\alpha} \alpha_m \cdots \alpha_{i+1} \otimes \alpha_i \otimes \alpha_{i-1} \cdots \alpha_1,$$

and extend by linearity to obtain a map $\Delta_\alpha\colon \mathbb{K}Q \to A \otimes \mathbb{K}Q_1 \otimes A$. We then define

$$\bar{\mu}_2(x \otimes \rho_\alpha \otimes y) = \sum_{\beta\in Q_1} x\Delta_\beta(\partial_\alpha W)y.$$

Finally, let

$$\bar{\mu}_3(x \otimes \omega_v \otimes y) = \sum_{\alpha\in\mathrm{out}(v)} x \otimes \rho_\alpha \otimes \alpha y - \sum_{\beta\in\mathrm{in}(v)} x\beta \otimes \rho_\beta \otimes y.$$

Definition 4.1. For an ice quiver with potential (Q, F, W), with associated frozen Jacobian algebra A, let $\mathbf{P}(A)$ be the complex of A-bimodules with non-zero terms

$$A \otimes \mathbb{K}Q_3^{\mathrm{m}} \otimes A \xrightarrow{\mu_3} A \otimes \mathbb{K}Q_2^{\mathrm{m}} \otimes A \xrightarrow{\mu_2} A \otimes \mathbb{K}Q_1 \otimes A \xrightarrow{\mu_1} A \otimes \mathbb{K}Q_0 \otimes A$$

and $A \otimes \mathbb{K}Q_0 \otimes A$ in degree 0, where $\mu_1 = \bar{\mu}_1$, and the maps μ_2 and μ_3 are obtained by restricting $\bar{\mu}_2$ and $\bar{\mu}_3$ to $A \otimes \mathbb{K}Q_2^{\mathrm{m}} \otimes A$ and $A \otimes \mathbb{K}Q_3^{\mathrm{m}} \otimes A$ respectively. As $\operatorname{out}(v) \cup \operatorname{in}(v) \subseteq Q_1^{\mathrm{m}}$ for any $v \in Q_0^{\mathrm{m}}$, the map μ_3 takes values in $A \otimes \mathbb{K}Q_2^{\mathrm{m}} \otimes A$ as claimed.

Let $\mu_0 \colon A \otimes A \to A$ denote the multiplication map. The following theorem explains when a frozen Jacobian algebra is bimodule internally 3-Calabi–Yau with respect to its frozen idempotent.

Theorem 4.2 ([?presslandinternally, Thm. 5.7])**.** *If A is a frozen Jacobian algebra such that*

$$0 \to \mathbf{P}(A) \xrightarrow{\mu_0} A \to 0$$

is exact, then A is bimodule internally 3-Calabi–Yau with respect to the frozen idempotent $e = \sum_{v \in F_0} e_v$.

Remark 4.3. By standard results on presentations of algebras, as in Butler–King [?butlerminimal], we have $\ker \mu_0 = \operatorname{im} \mu_1$ and $\ker \mu_1 = \operatorname{im} \mu_2$. Thus it is sufficient to check exactness in degrees -2 and -3.

Returning to our main goal, let (Q, W) be a quiver with potential, let $(\widetilde{Q}, \widetilde{F}, \widetilde{W})$ be the ice quiver with potential associated to (Q, W) in Definition 3.1, and let $A = A_{Q,W}$ be its frozen Jacobian algebra. We wish to show that $0 \to \mathbf{P}(A) \xrightarrow{\mu_0} A \to 0$ is exact. To do this, we assume a little more about (Q, W).

Definition 4.4. Let (Q, W) be a quiver with potential. A *positive grading* of (Q, W) is a a function $\deg \colon Q_1 \to \mathbb{Z}_{>0}$, such that W is homogeneous of degree d in the induced $\mathbb{Z}$-grading on $\mathbb{K}Q$.

Any $\mathbb{Z}$-grading of $\mathbb{K}Q$ in which W is homogeneous descends to a $\mathbb{Z}$-grading of the Jacobian algebra $\mathcal{J}(Q, W)$, or of the frozen Jacobian algebra $A = \mathcal{J}(Q, F, W)$ for any subquiver F of Q. If this grading is induced from a positive grading as in Definition 4.4, then the degree 0 part of A is isomorphic to $S = A/\mathfrak{m}(A)$.

If $W = 0$, then any assignment of a positive integer to each arrow defines a positive grading. Given a positive grading in which W has degree d, multiplying the degree of every arrow appearing in W by k gives a new positive grading in which W has degree kd. This procedure will be useful later, when we wish to extend such positive gradings $\mathbb{K}\widetilde{Q}$ in such a way that $\widetilde{W}$ becomes homogeneous; this may only be possible after some rescaling.

Note that a positive grading need not exist for an arbitrary quiver with potential (Q, W), for example if $W = c + c^2$ for some cycle c. However, we are really interested in the structure of the algebra $\mathcal{J}(Q, W)$, which only depends on (Q, W) up to right equivalence [?derksenquivers1, Defn. 4.2], and so it is enough that (Q, W) is right equivalent to some quiver with potential admitting a positive grading. This can happen even if (Q, W) does not itself admit such a grading. For example, the quiver with potential (Q, W) from Example 3.2 in which Q is a 3-cycle admits a positive grading in which every arrow has degree 1, but it is right equivalent to $(Q, cba + cbacba)$ which does not admit a positive grading.

Lemma 4.5. *Suppose (Q, W) admits a positive grading. Then $(\widetilde{Q}, \widetilde{W})$ admits a positive grading.*

Proof. If $W = 0$, then $(\widetilde{Q}, \widetilde{W})$ has the positive grading deg with

$$\deg a = 1, \qquad \deg \alpha_i = \deg \beta_i = \deg \delta_a = 1, \qquad \deg \delta_i = 2$$

for all $i \in Q_0$ and $a \in Q_1$. From now on, we assume $W \neq 0$.

Let $\deg_0$ be a positive grading of (Q, W); since $W \neq 0$, we have $\deg_0(W) > 0$. We first construct a positive grading deg of (Q, W) with the property that $\deg(W) - \deg(a) \geq 3$ for all $a \in Q_1$.

First, note that if $\deg_0(W) - \deg_0(a) \leq 0$ for some $a \in Q_1$, then a does not appear in W, since Q has no loops. Pick $K \in \mathbb{Z}$ such that $K \deg_0(W) - \deg_0(a) \geq 1$ for all $a \in Q_1$ not appearing in W; then defining

$$\deg_1(a) = \begin{cases} K \deg_0(a), & a \text{ appears in } W, \\ \deg_0(a), & \text{otherwise}, \end{cases}$$

we see that $\deg_1(W) = K \deg_0(W)$, and so $\deg_1(W) - \deg_1(a) \geq 1$ for all $a \in Q_1$.

Since $\deg_1(W) - \deg_1(a) \geq 1$ for all $a \in Q_1$, defining $\deg(a) = 3 \deg_1(a)$ for all $a \in Q_1$ gives the required potential.

Write $d = \deg(W)$. We now extend $\deg\colon Q_1 \to \mathbb{Z}$ to the arrows of $\widetilde{Q}$, by defining

$$\deg(\alpha_v) = \deg(\beta_v) = 1, \ \deg(\delta_v) = d - 2, \ \text{and} \ \deg(\delta_a) = d - 2 - \deg(a),$$

for each $v \in Q_0$ and $a \in Q_1$. Since $d - \deg(a) \geq 3$ for all $a \in Q_1$, all of these values are positive integers. It follows immediately from the definition of $\widetilde{W}$ that this potential is homogeneous, again of degree d, with respect to deg, and so this function is a positive grading for $(\widetilde{Q}, \widetilde{W})$. □

When $A = A_{Q,W}$ is graded in such a way that all arrows have positive degree, Broomhead [?broomheaddimer, Prop. 7.5] shows that the exactness of the complex

$$0 \to \mathbf{P}(A) \xrightarrow{\mu_0} A \to 0$$

is equivalent to the exactness of

$$0 \to \mathbf{P}(A) \otimes_A S \xrightarrow{\mu_0} S \to 0.$$

(The forward implication holds in general, since $\mathbf{P}(A) \xrightarrow{\mu_0} A$ is perfect as a complex of right A-modules, and so remains exact under $- \otimes_A M$ for any $M \in \operatorname{Mod} A$.) This latter complex decomposes along with S, so that its exactness is equivalent to the exactness of

$$0 \to \mathbf{P}(A) \otimes_A S_v \xrightarrow{\mu_0} S_v \to 0$$

for each $v \in \widetilde{Q}_0$, where S_v denotes the vertex simple left A-module at v. Thus when (Q, W), and hence $(\widetilde{Q}, \widetilde{W})$ by Lemma 4.5, admits a positive grading, we are able to reduce the problem of computing a bimodule resolution of A to the simpler problem of computing a projective resolution of each vertex simple left A-module.

The most complicated map in the complex $\mathbf{P}(A)$ is μ_2, so we wish to spell out $\mu_2 \otimes_A S_v$ explicitly. We have

$$A \otimes \mathbb{K}\widetilde{Q}_2^{\mathrm{m}} \otimes A \otimes_A S_v \cong \bigoplus_{a \in \mathrm{in}(v) \cap \widetilde{Q}_1^{\mathrm{m}}} Ae_{ta},$$

$$A \otimes \mathbb{K}\widetilde{Q}_1 \otimes A \otimes_A S_v \cong \bigoplus_{b \in \mathrm{out}(v)} Ae_{hb},$$

and under these isomorphisms we can write

$$(\mu_2 \otimes_A S_v)(x) = \sum_{a \in \mathrm{in}(v) \cap \widetilde{Q}_1^{\mathrm{m}}} \sum_{b \in \mathrm{out}(v)} x_a \partial_b^{\mathrm{r}} \partial_a W,$$

where ∂_b^{r}, called the *right derivative* with respect to b, is defined on paths by

$$\partial_b^{\mathrm{r}}(\alpha_n \cdots \alpha_1) = \begin{cases} \alpha_n \cdots \alpha_2, & \alpha_1 = b, \\ 0, & \alpha_1 \neq b \end{cases}$$

and extended linearly.

We now prove the necessary exactness for the complex $\mathbf{P}(A) \otimes_A S_v$. To do this, we break into two cases depending on whether v is mutable or frozen, and use heavily the explicit set R of defining relations for A given above (3.2).

Lemma 4.6. *Let $i \in Q_0$ be a mutable vertex of $\widetilde{Q}$. For any $x \in Ae_i$, if $x\beta_i = 0$ then $x = 0$. For any $y \in e_iA$, if $\alpha_i y = 0$ then $y = 0$.*

Proof. Let $\widetilde{x}$ be an arbitrary lift of x to $\mathbb{K}\widetilde{Q}e_i$. Now assume $x\beta_i = 0$, so $\widetilde{x}\beta_v \in \overline{\langle R \rangle}$. Since every term of $\widetilde{x}\beta_i$, when written in the basis of paths of $\widetilde{Q}$, ends with the arrow β_i, but no term of any element of R has a term ending with β_i, we must be able to write

$$\widetilde{x}\beta_i = \sum_j z_j \beta_i$$

for $z_j \in \overline{\langle R \rangle} e_i$. Comparing terms, we see that $\widetilde{x} = \sum_j z_j \in \overline{\langle R \rangle}$, and so $x = 0$ in A. The second statement is proved analogously. □

Proposition 4.7. *For $i \in Q_0$, the map $\mu_3 \otimes_A S_v \colon Ae_v \to \bigoplus_{a \in \mathrm{in}(v) \cap \widetilde{Q}_1^{\mathrm{m}}} Ae_{ta}$ is injective.*

Proof. We have

$$(\mu_3 \otimes_A S_i)(x) = \sum_{a \in \mathrm{in}(v) \cap \widetilde{Q}_1^{\mathrm{m}}} (-xa).$$

Since i is mutable, $\beta_i \in \mathrm{in}(i) \cap \widetilde{Q}_1^{\mathrm{m}}$ by the construction of $(\widetilde{Q}, \widetilde{F})$, and moreover it is the unique arrow in this set with tail at i^-. Thus if $(\mu_3 \otimes S_i)(x) = 0$, it follows by multiplying the above formula on the right by e_i^- that $x\beta_i = 0$, and so $x = 0$ by Lemma 4.6. □

Lemma 4.8. *Let $i \in Q_0$ be a mutable vertex of $\widetilde{Q}$. For each $a \in \mathrm{in}(i) \cap \widetilde{Q}_1^{\mathrm{m}} = (\mathrm{in}(i) \cap Q_1) \cup \{\beta_i\}$, pick $x_a \in Ae_{ta}$. If*

$$x_{\beta_i}\delta_i = \sum_{a \in \mathrm{in}(i) \cap Q_1} x_a \beta_{ta} \delta_a,$$

then there exists $y \in Ae_{ha}$ such that $x_a = ya$ for each $a \in \mathrm{in}(v) \cap \widetilde{Q}_1^{\mathrm{m}}$.

Proof. Pick a lift $\widetilde{x}_a \in \mathbb{K}\widetilde{Q}$ of each x_a. Writing

$$F = \widetilde{x}_{\beta_i}\delta_i - \sum_{a \in \mathrm{in}(v) \cap Q_1} \widetilde{x}_a \beta_{ta} \delta_a,$$

our assumption on the x_a is equivalent to $F \in \overline{\langle R \rangle}$. Since every term of F ends with either δ_i or $\beta_{ta}\delta_a$ for some $a \in \mathrm{in}(i) \cap Q_1$, and the only element of R including terms ending with these arrows is $\beta_i\delta_i - \sum_{a \in \mathrm{in}(i) \cap Q_1} a\beta_{ta}\delta_a$, we can write

$$F = z_i\delta_i + \sum_{a \in \mathrm{in}(i) \cap Q_1} z_a \beta_{ta}\delta_a + y\Big(\beta_i\delta_i - \sum_{a \in \mathrm{in}(i) \cap Q_1} a\beta_{ta}\delta_a\Big),$$

where $z_i \in \overline{\langle R \rangle} e_i^-$, $z_a \in \overline{\langle R \rangle} e_{ta}$ and $y \in \mathbb{K}\widetilde{Q}e_i^+$. Comparing terms in our two expressions for F, we see that

$$\widetilde{x}_{\beta_i} = z_i + y\beta_i,$$
$$\widetilde{x}_a = z_a + ya.$$

Since $z_i, z_a \in \overline{\langle R \rangle}$, when we pass to the quotient algebra $A = \mathbb{K}\widetilde{Q}/\overline{\langle R \rangle}$ we see that $x_{\beta_i} = y\beta_i$ and $x_a = ya$, as required. □

Proposition 4.9. *For $i \in Q_0$, we have* $\ker(\mu_2 \otimes_A S_i) = \mathrm{im}(\mu_3 \otimes_A S_i)$.

Proof. Since we already know that $\mathbf{P}(A)$ is a complex, it is enough to show that $\ker(\mu_2 \otimes_A S_i) \subseteq \mathrm{im}(\mu_3 \otimes_A S_i)$.

Let $x \in \bigoplus_{a \in \mathrm{in}(i) \cap \widetilde{Q}_1^m} Ae_{ta}$. Applying $\mu_2 \otimes S_i$ gives

$$\sum_{a \in \mathrm{in}(i) \cap \widetilde{Q}_1^{\mathrm{m}}} \sum_{b \in \mathrm{out}(i)} x_a \partial_b^r \partial_a W \in \bigoplus_{b \in \mathrm{out}(i)} Ae_{hb}.$$

Since α_i is the unique arrow in $\mathrm{out}(i)$ with head i^+, multiplying the above expression on the right by e_i^+ and using the explicit expressions for the relations $\partial_a W$ computed earlier gives

$$\sum_{a \in \mathrm{in}(i) \cap \widetilde{Q}_1^m} x_a \partial_{\alpha_i}^r \partial_a W = x_{\beta_i}\delta_i - \sum_{a \in \mathrm{in}(i) \cap Q_1} x_a \beta_{ta}\delta_a \in Ae_i^+.$$

So if x is in the kernel of $\mu_2 \otimes S_i$, then in particular we have

$$x_{\beta_i}\delta_i = \sum_{a \in \mathrm{in}(i) \cap Q_1} x_a \beta_{ta} \delta_a,$$

and so by Lemma 4.8 there exists $y \in A$ such that $x_a = ya$ for each a. It follows that $x = (\mu_3 \otimes_A S_i)(y)$, as required. □

Proposition 4.10. *If $v \in \widetilde{F}_0$ then $\mathbf{P}(A) \otimes_A S_v$ is a projective resolution of S_v.*

Proof. Since $v \in \widetilde{F}_0$, the complex $\mathbf{P}(A) \otimes_A S_v$ is zero in degree -3, so in view of Remark 4.3, it is only necessary for us to check that $\mu_2 \otimes S_v$ is injective. In fact, if $v = i^-$ for some $i \in Q_0$, then $\mathbf{P}(A) \otimes_A S_v$ is also zero in degree -2 since there are no unfrozen arrows in $\mathrm{in}(i^-)$, so we need only consider $v = i^+$ for some $i \in Q_0$.

Since $\mathrm{in}(i^+) \cap \widetilde{Q}_1^{\mathrm{m}} = \{\alpha_i\}$, we have $\mu_2 \otimes_A S_i^+ \colon Ae_i \to \bigoplus_{b \in \mathrm{out}(i)} Ae_{hb}$. Let $x \in Ae_i$. Then, computing as above, we have

$$(\mu_2 \otimes_A S_i^+)(x) = \sum_{b \in \mathrm{out}(i^+)} x\partial_b^{\mathrm{r}} \partial_{\alpha_i} W.$$

Assume this is 0. The unique arrow of $\mathrm{out}(i^+)$ with head i^- is δ_i, so multiplying on the right by e_i^- gives

$$0 = x\partial_{\delta_i}^{\mathrm{r}} \partial_{\alpha_i} W = x\beta_i.$$

By Lemma 4.6, it follows that $x = 0$, and $\mu_2 \otimes_A S_i^+$ is injective as required. □

Combining these results, we are able to establish the desired internal Calabi–Yau property for $A_{Q,W}$ whenever (Q, W) admits a positive grading.

Theorem 4.11. *If (Q, W) admits a positive grading, then $A = \mathcal{J}(\widetilde{Q}, \widetilde{F}, \widetilde{W})$ is bimodule internally 3-Calabi–Yau with respect to the frozen idempotent $e = \sum_{i \in Q_0} (e_i^+ + e_i^-)$.*

Proof. The combination of Propositions 4.7 and 4.9 shows that $\mathbf{P}(A) \otimes_A S_i$ is exact for any $i \in Q_0 = \widetilde{Q}_0^{\mathrm{m}}$, and Proposition 4.10 shows that the same is true for $v \in \widetilde{F}_0$, so the direct sum $\mathbf{P}(A) \otimes_A S \xrightarrow{\mu_0} S$ of all of these complexes is exact. By Lemma 4.5, we can grade A so that all arrows have positive degree. As already remarked, this allows us to apply Broomhead's result [?broomheaddimer, Prop. 7.5] to deduce exactness of $\mathbf{P}(A) \xrightarrow{\mu_0} A$ from that of $\mathbf{P}(A) \otimes_A S \xrightarrow{\mu_0} S$. Now the result follows from Theorem 4.2. □

As a corollary, we obtain the promised result (Theorem 2 in the introduction) that any quiver with potential admitting a positive grading may be realised as the 'interior' of an bimodule internally 3-Calabi–Yau frozen Jacobian algebra.

Corollary 4.12. *Let (Q, W) be a quiver with potential admitting a positive grading, and let $\underline{A} = \mathcal{J}(Q, W)$ be the corresponding Jacobian algebra. Then $\underline{A} = A_{Q,W}/\langle e \rangle$, where e is the frozen idempotent of $A_{Q,W}$, and $A_{Q,W}$ is internally 3-Calabi–Yau with respect to e.* □

We have no evidence that the existence of a positive grading is necessary to obtain the conclusion of Theorem 4.11; rather, this condition was imposed in order to make the necessary calculations more manageable. By analogy with Keller's result [?kellerdeformed, Thm. 6.3, Thm. A.12] that the Jacobian algebra of any quiver with potential is the 0-th homology of a bimodule 3-Calabi–Yau dg-algebra, we conjecture that this assumption is not in fact needed.

Conjecture 4.13. *The conclusion of Theorem 4.11, and hence also that of Corollary 4.12, remains valid without the assumption that (Q, W) admits a positive grading.*

In support of this conjecture, we prove injectivity of μ_3 directly, without any assumption on the existence of a positive grading.

Proposition 4.14. *The map $\mu_3 \colon A \otimes \widetilde{Q}_3^{\mathrm{m}} \otimes A \to A \otimes \widetilde{Q}_2^{\mathrm{m}} \otimes A$ is injective.*

Proof. We use the natural isomorphism

$$A \otimes \mathbb{K}\widetilde{Q}_3^{\mathrm{m}} \otimes A \xrightarrow{\sim} \bigoplus_{i \in Q_0} Ae_i \otimes_{\mathbb{K}} e_iA$$

given by $x \otimes \omega_i \otimes y \mapsto x \otimes y$. For each $i \in Q_0$, let $x_i = (x_i^j)_{j \in J_i}$ and $y_i = (y_i^j)_{j \in J_i}$ be finite sets of elements of Ae_i and e_iA respectively. These define an element

$$x_i \otimes y_i := \sum_{j \in J_i} x_i^j \otimes y_i^j$$

of $Ae_i \otimes_{\mathbb{K}} e_iA$, and all elements of $Ae_i \otimes_{\mathbb{K}} e_iA$ are of this form. Without loss of generality, i.e. without changing the value of $x_i \otimes y_i$, we may assume that $\{y_i^j : j \in J_i\}$ is a linearly independent set.

Now assume $\sum_{i \in Q_0} x_i \otimes y_i$ is in the kernel of μ_3. We aim to show that, in this case, $x_i \otimes y_i = 0$ for all $i \in Q_0$, and so in particular their sum is zero. Projecting onto the component $A \otimes \mathbb{K}\rho_{\beta_i} \otimes A = Ae_i \otimes_{\mathbb{K}} e_i^+ A$ of $A \otimes \widetilde{Q}_2^{\mathrm{m}} \otimes A$, we see that

$$x_i\beta_i \otimes y_i = \sum_{j \in J_i} x_i^j \beta_i \otimes y_i^j = 0.$$

Since the y_i^j are linearly independent, it follows that $x_i^j\beta_i = 0$ for all j. By Lemma 4.6, it follows that $x_i^j = 0$ for all j, and so $x_i \otimes y_i = 0$. □

5. The acyclic case

Let (Q, W) be a quiver with potential such that $\mathcal{J}(Q, W) = A_{Q,W}/\langle e \rangle$ is finite-dimensional. Then all of the assumptions of Theorem 2.6 are satisfied for $A_{Q,W}$, except possibly Noetherianity. Our goal in this section is to show that if Q is an acyclic quiver, then the algebra $A = A_Q = \mathcal{J}(\widetilde{Q}, \widetilde{F}, \widetilde{0})$ is finite-dimensional. In particular, this means that A is Noetherian, and so we may apply Theorem 2.6 to obtain a categorification of the polarised principal coefficient cluster algebra $\widetilde{\mathscr{A}}_Q$.

Lemma 5.1. *Let Q be any quiver, and let p be a path in $\widetilde{Q}$. Assume that p contains at least one arrow not in Q_1, and that $hp, tp \in Q_0$. Then p maps to zero under the projection $\mathbb{K}\widetilde{Q} \to A = \mathcal{J}(\widetilde{Q}, \widetilde{F}, \widetilde{0})$. Moreover, if q is a path of length at least 5 containing no arrows of Q_1, then q maps to zero under this projection.*

Proof. Let γ be the first arrow of p in $\widetilde{Q}_1 \setminus Q_1$; there is such an arrow by assumption. If γ has a predecessor in p, then this arrow is in Q_1, and so $t\gamma \in Q_0$. On the other hand, if γ is the first arrow of p, then $t\gamma = tp \in Q_0$ by assumption. By the construction of $\widetilde{Q}$, it follows that $\gamma = \alpha_i$ for some $i \in Q_0$.

Since $hp \in Q_0$, the path p cannot terminate with α_i, so it has a successor. Looking again at the combinatorics of $\widetilde{Q}$, the only options are δ_a for some $a \in Q_1$ with $ha = i$, or δ_i. We break into two cases.

First assume α_i is followed in p by δ_a for some $a \in Q_0$. This again cannot be the final arrow of p, since $hp \in Q_0$. The only arrow leaving $h\delta_a = i^-_{ta}$ is β_{ta}, so this must be the next arrow of p. But after projection to A, we have

$$\beta_{ta}\delta_a\alpha_{ha} = \partial_a W = 0$$

since $W = 0$, so p projects to zero.

In the second case, α_i is followed by δ_i. As in the previous case, δ_i must be followed in p by β_i. Again projecting to A, we have

$$\beta_i\delta_i\alpha_i = \Big(\sum_{\substack{a\in Q_1\\ ha=i}} a\beta_{ta}\delta_a\Big)\alpha_i = \sum_{\substack{a\in Q_1\\ ha=i}} a\beta_{ta}\delta_a\alpha_{ha} = 0,$$

and so again p projects to zero.

For the final statement, we have already shown that the paths $\beta_{ta}\delta_a\alpha_{ha}$ for some $a \in Q_1$ or $\beta_i\delta_i\alpha_i$ for some $i \in Q_0$ project to zero. Thus if q as in the statement does not project to zero, it must not contain either of these subpaths. However, the bipartite property of $\widetilde{F}$ means that any path of arrows not in Q_1 without either of these subpaths is itself a subpath of $\delta_x\alpha_i\beta_i\delta_y$ for some $x, y \in Q_0 \cup Q_1$ and $i \in Q_0$ with $hx = i = ty$, and so has length at most 4. □

Theorem 5.2. *Let Q be an acyclic quiver. Then $A = \mathcal{J}(\widetilde{Q}, \widetilde{F}, \widetilde{0})$ is finite-dimensional.*

Proof. We show that there are finitely many paths of $\widetilde{Q}$ determining non-zero elements of A. By Lemma 5.1, any path p of $\widetilde{Q}$ determining a non-zero element of A may not have any subpath with endpoints in Q_0 and containing an arrow outside Q_1. Thus we must have $p = q_2p'q_1$, where q_1 and q_2 feature no arrows of Q_1, and p' is a path in Q. Since Q is acyclic, there are only finitely many possibilities for p'. By Lemma 5.1 again, q_1 and q_2 have length at most 4, and so there are again only finitely many possibilities. □

We have now established everything we need in order to deduce part (i) and most of part (ii) of Theorem 1 from the introduction; precisely, we may prove:

Theorem 5.3. *Let Q be an acyclic quiver, and consider the frozen Jacobian algebra $A = A_Q = \mathcal{J}(\widetilde{Q}, \widetilde{F}, \widetilde{W})$. Let $e = \sum_{i\in Q_0}(e^+_i + e^-_i)$ be the frozen idempotent of A, let $B = B_Q = eAe$ be the boundary algebra, and let $T = eA$. Then T is a cluster-tilting object of the Frobenius cluster category $\mathrm{GP}(B)$, with $\mathrm{End}_B(T)^{\mathrm{op}} \cong A$ and $\underline{\mathrm{End}}_B(T)^{\mathrm{op}} \cong \mathbb{K}Q$. It follows that $\underline{\mathrm{GP}}(B) \simeq \mathcal{C}_Q$.*

Proof. Since Q is acyclic, Theorem 5.2 shows that A is finite-dimensional (so in particular it is Noetherian), and so is $\underline{A} = A/\langle e\rangle = \mathbb{K}Q$. Moreover, any assignment of a positive integer to each arrow of Q determines a positive grading on $(Q, 0)$, so A is bimodule internally 3-Calabi–Yau with respect to e by Theorem 4.11. Now all necessary conclusions follow from Theorem 2.6, except that the endomorphism algebra of any cluster-tilting object in $\mathrm{GP}(B)$ has global dimension at most 3. This is obtained from [?presslandinternally, Prop. 3.7], using that B is a finite-dimensional algebra, meaning that the relevant endomorphism algebras are also finite-dimensional, and so in particular Noetherian.

We obtain the final conclusion from Keller–Reiten's 'recognition' theorem [?kelleracyclic, Thm. 2.1]; $\underline{\mathrm{GP}}(B)$ is a 2-Calabi–Yau triangulated category admitting a cluster-tilting object T with endomorphism algebra $\mathbb{K}Q$, and therefore $\underline{\mathrm{GP}}(B)$ is triangle equivalent to the cluster category $\mathcal{C}_Q$. □

We note that most of the conclusions of Theorem 5.3 may still hold when Q has cycles, for example when Q is the 3-cycle and W the obvious potential (see Example 3.2 above and Example 9.3 below). In this case $A_{Q,W}$, although not finite-dimensional, is still Noetherian, as are the endomorphism algebras of all other cluster-tilting objects of $\mathrm{GP}(B_{Q,W})$ [?presslandinternally, Exa. 3.12]. Since $\mathcal{J}(Q,W)$ is finite-dimensional and can be positively graded by giving each arrow of Q degree 1, most of the argument of Theorem 5.3 goes through essentially unchanged. The main exception is that Keller–Reiten's recognition theorem no longer applies, so one must observe directly in this case that $\underline{\mathrm{GP}}(B) \simeq \mathcal{C}_{Q,W}$.

6. Mutation

Let Q be an acyclic quiver. In this section we show that if T and $\mu_k T$ are cluster-tilting objects of $\mathcal{E} = \mathrm{GP}(B_Q)$, mutation equivalent to the initial cluster-tilting object eA_Q, and related by mutation at an indecomposable summand T_k of T (see Proposition 2.8), the quiver of $\mathrm{End}_{\mathcal{E}}(\mu_k T)^{\mathrm{op}}$ is given by the Fomin–Zelevinsky mutation of the quiver of $\mathrm{End}_{\mathcal{E}}(T)^{\mathrm{op}}$ at the vertex corresponding to T_k, up to arrows between frozen vertices. This will be used to complete the proof of Theorem 1; see Theorem 6.13 below.

We will in fact prove more than this, and in greater generality, via a mild generalisation of work of Buan–Iyama–Reiten–Smith [?buanmutation]. Let T be a cluster-tilting object of a Frobenius category $\mathcal{E}$ such that there is an isomorphism $\Phi\colon \mathcal{J}(Q,F,W) \xrightarrow{\sim} \mathrm{End}_{\mathcal{E}}(T)^{\mathrm{op}}$ for some ice quiver with potential (Q,F,W), and let $T_k = \Phi(e_k)(T)$ be an indecomposable non-projective summand of T. Under certain conditions on $\mathcal{E}$, T and T_k, we show that the mutated cluster tilting object $\mu_k T = T/T_k \oplus T_k^*$ has $\mathrm{End}_{\mathcal{E}}(T)^{\mathrm{op}} \xrightarrow{\sim} \mathcal{J}(\mu_k(Q,F,W))$, for some appropriately defined mutation operation μ_k on ice quivers with potential. The quiver of $\mu_k(Q,F,W)$ coincides, up to arrows between frozen vertices and the addition of 2-cycles, with the Fomin–Zelevinsky mutation of Q at the vertex k corresponding to T_k. All of the necessary assumptions for this general result will turn out to hold when $\mathcal{E} = \mathrm{GP}(B_Q)$ for Q acyclic, T is a cluster-tilting object related to our initial object eA_Q by a finite sequence of mutations, and T_k is any non-projective indecomposable summand of T. Moreover, we may show directly that the quiver of $\mathrm{End}_{\mathcal{E}}(T)^{\mathrm{op}}$ has no 2-cycles, and thus obtain our desired result.

We begin by defining mutations of ice quivers with potential, following Derksen–Weyman–Zelevinsky [?derksenquivers1] in the case $F = \varnothing$ (see also [?buanmutation, §1.2], [?presslandfrobenius, §3.3]).

Definition 6.1. Let (Q,F,W) be an ice quiver with potential such that Q has no loops, and let $k\in Q_0^{\mathrm{m}}$ be a mutable vertex such that there are no 2-cycles of Q passing through k. We define the mutation $\mu_k(Q,F,W)=(Q',F',W')$ as follows. The quiver Q' is the output of the following procedure.

(0) Replace the vertex k by a k^*. (This purely notational change is to help distinguish the quivers Q and Q' later on.)
(1) Add an arrow $[ab]\colon i\to j$ for each pair of arrows $b\colon i\to k$ and $a\colon k\to j$ in Q.
(2) Replace each arrow $a\colon k\to j$ in Q by an arrow $a^*\colon j\to k^*$, and each arrow $b\colon i\to k$ in Q by an arrow $b^*\colon k^*\to i$.

Note that this is precisely the operation of Fomin–Zelevinsky mutation at k, excluding the final step of cancelling 2-cycles, and that Q' again has no loops, and no 2-cycles at k^*. The frozen subquiver F' of Q' is given by exactly the vertices and arrows of F, none of which were changed in the above procedure. (In particular, this means that the new arrows a^*, b^* and $[ab]$ are all unfrozen.) We define

$$W'=[W]+\sum_{\substack{a,b\in Q_1\\ ta=k=hb}}a^*[ab]b^*,$$

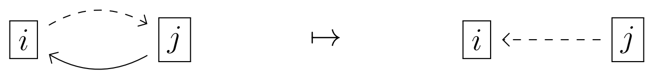
where $[W]$ is obtained from W by substituting $[ab]$ for ab each time the latter appears in a cycle of W.

For later use, we record the right derivatives of the relations defined by W'; these are calculated directly from the definition.

Lemma 6.2 (cf. [?buanmutation, Lem. 5.8])**.** *Let (Q,F,W) be an ice quiver with potential such that Q has no loops, let $k\in Q_0^{\mathrm{m}}$ be a mutable vertex such that no 2-cycles of Q are incident with k, and write $\mu_k(Q,F,W)=(Q',F',W')$. Let $a,b\in Q_1$ have $ta=k=hb$, and let $c,c'\in Q_1\cap Q_1'$. Then*

(i) $\partial^r_c\partial_{c'}W'=\partial^r_c\partial_{c'}W$,
(ii) $\partial^r_c\partial_{[ab]}W'=\partial^r_c\partial_{[ab]}[W]=\partial^r_c\partial^r_a\partial_bW$ *and* $\partial^r_{[ab]}\partial_cW'=\partial^r_{[ab]}\partial_c[W]=\partial^r_a\partial^r_b\partial_cW$,
(iii) $\partial^r_{[ab]}\partial_{a^*}W'=b^*$,
(iv) $\partial^r_{b^*}\partial_{[ab]}W'=a^*$,
(v) $\partial^r_{a^*}\partial_{b^*}W'=[ab]$, *and*
(vi) *For any other pair $d,d'\in Q_1'$, we have $\partial^r_d\partial_dW'=0$.* □

Remark 6.3. The potential W' above may contain terms of length 2 (but not of length 1, since Q' has no loops), meaning that some of the defining relations of $\mathcal{J}(Q',F',W')$ are non-admissible, and so the Gabriel quiver of $\mathcal{J}(Q',F',W')$ is $Q''\neq Q'$. Via a process of reduction from (Q',F',W'), similar to that of [?derksenquivers1, Thm. 4.6] for ordinary quivers with potential, we may find a frozen subquiver F'' of Q'' and a potential W'' such that $\mathcal{J}(Q',F',W')\cong\mathcal{J}(Q'',F'',W'')$. On the level of the ice quiver, this process involves deleting any 2-cycle appearing in W' and consisting of unfrozen arrows, and replaces any 2-cycle of W' consisting of one frozen and one unfrozen arrow with a single frozen arrow as below.



The interested reader can find more details in [?presslandfrobenius, §3.3]. Since we only define mutation at vertices not lying on 2-cycles, such reduction steps can be necessary in order to perform iterated mutation, but they will not be needed for the arguments that follow.

We now, following [?buanmutation], explain a result linking mutation of ice quivers with potential to mutation of cluster-tilting objects in Frobenius categories. Given an additive category $\mathcal{C}$, and objects $X, Y \in \mathcal{C}$, let $\operatorname{Rad}_{\mathcal{C}}(X,Y)$ denote the subspace of $\operatorname{Hom}_{\mathcal{C}}(X,Y)$ consisting of maps f such that $1_X - gf$ is invertible for all $g\colon Y \to X$. We then define $\operatorname{Rad}^m_{\mathcal{C}}(X,Y)$ to be the subspace of $\operatorname{Hom}_{\mathcal{E}}(X,Y)$ consisting of maps that may be written as a composition $f_m \circ \cdots \circ f_1$ with $f_i \in \operatorname{Rad}_{\mathcal{C}}(X_{i-1}, X_i)$ for some $X_i \in \mathcal{C}$ (so that necessarily $X_0 = X$ and $X_m = Y$). We extend this notation by

$$\operatorname{Rad}^0_{\mathcal{C}}(X,Y) := \operatorname{Hom}_{\mathcal{C}}(X,Y).$$

Note that, for any m, the subspace $\operatorname{Rad}^m_{\mathcal{C}}(X,X)$ is an ideal of $\operatorname{End}_{\mathcal{C}}(X)^{\mathrm{op}}$. Moreover, if $\mathcal{D} \subseteq \mathcal{C}$ is a full subcategory, then $\operatorname{Rad}^m_{\mathcal{D}}(X,Y) = \operatorname{Rad}^m_{\mathcal{C}}(X,Y)$ for $m = 0$ and $m = 1$, but this equality need not hold for $m > 2$. More information about the radical of a category may be found in [?assemelements, §A.3].

We will consider $\mathbb{K}$-linear categories $\mathcal{C}$ satisfying the conditions

(C1) $\mathcal{C}$ is Krull–Schmidt, and

(C2) for any non-zero basic object $X \in \mathcal{C}$, we have

  (A1) $\operatorname{End}_{\mathcal{C}}(X)^{\mathrm{op}} / \operatorname{Rad}_{\mathcal{C}}(X,X) \cong \mathbb{K}^n$ for some $n > 0$, and

  (A2) $\operatorname{End}_{\mathcal{C}}(X)^{\mathrm{op}} \cong \varprojlim_{m\geq 0} \operatorname{End}_{\mathcal{C}}(X)^{\mathrm{op}} / \operatorname{Rad}^m_{\mathcal{C}}(X,X)$.

For example, if B is a finite-dimensional Iwanaga–Gorenstein algebra, then $\operatorname{GP}(B)$ is a Frobenius category satisfying (C1) and (C2); indeed, (C2) is satisfied by any Hom-finite $\mathbb{K}$-linear category. The Grassmannian cluster categories introduced by Jensen–King–Su [?jensencategorification] also satisfy (C1) and (C2) providing one takes completions in all of the definitions (cf. [?jensencategorification, Rem. 3.3]), since in this case the endomorphism algebra of any basic object is a finitely generated $\mathbb{C}[[t]]$-module, meaning (A1) and (A2) hold.

Let $\mathcal{C}$ be a category satisfying (C1) and (C2), and let Q be a finite quiver. For each vertex $i \in Q_0$, choose an object $T_i \in \mathcal{C}$, and for each arrow $a\colon i \to j$ in Q, choose a morphism $\Phi a \in \operatorname{Hom}_{\mathcal{C}}(T_j, T_i)$. This data is equivalent to specifying an algebra homomorphism

$$\Phi\colon \mathbb{K}Q \to \operatorname{End}_{\mathcal{C}}(T)^{\mathrm{op}},$$

where $T = \bigoplus_{i \in Q_0} T_i$ [?buanmutation, Lem. 3.5], with $\Phi(e_i) = 1_{T_i}$ for each vertex idempotent e_i. Let R be a finite subset of the arrow ideal $\mathfrak{m}(\mathbb{K}Q)$ such that each $r \in R$ is basic, meaning it is a formal linear combination of paths of Q with the same head and tail, and let I denote the closure of the ideal generated by R. For example, the set of cyclic derivatives of a potential on Q is a set of basic elements. Buan–Iyama–Reiten–Smith [?buanmutation, Prop. 3.6] characterise when the homomorphism Φ above induces an isomorphism

$$\Phi\colon \mathbb{K}Q/I \xrightarrow{\sim} \operatorname{End}_{\mathcal{C}}(T)^{\mathrm{op}}$$

in terms of certain complexes in $\operatorname{add} T$, depending on Φ, Q and R, being right 2-almost split, a definition we now recall.

Definition 6.4 ([?buanmutation, Defn. 4.4])**.** Let $\mathcal{C}$ be a category satisfying (C1) and (C2), and let $T \in \mathcal{C}$ be any object. Let

$$U_1 \xrightarrow{f_1} U_0 \xrightarrow{f_0} X$$

be a complex in $\operatorname{add} T$, and consider the induced sequence

$$\operatorname{Hom}_{\mathcal{C}}(T, U_1) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(T, U_0) \longrightarrow \operatorname{Rad}_{\mathcal{C}}(T, X) \longrightarrow 0.$$

We say that f_0 is *right almost split* in $\operatorname{add} T$ if this induced sequence is exact at $\operatorname{Rad}_{\mathcal{C}}(T, X)$, that f_1 is a *pseudo-kernel* of f_0 in $\operatorname{add} T$ if this induced sequence is exact at $\operatorname{Hom}_{\mathcal{C}}(T, U_0)$, and that the sequence (f_1, f_0) is *right* 2*-almost split* if both of these conditions hold simultaneously.

We define *left almost split* maps, *pseudo-cokernels* and *left* 2*-almost split* sequences in $\operatorname{add} T$ dually, and call a complex

$$Y \xrightarrow{f_2} U_1 \xrightarrow{f_1} U_0 \xrightarrow{f_0} X$$

weak 2*-almost split* in $\operatorname{add} T$ if (f_1, f_0) is a right 2-almost split sequence in $\operatorname{add} T$ and (f_2, f_1) is a left 2-almost split sequence in $\operatorname{add} T$.

To establish our isomorphisms, we will use [?buanmutation, Prop. 3.3] (see also [?buanmutation, Prop. 3.6], which is the same result in more categorical language). The following statement specialises this proposition to the case of frozen Jacobian algebras.

Proposition 6.5 (cf. [?buanmutation, Prop. 3.3])**.** *Let (Q, F, W) be an ice quiver with potential, $\mathcal{C}$ be an additive category satisfying (C1) and (C2), and $\Phi\colon \mathbb{K}Q \to \operatorname{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ an algebra homomorphism. Write $T_i = \Phi(e_i)(T)$. Then the following are equivalent:*

(i) Φ *induces an isomorphism $\mathcal{J}(Q, F, W) \xrightarrow{\sim} \operatorname{End}_{\mathcal{C}}(T)^{\mathrm{op}}$,*

(ii) *for every $i \in Q_0$, the complex*

$$\bigoplus_{\substack{b \in Q_1^{\mathrm{m}} \\ hb = i}} T_{tb} \xrightarrow{\Phi\partial_a^r\partial_b W} \bigoplus_{\substack{a \in Q_1 \\ ta = i}} T_{ha} \xrightarrow{\Phi a} T_i \tag{6.1}$$

is right 2-almost split in $\operatorname{add} T$, and

(iii) *for every $i \in Q_0$, the complex*

$$T_i \xrightarrow{\Phi b} \bigoplus_{\substack{b \in Q_1 \\ hb = i}} T_{tb} \xrightarrow{\Phi\partial_b^l\partial_a W} \bigoplus_{\substack{a \in Q_1^{\mathrm{m}} \\ ta = i}} T_{ha} \tag{6.2}$$

is left 2*-almost split in* $\operatorname{add} T$.

Remark 6.6. If i is a mutable vertex, then the sequences (6.1) and (6.2) glue together into a weak 2-almost split sequence in $\operatorname{add} T$ (see [?buanmutation, Lem. 4.1] for the equality $\partial_a^r \partial_b W = \partial_a^l \partial_b W$). Thus in the context of [?buanmutation, §5], which deals with ordinary Jacobian algebras, it is both possible and convenient to phrase assumptions and conclusions in terms of the existence of such weak 2-almost split sequences. Since this symmetry breaks down at frozen vertices, we must make a choice, and we choose to use right 2-almost split sequences in these cases.

Under the notation and assumptions of Proposition 6.5, let $k \in Q_0^{\mathrm{m}}$ be a mutable vertex. Let $T_k^* \in \mathcal{C}$ be an object not in $\operatorname{add} T$, and write $\mu_k T = T/T_k \oplus T_k^*$. We make the following assumptions, labelled for consistency with the corresponding assumptions of [?buanmutation, §5.2]. Our assumptions differ from these only by additional conditions at frozen vertices in (O) and (IV), and conventions on composing maps.

(O) The map Φ induces an isomorphism $\mathcal{J}(Q,F,W) \xrightarrow{\sim} \operatorname{End}_{\mathcal{C}}(T)^{\mathrm{op}}$. By Proposition 6.5, this condition may be phrased equivalently as follows: for every $i \in Q_0^{\mathrm{m}}$, the complex

$$T_i \xrightarrow{\Phi b} \bigoplus_{\substack{b \in Q_1 \\ hb=i}} T_{tb} \xrightarrow{\Phi \partial_a^r \partial_b W} \bigoplus_{\substack{a \in Q_1 \\ ta=i}} T_{ha} \xrightarrow{\Phi a} T_i$$

is a weak 2-almost split sequence in $\operatorname{add} T$, which we abbreviate to

$$T_i \xrightarrow{f_{i2}} U_{i1} \xrightarrow{f_{i1}} U_{i0} \xrightarrow{f_{i0}} T_i,$$

and for each $i \in F_0$, the complex

$$\bigoplus_{\substack{b \in Q_1^m \\ hb=i}} T_{tb} \xrightarrow{\Phi \partial_a^r \partial_b W} \bigoplus_{\substack{a \in Q_1 \\ ta=i}} T_{ha} \xrightarrow{\Phi a} T_i$$

is a right 2-almost split sequence in $\operatorname{add} T$, which we abbreviate to

$$U_{i1} \xrightarrow{f_{i1}} U_{i0} \xrightarrow{f_{i0}} T_i.$$

(I) There exist complexes

$$T_k \xrightarrow{f_{k2}} U_{k1} \xrightarrow{h_k} T_k^*,$$
$$T_k^* \xrightarrow{g_k} U_{k0} \xrightarrow{f_{k0}} T_k$$

in $\mathcal{C}$ such that $f_{k1} = g_k h_k$.

(II) The complex

$$T_k^* \xrightarrow{g_k} U_{k0} \xrightarrow{f_{k0} f_{k2}} U_{k1} \xrightarrow{h_k} T_k^*$$

is a weak 2-almost split sequence in $\operatorname{add}(\mu_k T)$.

(III) The sequences

$$\mathrm{Hom}_{\mathcal{C}}(T_k^*, T_k^*) \xrightarrow{h_k} \mathrm{Hom}_{\mathcal{C}}(U_{k1}, T_k^*) \xrightarrow{f_{k2}} \mathrm{Hom}_{\mathcal{C}}(T_k, T_k^*),$$
$$\mathrm{Hom}_{\mathcal{C}}(T_k^*, T_k^*) \xrightarrow{g_k} \mathrm{Hom}_{\mathcal{C}}(T_k^*, U_{k0}) \xrightarrow{f_{k0}} \mathrm{Hom}_{\mathcal{C}}(T_k^*, T_k),$$

induced from those of (I) by applying $\mathrm{Hom}_{\mathcal{C}}(-, T_k^*)$ and $\mathrm{Hom}_{\mathcal{C}}(T_k^*, -)$ respectively, are exact.

(IV) For all $i \in Q_0$, we have $T_k \notin (\operatorname{add} U_{i1}) \cap (\operatorname{add} U_{i0})$. For $i \in Q_0^{\mathrm{m}}$ the sequences

$$\mathrm{Hom}_{\mathcal{C}}(T_k^*, U_{i1}) \xrightarrow{f_{i1}} \mathrm{Hom}_{\mathcal{C}}(T_k^*, U_{i0}) \xrightarrow{f_{i0}} \mathrm{Hom}_{\mathcal{C}}(T_k^*, T_i),$$
$$\mathrm{Hom}_{\mathcal{C}}(U_{i0}, T_k^*) \xrightarrow{f_{i1}} \mathrm{Hom}_{\mathcal{C}}(U_{i1}, T_k^*) \xrightarrow{f_{i2}} \mathrm{Hom}_{\mathcal{C}}(T_i, T_k^*),$$

obtained by applying $\mathrm{Hom}_{\mathcal{C}}(T_k^*, -)$ and $\mathrm{Hom}_{\mathcal{C}}(-, T_k^*)$ respectively to the weak 2-almost split sequence from (O), are exact. For all $i \in F_0$, the sequence

$$\mathrm{Hom}_{\mathcal{C}}(T_k^*, U_{i1}) \xrightarrow{f_{i1}} \mathrm{Hom}_{\mathcal{C}}(T_k^*, U_{i0}) \xrightarrow{f_{i0}} \mathrm{Hom}_{\mathcal{C}}(T_k^*, T_i),$$

obtained by applying $\mathrm{Hom}_{\mathcal{C}}(T_k^*, -)$ to the right 2-almost split sequence from (O), is exact.

Lemma 6.7. *Let $\mathcal{E}$ be a stably 2-Calabi–Yau Frobenius category satisfying (C1) and (C2), (Q, F, W) an ice quiver with potential such that Q has no loops, and $\Phi\colon \mathbb{K}Q \to \mathrm{End}_{\mathcal{E}}(T)^{\mathrm{op}}$ an algebra homomorphism. Assume that T is cluster-tilting in $\mathcal{E}$, that $T_i = \Phi(e_i)(T)$ is not projective-injective when $i \in Q_0^{\mathrm{m}}$, and that Φ induces an isomorphism*

$$\Phi\colon \mathcal{J}(Q, F, W) \xrightarrow{\sim} \mathrm{End}_{\mathcal{E}}(T)^{\mathrm{op}}.$$

Let $k \in Q_0^{\mathrm{m}}$ be such that there are no 2-cycles in Q incident with k. Then there exists $T_k^* \notin \operatorname{add} T$ such that Φ, T and T_k^* satisfy the assumptions (O)–(IV).

Proof. By the assumptions on Φ, we have that T_k is an indecomposable non-projective-injective summand of the cluster-tilting object T. Since $\mathcal{E}$ is stably 2-Calabi–Yau and Q has no loops or 2-cycles at k, we may take T_k^* as in Proposition 2.8. For this choice of T_k^*, most of our desired statements are proved in [?buanmutation, Lem. 5.7]. Note in particular that the complexes in (I) are in fact the short exact sequences from Proposition 2.8; we will use this below. It remains to check the statements of (O) and (IV) dealing with frozen vertices.

The existence of the required right 2-almost split sequence in (O) follows from the statement (i) $\Longrightarrow$ (ii) of Proposition 6.5. Since there are no 2-cycles of Q incident with k, the statement that $T_k \notin (\operatorname{add} U_{i1}) \cap (\operatorname{add} U_{i0})$ holds when i is frozen exactly as when

i is unfrozen. For the remaining statement in (IV), consider the diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathrm{Hom}_{\mathcal{E}}(T_k, U_{i1}) & \longrightarrow & \mathrm{Hom}_{\mathcal{E}}(T_k, U_{i0}) & \longrightarrow & \mathrm{Hom}_{\mathcal{E}}(T_k, T_i) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
(6.3) & & \mathrm{Hom}_{\mathcal{E}}(U_{k0}, U_{i1}) & \longrightarrow & \mathrm{Hom}_{\mathcal{E}}(U_{k0}, U_{i0}) & \longrightarrow & \mathrm{Hom}_{\mathcal{E}}(U_{k0}, T_i) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathrm{Hom}_{\mathcal{E}}(T_k^*, U_{i1}) & \longrightarrow & \mathrm{Hom}_{\mathcal{E}}(T_k^*, U_{i0}) & \longrightarrow & \mathrm{Hom}_{\mathcal{E}}(T_k^*, T_i) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
\end{array}$$

in which the lowest non-zero row is the sequence we wish to prove is exact. The columns are obtained by applying $\mathrm{Hom}_{\mathcal{E}}(-, X)$ to the short exact sequence

$$0 \longrightarrow T_k^* \xrightarrow{g_k} U_{k0} \xrightarrow{f_{k0}} T_k \longrightarrow 0$$

for various $X \in \operatorname{add} T$; since T is cluster-tilting, we have $\mathrm{Ext}^1_{\mathcal{E}}(T_k, X) = 0$ in each case, and so these columns are short exact sequences. The rows are obtained by applying $\mathrm{Hom}_{\mathcal{E}}(Y, -)$ to the complex

$$U_{i1} \xrightarrow{f_{i1}} U_{i0} \xrightarrow{f_{i0}} T_i,$$

which we have already shown is right 2-almost split in $\operatorname{add} T$, for various $Y \in \mathcal{E}$. In the case of the first two rows, we even take $Y \in \operatorname{add} T$; it then follows immediately from the definition of right 2-almost splitness that the second row is exact. Exactness of the first row follows similarly, using that $T_k \not\cong T_i$ to see that

$$\mathrm{Hom}_{\mathcal{E}}(T_k, T_i) = \mathrm{Rad}_{\mathcal{E}}(T_k, T_i),$$

so that we also have exactness at $\mathrm{Hom}_{\mathcal{E}}(T_k, T_i)$. Exactness of the lowest row now follows by viewing the diagram (6.3) as a short exact sequence of chain complexes, and passing to the long-exact sequence in cohomology. □

Example 6.8. We pick out three families of Frobenius cluster category for which some cluster-tilting objects are known to have endomorphism algebra isomorphic to a frozen Jacobian algebra $\mathcal{J}(Q, F, W)$ in which Q has no loops or 2-cycles, to which Lemma 6.7 applies. (For cases (ii) and (iii), proofs that the categories are indeed Frobenius cluster categories can be found in [?presslandinternally, Eg. 3.11–12].)

(i) For the Frobenius cluster category $\mathrm{GP}(B_Q)$ constructed in this paper from the data of an acyclic quiver Q, the initial cluster-tilting object $T = eA$ has endomorphism algebra $\mathrm{End}_B(T)^{\mathrm{op}} \cong \mathcal{J}(\widetilde{Q}, \widetilde{F}, \widetilde{W})$ by Theorem 5.3.

(ii) Buan–Iyama–Reiten–Scott associate Frobenius cluster categories $\mathcal{C}_w$ to elements w of Coxeter groups. Each reduced expression $\mathbf{i}$ for w in terms of simple reflections determines a cluster-tilting object $T_{\mathbf{i}} \in \mathcal{C}_w$, and Buan–Iyama–Reiten–Smith have shown that $\operatorname{End}_{\mathcal{C}_w}(T_{\mathbf{i}})^{\mathrm{op}}$ is isomorphic to frozen Jacobian algebra, constructed combinatorially from $\mathbf{i}$ [?buanmutation, Thm. 6.6].

(iii) Jensen–King–Su [?jensencategorification] describe a Frobenius cluster category $\operatorname{CM}(B_{k,n})$ categorifying Scott's cluster algebra structure on the homogeneous coordinate ring of the Grassmannian of k-dimensional subspaces of $\mathbb{C}^n$ [?scottgrassmannians]. A (k,n) Postnikov diagram D determines both a cluster of Plücker coordinates in the cluster algebra, and a cluster-tilting object $T_D \in \operatorname{CM}(B_{k,n})$. Baur–King–Marsh show that $\operatorname{End}_{B_{k,n}}(T_D)^{\mathrm{op}}$ is isomorphic to a frozen Jacobian algebra, constructed combinatorially from D [?baurdimer, Thm. 10.3].

Under the notation and assumptions of Proposition 6.5, let $k \in Q_0^{\mathrm{m}}$ be a mutable vertex. Choose $T_k^* \notin \operatorname{add} T$ and write $\mu_k T = T/T_k \oplus T_k^*$. Assume (O)–(IV). By (IV), there are no 2-cycles in Q incident with k, so we may take $(Q', F', W') = \mu_k(Q, F, W)$. We now define an algebra homomorphism

$$\Phi' \colon \mathbb{K}Q' \to \operatorname{End}_{\mathcal{C}}(\mu_k T)^{\mathrm{op}}$$

by choosing a summand of $\mu_k T$ for each $i \in Q_0'$ and a map $\Phi' a \colon T_j \to T_i$ for each arrow $a \colon i \to j$ in Q_1', as follows. By assumption,

$$T/T_k = \bigoplus_{\substack{i \in Q_0' \\ i \neq k^*}} T_i,$$

noting that $Q_0' \setminus \{k^*\} = Q_0 \cap Q_0'$. We complete the assignment of summands of $\mu_k T$ to vertices of Q' by associating T_k^* to the vertex k^*. On arrows, we define Φ' as follows.

(i) If a is an arrow common to Q and Q', then we take $\Phi' a = \Phi a$.

(ii) On arrows $[ab]$ of Q', define $\Phi'[ab] = \Phi b \circ \Phi a$.

(iii) Recall that by assumption (I) we have maps

$$g_k \colon T_k^* \to \bigoplus_{\substack{a \in Q_1 \\ ta = k}} T_{ha}, \qquad h_k \colon \bigoplus_{\substack{b \in Q_1 \\ hb = k}} T_{tb} \to T_k^*.$$

If $a \in Q_1$ has $ta = k$, define $\Phi' a^*$ to be the component of g_k indexed by a, and if $b \in Q_1$ has $hb = k$, define $\Phi' b^*$ to be the component of $-h_k$ indexed by b.

We are now able to state the main result of this section.

Theorem 6.9. *Under the assumptions and notation of the preceding paragraph, we have an induced isomorphism*

$$\Phi' \colon \mathcal{J}(Q', F', W') \xrightarrow{\sim} \operatorname{End}_{\mathcal{C}}(\mu_k T)^{\mathrm{op}}.$$

Proof. We wish to apply the statement (ii) $\implies$ (i) of Proposition 6.5, so it suffices to show, for each $i \in Q'_0$, that the sequence

$$\bigoplus_{\substack{d\in Q'^{\mathrm{m}}_1\\ hd=i}} T_{td} \xrightarrow{\Phi'\partial^r_c\partial_d W'} \bigoplus_{\substack{c\in Q'_1\\ tc=i}} T_{hc} \xrightarrow{\Phi' c} T_i \tag{6.4}$$

is right 2-almost split in $\operatorname{add}(\mu_k T)$. When i is mutable, this follows from [?buanmutation, Thm. 5.6], so we need only deal with the case $i \in F'_0 = F_0$. Our argument follows closely the proof of [?buanmutation, Lem. 5.10], using freely computations of the derivatives $\partial^r_c\partial_d W'$ from Lemma 6.2. We treat elements of direct sums as column vectors, with maps acting as matrices from the left; this convention is transposed from that of [?buanmutation].

Let $i \in F'_0 = F_0$. Since Q has no 2-cycles incident with k, either there is no arrow $k \to i$ in Q, or there is no arrow $i \to k$ in Q. In the first case, the sequence (6.4) has the form

$$\begin{matrix}\Bigg(\bigoplus\limits_{\substack{b\in Q_1\\ b\colon i\to k}} T^*_k\Bigg)\\ \oplus \\ \Bigg(\bigoplus\limits_{\substack{d\in Q^{\mathrm{m}}_1\\ hd=i}} T_{td}\Bigg)\end{matrix} \xrightarrow{\quad x \quad} \begin{matrix}\Bigg(\bigoplus\limits_{\substack{a,b\in Q_1\\ ta=k\\ b\colon i\to k}} T_{ha}\Bigg)\\ \oplus \\ \Bigg(\bigoplus\limits_{\substack{c\in Q_1\\ hc\neq k\\ tc=i}} T_{hc}\Bigg)\end{matrix} \xrightarrow{\left(\Phi'[ab]\ \ \Phi' c\right)} T_i, \tag{6.5}$$

where the direct sums are divided so that the upper portion consists of the contribution from arrows in $Q'_1 \setminus Q_1$, and x is given by the matrix

$$x = \begin{pmatrix} \Phi' a^* & \Phi'\partial^r_{[ab]}\partial_d[W] \\ 0 & \Phi'\partial^r_c\partial_d[W] \end{pmatrix}.$$

First we note that this is a complex, since

$$\sum_{\substack{a\in Q_1\\ ta=k}} \Phi'[ab]\Phi' a^* = \Phi b f_{k0} g_k = 0,$$

$$\sum_{\substack{a,b\in Q_1\\ ta=k\\ b\colon i\to k}} \Phi'[ab]\Phi'\partial^r_{[ab]}\partial_d[W] + \sum_{\substack{c\in Q_1\\ hc\neq k\\ tc=i}} \Phi' c\Phi'\partial^r_c\partial_d[W] = \Phi\partial_d W = 0$$

for each $b\colon i \to k$ in Q_1 and $d \in Q^{\mathrm{m}}_1$ with $hd = i$. Let ℓ be the number of arrows $i \to k$ in Q. Then we have $U_{i0} = T^\ell_k \oplus U''_{i0}$ with $T_k \notin \operatorname{add} U''_{i0}$, and the maps f_{i0} and f_{i1} from the right 2-almost split sequence of (O) decompose as

$$f_{i0} = \begin{pmatrix} f'_{i0} & f''_{i0} \end{pmatrix} : T^\ell_k \oplus U''_{i0} \to T_i,$$

$$f_{i1} = \begin{pmatrix} f'_{i1} \\ f''_{i1} \end{pmatrix} : U_{i1} \to T^\ell_k \oplus U_{i0}.$$

We may then rewrite (6.5) as

$$T_k^{*\ell}\oplus U_{i1}\xrightarrow{\begin{pmatrix} g_k^\ell & t\\ 0 & f''_{i1}\end{pmatrix}} U_{k0}^\ell\oplus U''_{i0}\xrightarrow{\left(\begin{smallmatrix} f'_{i0}f^\ell_{k0} & f''_{i0}\end{smallmatrix}\right)} T_i,$$

where $f^\ell_{k0}t=f'_{i1}$.

Next we show that $\left(\begin{smallmatrix} f'_{i0}f^\ell_{k0} & f''_{i0}\end{smallmatrix}\right)$ is right almost split in $\operatorname{add}\mu_kT$. Let $p\in\operatorname{Rad}_{\mathcal{C}}(T/T_k,T_i)$. Since $f_{i0}=\left(\begin{smallmatrix} f'_{i0} & f''_{i0}\end{smallmatrix}\right)$ is right almost split in $\operatorname{add}T$, there exists $\left(\begin{smallmatrix} p_1\\ p_2\end{smallmatrix}\right)\colon T/T_k\to T_k^\ell\oplus U''_{i0}$ such that $p=f'_{i0}p_1+f''_{i0}p_2$. Moreover, since f_{k0} is right almost split in $\operatorname{add}T$, there exists $q\colon T/T_k\to U^\ell_{k0}$ such that $p_1=f^\ell_{k0}q$, and so

$$p=f'_{i0}f^\ell_{k0}q+f''_{i0}p_2$$

factors through $\left(\begin{smallmatrix} f'_{i0}f^\ell_{k0} & f''_{i0}\end{smallmatrix}\right)$ as required. On the other hand, if $p\in\operatorname{Rad}_{\mathcal{C}}(T_k^*,T_i)$, then since g_k is left almost split in $\operatorname{add}(\mu_kT)$ there exists $q\colon U_{k0}\to T_i$ such that $p=qg_k$. Since there are no arrows $k\to i$ in Q, there are no summands of U_{k0} isomorphic to T_i, and so $q\in\operatorname{Rad}_{\mathcal{C}}(U_{k0},T_i)$. Since $U_{k0}\in\operatorname{add}(T/T_k)$, we see as above that q, and therefore p, factors through $\left(\begin{smallmatrix} f'_{i0}f^\ell_{k0} & f''_{i0}\end{smallmatrix}\right)$.

Now we show that $\begin{pmatrix} g_k^\ell & t\\ 0 & f''_{i1}\end{pmatrix}$ is a pseudo-kernel of $\left(\begin{smallmatrix} f'_{i0}f^\ell_{k0} & f''_{i0}\end{smallmatrix}\right)$ in $\operatorname{add}\mu_kT$. By (III) and (IV) we have exact sequences

$$\operatorname{Hom}_{\mathcal{C}}(\mu_kT,T_k^*)\xrightarrow{g_k}\operatorname{Hom}_{\mathcal{C}}(\mu_kT,U_{k0})\xrightarrow{f_{k0}}\operatorname{Hom}_{\mathcal{C}}(\mu_kT,T_k)\tag{6.6}$$

and

$$\operatorname{Hom}_{\mathcal{C}}(\mu_kT,U_{i1})\xrightarrow{\begin{pmatrix} f'_{i1}\\ f''_{i1}\end{pmatrix}}\operatorname{Hom}_{\mathcal{C}}(\mu_kT,T_k^\ell\oplus U''_{i0})\xrightarrow{\left(\begin{smallmatrix} f'_{i0} & f''_{i0}\end{smallmatrix}\right)}\operatorname{Hom}_{\mathcal{C}}(\mu_kT,T_i).\tag{6.7}$$

Now if $\left(\begin{smallmatrix} p_1\\ p_2\end{smallmatrix}\right):\mu_kT\to U^\ell_{k0}\oplus U''_{i0}$ satisfies

$$0=\begin{pmatrix} f'_{i0}f^\ell_{k0} & f''_{i0}\end{pmatrix}\begin{pmatrix} p_1\\ p_2\end{pmatrix}=\begin{pmatrix} f'_{i0} & f''_{i0}\end{pmatrix}\begin{pmatrix} f^\ell_{k0} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} p_1\\ p_2\end{pmatrix},$$

then by exactness of (6.7) there exists $q\colon\mu_kT\to U_{i1}$ such that

$$\begin{pmatrix} f_{i1'}\\ f''_{i1}\end{pmatrix}q=\begin{pmatrix} f^\ell_{k0} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} p_1\\ p_2\end{pmatrix}.$$

It follows that $f^\ell_{k0}p_1=f'_{i1}q$ and $p_2=f''_{i1}q$. In particular,

$$f^\ell_{k0}(p_1-tq)=f'_{i1}q-f_{i1}q=0,$$

so by exactness of (6.6) there exists $r\colon\mu_kT\to T_k^{*\ell}$ such that $p_1-tq=g^\ell_kr$. It follows that

$$\begin{pmatrix} p_1\\ p_2\end{pmatrix}=\begin{pmatrix} g^\ell_k & t\\ 0 & f''_{i1}\end{pmatrix}\begin{pmatrix} r\\ q\end{pmatrix}$$

factors through $\begin{pmatrix} g_k^\ell & t\\ 0 & f''_{i1}\end{pmatrix}$, as required. This completes the proof that (6.4) is right 2-almost split when there are no arrows $k\to i$ in Q.

Now assume instead that there are no arrows $i \to k$ in Q. In this case, the sequence (6.4) has the form

$$\begin{array}{c}\Big(\bigoplus_{\substack{a,b\in Q_1\\ hb=k\\ a\colon k\to i}} T_{tb}\Big)\\ \oplus \\ \Big(\bigoplus_{\substack{d\in Q_1^{\mathrm{m}}\\ hd=i\\ td\neq k}} T_{td}\Big)\end{array} \xrightarrow{\;y\;} \begin{array}{c}\Big(\bigoplus_{\substack{a\in Q_1\\ a\colon k\to i}} T_k^*\Big)\\ \oplus\\ \Big(\bigoplus_{\substack{c\in Q_1\\ tc=i}} T_{hc}\Big)\end{array} \xrightarrow{(\Phi' a^*\ \ \Phi' c)} T_i, \tag{6.8}$$

where

$$y = \begin{pmatrix} \Phi' b^* & 0 \\ \Phi'\partial^r_{[ab]}\partial_c[W] & \Phi'\partial^r_d\partial_c[W]\end{pmatrix}.$$

We see using (I) that this is a complex, since

$$\sum_{\substack{a\in Q_1\\ a\colon k\to i}} \Phi' a^*\Phi' b^* + \sum_{\substack{c\in Q_1\\ tc=i}} \Phi' c\Phi'\partial^r_{[ab]}\partial_c[W] = (-g_k h_k + \Phi\partial^r_a\partial_b W)|^{T_i}_{T_{tb}} = (-f_{k1}+f_{k1})|^{T_i}_{T_{tb}} = 0,$$

$$\sum_{\substack{c\in Q_1\\ tc=i}} \Phi' c\Phi'\partial^r_d\partial_c[W] = \Phi\partial_d W = 0$$

for each pair $a, b \in Q_1$ with $hb = k$ and $a\colon k \to i$, and each $d \in Q_1^{\mathrm{m}}$ with $hd = i$ and $td \neq k$. (The notation after the first equality sign on the first line refers to taking the component $T_{tb} \to T_i = T_{ha}$ indexed by the pair (a, b).) Let ℓ be the number of arrows $k \to i$ in Q. Then $U_{i1} = T_k^\ell \oplus U''_{i1}$, where $T_k \notin \operatorname{add} U''_{i1}$, and f_{i1} decomposes as

$$f_{i1} = \begin{pmatrix} f'_{i1} & f''_{i1}\end{pmatrix} : T_k^\ell \oplus U''_{i1} \to U_{i0}.$$

We may then rewrite (6.8) as

$$U_{k1}^\ell \oplus U''_{i1} \xrightarrow{\begin{pmatrix} -h_k^\ell & 0 \\ s & f_{i1''}\end{pmatrix}} T_k^{*\ell} \oplus U_{i0} \xrightarrow{(u\ f_{i0})} T_i,$$

where $sf_{k2}^\ell = f'_{i1}$ and $f_{i0}s = uh_k^\ell$.

Before showing that this sequence is right 2-almost split in $\operatorname{add}\mu_k T$, we establish that the map $u\colon T_k^\ell \to T_i$, whose components are given by $\Phi' a^*$ for the ℓ arrows $a\colon k \to i$, induces a bijection

$$u\colon \operatorname{Hom}_{\mathcal{C}}(T_k^*, T_k^{*\ell})/\operatorname{Rad}_{\mathcal{C}}(T_k^*, T_k^{*\ell}) \xrightarrow{\sim} \operatorname{Rad}_{\mathcal{C}}(T_k^*, T_i)/\operatorname{Rad}^2_{\operatorname{add}(\mu_k T)}(T_k^*, T_i). \tag{6.9}$$

By (C2), we have $\operatorname{Hom}_{\mathcal{C}}(T_k^*, T_k^*)/\operatorname{Rad}_{\mathcal{C}}(T_k^*, T_k^*) \cong \mathbb{K}$, spanned by the class of the identity, so it is sufficient to show that the ℓ maps $\Phi' a^*$ for $a\colon k \to i$ form a basis of $\operatorname{Rad}_{\mathcal{C}}(T_k^*, T_i)/\operatorname{Rad}^2_{\operatorname{add}(\mu_k T)}(T_k^*, T_i)$. These maps are some of the components of g_k, which is left almost split in $\operatorname{add}(\mu_k T)$ by (II), meaning that its components span $\operatorname{Rad}_{\mathcal{C}}(T_k^*, U_{k0})/\operatorname{Rad}^2_{\operatorname{add}(\mu_k T)}(T_k^*, U_{k0})$. Since there is no 2-cycle of Q incident with k, we have $f_{k0}f_{k2} \in \operatorname{Rad}_{\mathcal{C}}(U_{k0}, U_{k1})$, from which it follows that g_k is also left minimal, i.e. that its components are linearly independent in $\operatorname{Rad}_{\mathcal{C}}(T_k^*, U_{k0})/\operatorname{Rad}^2_{\operatorname{add}(\mu_k T)}(T_k^*, U_{k0})$, hence a basis. Restricting to the summands of U_{k0} isomorphic to T_i then gives the desired result.

We may now show that $\begin{pmatrix} u & f_{i0} \end{pmatrix}$ is right almost split in $\operatorname{add}(\mu_k T)$. Since f_{i0} is right almost split in $\operatorname{add} T$ by (O), for any $p \in \operatorname{Rad}_{\mathcal{C}}(T/T_k, T_i)$ there exists $p' \colon T/T_k \to U_{i0}$ such that

$$p = f_{i0}p' = \begin{pmatrix} u & f_{i0} \end{pmatrix} \begin{pmatrix} 0 \\ p' \end{pmatrix}.$$

On the other hand, if $p \in \operatorname{Rad}_{\mathcal{C}}(T_k^*, T_i)$, then by (6.9) there exists $p_1 \in \operatorname{Hom}_{\mathcal{C}}(T_k^*, T_k^{*\ell})$ such that $p - up_1 \in \operatorname{Rad}^2_{\operatorname{add}(\mu_k T)}(T_k^*, T_i)$. Since g_k is left almost split in $\operatorname{add}(\mu_k T)$, there exists $q \colon U_{k0} \to T_i$ such that $p - up_1 = qg_k$. Now, using again that f_{i0} is right almost split in $\operatorname{add} T$, there exists $r \colon U_{k0} \to U_{i0}$ such that $q = f_{i0}r$, so that

$$p = up_1 + f_{i0}rg_k = \begin{pmatrix} u & f_{i0} \end{pmatrix} \begin{pmatrix} p_1 \\ rg_k \end{pmatrix}$$

factors through $\begin{pmatrix} u & f_{i0} \end{pmatrix}$ as required.

Finally, we show that $\begin{pmatrix} -h_k^\ell & 0 \\ s & f_{i1''} \end{pmatrix}$ is a pseudo-kernel of $\begin{pmatrix} u & f_{i0} \end{pmatrix}$ in $\operatorname{add}(\mu_k T)$. Assume that $\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \colon T' \to T_k^{*\ell} \oplus U_{i0}$ satisfies

$$\begin{pmatrix} u & f_{i0} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = 0.$$

We first show that p_1 factors through h_k^ℓ. To do this, we first observe that $p_1 \in \operatorname{Rad}_{\mathcal{C}}(T', T_k^{*\ell})$, for which it suffices to consider the case $T' = T_k^*$. We then have $p_2 \in \operatorname{Hom}_{\mathcal{C}}(T_k^*, U_{i0}) = \operatorname{Rad}_{\operatorname{add}(\mu_k T)}(T_k^*, U_{i0})$, and $f_{i0} \in \operatorname{Rad}_{\operatorname{add} T}(U_{i0}, T_i) = \operatorname{Rad}_{\operatorname{add}(\mu_k T)}(U_{i0}, T_i)$ by (O) and the assumption that there are no arrows $i \to k$ in Q, so that $U_{i0} \in \operatorname{add}(\mu_k T)$. It follows that

$$up_1 = -f_{i0}p_2 \in \operatorname{Rad}^2_{\operatorname{add}(\mu_k T)}(T_k^*, T_i)$$

so by (6.9) we have $p_1 \in \operatorname{Rad}_{\mathcal{C}}(T_k^*, T_k^{*\ell})$ as required. Now since h_k is right almost split in $\operatorname{add}(\mu_k T)$ by (II), there exists $q \colon T' \to U_{k1}^\ell$ such that $p_1 = h_k^\ell q$.

By (III) and (IV) we have an exact sequence

$$\operatorname{Hom}_{\mathcal{C}}(\mu_k T, T_k^\ell \oplus U''_{i0}) \xrightarrow{\begin{pmatrix} f'_{i1} & f''_{i1} \end{pmatrix}} \operatorname{Hom}_{\mathcal{C}}(\mu_k T, U_{i0}) \xrightarrow{f_{i0}} \operatorname{Hom}_{\mathcal{C}}(\mu_k T, T_i). \tag{6.10}$$

Since $f_{i0}(p_2 + sq) = f_{i0}p_2 + uh_k^\ell q = 0$, it follows from (6.10) that there exists $\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \colon T' \to T_k^\ell \oplus U_{i0}$ such that

$$p_2 + sq = \begin{pmatrix} f_{i1'} & f''_{i1} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}.$$

We therefore have

$$p_2 = -sq + f'_{i1}q_1 + f''_{i1}q_2 = s(f_{k2}^\ell q_1 - q) + f''_{i1}q_2,$$

and so

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} -h_k^\ell & 0 \\ s & f''_{i1} \end{pmatrix} \begin{pmatrix} f_{k2}^\ell q_1 - q \\ q_2 \end{pmatrix}$$

factors through $\left(\begin{smallmatrix} -h_k^\ell & 0 \\ s & f_{i1''} \end{smallmatrix}\right)$, as required. This shows that (6.4) is right 2-almost split when there are no arrows $i \to k$ in Q, completing the proof. □

Proposition 6.10 (cf. [?buancluster, Prop. II.1.11])**.** *Let $\mathcal{E}$ be a Hom-finite Frobenius cluster category, and let $T \in \mathcal{E}$ be a cluster-tilting object. If the quiver of $\operatorname{End}_{\mathcal{E}}(T)^{\mathrm{op}}$ has no loops at its mutable vertices, then it has no 2-cycles incident with its mutable vertices.*

Proof. Let $A = \operatorname{End}_{\mathcal{E}}(T)^{\mathrm{op}}$ and let

$$S_k = \operatorname{Hom}_{\mathcal{E}}(T, T_k)/\operatorname{Rad}_{\mathcal{E}}(T, T_k) \cong \mathbb{K}$$

be the simple A-module corresponding to a non-projective summand T_k of T. We show that $\operatorname{Ext}^2_A(S_k, S_k) = 0$. Consider the exchange sequences

$$0 \longrightarrow T_k \xrightarrow{f_{k2}} U_{k1} \xrightarrow{h_k} T_k^* \longrightarrow 0,$$
$$0 \longrightarrow T_k^* \xrightarrow{g_k} U_{k0} \xrightarrow{f_{k0}} T_k \longrightarrow 0,$$

for T_k. Applying $\operatorname{Hom}_{\mathcal{E}}(T, -)$ to these sequences and gluing gives a projective resolution

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{E}}(T, T_k) \longrightarrow \operatorname{Hom}_{\mathcal{E}}(T, U_{k1}) \longrightarrow \operatorname{Hom}_{\mathcal{E}}(T, U_{k0}) \longrightarrow \operatorname{Hom}_{\mathcal{E}}(T, T_k) \longrightarrow C \longrightarrow 0$$

of $C = \operatorname{coker}(\operatorname{Hom}_{\mathcal{E}}(T, f_{k0}))$. Since A has no loops, we have $C \cong S_k$ (cf. [?geissrigid, Lem. 6.1]), and T_k is not a summand of U_{k1}, so $\operatorname{Ext}^2_A(S_k, S_k) = 0$. The result then follows by [?geissrigid, Prop. 3.11], using that $\operatorname{gl.dim} A < \infty$ (since $\mathcal{E}$ is a Frobenius cluster category) and that A is finite-dimensional. □

Corollary 6.11. *Let $\mathcal{E}$ be a Frobenius cluster category, and assume that there is a mutation class of cluster-tilting objects in $\mathcal{E}$ such that the quivers of endomorphism algebras of these objects have no 2-cycles incident with their mutable vertices. Let $T \in \mathcal{E}$ be a cluster-tilting object in this class such that $\operatorname{End}_{\mathcal{E}}(T)^{\mathrm{op}} \cong \mathcal{J}(Q, F, W)$ for some quiver Q with no loops or 2-cycles incident with its mutable vertices. Then for any cluster-tilting object $T' = \mu_{k_\ell} \cdots \mu_{k_1} T$ mutation equivalent to T, there is an isomorphism $\operatorname{End}_{\mathcal{E}}(T')^{\mathrm{op}} \cong \mathcal{J}(Q', F', W')$ for some quiver Q' which has no loops and, after removing arrows between frozen vertices, coincides with the Fomin–Zelevinsky mutation $\mu_{k_\ell} \cdots \mu_{k_1} Q$ of Q.*

Proof. First assume $T' = \mu_k T$. By Theorem 6.9, we have $\operatorname{End}_{\mathcal{E}}(T')^{\mathrm{op}} \cong \mathcal{J}(Q'_0, F'_0, W'_0)$ for $(Q'_0, F'_0, W'_0) = \mu_k(Q, F, W)$. By construction, the quiver Q'_0 has no loops, and differs from $\mu_k Q$ only by arrows between frozen vertices and the possible addition of 2-cycles, of which $\mu_k Q$ has none. By using reduction, i.e. using the defining relations of

$\mathcal{J}(Q_0',F_0',W_0')$ to cancel redundant arrows (see Remark 6.3), we can find an ice quiver with potential (Q',F',W') such that

$$\mathcal{J}(Q',F',W')\cong\mathcal{J}(Q_0',F_0',W_0')\cong\operatorname{End}_{\mathcal{E}}(T')^{\mathrm{op}}$$

and Q' is the Gabriel quiver of these three isomorphic algebras. By assumption, the quiver Q' has no 2-cycles incident with its mutable vertices, so it must agree with μ_kQ up to arrows between frozen vertices, and it has no loops since this was already true of Q_0'. Thus mutation is defined at every mutable vertex of (Q',F',W'), and so the general case follows by induction. □

Consider again the Frobenius cluster categories from Example 6.8 of types (i) and (ii). By Proposition 6.10 and Corollary 6.11, the endomorphism algebra of any cluster-tilting object within the mutation class of those referred to in Example 6.8 has endomorphism algebra isomorphic to a frozen Jacobian algebra, and moreover mutation of cluster-tilting objects commutes with Fomin–Zelevinsky mutation of quivers within these classes.

The argument above does not apply to the Grassmannian cluster categories of Example 6.8(iii)—since these are Hom-infinite, we may not use [?geissrigid, Prop. 3.11] in the final step of the proof of Proposition 6.10. However, we may replace this proposition by the following ad hoc argument, and then use Corollary 6.11 to draw the same conclusion as in cases (i) and (ii).

Proposition 6.12. *Let* $\mathrm{CM}(B)$ *be a Grassmannian cluster category* [?jensencategorification], *and let* $T\in\mathrm{CM}(B)$ *be a cluster-tilting object. If the quiver of* $\operatorname{End}_{\mathcal{E}}(T)^{\mathrm{op}}$ *has no loops at its mutable vertices, then it has no* 2*-cycles incident with its mutable vertices.*

Proof. By [?jensencategorification, Thm. 4.5], there is an exact functor $\pi\colon\ \mathrm{CM}(B)\to\operatorname{Sub}Q_k$, which is a quotient by the ideal generated by an indecomposable projective B-module P_n. Here $\operatorname{Sub}Q_k$ denotes the exact category of submodules of an injective module Q_k for the preprojective algebra of type A_{n-1}, see [?geisspartial, §3], and is a Hom-finite Frobenius cluster category [?presslandinternally, Eg. 3.11] (in fact, it is even one of the categories $\mathcal{C}_w$ considered in [?buancluster]; cf. [?geisskacmoody, Lem. 17.2]).

As such, πT is a cluster-tilting object in $\operatorname{Sub}Q_k$, and the quiver of $\operatorname{End}_{\operatorname{Sub}Q_k}(\pi T)^{\mathrm{op}}$ is obtained from that of $\operatorname{End}_B(T)^{\mathrm{op}}$ by deleting the vertex corresponding to the summand P_n of T, and all incident arrows. Thus this smaller quiver has no loops, and so by Proposition 6.10 it has no 2-cycles. It follows that any 2-cycles in the quiver of $\operatorname{End}_B(T)^{\mathrm{op}}$ incident with a mutable vertex must also be incident with the vertex corresponding to P_n.

However, because of the cyclic symmetry of the algebra B, the same argument applies when replacing P_n by one of the $n-1$ other indecomposable projective B-modules, giving another quotient functor $\pi'\colon\ \mathrm{CM}(B)\to\operatorname{Sub}Q_k$ (typically with $\pi'T\not\cong\pi T$). This allows us to also rule out any 2-cycles in the quiver of $\operatorname{End}_B(T)^{\mathrm{op}}$ between a mutable vertex and that corresponding to P_n. □

We end this section with a proof of part (ii) of Theorem 1.

Theorem 6.13. *Let Q be an acyclic quiver. Then there is a bijection between cluster-tilting objects of* $\mathrm{GP}(B_Q)$ *and seeds of the cluster algebra* $\widetilde{\mathscr{A}_Q}$*, commuting with mutation, such that the ice quiver of the endomorphism algebra of each cluster-tilting object agrees, up to arrows between frozen vertices, with the ice quiver of the corresponding seed.*

Proof. Let $T = eA_Q$ be the initial cluster-tilting object. By Corollary 6.11, the quiver of $\mathrm{End}_{B_Q}(T')^{\mathrm{op}}$ has no loops or 2-cycles incident with its mutable vertices whenever T' is a cluster-tilting object mutation equivalent to T. It follows that, within this mutation class, every cluster-tilting object may be mutated at all of its non-projective indecomposable summands.

Now for any sequence $k_m, \dots, k_1$ of vertices of Q, we associate the cluster-tilting object $\mu_{k_m} \cdots \mu_{k_1} T$, which is well-defined by our initial observations, to the seed $\mu_{k_m} \cdots \mu_{k_1} \widetilde{s}_0$ of $\widetilde{\mathscr{A}_Q}$. By construction, this map commutes with mutation, and by Corollary 6.11 again, the quiver of a cluster-tilting object T' agrees, up to arrows between frozen vertices with the corresponding seed. Since cluster-tilting objects of $\mathrm{GP}(B_Q)$ are in bijection with those of the stable category $\underline{\mathrm{GP}}(B_Q) \simeq \mathcal{C}_Q$ (see Theorem 5.3), the fact that this assignment is a bijection follows from [?buanclusters, Thm. A.1]. □

7. Boundary algebras

Whenever we can construct a categorification of $\widetilde{\mathscr{A}_Q}$ via Theorem 2.6, such as when Q is acyclic, the objects of this category are modules for the idempotent subalgebra $B_{Q,W}$ of $A_{Q,W}$ determined by the frozen vertices, so we wish to describe this subalgebra more explicitly. In this section we will present B_Q via a quiver with relations, in the case that Q is acyclic.

Recall that the double quiver $\overline{Q}$ of a quiver Q has vertex set Q_0, and arrows $Q_1 \cup Q_1^\vee$, where $Q_1^\vee = \{\alpha^\vee : \alpha \in Q_1\}$. The head and tail maps agree with those of Q on Q_1, and are defined by $h\alpha^\vee = t\alpha$ and $t\alpha^\vee = h\alpha$ on $Q_1^\vee$. The preprojective algebra of Q is

$$\Pi(Q) = \mathbb{K}\overline{Q}/\Big(\sum_{\alpha \in Q_1} [\alpha, \alpha^\vee]\Big)$$

and, up to isomorphism, depends only on the underlying graph of Q. We begin with the following very general statement for frozen Jacobian algebras, which reveal some of the relations of $B_{Q,W}$ for an arbitrary quiver with potential (Q, W).

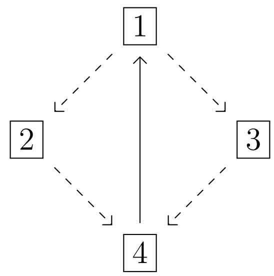
Proposition 7.1. *Let (Q, F, W) be an ice quiver with potential, let $A = \mathcal{J}(Q, F, W)$ and let $B = eAe$ for e the frozen idempotent be the boundary algebra of A. Then there is a map $\pi \colon \Pi(F) \to B$ given by $\pi(e_i) = e_i$ for all $i \in F_0$, and $\pi(\alpha) = \alpha$, $\pi(\alpha^\vee) = \partial_\alpha W$ for all $\alpha \in F_1$.*

Proof. It suffices to check that $\pi(\sum_{\alpha \in F_1} [\alpha, \alpha^\vee]) = 0$, i.e. that

$$\sum_{\alpha \in F_1} [\alpha, \partial_\alpha W] = 0.$$

By construction, for any $v \in Q_0$ we have
$$\sum_{\alpha \in \mathrm{in}(v)} \alpha \partial_\alpha W = \sum_{\beta \in \mathrm{out}(v)} \partial_\beta W \beta$$
in $\mathbb{K}Q$. Projecting to A and summing over vertices, we see that
$$0 = \sum_{\alpha \in Q_1} [\alpha, \partial_\alpha W] = \sum_{\alpha \in Q_1^{\mathrm{m}}} [\alpha, \partial_\alpha W] + \sum_{\alpha \in F_1} [\alpha, \partial_\alpha W] = \sum_{\alpha \in F_1} [\alpha, \partial_\alpha W],$$
where the final equality holds since $\partial_\alpha W = 0$ in A whenever $\alpha \in Q_1^{\mathrm{m}}$. □

Remark 7.2. Familiarity with the constructions of [?geisspartial], [?buancluster] and [?jensencategorification] may make it tempting to conjecture that the map π in Proposition 7.1 is surjective, at least when $\mathcal{J}(Q, F, W)$ is bimodule internally 3-Calabi–Yau, but this is not the case. A small explicit counterexample is



where the frozen subquiver is indicated by boxed arrows and dashed arrows as usual, and the potential is given by the difference of the two 3-cycles, so that the relations are generated by setting equal the two length two paths from 1 to 4. In this case the boundary algebra and the frozen Jacobian algebra agree, and the arrow from 4 to 1 is not in the image of $\pi \colon \Pi(F) \to B$. Since this algebra can be graded with every arrow in degree 1, one can again check that the bimodule complex gives a resolution via the more straightforward task of computing a projective resolution of each simple module. We will see in this section that π fails to be surjective for our frozen Jacobian algebras $\mathcal{J}(\widetilde{Q}, \widetilde{F}, \widetilde{W})$ whenever Q is acyclic with a path of length at least 2; see Example 7.4 below.

We now turn to our description of $B = B_Q$. Acyclicity of Q means that $A_Q/\langle e \rangle = \mathbb{K}Q$ has a natural basis, given by the paths of Q, each of which will determine an arrow of the quiver Q_B of B.

Definition 7.3. Let Q be an acyclic quiver, and consider the frozen subquiver $\widetilde{F}$ of $\widetilde{Q}$, which has vertex set
$$\widetilde{F}_0 = \{i^+, i^- : i \in Q_0\}$$
and arrows
$$\delta_i \colon i^+ \to i^-$$
$$\delta_a \colon ha^+ \to ta^-$$

for each $i \in Q_0$ and $a \in Q_1$. We define a quiver Q_B by adjoining to $\widetilde{F}$ an arrow

$$\delta_p^\vee \colon tp^- \to hp^+$$

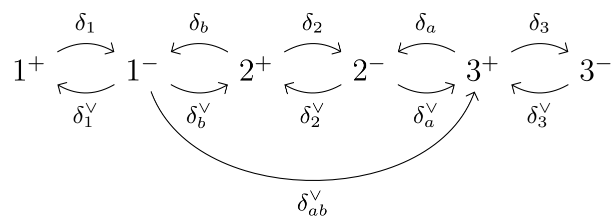
for each path p of Q.

As usual, if $p = e_i$ is the trivial path at $i \in Q_0$, we write $\delta_p^\vee = \delta_i^\vee$ to avoid a double subscript. The double quiver of $\widetilde{F}$ appears as the subquiver of Q_B obtained by excluding the arrows $\delta_p^\vee$ for p of length at least two; the notation for the arrows of Q_B is chosen to be consistent with that used earlier for the arrows of this double quiver.

Example 7.4. Let

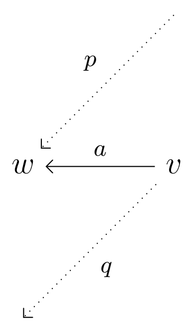
$$Q = \; 1 \xrightarrow{\;b\;} 2 \xrightarrow{\;a\;} 3$$

be a linearly oriented quiver of type A_3. Then Q_B is the following quiver.

$Q_B =$ 

Before we describe the ideal I of relations such that $\mathbb{K}Q_B/I \cong B$, we need an additional definition.

Definition 7.5. Let Q be a quiver. A *zig-zag* in Q is a triple (q, a, p), where p and q are paths in Q, and $a \in Q_1$ is an arrow, such that $hp = ha$ and $tq = ta$. Thus if $a \colon v \to w$ is an arrow of Q, a zig-zag involving a is some configuration



where the dotted arrows denote paths. We call the zig-zag *strict* if $p \neq ap'$ for any path p' and $q \neq q'a$ for any path q', but do not exclude these possibilities in general. If $z = (q, a, p)$ is a zig-zag, then we define $hz = hq$ and $tz = tp$.

We now write down what will turn out to be a set of generating relations for B, having three flavours, as follows.

(i) For each path p of Q, let

$$r_1(p) = \delta_p^\vee \delta_{tp} - \sum_{\substack{a \in Q_1 \\ ha = tp}} \delta_{pa}^\vee \delta_a.$$

(ii) For each path p of Q, let

$$r_2(p) = \delta_{hp}\delta_p^\vee - \sum_{\substack{a\in Q_1\\ ta=hp}} \delta_a\delta_{ap}^\vee.$$

(iii) For each zig-zag (q,a,p) of Q, let

$$r_3(q,a,p) = \delta_q^\vee\delta_a\delta_p^\vee.$$

We write I for the closure of the ideal of $\mathbb{K}Q_B$ generated by the union of these three sets of relations. This generating set is usually not minimal, as for certain zig-zags (q,a,p), the relation $r_3(q,a,p)$ may already lie in the ideal generated by relations of the form r_1 and r_2. For example, if $a\in Q_1$ is an arrow such that $i=ta$ is incident with no other arrows of Q, then

$$r_1(a) = \delta_a^\vee\delta_i,$$
$$r_2(e_i) = \delta_i\delta_i^\vee - \delta_a\delta_a^\vee,$$

so in $\mathbb{K}Q_B/\langle r_1(a), r_2(e_i)\rangle$ we already have

$$r(a,a,a) = \delta_a^\vee\delta_a\delta_a^\vee = \delta_a^\vee\delta_i\delta_i^\vee = 0.$$

However, we obtain a combinatorially more straightforward presentation by including all relations of the form $r_3(q,a,p)$, rather than characterising and excluding the redundant ones. One can check that if (q,a,p) is a strict zig-zag, then $r_3(q,a,p)$ is not redundant, but this condition is not necessary; if Q is a linearly oriented quiver of type A_4 and a is the middle arrow, then the non-strict zig-zag (a,a,a) yields an irredundant relation.

Remark 7.6. When $p=e_i$ is a vertex idempotent, the relations $r_1(e_i)$ and $r_2(e_i)$ reduce to 'preprojective' relations on the double quiver of $\widetilde{F}$, of the form predicted by Proposition 7.1. Each arrow $a\in Q_1$ is part of the trivial strict zig-zag (e_{ta},a,e_{ha}), and so contributes an irredundant relation $r_3(e_{ta},a,e_{ha}) = \delta_{ta}^\vee\delta_a\delta_{ha}^\vee$.

Let $\Phi\colon \mathbb{K}Q_B\to A_Q$ be the map given by the identity on the vertices of Q_B and the arrows δ_i and δ_a for $i\in Q_0$ and $a\in Q_1$; this makes sense as these are subsets of the vertices and arrows of $\widetilde{Q}$. On the remaining arrows $\delta_p^\vee$ of Q_B, we define

$$\Phi(\delta_p^\vee) = \alpha_{hp}p\beta_{tp}.$$

Proposition 7.7. *The map Φ induces a well-defined map $\Phi\colon Q_B/I\to B$.*

Proof. Since Φ sends every vertex or arrow of Q_B to (the image in A_Q of) a path of $\widetilde{Q}$ with frozen head and tail, it takes values in B. It remains to check that it is zero on each of the generating relations of I, which we do by explicit calculation as follows. Let

p be a path in Q. Then

$$\begin{aligned}\Phi(r_1(p)) &= \alpha_{hp}p\beta_{tp}\delta_{tp} - \sum_{\substack{a\in Q_1\\ ha=tp}} \alpha_{hp}pa\beta_{ta}\delta_a \\ &= \alpha_{hp}p(\partial_{\alpha_{tp}}W) = 0, \\ \Phi(r_2(p)) &= \delta_{hp}\alpha_{hp}p\beta_{tp} - \sum_{\substack{a\in Q_1\\ ta=hp}} \delta_a\alpha_{ha}ap\beta_{tp} \\ &= (\partial_{\beta_{hp}}W)p\beta_{tp} = 0.\end{aligned}$$

If (q,a,p) is a zig-zag, then

$$\Phi(r_3(q,a,p)) = \alpha_{hr}q\beta_{ta}\delta_a\alpha_{ha}p\beta_{tp} = 0,$$

since

$$0 = \partial_a\widetilde{W} = \partial_a W - \beta_{ta}\delta_a\alpha_{ha} = -\beta_{ta}\delta_a\alpha_{ha}$$

by acyclicity of Q. □

Theorem 7.8. *Let Q be an acyclic quiver. Then the map $\Phi\colon Q_B/I \to B_Q$, where Φ, Q_B and I are all defined as above, is an isomorphism.*

Proof. We begin by showing surjectivity. As in the proof of Theorem 5.2, we may use Lemma 5.1 to see that any path in $\widetilde{Q}$ determining a non-zero element of A has the form

$$p = q_1p'q_2$$

where q_1 and q_2 contain no arrows of Q_1, and p' is a path of Q. If p has frozen head and tail, then q_1 and q_2 must be non-zero, so we even have

$$p = q_1'\alpha_{hp'}p'\beta_{tp'}q_2' = q_1'\Phi(\delta_{p'}^\vee)q_2'$$

Now q_1' and q_2' are, like p, paths of $\widetilde{Q}$ with frozen head or tail, but with the additional property that they include no arrows of Q_1. Let q be such a path. If q contains an arrow β_i for some $i\in Q_0$, then this arrow cannot be the final arrow of q, since its head is unfrozen, so it must be followed by the arrow α_i, as this is the only arrow outside of Q_1 that composes with β_i. It follows that q is either a vertex idempotent $e_i^\pm = \Phi(e_i^\pm)$, or is formed by composing paths of the form $\delta_i = \Phi(\delta_i)$ for $i\in Q_0$, $\delta_a = \Phi(\delta_a)$ for $a\in Q_1$, or $\alpha_i\beta_i = \Phi(\delta_i^\vee)$ for $i\in Q_0$, and so is in the image of Φ.

We conclude that image in A of any path of $\widetilde{Q}$ with frozen head or tail lies in the image of Φ. Since such classes span $B = B_Q$, we see that Φ is surjective.

To complete the proof, we will use [?buanmutation, Prop. 3.3], stated earlier for Jacobian algebras as Proposition 6.5. In this context, [?buanmutation, Prop. 3.3] states that Φ is an isomorphism if and only if the sequences

$$\bigoplus_{\substack{p\text{ path}\\ tp=i}} Be_{hp}^+ \xrightarrow{\ f\ } Be_i^- \oplus \Big(\bigoplus_{\substack{a\in Q_1\\ ha=i}} Be_{ta}^-\Big) \xrightarrow{(\cdot\delta_i,\cdot\delta_a)} \mathfrak{m}(Be_i^+) \longrightarrow 0 \tag{7.1}$$

and

$$(7.2)\qquad \Big(\bigoplus_{\substack{p\text{ path}\\ tp=i}} Be^-_{hp}\Big)\oplus\Big(\bigoplus_{\substack{z\text{ zig-zag}\\ tz=i}} Be^+_{hz}\Big)\xrightarrow{\ g\ }\bigoplus_{\substack{p\text{ path}\\ tp=i}} Be^+_{hp}\xrightarrow{\cdot\Phi(\delta_p^\vee)}\mathfrak{m}(Be^-_i)\longrightarrow 0$$

are exact for all $i\in Q_0$. Here the left-most maps in each sequence are obtained from our generators of I by right-differentiation and the application of Φ, as proscribed in [?buanmutation], so they act on components by

$$\begin{aligned} f(ye^+_{hp}) &= y\Phi(\delta_p^\vee)e^-_i-\sum_{\substack{a\in Q_1\\ ha=i}} y\Phi(\delta_{pa}^\vee)e^-_{ta},\\ g(ye^-_{hp}) &= y\delta_{hp}e^+_{hp}-\sum_{\substack{a\in Q_1\\ ta=hp}} y\delta_a e^+_{h(ap)}\qquad (7.3)\\ g(ye^+_{hz}) &= y\Phi(\delta_q^\vee)\delta_a e^+_{hp},\ \text{where } z=qa^{-1}p. \end{aligned}$$

Note that the sums in the above expressions are formal, and we have tried to resolve the usual ambiguity about which summand contains each term via the notation for the idempotent on the right. For example, the term $-y\delta_a e^+_{h(ap)}$ of $g(ye^-_{hp})$ lies in the summand $Be^+_{h(ap)}$ of the codomain of g indexed by the path ap. We stress that this means that, for example, $-y\delta_a e^+_{h(ap)}$ and $-y\delta_a e^+_{h(bp)}$ should be interpreted as distinct (and even linearly independent) when $a\neq b$, even if $ha=hb$, since they lie in two different (albeit possibly isomorphic) summands of the codomain.

Since Φ is well-defined and surjective, sequences (7.1) and (7.2) are complexes and exact at $\mathfrak{m}(Be^+_i)$ and $\mathfrak{m}(Be^-_i)$ respectively, so we need only check exactness at the next term to the left in each case.

We proceed as in Section 4, using our explicit set R of relations for A_Q (3.2), and begin with (7.1). Let $x_i\in e\mathbb{K}\widetilde{Q}e^-_i$ and $x_a\in e\mathbb{K}\widetilde{Q}e^-_{ta}$ for each $a\in Q_1$ with $ha=i$. Assume

$$x_i\delta_i+\sum_{\substack{a\in Q_1\\ ha=i}} x_a\delta_a\in\overline{\langle R\rangle},$$

so that this expression projects to 0 in A. Since the only generating relation $\partial_{\alpha_i}W$ with terms ending in δ_i or δ_a for $a\in Q_1$ with $ha=i$ is $\partial_{\alpha_i}W$, it follows that in $\mathbb{K}\widetilde{Q}$, we have

$$x_i\delta_i+\sum_{\substack{a\in Q_1\\ ha=i}} x_a\delta_a=z_i\delta_i+\sum_{\substack{a\in Q_1\\ ha=i}} z_a\delta_a+y\Big(\beta_i\delta_i-\sum_{\substack{a\in Q_1\\ ha=i}} a\beta_{ta}\delta_a\Big)$$

for $z_i,z_a\in\overline{\langle R\rangle}$ and $y\in\mathbb{K}\widetilde{Q}$. Comparing terms, we see that

$$\begin{aligned} x_i &= z_i+y\beta_i,\\ x_a &= z_a-ya\beta_{ta}. \end{aligned}$$

Since hx_i and hx_a are frozen, but $h\beta_v$ is unfrozen for all $v \in Q_0$, we must have

$$y = \sum_{\substack{p \text{ path} \\ tp=i}} y_p \alpha_{hp} p.$$

for some $y_p \in Be^+_{hp}$. Projecting to B, we have

$$x_i = \sum_{\substack{p \text{ path} \\ tp=i}} y_p \Phi(\delta_p^\vee),$$

$$x_a = \sum_{\substack{p \text{ path} \\ tp=i}} -y_p \Phi(\delta_{pa}^\vee),$$

Thus the y_p give the required preimage of x_i and the x_a under the map f, and sequence (7.1) is exact.

Now we turn to (7.2). For each path p with $tp = i$, pick $x_p \in e\mathbb{K}\widetilde{Q}e^+_{hp}$, and assume that

$$\sum_{\substack{p \text{ path} \\ tp=i}} x_p \Phi(\delta_p^\vee) = 0$$

in A_Q, or equivalently that

$$\sum_{\substack{p \text{ path} \\ tp=i}} x_p \alpha_{hp} p \beta_i \in \overline{\langle R \rangle}.$$

By comparison with the generating relations, we see that we may write

$$\sum_{\substack{p \text{ path} \\ tp=i}} x_p \alpha_{hp} p \beta_i = \sum_{\substack{p \text{ path} \\ tp=i}} \Bigg(z_p \alpha_{hp} p \beta_i + y_p \Big(\delta_{hp} \alpha_{hp} - \sum_{\substack{b \in Q_1 \\ tb=hp}} \delta_b \alpha_{hb} b \Big) p \beta_i - \sum_{\substack{a \in Q_1 \\ ha=hp}} y_{a,p} (\beta_{ta} \delta_a \alpha_{ha}) p \beta_i \Bigg),$$

for some $z_p \in \overline{\langle R \rangle}$, and $y_p, y_{a,p} \in e\mathbb{K}\widetilde{Q}$. Note that either $p = e_i$, or we may write $p = br$ for some arrow b and path r. By comparing terms, we deduce that after projection to B we have

$$x_i = y_i \delta_i - \sum_{\substack{a \in Q_1 \\ ha=i}} y_{a,e_i} \beta_{ta} \delta_a,$$

$$x_{br} = y_{br} \delta_{hb} - y_r \delta_b - \sum_{\substack{a \in Q_1 \\ ha=hb}} y_{a,br} \beta_{ta} \delta_a,$$

Since $y_{a,p} \in e\mathbb{K}\widetilde{Q}$, but $h\beta_{ta} = ta \in Q_0$, we must have

$$y_{a,p} = \sum_{\substack{q \text{ path} \\ tq=ta}} y_{q,a,p} \alpha_{hq} q$$

for some $y_{q,a,p} \in e\mathbb{K}\widetilde{Q}e^+_{hq}$. Since the triple $z = (q, a, p)$ occurring in a subscript here satisfies $ha = hp$ and $ta = tq$, it is a zig-zag. One may then calculate explicitly using (7.3) that the y_p and $y_z = y_{q,a,p}$ give a preimage of the x_p under g. □

By similar methods, we may obtain the following curious property of the category $\mathrm{GP}(B_Q)$.

Proposition 7.9. *Assume Q is an acyclic quiver with no isolated vertices, and let $P^+ = \bigoplus_{k\in Q_0} B_Q e_k^+$. Then $\mathrm{GP}(B_Q) \subseteq \mathrm{Sub}(P^+)$.*

Proof. Write $B = B_Q$. Since $\mathrm{GP}(B)$ is a Frobenius category with injective objects those in $\operatorname{add} B$, it suffices to show that $B = B \in \mathrm{Sub}(P^+)$, or that $Be_k^{\pm} \in \mathrm{Sub}(P^+)$ for each k. Since this is true of Be_k^+ by definition of P^+, it remains to show $Be_k^- \in \mathrm{Sub}(P^+)$ for each k. Since Q has no isolated vertices, k cannot be both a source and a sink.

First assume k is not a source in Q. Then the map $Be_k^- \to Be_k^+$ given by right multiplication by δ_k is injective as follows. If $x \in Be_k^-$ satisfies $x\delta_k = 0$ in Be_k^+, then lifting to $\mathbb{K}Q_B$ and using the explicit generating set of I, we have

$$x\delta_k = \sum_{\substack{p \text{ path}\\ tp=k}} y_p\Big(\delta_p^\vee \delta_k - \sum_{\substack{a\in Q_1\\ ha=k}} \delta_{pa}^\vee \delta_a\Big).$$

Since k is not a source, the sum over arrows on the right-hand side is non-empty, so comparing coefficients shows that $y_p \in I$ for all p, and hence $x = 0$.

Now assume k is not a sink. Pick $a \in Q_1$ with $ta = k$. Then the map $Be_k^- \to Be_{ha}^+$ given by right multiplication by δ_a is injective as follows. If $x \in Be_k^-$ satisfies $x\delta_a = 0$ in Be_k^+, then lifting to $\mathbb{K}Q_B$ and using our explicit relations, we have

$$x\delta_a = \sum_{\substack{q \text{ path}\\ tq=ha}} z_q\Big(\delta_q^\vee \delta_{ha} - \sum_{\substack{b\in Q_1\\ hb=ha}} \delta_{qa}^\vee \delta_a\Big),$$

and comparing coefficients shows $z_q \in I$ for all q, so $x = 0$. □

Example 9.2 below shows that, in general, $\mathrm{GP}(B_Q) \neq \mathrm{Sub}(P^+)$. If Q does have an isolated vertex k, then this corresponds to a direct summand $\mathcal{C}$ of $\mathrm{GP}(B_Q)$ equivalent to $\operatorname{mod}\Pi$ for Π the preprojective algebra of type A_2, with indecomposable objects $S_k^\pm$ and $P_k^\pm$. It follows that $S_k^-, P_k^- \notin \mathrm{Sub}(P^+)$.

8. Gradings, indices and c-vectors

Let Q be an acyclic quiver. In this section, we show that $\mathrm{GP}(B_Q)$ may be equipped with the structure of a $\mathbb{Z}^n$-graded Frobenius cluster category [?grabowskigradedfrobenius, Defn. 3.7], corresponding to the natural grading of the principal coefficient cluster algebra $\mathscr{A}_Q^\bullet$ described by Fomin–Zelevinsky [?fomincluster4, §6]. By [?grabowskigradedfrobenius, Thm. 3.12], the data of this grading, when restricted to an initial seed, is equivalent to the data of a group homomorphism $\mathrm{K}_0(\mathrm{GP}(B_Q)) \to \mathbb{Z}^n$, where $\mathrm{K}_0(\mathrm{GP}(B_Q))$ denotes the Grothendieck group of the exact category $\mathrm{GP}(B_Q)$. We will be able to describe this group homomorphism in (almost) purely homological terms. The material in this section is largely well-known or elementary, but is used to illustrate how one can pass information back and forth between a cluster algebra and its categorification, and how $\mathrm{GP}(B_Q)$ may serve as a model for the principal coefficient cluster algebra $\mathscr{A}_Q^\bullet$.

First we briefly recall how to specify a grading on a cluster algebra, and how to interpret it categorically. Here we will only consider $\mathbb{Z}^n$-gradings, where n is the rank of the cluster algebra; a more detailed explanation for gradings by arbitrary abelian groups has been given by Grabowski and the author [?grabowskigradedfrobenius].

To give a $\mathbb{Z}^n$-grading of a rank n cluster algebra (i.e. a grading of the underlying algebra in which all cluster variables are homogeneous elements), it suffices to pick a seed s with (extended) $m\times n$ integer exchange matrix $\widetilde{b}$, and an $m\times n$ integer matrix g such that $\widetilde{b}^t\widetilde{g}=0$ [?grabowskigraded, Defn. 3.1]. The i-th row of $\widetilde{g}$ is the degree of the cluster variable x_i (which is frozen if $i>n$), and the compatibility condition ensures that all of the exchange relations are homogeneous. If b is skew-symmetric, so we may think of it as a quiver, the compatibility condition is equivalent to requiring, for each $1\leq k\leq n$, that the sum of degrees of cluster variables x_i at the tails of arrows with head at x_k is equal to the sum of degrees of cluster variables x_j at the heads of arrows with tail at x_k.

Now assume that the upper $n\times n$ submatrix (principal part) of $\widetilde{b}$ is skew-symmetric, and let Q be ice quiver corresponding to $\widetilde{b}$. If $\mathcal{E}$ is a Krull–Schmidt Frobenius cluster category, and $T\in\mathcal{E}$ is a cluster-tilting object such that the quiver of $A=\operatorname{End}_{\mathcal{E}}(T)^{\mathrm{op}}$ agrees with Q up to arrows with frozen vertices, then one can interpret a grading g as the element

$$G=\sum_{i=1}^{m}\widetilde{g}_i\otimes[S_i]\in\mathbb{Z}^n\otimes_{\mathbb{Z}}\mathrm{K}_0(\mathrm{fd}(A))=\mathrm{K}_0(\mathrm{fd}\,A)^n,$$

where $\mathrm{K}_0(\mathrm{fd}\,A)$ is the Grothendieck group of finite-dimensional A-modules, and $\widetilde{g}_i$ denotes the i-th row of $\widetilde{g}$. The matrix identity $\widetilde{b}^t\widetilde{g}=0$ implies, on the level of Grothendieck groups, that

$$\langle M,G\rangle=0$$

for all $M\in\operatorname{mod}\underline{A}$, where $\underline{A}=\underline{\operatorname{End}}_{\mathcal{E}}(T)^{\mathrm{op}}$ and $\langle -,-\rangle\colon\mathrm{K}_0(\operatorname{mod}A)\times\mathrm{K}_0(\mathrm{fd}\,A)^n\to\mathbb{Z}^n$ is induced from the Euler form of A by tensor product with $\mathbb{Z}^n$. This form is well-defined since, by the assumption that A is a Frobenius cluster category, $\operatorname{gl.dim}A\leq 3$. Moreover, any $G\in\mathrm{K}_0(\mathrm{fd}\,A)^n$ with the above property arises from a unique grading of the cluster algebra in this way.

Now by [?grabowskigradedfrobenius, Thm. 3.12], given $G\in\mathrm{K}_0(\mathrm{fd}\,A)^n$ as above, the map

$$\deg_G\colon[X]\mapsto\langle\operatorname{Hom}_{\mathcal{E}}(T,X),G\rangle$$

is a group homomorphism $\mathrm{K}_0(\mathcal{E})\to\mathbb{Z}^n$, and, having fixed T, all such homomorphisms arise in this way for some unique G. Thus the $\mathbb{Z}^n$-gradings of the cluster algebra generated by S are in bijection with $\operatorname{Hom}_{\mathbb{Z}}(\mathrm{K}_0(\mathcal{E}),\mathbb{Z}^n)$.

We now return to the case of polarised principal coefficient cluster algebras. Let (Q,W) be a quiver with potential such that $A=A_{Q,W}$ is Noetherian and the category $\mathrm{GP}(B_{Q,W})$ is a Krull–Schmidt Frobenius cluster category, such as if Q is acyclic. We abbreviate $B=B_{Q,W}$. To write down matrices unambiguously, we pick a labelling of the vertices of Q by $1,\dots,n$, so that the vertices of $\widetilde{Q}$, the ice quiver of our preferred initial seed of $\widetilde{\mathscr{A}_Q}$, are labelled by $1,\dots,n,1^+,\dots,n^+,1^-,\dots,n^-$. The rows of the $3n\times n$ matrices we are about to write are indexed by these labels, in this order, and their columns are indexed by $1,\dots,n$.

Let b be the skew-symmetric matrix associated to Q. Then the extended exchange matrix of our preferred initial seed $\widetilde{s}_0$ for $\widetilde{\mathscr{A}_Q}$ is

$$\widetilde{b}=\begin{pmatrix}b\\1_n\\-1_n\end{pmatrix}.$$

The principal coefficient grading of $\widetilde{\mathscr{A}_Q}$ is defined via the matrix

$$\widetilde{g}=\begin{pmatrix}1_n\\b\\0_n\end{pmatrix}.$$

It is straightforward to check that this is indeed a grading;

$$\widetilde{b}^t\widetilde{g}=b^t+b=0,$$

since b is skew-symmetric. Note that if one sets the cluster variables x_i^- of $\widetilde{\mathscr{A}_Q}$ to 1, obtaining the principal coefficient cluster algebra $\mathscr{A}_Q^\bullet$, then this quotient map is homogeneous for the grading of the target cluster algebra defined by Fomin–Zelevinsky [?fomincluster4, §6], which is given by the first $2n$ rows of $\widetilde{g}$. This grading extends to the Laurent polynomial ring R generated by the x_i and $x_i^\pm$, since $\widetilde{\mathscr{A}_Q}$ is a subring containing these generators. The degree of a homogeneous element of R is called its g-vector [?fomincluster4, §6].

Let $T=eA\in\mathrm{GP}(B)$ be the cluster-tilting object of $\mathrm{GP}(B)$ from Theorem 2.6. In $\underline{\mathrm{GP}}(B)$, we have

$$T\cong\bigoplus_{i=1}^n T_i,$$

where $T_i=eAe_i$. Let $G\in\mathrm{K}_0(\mathrm{fd}\,A)^n$ be the element corresponding to the grading $\widetilde{g}$, and $\deg_G\colon\mathrm{K}_0(\mathrm{GP}(B))\to\mathbb{Z}^n$ the associated group homomorphism.

For any $X\in\mathrm{GP}(B)$, its cluster character C_X^T with respect to T (defined by Fu–Keller [?fucluster, Thm. 3.3], see also [?grabowskigradedfrobenius, §3]) is a homogenous element of R, with degree equal to $\deg_G(X)$ [?grabowskigradedfrobenius, Prop. 3.11]. If X has no non-zero projective summands, Fu–Keller [?fucluster, Prop. 6.2] (see also Plamondon [?plamondonclusteralgebras, Prop. 3.6]) show how to compute the g-vector of C_X^T homologically in the stable category $\underline{\mathrm{GP}}(B)$, as we now recall.

Let $X\in\mathrm{GP}(B)$ have no non-zero projective summands. Pick a triangle

$$T^{m_X}\longrightarrow T^{p_X}\xrightarrow{f}X\longrightarrow\Omega^{-1}T^{m_j}\tag{8.1}$$

of $\underline{\mathrm{GP}}(B)$. We use monomial notation, so that

$$T^x=\bigoplus_{i=1}^n T_i^{x_i}.$$

Since T is cluster-tilting, such a triangle can be obtained by choosing f to be a right $(\operatorname{add} T)$-approximation, this property being equivalent to having domain and mapping cylinder in $\operatorname{add} T$. Then the index of X with respect to T is by definition the vector $\operatorname{ind}_T(X) = p_X - m_X \in \mathbb{Z}^n$. This quantity is independent of the choice of f, although p_X and m_X individually are not. By [?fucluster, Prop. 6.2], we then have

$$\deg_G(X) = \operatorname{ind}_T(X).$$

This applies in particular to the non-projective rigid indecomposable objects of $\operatorname{GP}(B)$, which are in bijection with the unfrozen cluster variables of $\mathscr{A}$ via $X \mapsto C_X^T$ when Q is acyclic, and can be exploited to compute $\deg_G(X)$ for any $X \in \operatorname{GP}(B)$, using the explicit degrees of the projective objects $eAe_i^{\pm}$ given by the lower $2n$ rows of $\widetilde{g}$ and the fact that $\deg_G$ is additive on direct sums.

Remark 8.1. By choosing a suitable map $P \to X$, where P is projective, the triangle (8.1) may always be lifted to a short exact sequence

$$0 \longrightarrow T^{m_X} \longrightarrow T^{p_X} \oplus P \longrightarrow X \longrightarrow 0$$

in $\operatorname{GP}(B)$, from which it follows that in $\operatorname{K}_0(A)$ we have

$$[\operatorname{Hom}_B(T, X)] = [\operatorname{Hom}_B(T, T^{p_X})] + [\operatorname{Hom}_B(T, P)] - [\operatorname{Hom}_B(T, T^{m_X})],$$

and all of the A-modules on the right-hand side are projective. We have

$$\operatorname{ind}_T(X)_i = (p_X - m_X)_i = \langle \operatorname{Hom}_B(T, T^{p_X}), S_i\rangle - \langle \operatorname{Hom}_B(T, T^{m_X}), S_i\rangle$$

for all $1 \leq i \leq n$, and

$$\begin{aligned}\deg_G(X)_i &= (\langle \operatorname{Hom}_B(T, T^{p_X}), G\rangle - \langle \operatorname{Hom}_B(T, T^{m_X}), G\rangle + \langle \operatorname{Hom}_B(T, P), G\rangle)_i \\ &= \langle \operatorname{Hom}_B(T, T^{p_X}), S_i\rangle - \langle \operatorname{Hom}_B(T, T^{m_X}), S_i\rangle + (\langle \operatorname{Hom}_B(T, P), G\rangle)_i,\end{aligned}$$

from the definition of G and the fact that T^{p_X} and T^{m_X} have no projective summands in $\operatorname{GP}(B)$. We therefore deduce from the identity $\operatorname{ind}_T(X) = \deg_G(X)$ that

$$\langle \operatorname{Hom}_B(T, P), G\rangle = 0,$$

giving a linear relation between the rows of the (unextended) exchange matrix b, with coefficients determined by the multiplicity of Be_i^+ as a summand of P. When b has full rank, it follows that it is always possible to choose $P \in \operatorname{add} P^-$, where $P^- = \bigoplus_{i\in Q_0} Be_i^-$. We conjecture that this is in fact true in general. We can observe this property directly in Examples 9.2 and 9.3 below, in which the exchange matrices do not have full rank. Conversely, establishing that one can always choose $P \in \operatorname{add} P^-$ would provide a new proof of the identity $\deg(C_X^T) = \operatorname{ind}_T(X)$.

Having understood the grading $\widetilde{g}$ globally, as a function on $\operatorname{K}_0(\operatorname{GP}(B))$, we may now recover local information for any cluster-tilting object, as follows. Let $T' = \bigoplus_{i=1}^n T_i' \oplus B$, with T_i' indecomposable and non-projective, be a cluster-tilting object of $\operatorname{GP}(B)$, and write $A' = \mathrm{End}_B(T')^{\mathrm{op}}$. Let $\widetilde{b}'$ be the extended exchange matrix associated to the ice quiver Q' of A', and let

$$\widetilde{g}' = \begin{pmatrix} g' \\ b \\ 0_n \end{pmatrix}$$

be the $3n \times n$ matrix whose first n rows are given by

$$\deg_G(T'_i) = \langle \mathrm{Hom}_B(T, T'_i), G \rangle = \mathrm{ind}_T(T'_i).$$

One can check as in [?grabowskigradedfrobenius, §3] that the corresponding element $G' \in \mathrm{K}_0(\mathrm{fd}\, A')^n$ satisfies $\langle M, G' \rangle = 0$ for all $M \in \mathrm{mod}\, \underline{\mathrm{End}}_B(T')^{\mathrm{op}}$, and the induced function $\mathrm{K}_0(\mathrm{GP}(B)) \to \mathbb{Z}^n$ coincides with $\deg_G$. In particular, this means that $\widetilde{g}'$ is a grading of the cluster algebra associated to Q', and so

$$(\widetilde{b}')^t \widetilde{g}' = 0.$$

Writing

$$\widetilde{b}' = \begin{pmatrix} b' \\ c' \\ d' \end{pmatrix},$$

where b', c' and d' are $n \times n$ submatrices, and using that b' is skew-symmetric, we conclude that

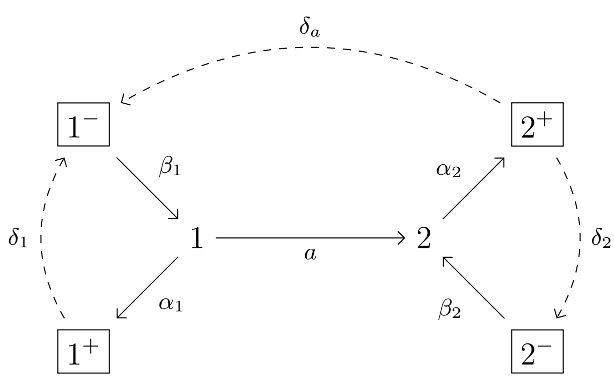
$$b' g' = (c')^t b.$$

If Q is acyclic, then by Theorem 6.13, the matrix $\widetilde{b}$ is the extended exchange matrix of a seed of $\widetilde{\mathscr{A}_Q}$, and the submatrix c' is by definition the matrix of c-vectors of this seed. Thus in this case we recover (the transpose of) an identity of Fomin–Zelevinsky [?fomincluster4, 6.14] (see also [?nakanishitropical, Rem. 2.1]), noting for comparison that it is the rows, not columns, of our matrix g' that are g-vectors.

9. Examples

Example 9.1. Let Q be an A_2 quiver, so, as computed in Example 3.2,

$(\widetilde{Q}, \widetilde{F}) =$



and

$$\widetilde{W} = \beta_1\delta_1\alpha_1 + \beta_2\delta_2\alpha_2 - a\beta_1\delta_a\alpha_2.$$

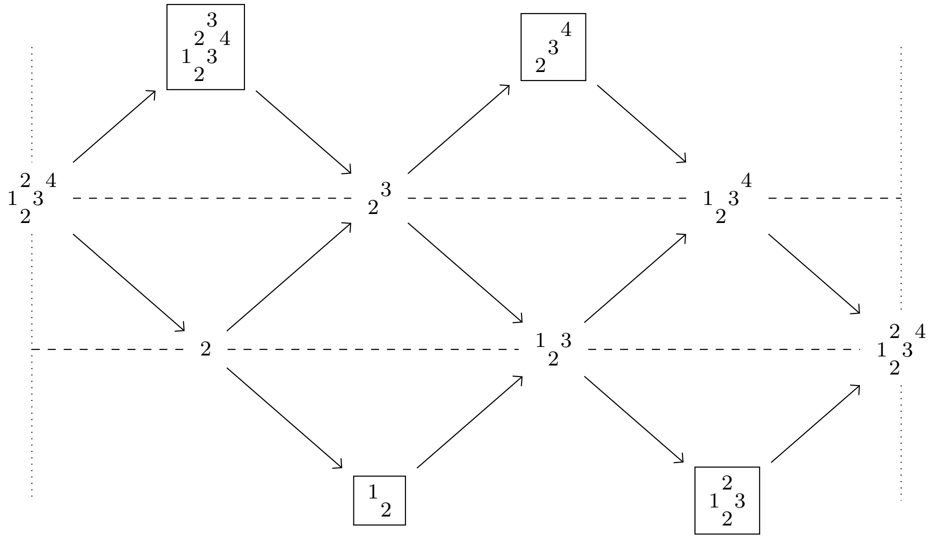


FIGURE 1. The Auslander–Reiten quiver of $\mathrm{GP}(B_Q)$ for Q of type A_2.

Then $A_Q = \mathcal{J}(\widetilde{Q}, \widetilde{F}, \widetilde{W})$, and its boundary algebra is $B_Q \cong \mathbb{K}Q_B/I$ for

$$Q_B = \quad 1 \underset{\alpha^\vee}{\overset{\alpha}{\rightleftarrows}} 2 \underset{\beta^\vee}{\overset{\beta}{\rightleftarrows}} 3 \underset{\gamma^\vee}{\overset{\gamma}{\rightleftarrows}} 4$$

and

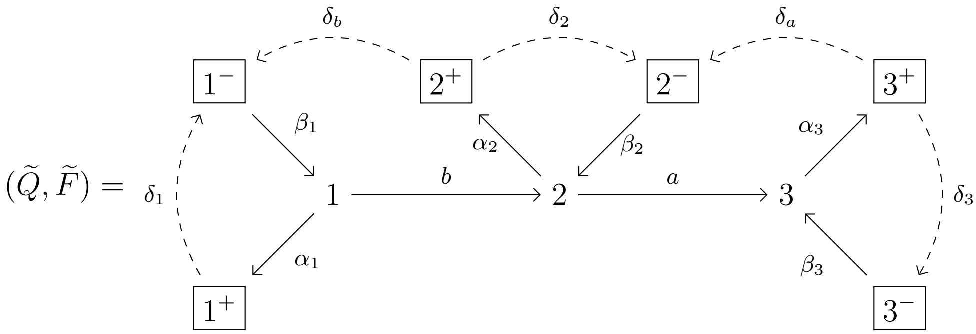
$$I = \langle \alpha^\vee\alpha,\ \alpha\alpha^\vee - \beta^\vee\beta,\ \beta\beta^\vee - \gamma^\vee\gamma,\ \gamma\gamma^\vee,\ \beta\alpha,\ \gamma\beta,\ \alpha^\vee\beta^\vee\gamma^\vee \rangle.$$

Here we have relabelled the vertices by mapping the ordered set $(1^+, 1^-, 2^+, 2^-)$ onto $(1, 2, 3, 4)$ in the unique order preserving way. The Auslander–Reiten quiver of $\mathrm{GP}(B)$ is shown in Figure 1, where we identify the left and right sides of the picture so that the quiver is drawn on a Möbius band. To calculate the objects of $\mathrm{GP}(B_Q)$ it is useful to observe that, in this example, B_Q is 1-Iwanaga–Gorenstein, and so $\mathrm{GP}(B_Q) = \mathrm{Sub}(B_Q)$. The stable category $\underline{\mathrm{GP}}(B_Q)$ is the cluster category of type A_2, as expected. The cluster tilting object

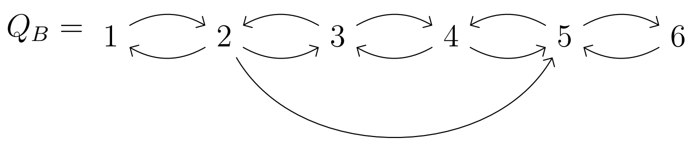
$$T = {}_{2}{}^{3} \oplus {}^{1}{}_{2}{}^{3} \oplus B_Q$$

of $\mathrm{GP}(B_Q)$ has endomorphism algebra A_Q, and corresponds to the initial seed of the cluster algebra with polarised principle coefficients associated to our initial A_2 quiver Q.

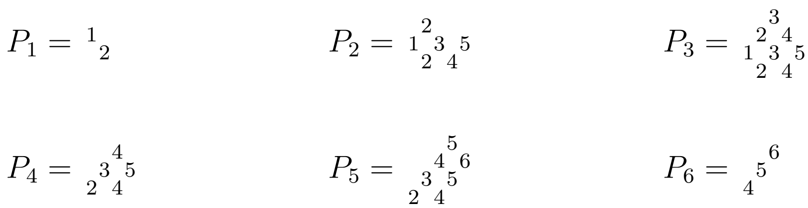
Example 9.2. Let Q be a linearly oriented quiver of type A_3. We may then compute



Relabelling vertices similarly to Example 9.1, the boundary algebra B_Q has quiver



as computed before in Example 7.4. Explicit relations can be written down as in Section 7, but here we will simply give radical filtrations for the projective modules. Note that despite the 'geographical' separation of 2 and 5 in these filtrations, the arrow $2 \to 5$ always acts as a vector space isomorphism from the 1-dimensional subspace of e_2B_Q indicated by a 2 in the filtration to the 1-dimensional subspace of e_5B_Q indicated by a 5 in the row below, when this configuration occurs.



In this case the Gorenstein dimension of B_Q is 2; the indecomposable projective P_2 has injective dimension 2, while all others have injective dimension 1. (This is in fact the first example known to the author of a Frobenius cluster category $\mathrm{GP}(B)$, with $\underline{\mathrm{GP}}(B) \neq 0$, for which the Gorenstein dimension of B is greater than 1.) The Auslander–Reiten quiver of $\mathrm{GP}(B_Q)$ is shown, again on a Möbius band, in Figure 2.

Example 9.3. Applying our construction to the quiver with potential (Q, W) from Example 3.2 with Q a 3-cycle (which we may do, since while A is not finite-dimensional in this case, it is still Noetherian) yields, as observed above, the Grassmannian cluster category $\mathrm{GP}(B_{Q,W}) = \mathrm{CM}(B_{2,6})$ [?jensencategorification]. This is a Hom-infinite category, and the Gorenstein projective B-modules are all infinite-dimensional. Representing these modules by Plücker labels as in [?jensencategorification], the Auslander–Reiten quiver of $\mathrm{GP}(B_{Q,W})$ is shown, on the now familiar Möbius band, in Figure 3. In this case, the quiver of the endomorphism algebra of the object

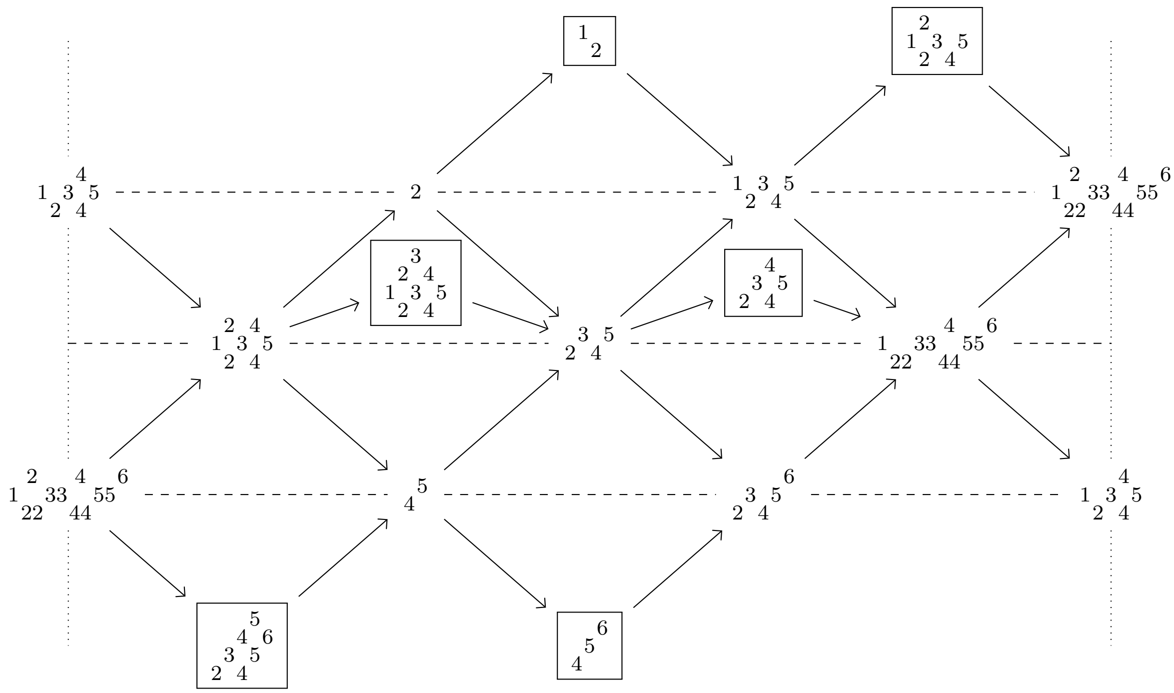


FIGURE 2. The Auslander–Reiten quiver of $\mathrm{GP}(B_Q)$ for Q linearly oriented of type A_3. Note that there are more than just the usual mesh relations corresponding to Auslander–Reiten sequences; the length two path from P_2 to P_5 represents the zero map.

$$13 \oplus 15 \oplus 35 \oplus B_{Q,W}$$

is $\widetilde{Q}$, as is the quiver of the (isomorphic) endomorphism algebra of

$$24 \oplus 26 \oplus 46 \oplus B_{Q,W}.$$

We note that the stable category $\underline{\mathrm{GP}}(B_{Q,W})$ is equivalent to the cluster category $\mathcal{C}_{Q,W} \simeq \mathcal{C}_{Q'}$ where Q' is any orientation of the Dynkin diagram A_3. Thus all of the conclusions of Theorem 5.3 (replacing $\mathbb{K}Q$ by $\mathcal{J}(Q,W)$ and $\mathcal{C}_Q$ by $\mathcal{C}_{Q,W}$) still hold for this example, despite the failure of acyclicity.

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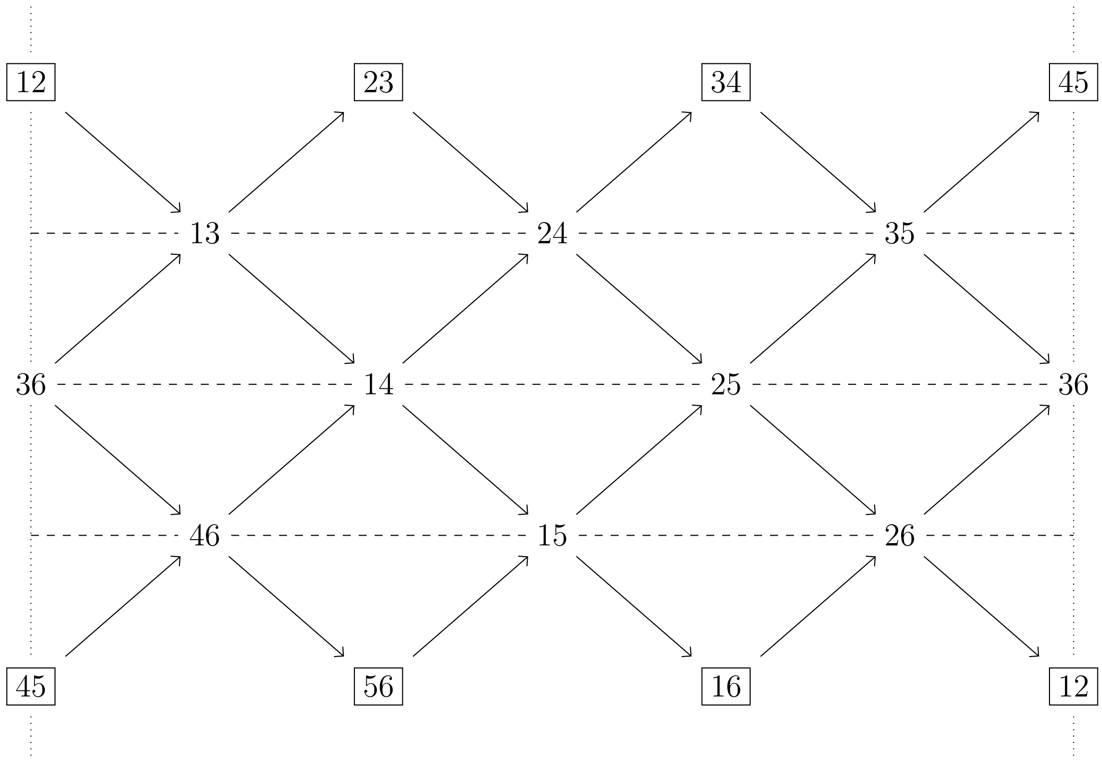

Figure 3. The Auslander–Reiten quiver of the Grassmannian cluster category $\mathrm{GP}(B_{Q,W}) = \mathrm{CM}(B_{2,6})$, where (Q, W) is a 3-cycle and its usual potential.

mutations between the Grassmannian cluster category and the Grassmannian cluster algebra had not been previously established.

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