A GENERAL ALGEBRAIC APPROACH TO THE PROBLEM OF DESCRIBING STABLE HOMOTOPY GROUPS OF SPHERES

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MPI / 94-19

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(A report on the seminar MPI)

9. 2. 1994

I would like to begin with a history of the question. The problem of calculation the homotopy groups $\pi_m(S^n)$ of spheres S^n is very old and very difficult one in algebraic topology.

It is not so difficult to see that $\pi_m(S^n) = 0$ if $m < n$ and $\pi_n(S^n) \cong \mathbb{Z}$. First nontrivial calculations were made by Hopf in 1935 y. [1]. He constructed a homomorphism $H : \pi_{2n-1}(S^n) \to \mathbb{Z}$ and proved that if $n = 2k$ then it's image contains $2\mathbb{Z}$ and if $n = 2,4,8$ then there exist h_1, h_2, h_3 in the corresponding homotopy groups for wich $H(h_1) = H(h_2) = H(h_3) = 1$. These elements are called Hopf elements.

Next step in the calculations of the homotopy groups of spheres was made by Freudenthal in 1937 [2]. He constructed a suspension homomorphism $S : \pi_m(S^n) \to \pi_{m+1}(S^{n+1})$ wich corresponds to homotopy classes $[f]$ of mappings $f : S^m \to S^n$ the homotopy classes $[Sf]$ of a suspension $Sf : S^{m+1} = SS^m \to SS^n = S^{n+1}$. It was proved that this homomorphism is an isomorphism if $m < 2n-1$ and epimorphism if $m = 2n-1$. Using this homomorphism Freudenthal showed that there is an isomorphism $\pi_{n+1} \cong \mathbb{Z}/2, n \geqq 3$. The generator elements of these groups are denoted η_n, where $\eta_{n+1} = S\eta_n$ and $\eta_3 = Sh_1$. Thus all generator elements in $\pi_{n+1}(S^n)$ may be obtained as the suspensions over Hopf element h_1.

Freudental results showed that homotopy groups $\pi_{n+k}(S^n)$ with sufficiently large $n(n > k+1)$ do not depend on k and it was the reason to consider so called stable homotopy groups $\sigma_k = \lim \pi_{n+k}(S^n)$.

In 1950 y. Whitehead has calculated $\pi_{n+2}(S^n) : \pi_{n+2}(S^n) \cong \mathbb{Z}/2, n \geq 2$, [3]. For it he defined a composition product $\circ : \pi_m(S^n) \times \pi_k(S^m) \to \pi_k(S^n)$ wich corresponds to the homotopy classes $[f_1], [f_2]$ of mappings $f_1 : S^m \to S^n, f_2 : S^k \to S^m$ the homotopy class $[f_1 \circ f_2]$ of its composition $f_1 \circ f_2 : S^k \to S^n$. Taking the composition $\eta_n \circ \eta_{n+1}$ of Freudenthal elements he obtained a generator element in $\pi_{n+2}(S^n)$.

It is a remarkable thing that all these and others calculations in that time were produced with the help of Hopf elements. New elements in homotopy groups were obtained by applying a suspension homomorphism and Whitehead product. There was a question: may be all elements in the homotopy groups can be obtained in such manner? The answer was - no.

To find new elements in homotopy groups of spheres Toda in 1962 y. [4] introduced the new operations called higher composition products, wich are partial defined and multivalued operations $\pi_{m_1}(S^n) \times ... \times \pi_{m_i}(S^{m_{i-1}}) \to \pi_{m_i+i-2}(S^n)$ and with the help of these operations Toda has calculated the homotopy groups of spheres $\pi_{n+k}(S^n)$ for $k \leq 19$.

Consider a definition of the first such operations $<\ ,\ ,\ >: \pi_m(S^n) \times \pi_k(S^m) \times \pi_l(S^k) \to \pi_{l+1}(S^n)$ called a secondary composition product. It is defined only on the triples $(\alpha_1, \alpha_2, \alpha_3)$ of the elements $\alpha_1 \in \pi_m(S^n), \alpha_2 \in \pi_k(S^m), \alpha_3 \in \pi_l(S^k)$ for wich $\alpha_1 \circ \alpha_2 = 0, \alpha_2 \circ \alpha_3 = 0$. Let $f_1 : S^m \to S^n, f_2 : S^k \to S^m, f_3; S^l \to S^k$ are the representatives in corresponding homotopy classes $\alpha_1, \alpha_2, \alpha_3$. Then $f_1 \circ f_2 \sim 0$ and there exists a contracting homotopy $h_1 : E^{k+1} \to S^n$. Therefore we have a map $h_1 \circ Ef_3 : E^{l+1} \to S^n$. From the other side $f_2 \circ f_3 \sim 0$ and there exists a contracting homotopy $h_2 : E^{l+1} \to S^m$. Therefore we have a map $f_1 \circ h_2 : E^{l+1} \to S^n$. These two maps $h_1 \circ Ef_3, f_1 \circ h_2 : E^{l+1} \to S^n$ coincide on the boundary of E^{l+1} and give the mappings of two $l+1$ semispheres to S^n. So they define a map $S^{l+1} \to S^n$ wich homotopy class denotes $< \alpha_1, \alpha_2, \alpha_3 >$ and it is the desirable value of secondary composition product. Of course this definition depends on the choice of the represantatives f_1, f_2, f_3 and homotopies h_1, h_2. Therefore indeed the secondary composition products are multivalued operations.

In 1968 y. Cohen showed that Whitehead and Toda products are sufficient for obtaining any elements in stable homotopy groups of spheres from the Hopf elements. More precisely Hopf elements are indecomposable in stable homotopy groups of spheres and all other elements are decomposable. It means they may be obtained from Hopf elements by applying Whitehead and Toda products.

So to describe stable homotopy groups of spheres it remained to find the relations between Whitehead and Toda poducts. Some of such relations for secondary operations $<\ ,\ ,\ >: \sigma_k \times \sigma_m \times \sigma_n \to \sigma_{k+m+n+1}$ were founded by Toda. Namely

1.$< \alpha, \beta, \gamma > \circ \delta \subset < \alpha, \beta, \gamma \circ \delta >$; 2.$< \alpha, \beta, \gamma \circ \delta > \subset < \alpha, \beta \circ \gamma, \delta >$;

3.$< \alpha \circ \beta, \gamma, \delta > \subset < \alpha, \beta \circ \gamma, \delta >$; 4.$\alpha \circ < \beta, \gamma, \delta > \subset (-1)^\alpha < \alpha \circ \beta, \gamma, \delta >$;

5.$0 \in << \alpha, \beta, \gamma >, \delta, \epsilon > + (-1)^\alpha < \alpha, < \beta, \gamma, \delta >, \epsilon > +$

$+(-1)^{\alpha+\beta} < \alpha, \beta, < \gamma, \delta, \epsilon >>$;

6.$< \alpha, \beta, \gamma >= (-1)^{\alpha\beta+\alpha\gamma+\beta\gamma+1} < \gamma, \beta, \alpha >$;

7.$0 \in < \alpha, \beta, \gamma > + (-1)^{\alpha\gamma+\beta\gamma} < \gamma, \alpha, \beta > + (-1)^{\alpha\gamma+\alpha\beta} < \beta, \gamma, \alpha >$.

The relations 1-5 are some kind of associativity relations between Whitehead and Toda products. The relations 6-7 are some kind of commutativity relations for Toda products.

Indeed it is very difficult to work with partial defined and multivalued Toda operations and to look for the relations between them. Our aim was to find convenient language and to give an approach to the problem of describing stable homotopy groups of spheres. More precisely its two component.

Much more convenient language for this purpose is the language of A_∞- structures introduced by Stasheff in 1963 y. [6]. We recall that a graded module A is called A_∞- algebra (or Stasheff algebra) if there are given the operations $\pi_i : A^{\otimes i+2} \to A$ of dimension i such that for any $n \geq 0$ the following relations are satisfied

$$\sum_{i=0}^{n}(-1)^{\epsilon}\pi_i(1\otimes\ldots\otimes\pi_{n-i}\otimes\ldots\otimes 1)=0$$

where the sum is taken also over all places of π_{n-i}

For example if $n=0$ we obtain associativity relation for π_0: $\pi_o(\pi_0\otimes 1)-\pi_0(1\otimes\pi_0)=0$. If $n=1$ we obtain the relation $\pi_0(\pi_1\otimes 1+1\otimes\pi_1)-\pi_1(\pi_0\otimes 1\otimes 1-1\otimes\pi_0\otimes 1+1\otimes 1\otimes\pi_0)=0$, wich is some kind of associativity relation between π_0 and π_1.(Compare with Toda relations 1-4).

Indeed A_∞-algebra structure consists of the operations π_i and all possible associativity relations between them. Our first result is the next

Theorem 1. *On the stable homotopy groups of spheres (more precisely on its associated $\mathbb{Z}/2$ module) there is a structure of A_∞-algebra wich defines Whitehead and Toda products.*

The construction of A_∞- algebra structure on stable homotopy groups of spheres may be obtained in two ways - algebraic and geometric. Here we will not touch algebraic construction since it needs developted algebraic methods such as B - constructions and functional homology operations. The geometric construction repeats Toda construction with some modifications. For the operation $\pi_1:\pi_m(S^n)\times\pi_k(S^m)\times\pi_l(S^k)\to\pi_{l+1}(S^n)$ it was produced by Baues [7]. Recall it.

Let $\{S^m,S^n\}$ be a space of continuous mappings from S^m to S^n preserving base points, $\eta:\{S^m,S^n\}\to\pi_m(S^n)$ be a projection. Fix a map $\xi:\pi_m(S^n)\to\{S^m,S^n\}$ of choosing representatives in homotopy classes. Then $\eta\circ\xi=Id$ and for any $f:S^m\to S^n,\xi\circ\eta(f)$ will be homotopic to f. Fix also such homotopy $h(f)$. If $\alpha_1\in\pi_m(S^n),\alpha_2\in\pi_k(S^m)$ then $\alpha_1\circ\alpha_2=\eta(\xi(\alpha_1)\circ\xi(\alpha_2))$. The elements $\xi(\alpha_1\circ\alpha_2)$ and $\xi(\alpha_1)\circ\xi(\alpha_2)$ are homotopic. Corresponding homotopies are $h(\xi(\alpha_1)\circ\xi(\alpha_2))$. We denote it $\xi_1(\alpha_1,\alpha_2)$. Let now α_1,α_2 be as above and $\alpha_3\in\pi_l(S^k)$. Consider the homotopies $\xi_1(\alpha_1,\alpha_2)\circ\xi(\alpha_3),\xi(\alpha_1)\circ\xi_1(\alpha_2,\alpha_3),\xi_1(\alpha_1\circ\alpha_2,\alpha_3),\xi_1(\alpha_1,\alpha_2\circ\alpha_3)$. They define a loop in the space $\{S^l,S^n\}$ and give a map $S^{l+1}\to S^n$ wich homotopy class denotes $\pi_1(\alpha_1,\alpha_2,\alpha_3)\in\pi_{l+1}(S^n)$. Corresponding homotopy between $\xi\pi_1(\alpha_1,\alpha_2,\alpha_3)$ and initial loop denotes $\xi_2(\alpha_1,\alpha_2,\alpha_3)$.

Let now $\alpha_1,\alpha_2,\alpha_3$ be as above and $\alpha_4\in\pi_q(S^l)$. Construct the operation $\pi_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\in\pi_{q+2}(S^n)$. For it consider the mappings $\xi_2(\alpha_1,\alpha_2,\alpha_3)\circ\xi(\alpha_4)$, $\xi_1(\alpha_1,\alpha_2)\circ\xi_1(\alpha_3,\alpha_4)$, $\xi(\alpha_1)\circ\xi_2(\alpha_2,\alpha_3,\alpha_4)$, $\xi_2(\alpha_1\circ\alpha_2,\alpha_3,\alpha_4)$, $\xi_2(\alpha_1,\alpha_2\circ\alpha_3,\alpha_4)$, $\xi_2(\alpha_1,\alpha_2,\alpha_3\circ\alpha_4)$, $\xi_1(\alpha_1,\pi_1(\alpha_2,\alpha_3,\alpha_4))$, $\xi_1(\pi_1(\alpha_1,\alpha_2,\alpha_3),\alpha_4)$. They define a map of two dimensional cell wich boundary consists of $\xi(\pi_1(\alpha_1,\alpha_2,\alpha_3)\circ\alpha_4)$,$\xi(\alpha_1\circ\pi_1(\alpha_2,\alpha_3,\alpha_4))$,$\xi\pi_1(\alpha_1\circ\alpha_2,\alpha_3,\alpha_4)$,$\xi\pi_1(\alpha_1,\alpha_2\circ\alpha_3,\alpha_4)$,$\xi\pi_1(\alpha_1,\alpha_2,\alpha_3\circ\alpha_4)$. It means that the sum of these elements is homotopic to zero. But we fixed the homotopies. Thus we have once more mapping wich together with otheres gives a mapping of two dimensional sphere into $\{S^q,S^n\}$. It generates a map from S^{l+2} to S^n and its homotopy class $\pi_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ will be the desirable value of the operation π_2. Next steps by the geometric way are more and more complicated.

There is one more unpleasant thing with geometric way. Constructed operations π_1 and π_2 are not additive and we indeed can't use it for A_∞-algebra structure. To do these operation additive we must pass from stable homotopy groups of spheres to its associated $\mathbb{Z}/2$ module or use in the definition of A_∞-algebra structure more comlicated notion of A_∞ -additive operations.

So we will describe stable homotopy groups of spheres using A_∞-algebra structure. But what it means to describe A_∞-algebras? Usual algebras can be described with the help of indecomposable elements and the relations between its products. What it means decomposable and indecomposable elements for A_∞-algebra. It is wrong to define decomposable elements for A_∞-algebra A as such elements wich belong to the images of some products $\pi_i : A^{\otimes(i+2)} \to A$. For giving right definition we must simply translate the definition of decomposable elements for the algebras with Massey products [8] to the language of A_∞-algebras. For it we define a notion of Massey sequence for A_∞-algebra A.

The sequence $(x^2, ..., x^n)$ of the elements $x^i \in A^{\otimes i}$ is called Massey sequence if the following relations are satisfied

$$(\pi_0 \otimes 1 \otimes ... \otimes 1 + ... + 1 \otimes ... \otimes 1 \otimes \pi_0)(x^n) = 0$$

$$(\pi_1 \otimes 1 \otimes ... \otimes 1 + ... + 1 \otimes ... \otimes 1 \otimes \pi_1)(x^n) = (\pi_0 \otimes ... \otimes 1 + ... + 1 \otimes ... \otimes \pi_0)(x^{n-1})$$

....

$$(\pi_{n-3} \otimes 1 + 1 \otimes \pi_{n-3})(x^n) = (\pi_{n-4} \otimes 1 + 1 \otimes \pi_{n-4})(x^{n-1}) + ... + (\pi_0 \otimes 1 + 1 \otimes \pi_0)(x^3)$$

For such Massey sequence $(x^2, ..., x^n)$ we define Massey product $\mu(x^2, ..., x^n)$ by putting $\mu(x^2, ..., x^n) = \pi_0(x^2) + ... + \pi_{n-2}(x^n)$. The element $x \in A$ is called decomposable if it is the image of some Massey product. It means that there exist Massey sequence $(x^2, ..., x^n)$ such that $\mu(x^2, ..., x^n) = x$. The elements wich are not decomposable call indecomposable elements.

A reformulation of Cohen result on the language of A_∞-algebras gives us the next theorem.

Theorem 2. *Hopf elements h_0, h_1, h_2, h_3 are indecomposable in stable homotopy groups of spheres σ_* considered as A_∞- algebra. All other elements in σ_* are decomposable and therefore Hopf elements are the generator elements in stable homotopy groups of spheres.*

Thus it remains to find the relations in σ_* considered as A_∞-algebra. The answer to this question gives next theorem

Theorem 3. *Let K denote Milnor coalgebra (dual to Steenrod algebra) [9], $A_\infty(h)$ - free A_∞- algebra generated by the Hopf elements h_i. Then there is a monomorphism $\psi : K \to A_\infty(h)$ such that all required relations have the form $\psi(x) = 0$.*

Indeed the formulas for ψ have complicated inductive form. There is another way to describe such relations. To do this note that A_∞-algebra structure includes all possible associativity relations. Consider the question about commutativity relations. Using what language it must be described? The language of A_∞-algebras is not so well for it. More convenient one is the language of E_∞-algebras introduced by May in 1972,[10]. E_∞-algebra structure includes A_∞- algebra structure but containes more operations and commutativity relations between them. Besides the operations π_i of A_∞-algebra structure there are for example cup-i products $\cup_i$. To give an exactly definition of E_∞- algebra structure it needs a notion of operad [10]. Our last result about describing stable homotopy groups of spheres is the next theorem

Theorem 4. *On stable homotopy groups of spheres there is E_∞-algebra structure, wich extend considered above A_∞-algebra structure. There is only one indecomposable element - h_0, all other elements are decomposable. In particulary $h_1 = h_0 \cup_1 h_0, h_2 = h_1 \cup_1 h_1, h_3 = h_2 \cup_1 h_2$. All relations in stable homotopy groups of spheres follows from E_∞-algebra structure on it.*

In the conclusion I must say that these results are not a calculation of stable homotopy groups of spheres. They give only an approach for describing it and may be in future for its calculation.

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