Twisted Affine Poisson Structures, Decompositions of Lie Algebras, and the Classical Yang-Baxter Equation

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Abstract.

We show that affine Poisson structures on Lie groups may be deformed by means of the morphisms of their (left and right) tangent Lie bialgebras. These twisted structures are then used to "integrate" the double Lie algebras for (non-skew) graded R-matrices on semi-simple Lie algebras and to formulate the Hamiltonian theory of the associated generalized Lax equations. We construct a symplectic dual pair for the twisted brackets. The explicit description of the symmetric morphisms of a natural Lie bialgebra yields a large class of examples of twisted affine Poisson structures as well as of new (non-graded) R-matrices on semi-simple Lie algebras.

1. Introduction.

Poisson Lie groups (first defined in [D 83], see also [D 86]) and the classical Yang-Baxter equation (first derived in [S 79], see also [S 82], [RF 83], and the book [FT 87]) as well as its modified form (**MCYB**) (see [S-T-S 83] and [S-T-S 85]) have, in recent years, been shown to play a central role in the theory of classical integrable systems.

A Poisson Lie group is defined as a Lie group $G\,(T_eG \simeq \mathcal{G}\,)$ equipped with a Poisson structure compatible with the group multiplication. Such a Poisson bracket vanishes at the identity e of G, and its linearization at e equips $(\mathcal{G}^\star,\, \mathcal{G})$ (here, $\mathcal{G}^\star$ is the dual of $\mathcal{G}$) with the structure of a Lie bialgebra (i.e. the $\mathcal{G}^\star$ Lie co-bracket is a $1-\mathcal{G}$ cocycle). Drinfel'd has proved the following theorem.

Theorem 1.1[D 83]

If G is connected 1-connected, Cat (Poisson Lie group) $\simeq$ Cat (Lie bialgebra).

Let G be semi-simple and let $(\ ,\ )$ be the Killing form on $\mathcal{G}$. Lie bialgebras are related to (**MCYB**) as follows:

Let $\zeta,\, \eta \in \mathcal{G}$. To any solution $R \in \mathrm{End}(\mathcal{G})$ of

$$[R(\zeta),\, R(\eta)] - R([R(\zeta),\, \eta] + [\zeta,\, R(\eta)]) = -[\zeta, \eta] \qquad \textbf{(MCYB)}$$

we may associate the double Lie algebra $(\mathcal{G}_R \equiv (\mathcal{G},\, [\ ,\ ]_R),\, \mathcal{G})$ where $[\zeta,\, \eta]_R = \frac{1}{2}([R(\zeta),\, \eta] + [\zeta,\, R(\eta)])$. When R is skew-symmetric (w.r.t. $(\ ,\ )$) the latter double is a Lie bialgebra.

The following natural question thus arises: If, in Theorem 1.1, we enlarge "Lie bialgebra" to "double Lie algebra", how should we enlarge "Poisson Lie group", with the further requirement that, for the purposes of integrable system theory, these new Poisson algebras retain the essential features of Poisson Lie groups ?

For a class of (non-skewsymmetric) solutions of (**MCYB**) (containing the Iwasawa and root space decompositions) such Poisson algebras were constructed in [LP 89]. This was achieved by means of twisted Poisson brackets parametrized by the set of tangent Lie bialgebra morphisms (see [LP 91]).

One of the main results of the present paper is that such Poisson algebras may also be defined for the class of graded solutions of (**MCYB**) (see [S-T-S 83]) associated with the generalized Gauss decompositions (see (3.10) and (3.11)). Our basic observation (see Theorem 3.1) is that affine Poisson structures (studied in [DS 91], see also [Lu 90], [W 90], and references therein) may also be twisted by means of morphisms of their left and right tangent Lie bialgebras. Moreover, in some cases, these twisted affine structures may be linearized at e of G. Specializing to the generalized Gauss decompositions, the result (see Theorem 3.2) is the construction of a family of Poisson algebras $\{\,,\,\}_{(\Gamma,C,\tau)}$, where Γ is any subset of simple roots of $\mathcal{G}$, $C \in \operatorname{End}(\mathcal{H}_\Gamma^\perp)$ (see (3.3)) and τ is a suitable orthogonal automorphism of $\mathcal{G}$. For these brackets, the R-matrix approach to generalized Lax equations is feasible (as in [S-T-S 85] and [LP 89]). In particular, when formulated on the product $G^N = G \times \cdots \times G$ (N is the number of lattice sites) they provide the (non-ultralocal) Hamiltonian representation of the lattice Lax systems associated with the graded R-matrices. When τ is taken to be the identity, the family $\{\,,\,\}_{(\Gamma,C,1)}$ of twisted affine Poisson brackets is linearizable and interpolates (see Remark 3.3 a) between the pair of Poisson Lie groups in duality associated with the Lie bialgebras $(\mathcal{G}_{J^+}, \mathcal{G})$ and $(\mathcal{G}, \mathcal{G}_{J^+})$, where $J^+ = \Pi_{\mathcal{N}^+} - \Pi_{\mathcal{N}^-}$, $\mathcal{G} = \mathcal{N}^+ \oplus \mathcal{H} \oplus \mathcal{N}^-$ is a root space decomposition, and $\Pi_{\mathcal{A}}$ denotes projection onto the Lie subalgebra $\mathcal{A}$.

In section 3.2, we proceed to construct a local symplectic dual pair (see Theorem 3.3) for the twisted affine structures. This is achieved, in a way similar to that in [LP 89], by twisting the dual pair for affine Poisson brackets described in [Lu 90] – suitably restated in the language of [S-T-S 85] for semi-simple Lie groups (see Comment b at the end of section 3.2).

Section 4 is devoted to questions of classification. Our main result is Proposition 4.1 where we characterize the set of symmetric morphisms of the Lie bialgebra $(\mathcal{G}_{J^+}, \mathcal{G})$ subject to a (natural) additional constraint (see (4.1)). Here, the relevant parameters are quadruples $(\Gamma^+, \Gamma^-, \sigma^+, \sigma^-)$ where $\Gamma^\pm$ are mutually orthogonal subsets of simple roots of $\mathcal{G}$ and $\sigma^\pm$ are orthogonal involutions of $\Gamma^\pm$. This allows us to give (see Proposition 4.2) an explicit list (up to (4.1)) of the linearizable twisted Poisson algebras (introduced in [LP 89]) associated with $(\mathcal{G}_{J^+}, \mathcal{G})$ (and hence, of the R-matrices which arise in their tangent double Lie algebras). In addition (see the lemmas 4.2, 4.3, and Corollary 4.1) this knowledge enables us to construct a large number of natural examples of twisted affine Poisson algebras. In this section, the presentation of our results is influenced by [BD 84].

2. Preliminaries.

In this section, we briefly recall some basic results and constructs which we shall use in the rest of the paper. The material presented here is mainly taken from [D 83], [S-T-S 83], and [S-T-S 85].

$(\mathcal{G},[\,,\,])$ denotes a Lie algebra equipped with a nondegenerate ad-invariant pairing $(\,,\,)$, $R\in$ End$(\mathcal{G})$ a solution of (**MCYB**) with associated double Lie algebra $(\mathcal{G}_R,\,\mathcal{G})$. The following observation due to Semenov-Tian-Shansky [S-T-S 83] is basic.

Lemma 2.1

The linear maps $R_\pm := \frac{1}{2}(R\pm 1) : \mathcal{G}_R \longrightarrow \mathcal{G}$ *are Lie algebra homomorphisms.*

Let G and G_R be local Lie groups corresponding to the Lie algebras $\mathcal{G}_R$ and $\mathcal{G}$. By Lemma 2.1, there exist unique Lie group homomorphisms

$$\widehat{R_\pm} : G_R \longrightarrow G \tag{2.1}$$

such that $T_e\widehat{R_\pm} = R_\pm$. For $g\in G_R$, we shall write $g_\pm := \widehat{R_\pm}(g)$. Now, consider the map

$$m : G_R \longrightarrow G, \quad g \longmapsto g_+g_-^{-1} \tag{2.2}$$

Since $T_em = R_+ - R_- = Id$, m is a local diffeomorphism. For $x\in G$, we shall write $x = x_+x_-^{-1}$. Note that, when read via the map m, the group operation $\star$ in G_R is given by $x\star y = x_+y\,x_-^{-1}$.
Assume now that R is skew-symmetric w.r.t. $(\,,\,)$. The square of the Lie bialgebra $(\mathcal{G}_R,\,\mathcal{G})$ is then defined as follows: Set $\delta = \mathcal{G}\oplus\mathcal{G}$, and equip it with the ad-invariant pairing

$$\big\langle(\zeta_1,\,\eta_1),\,(\zeta_2,\,\eta_2)\big\rangle := (\zeta_1,\,\zeta_2) - (\eta_1,\,\eta_2), \quad \zeta_i,\eta_i\in\mathcal{G},\, i=1,2. \tag{2.3}$$

Let ${}^\delta\mathcal{G}\subset\delta$ be the diagonal Lie subalgebra. If we embed $\mathcal{G}_R\hookrightarrow\delta$ via $\zeta\mapsto(R_+(\zeta),\,R_-(\zeta))$, then, as a linear space, $\delta ={}^\delta\,\mathcal{G}\oplus\mathcal{G}_R$. The associated solution of (**MCYB**)

$$R_\delta = \Pi_{{}^\delta\mathcal{G}} - \Pi_{\mathcal{G}_R} \tag{2.4}$$

is skew-symmetric w.r.t. $\langle\,,\,\rangle$, and the resulting Lie bialgebra $(\delta_{R_\delta},\,\delta)$ is called the square of $(\mathcal{G}_R,\mathcal{G})$. Note that $\delta_{R_\delta} ={}^\delta\,\mathcal{G}\ominus\mathcal{G}_R$ (Lie algebra anti-direct sum).

Let us now recall the precise definition of a Poisson Lie group.

Definition 2.1 [D 83]

A Lie group G equipped with a Poisson structure is called a Poisson Lie group if group multiplication is a Poisson map from $G\times G$ (equipped with the product structure) onto G.

For $\varphi\in C^\infty(G)$, the right and left gradients $D\varphi$, $D'\varphi\in\mathcal{G}$ are defined by

$$(D\varphi(g),\,\zeta) = \frac{d}{dt}_{|_{t=0}}\varphi(e^{t\zeta}g), \quad (D'\varphi(g),\,\zeta) = \frac{d}{dt}_{|_{t=0}}\varphi(ge^{t\zeta}). \tag{2.5}$$

Let J_1 and $J_2 \in$ End $\mathcal{G}$ be skew-symmetric (w.r.t.(,)) solutions of (**MCYB**).

Theorem 2.1 [S-T-S 85]

The formula

$$\{\varphi, \psi\}_{(-J_1, J_2)} = -\frac{1}{2}(J_1(D\varphi), D\psi) + \frac{1}{2}(J_2(D'\varphi), D'\psi)$$

defines a Poisson structure on G.

Note that $\{\, ,\, \}_{(-J_1, J_2)}$ vanishes at the identity e of G if and only if $J_1 = J_2$, in which case it equips G with the structure of a Poisson Lie group whose tangent Lie bialgebra is $(\mathcal{G}_{J_1}, \mathcal{G})$. The bracket $\{\, ,\, \}_{(-J_1, J_1)}$ is known as the Sklyanin bracket. When $J_1 \neq J_2$, the Poisson structures of Theorem 1.1 are examples of affine Poisson structures studied in [DS 91]. In the following sections we shall use, as in [S-T-S 85], the notation $G_{(-J_1, J_2)} := (G, \{\, ,\, \}_{(-J_1, J_2)})$.

Definition 2.2

Let G be a Poisson Lie group, M a Poisson manifold. A Lie group action $L : G \times M \longrightarrow M$ is called a Poisson Lie group action if it is a Poisson map from $G \times M$ (equipped with the product structure) onto M.

Let

$$([\zeta, \eta]_\star, \chi) = d\{\varphi, \psi\}_G(e).\chi,$$

$(\zeta, \chi) = d\varphi(e).\chi$, $(\eta, \chi) = d\psi(e).\chi$, $\chi \in \mathcal{G}$, be the Lie algebra tangent to the Poisson Lie group $(G, \{\, ,\, \}_G)$.

We shall need the following proposition.

Proposition 2.1 [S-T-S 85]

Let $L : G \times M \longrightarrow M$ be a Poisson group action. Let *$H \subset G$ be a connected Lie subgroup with Lie algebra $\mathcal{H}$. If $[\mathcal{H}^\perp, \mathcal{H}^\perp]_\star \subset \mathcal{H}^\perp$, then the algebra of $H-$invariant functions is a Lie subalgebra of $(C^\infty(M), \{\, ,\, \}_M)$, i.e. $L \mid H \times M$* is admissible. *In this case, there exists a unique Poisson structure on the quotient $H\backslash M$ such that the projection is a Poisson map.*

To close this section, we recall the definition of a symplectic dual pair due to A.Weinstein [W 83].

Definition 2.3

A pair of constant rank Poisson maps $P_1 \xleftarrow{\pi_1} S \xrightarrow{\pi_2} P_2$ from the symplectic manifold S to the Poisson manifolds P_1 and P_2 is called a dual pair if either of the following equivalent conditions is satisfied:

(i) $\pi_1^\star C^\infty(P_1)$ and $\pi_2^\star C^\infty(P_2)$ are mutual centralizers in $C^\infty(S)$

(ii) at each $x \in S$, $Ker\, T_x\pi_1 = (Ker\, T_x\pi_2)^\perp$.

The dual pair is said to be full if π_1, π_2 are submersions onto P_1 and P_2.

3. Twisted affine Poisson algebras.

3.1 Twisted affine Poisson algebras and the generalized Gauss decomposition.

In this subsection we define the twisted affine Poisson structures and use them to 'integrate' the double Lie algebras associated with a class of graded R-matrices on semisimple Lie algebras.

$\mathcal{G}$ denotes a Lie algebra equipped with a nondegenerate ad-invariant pairing $(\,,\,)$, G a Lie group with Lie algebra $\mathcal{G}$. If $L \in \text{End}\,\mathcal{G}$, the adjoint operator $L^\star \in \text{End}\,\mathcal{G}$ is defined by $(L^\star(\zeta),\,\eta) := (\zeta,\,L(\eta))$, $\zeta,\eta \in \mathcal{G}$. We consider triples $J_1,\,J_2,\,\Psi \in \text{End}\,\mathcal{G}$ subject to the basic assumption

$$\begin{array}{c} J_1 \text{ and } J_2 \text{ are skew-symmetric solutions of } (\mathbf{MCYB})\,, \\ \Psi \in \text{Hom}^+(\mathcal{G}_{J_1},\,\mathcal{G}) \text{ is such that } \Psi^\star \in \text{Hom}^+(\mathcal{G}_{J_2},\,\mathcal{G})\,. \end{array} \qquad (A)$$

Here and below, if $\mathcal{A}$, $\mathcal{B}$ are two Lie algebras (groups), $\text{Hom}^{(-)+}(\mathcal{A},\,\mathcal{B})$ denotes the set of Lie algebra (group) (anti-) homomorphisms.

Theorem 3.1

Let $\varphi,\psi \in C^\infty(G)$. Under the assumption (A), the formula

$$\{\varphi,\,\psi\}_{(-J_1,J_2,\Psi)} = -\frac{1}{2}(J_1(D(\varphi)),\,D\psi) + \frac{1}{2}(J_2(D'(\varphi)),\,D'\psi)$$

$$+\,(\Psi(D\varphi),\,D'\psi) - (\Psi^\star(D'\varphi),\,D\psi)$$

defines a Poisson structure on G.

Theorem 3.1 may be checked by a direct computation which we omit here as we shall describe the reduction theory for the bracket in subsection 3.2.♠

Remarks 3.1

a) If ι_G denotes the inversion map on G, then $\{\varphi\circ\iota_G,\,\psi\circ\iota_G\}_{(-J_1,J_2,\Psi)} = -\{\varphi,\,\psi\}_{(-J_2,J_1,\Psi^\star)}\circ\iota_G$.

b) $\{\,,\,\}_{(-J_1,J_2,\Psi)}$ vanishes at the identity of G iff $J_2 = J_1 + 2(\Psi^\star - \Psi)$, in which case it linearizes to the Lie algebra $\mathcal{G}_{J_2+2\Psi} \equiv \mathcal{G}_{J_1+2\Psi^\star}$.

c) When $J_1 = J_2$, Theorem 3.1 reduces to Theorem 2 Sect.3 of [LP 89].

d) Theorem 3.1 may be easily restated for arbitrary affine Poisson structures as defined in [DS 91]. We leave this to the interested reader.

d) As in [S-T-S- 83] (see also [LP 89]), one may define a formal analogue of Theorem 3.1 on associative algebras.

We now apply Theorem 3.1 to the generalized Gauss decomposition. To this end we begin by setting up some Lie algebraic notations and by defining the basic Lie subalgebras and decompositions needed to state our next theorem.

Let $(\mathcal{G}, [\,,\,])$ denote a split real (or complex) semi-simple Lie algebra with Killing form $(\,,\,)$. Fix a Cartan subalgebra $\mathcal{H} \subset \mathcal{G}$ and respectively denote by Φ, $\Delta \subset \Phi$, and $\Phi^{\pm} \subset \Phi$, the system of roots, a fixed basis of simple roots, and the corresponding positive/negative root subsets. For $\alpha \in \Phi$, $\mathcal{G}^{\alpha} \subset \mathcal{G}$ denotes the associated root subspace, and let $\alpha^{\star}$ be the unique element of $\mathcal{H}$ such that $(\alpha^{\star}, H) = \alpha(H)$ for all $H \in \mathcal{H}$. Let $\beta \in \Phi$. We set $(\alpha, \beta) := (\alpha^{\star}, \beta^{\star})$, and let $H_{\alpha} := 2\frac{\alpha^{\star}}{(\alpha, \alpha)}$.
Set

$$\mathcal{N}^{\pm} = \amalg_{\alpha\in\Phi^{\pm}}\, \mathcal{G}^{\alpha}. \tag{3.1}$$

For $\Gamma \subset \Delta$, we denote by $\widehat{\Gamma} \subset \Phi$ the set of roots supported by Γ, and consider the following Lie subalgebras of $\mathcal{G}$:

$$\mathcal{H}_{\Gamma} := \amalg_{\alpha\in\Gamma}\, k\, H_{\alpha} \subset \mathcal{H}, \text{ where } k \text{ is either R or C}, \tag{3.2}$$

$$\mathcal{H}_{\Gamma}^{\perp} := \text{ the orthogonal complement of } \mathcal{H}_{\Gamma} \text{ w.r.t. } (\,,\,)|_{\mathcal{H}\times\mathcal{H}} (\mathcal{H} = \mathcal{H}_{\Gamma} \oplus \mathcal{H}_{\Gamma}^{\perp}), \tag{3.3}$$

$$\mathcal{L}_{\Gamma} := \mathcal{H} \oplus \amalg_{\alpha\in\widehat{\Gamma}}\, \mathcal{G}^{\alpha}, \tag{3.4}$$

$$\mathcal{N}_{\Gamma}^{\pm} := \amalg_{\alpha\in(\Phi^{\pm}\setminus\widehat{\Gamma}\cap\Phi^{\pm})}\, \mathcal{G}^{\alpha}, \tag{3.5}$$

$$\mathcal{P}_{\Gamma}^{\pm} := \mathcal{L}_{\Gamma} \oplus \mathcal{N}_{\Gamma}^{\pm}. \tag{3.6}$$

Here, and henceforth, $\oplus$ denotes vector space direct sum.

We recall that $\mathcal{P}_{\Gamma}^{\pm}$ are the standard parabolic subalgebras supported by Γ. Their Levi component $\mathcal{L}_{\Gamma}$ and nilpotent radicals $\mathcal{N}_{\Gamma}^{\pm}$ satisfy

$$[\mathcal{L}_{\Gamma}, \mathcal{H}_{\Gamma}^{\perp}] = 0 \quad (\mathcal{H}_{\Gamma}^{\perp} \text{ is the center of } \mathcal{L}_{\Gamma}), \tag{3.7}$$

$$[\mathcal{N}_{\Gamma}^{\pm}, \mathcal{L}_{\Gamma}] \subset \mathcal{N}_{\Gamma}^{\pm}. \tag{3.8}$$

Note that $\mathcal{L}_{\emptyset} = \mathcal{H}$, $\mathcal{N}_{\emptyset}^{\pm} = \mathcal{N}^{\pm}$, and $\mathcal{P}_{\emptyset}^{\pm} = \mathcal{B}^{\pm}$ (opposite Borel subalgebras), while $\mathcal{L}_{\Delta} = \mathcal{G} = \mathcal{P}_{\Delta}^{\pm}$, and $\mathcal{N}_{\Delta}^{\pm} = 0$.
We shall consider the decomposition

$$\mathcal{G} = \mathcal{P}_{\Gamma}^{+} \oplus \mathcal{N}_{\Gamma}^{-}, \tag{3.9}$$

which further splits as

$$\mathcal{G} = \mathcal{N}_{\Gamma}^{+} \oplus \mathcal{L}_{\Gamma} \oplus \mathcal{N}_{\Gamma}^{-}. \tag{3.10}$$

(3.8) then says that the latter decomposition is triangular; we shall refer to it as the generalized Gauss decomposition.

Notations

As in the previous sections, if the Lie subalgebra $\mathcal{A} \subset \mathcal{G}$ appears as a factor in a direct sum decomposition of $\mathcal{G}$, $\Pi_{\mathcal{A}}$ denotes the projection onto $\mathcal{A}$. If $L \in \text{End}\,\mathcal{A}$, we use the same letter to denote its natural extension to $\mathcal{G}$.

Let $\widehat{T} \in \text{End}\,\mathcal{L}_{\Gamma}$, $C \in \text{End}\,\mathcal{H}_{\Gamma}^{\perp}$. We shall need the following lemma whose proof follows easily from (3.7) and (3.8).

Lemma 3.1

a) $T = \Pi_{\mathcal{N}_\Gamma^+} - \Pi_{\mathcal{N}_\Gamma^-} + \widehat{T}$ *satisfies* (**MCYB**) *on* $\mathcal{G}$ *if and only if* $\widehat{T}$ *satisfies* (**MCYB**) *on* $\mathcal{L}_\Gamma$.
b) *If* $\widehat{T}$ *satisfies* (**MCYB**) *on* $\mathcal{L}_\Gamma$ *, so does* $\widehat{T} + C$ *.* ♠

The basic family of R-matrices which we shall consider is then given by (set $\widehat{T} = \Pi_{\mathcal{L}_\Gamma} + C$ in Lemma 3.1)

$$R_{(\Gamma,C)} = \Pi_{\mathcal{P}_\Gamma^+} - \Pi_{\mathcal{N}_\Gamma^-} + C. \tag{3.11}$$

For $\mathcal{G}$ complex, Semenov-Tian-Shansky has proved (see Prop. (26) in [STS 83]) that the R-matrices (3.11) exhaust the class of generic graded (w.r.t. root height) solutions of (**MCYB**) subordinate to the pair ($\mathcal{P}_\Gamma^+$, $\mathcal{P}_\Gamma^-$) ; i.e. such that $R_\pm(\mathcal{G}) \subset \mathcal{P}_\Gamma^\pm$ and $\operatorname{Ker} R_\pm \supset \mathcal{N}_\Gamma^\mp$.
Let $\mathcal{L}_\Gamma^\pm = \mathcal{L}_\Gamma \cap \mathcal{N}^\pm$. In order to relate the R-matrices (3.11) to the Poisson structures of Theorem 3.1, we further define the maps $J^\pm$ and $\Psi^\pm \in \operatorname{End}\mathcal{G}$ by

$$J^+ = \Pi_{\mathcal{N}_\Gamma^+} - \Pi_{\mathcal{N}_\Gamma^-} + (\Pi_{\mathcal{L}_\Gamma^+} - \Pi_{\mathcal{L}_\Gamma^-}) \equiv \Pi_{\mathcal{N}^+} - \Pi_{\mathcal{N}^-}\,, \tag{3.12}$$

$$J^- = \Pi_{\mathcal{N}_\Gamma^+} - \Pi_{\mathcal{N}_\Gamma^-} - (\Pi_{\mathcal{L}_\Gamma^+} - \Pi_{\mathcal{L}_\Gamma^-})\,, \tag{3.13}$$

$$\Psi^\pm = \Pi_{\mathcal{L}_\Gamma^\pm} + \frac{\Pi_{\mathcal{H}}}{2}\,. \tag{3.14}$$

Note that

$$R_{(\Gamma,C=0)} = J^+ + 2\Psi^- = J^- + 2\Psi^+. \tag{3.15}$$

Finally, set $A_C = \frac{C-C^\star}{2}$, $S_C = \frac{C+C^\star}{2}$,where $C^\star$ is the adjoint of C w.r.t. $(\,,\,)|_{\mathcal{H}\times\mathcal{H}}$.
By Lemma 3.1, $J^\pm + A_C$ are (skew-symmetric) solutions of (**MCYB**).
The following relations between these maps play a basic role.

Lemma 3.2

$\Psi^\pm + \frac{S_C}{2} \in \operatorname{Hom}^\pm(\mathcal{G}_{J^+ + A_C}, \mathcal{G}) \cap \operatorname{Hom}^\mp(\mathcal{G}_{J^- + A_C}, \mathcal{G})$.
Proof:
First, observe that, viewed as an element of $\operatorname{End}\mathcal{L}_\Gamma$,

$$\Psi^\pm = \frac{\pm 1}{2}((\Pi_{\mathcal{L}_\Gamma^+} - \Pi_{\mathcal{L}_\Gamma^-}) \pm Id_{\mathcal{L}_\Gamma}). \qquad (\star)$$

For $\zeta\,,\eta\, \in \mathcal{G}$, we have

$$\begin{aligned}
&(\Psi^+ + \tfrac{S_C}{2})\,([\zeta,\,\eta]_{J^\pm + A_C}) \\
&= (\Psi^+ + \tfrac{S_C}{2})\,([\Pi_{\mathcal{L}_\Gamma}\zeta,\,\Pi_{\mathcal{L}_\Gamma}\eta]_{\pm(\Pi_{\mathcal{L}_\Gamma^+} - \Pi_{\mathcal{L}_\Gamma^-}) + A_C}), \quad \text{by (3.8)}, \\
&= \Psi^+\,([\Pi_{\mathcal{L}_\Gamma}\zeta,\,\Pi_{\mathcal{L}_\Gamma}\eta]_{\pm(\Pi_{\mathcal{L}_\Gamma^+} - \Pi_{\mathcal{L}_\Gamma^-})}), \quad \text{by (3.7)}, \\
&= \pm\,[\Psi^+\zeta,\,\Psi^+\eta], \quad \text{by } (\star) \text{ and Lemma 2.1},
\end{aligned}$$

$$= \pm [(\Psi^+ + \tfrac{S_C}{2})\zeta,\, (\Psi^+ + \tfrac{S_C}{2})\eta], \quad \text{by (3.7).}$$

Proceed identically for Ψ^-.♠

Remark 3.2

Lemma 3.2 will be generalized in Corollary 4.1 (see also Example 4.2).

For later purposes (see Theorem 3.2 below and the remark thereafter), we let, as in [S-T-S 85], $\tau \in \operatorname{Aut} G$ be an automorphism whose induced map on $\mathcal{G}$ (denoted by the same letter) is orthogonal and commutes with all previously defined linear maps on $\mathcal{G}$.

If we put

$$\begin{gathered} J_1 = J^- + A_C,\ J_2 = J^+ + A_C, \\ \Psi = \tau \circ (\,\Psi^- + \frac{S_C}{2}\,), \text{ so that } \Psi^\star = \tau^{-1} \circ (\,\Psi^+ + \frac{S_C}{2}\,), \end{gathered} \tag{3.16}$$

then, by Lemma 3.2, the hypothesis of Theorem 3.1 is satisfied.

The following theorem shows that the Lie algebra $\mathcal{G}_{R_{(\Gamma,C)}}$ is tangent to a twisted affine Poisson structure and provides the Hamiltonian description of the generalized Lax equations on G associated with $R_{(\Gamma,C)}$.

Theorem 3.2 *(Poisson algebra for the generalized Gauss decomposition.)*

Let $\varphi,\psi \in C^\infty(G)$.
(a) The formula

$$\begin{aligned} \{\varphi,\psi\}_{(\Gamma,C,\tau)} &= -\tfrac{1}{2}\,((J^- + A_C)(D\varphi),\, D\psi\,) + \tfrac{1}{2}\,((J^+ + A_C)(D'\varphi),\, D'\psi\,) \\ &\quad + \,(\tau \circ (\Psi^- + \tfrac{S_C}{2})(D\varphi),\, D'\psi\,) - (\tau^{-1} \circ (\Psi^+ + \tfrac{S_C}{2})(D'\varphi),\, D\psi\,) \end{aligned}$$

defines a Poisson structure on G. When $\tau \in \operatorname{Aut} G$ is the identity map, the bracket vanishes at the identity element of G and linearizes to the Lie algebra $\mathcal{G}_{R_{(\Gamma,C)}}$.

(b) If $\varphi \in C^\infty(G)$ is invariant under twisted conjugation $g \longrightarrow h\,g\,(\tau(h))^{-1}$, $g, h \in G$, the equation of motion defined by the Hamiltonian φ in the structure $\{\,,\,\}_{(\Gamma,C,\tau)}$ is given by

$$g^\bullet = \frac{1}{2}\, T_e R_g\,(\,R_{(\Gamma,C)}(D\varphi(g))\,) - \frac{1}{2}\, T_e L_g\,(\,\tau \circ R_{(\Gamma,C)}\,(D\varphi(g))\,).$$

(c) Let $h_\pm(t) = \widehat{R}_{(\Gamma,C)_\pm}(h(t))$ (see(2.1)), $h(t) \in G_{R_{(\Gamma,C)}}$, be the solution of the factorization problem

$$exp\,(-t\,D\varphi(g_0)) = h_+(t)^{-1}\,h_-(t),$$

for those values of t for which the left-hand side is in the image of the map m (see 2.2). Then the solution of the initial value problem associated with the equation in (b) is given by

$$g(t) = h_+(t)\,g_0\,\tau(h_+(t)^{-1}) = h_-(t)\,g_0\,\tau(h_-(t)^{-1}).$$

(d) Functions which are invariant under twisted conjugation commute in $\{\,,\,\}_{(\Gamma,C,\tau)}$.

The claims (b), (c), and (d) above are the analogues of those for Poisson Lie groups in [S-T-S 85] and their twisted extensions in [LP 89]. They may be verified by direct computations which we shall omit here (see however Remark 3.3 b).♠

Remarks 3.3

a) Assume that $\tau = \mathrm{Id}_G$. It is then instructive to consider the above Poisson algebra in two special cases:
1) When $\Gamma = \emptyset$, $J^+ = J^-$, and $\Psi^\pm = \frac{\Pi_{\mathcal{H}}}{2}$. Since $\mathcal{H}_\emptyset^\perp = \mathcal{H}$, we may choose $C = -\Pi_{\mathcal{H}}$ for which $A_C = 0$, and hence $S_C = C$. The bracket then reduces to

$$\{\varphi, \psi\}_{(\emptyset, -\frac{\Pi_{\mathcal{H}}}{2}, 1)} = -\frac{1}{2}(J^+(D\varphi), D\psi) + \frac{1}{2}(J^+(D'\varphi), D'\psi),$$

i.e. to the Poisson Lie group structure associated with the Lie bialgebra $(\mathcal{G}_{J^+}, \mathcal{G})$.
2) On the other hand, when $\Gamma = \Delta$, $J^+ = -J^-$, $\Psi^\pm = \pm(J^+)_\pm$. Note that, since $\mathcal{H}_\Delta^\perp = 0$, C must vanish. The bracket thus reduces to

$$\{\varphi, \psi\}_{(\Delta, 0, 1)} = \tfrac{1}{2}(J^+(D\varphi), D\psi) + \tfrac{1}{2}(J^+(D'\varphi), D'\psi) - ((J^+)_-(D\varphi), D'\psi) - ((J^+)_+(D'\varphi), D\psi)$$

i.e. to the Poisson Lie group associated with the Lie bialgebra $(\mathcal{G}, \mathcal{G}_{J^+})$.

Thus, $\{\,,\,\}_{(\Gamma, C, 1)}$ provides a family of linearizable twisted affine Poisson algebras which suitably interpolates between the pair of Poisson Lie groups in duality associated with J^+.

b) In the appendix we indicate how, by extending an argument in [STS 85], one may describe part (c) of Theorem 3.2 geometrically.

c) As in [STS 85], Theorem 3.2 allows us to formulate the Hamiltonian theory of the lattice Lax systems associated with the graded R-matrices $R_{(\Gamma, C)}$. For this purpose, we need to specialize Theorem 3.2 as follows: As Lie group, take $G^N = G \times \cdots \times G$ (N is the number of lattice sites) and equip the Lie algebra $\mathcal{G}^N$ with the ad-invariant pairing

$$(\bar\zeta, \bar\eta) = \sum_{i=1}^{N} (\zeta_i, \eta_i), \quad \bar\zeta = (\zeta_1, \ldots, \zeta_N), \bar\eta = (\eta_1, \ldots, \eta_N) \in \mathcal{G}^N.$$

Choose $\bar\tau \in \mathrm{Aut}\ G^N$ as

$$\bar\tau(g_1, g_2, \ldots, g_N) = (g_2, \ldots, g_N, g_1).$$

Finally, extend the linear operators $J^\pm, \Psi^\pm, C$ componentwise to $\mathcal{G}^N$.
We shall however leave out the explicit restatement of Theorem 3.3 using the above choices. Indeed, aside from the expression of the bracket, which, as in [LP 89] but unlike in [S-T-S 85], is neither a product structure nor linearizable at the identity of G^N, the statement is identical to those in the latter references.

3.2 Symplectic dual pair for twisted affine Poisson algebras.

We devote this subsection to the reduction theory of the twisted affine Poisson algebras. The reader will notice that the construction below is a natural extension of that given in [LP 89].

In what follows, as in Section 2, we shall deal with local Lie groups without further notational specification.

Let $D = G \times G$. As symplectic manifold, we take $D_{(J_{1_\delta}, J_{2_\delta})}$ (see Section 2), and consider the natural left (L) and right (R) Poisson Lie group actions:

$$L : D_{(J_{1_\delta}, -J_{1_\delta})} \times D_{(J_{2_\delta}, -J_{2_\delta})} \times D_{(J_{1_\delta}, J_{2_\delta})} \longrightarrow D_{(J_{1_\delta}, J_{2_\delta})}$$

$$(\widehat{g}, \widehat{h}, \widehat{x}) \longmapsto \widehat{g} \cdot \widehat{x} \cdot \widehat{h}^{-1},$$

and

$$R : D_{(J_{1_\delta}, J_{2_\delta})} \times D_{(-J_{1_\delta}, J_{1_\delta})} \times D_{(-J_{2_\delta}, J_{2_\delta})} \longrightarrow D_{(J_{1_\delta}, J_{2_\delta})}$$

$$(\widehat{x}, \widehat{g}, \widehat{h}) \longmapsto \widehat{g}^{-1} \cdot \widehat{x} \cdot \widehat{h}, \quad \widehat{g},\ \widehat{h},\ \widehat{x} \in D.$$

Let $\widehat{\Psi} \in \mathrm{Hom}^+(G_{J_1}, G)$ and $\widehat{\Psi^\star} \in \mathrm{Hom}^+(G_{J_2}, G)$ be the unique Lie group homomorphisms with $T_e\widehat{\Psi} = \Psi$, $T_e\widehat{\Psi^\star} = \Psi^\star$, and embed G_{J_1} and G_{J_2} in D via

$$i_l : G_{J_1} \hookrightarrow D_{(J_{1_\delta}, -J_{1_\delta})} \times D_{(J_{2_\delta}, -J_{2_\delta})}$$

$$g \longmapsto (\widehat{J_{1_+}}(g), \widehat{J_{1_-}}(g), \widehat{\Psi}(g), \widehat{\Psi}(g)),$$

and

$$i_r : G_{J_2} \hookrightarrow D_{(-J_{1_\delta}, J_{1_\delta})} \times D_{(-J_{2_\delta}, J_{2_\delta})}$$

$$h \longmapsto (\widehat{\Psi^\star}(h), \widehat{\Psi^\star}(h), \widehat{J_{2_+}}(h), \widehat{J_{2_-}}(h)).$$

We denote by l and r the left $G_{J_1}-$ and right $G_{J_2}-$ actions on D obtained by restricting L and R to $i_l(G_{J_1})$ and $i_r(G_{J_2})$.

Lemma 3.3

The actions l and r are admissible, i.e. left $G_{J_1}-$ and right $G_{J_2}-$ invariant functions form Lie subalgebras of $(C^\infty(D), \{\, ,\, \}_{(J_{1_\delta}, J_{2_\delta})})$.

Proof:

We do it for the right action r.

The Lie bialgebra tangent to $D_{(-J_{1_\delta}, J_{1_\delta})} \times D_{(-J_{2_\delta}, J_{2_\delta})}$ is $(\delta_{J_{1_\delta}} \oplus \delta_{J_{2_\delta}}, \delta \oplus \delta)$.

Therefore, if we embed $\mathcal{G}_{J_2}$ into $\delta \oplus \delta$ via

$$T_e i_r : \mathcal{G}_{J_2} \longrightarrow \delta \oplus \delta$$

$$\zeta \longmapsto (\Psi^\star(\zeta), \Psi^\star(\zeta), J_{2_+}(\zeta), J_{2_-}(\zeta)),$$

by Proposition 2.1, to show that r is admissible, it suffices to verify that $\mathcal{G}_{J_2}^{\perp} \subset \delta_{J_{1_\delta}} \oplus \delta_{J_{2_\delta}}$ is a Lie subalgebra. Here, $\mathcal{G}_{J_2}^{\perp}$ is the orthogonal complement of $\mathcal{G}_{J_2}$ w.r.t. the pairing

$$\langle\langle(\zeta_1,\eta_1,\zeta_2,\eta_2),(\zeta_1',\eta_1',\zeta_2',\eta_2')\rangle\rangle = \langle(\zeta_1,\eta_1),\, (\zeta_1',\eta_1')\rangle \,+\, \langle(\zeta_2,\eta_2),\, (\zeta_2',\eta_2')\rangle\,, \zeta_i,\eta_i,\zeta_i'\eta_i' \in \mathcal{G}\,.$$

Now, upon using simultaneously

$$\delta_{J_{1_\delta}} =^{\delta} \mathcal{G} \ominus \mathcal{G}_{J_1}, \quad \delta_{J_{2_\delta}} =^{\delta} \mathcal{G} \ominus \mathcal{G}_{J_2} \quad \text{(Lie algebra anti-direct sums)}\,,$$

we find $\mathcal{G}_{J_2}^{\perp} = \{((X,X) + (J_{1_+}Y, J_{1_-}Y),\, (X',X') + (J_{2_+}Y', J_{2_-}Y'))\,|X,\, X' \in \mathcal{G},\, Y \in \mathcal{G}_{J_1}, Y' \in \mathcal{G}_{J_2},\, \Psi(Y) = -X'\}\,.$
Hence, to conclude it remains to check that

$$\Psi(Y_1) = -X_1', \quad \Psi(Y_2) = -X_2'$$

implies

$$\Psi(-[Y_1,Y_2]_{J_1}) = -[X_1',X_2']\,.$$

But this holds since, by assumption, $\Psi \in \mathrm{Hom}^+(\mathcal{G}_{J_1},\mathcal{G})$.♠

By Lemma 3.3, there is a unique Poisson structure on the left (right) quotient, such that the projection

$$\pi_l\,(\pi_r):\, D_{(J_{1_\delta},J_{2_\delta})} \longrightarrow G_{J_1}\backslash D \quad (\,D/G_{J_2}\,)$$

is a Poisson map. If we now identify $G_{J_1}\backslash D$ and D/G_{J_2} with ${}^{\delta}G \simeq G$, we have

$$\pi_l(x,\, y) = ((xy^{-1})_{+_1})^{-1}\, x\, \widehat{\Psi} \circ m_1^{-1}(xy^{-1}), \tag{3.17}$$

$$\pi_r(x,\, y) = (\widehat{\Psi^*} \circ m_2^{-1}(x^{-1}y))^{-1}\, x\, (x^{-1}y)_{+_2}\,. \tag{3.18}$$

Here, the symbols $+_1$ and $+_2$ refer to the factorisations relative to J_1 and J_2, and $m_{1(2)}$: $G_{J_{1(2)}} \longrightarrow G$ are the isomorphisms (2.2).

Lemma 3.4

a) The reduced Poisson bracket on D/G_{J_2}, ($G_{J_1}\backslash D$) coincides (up to sign) with the twisted affine Poisson structure of Theorem 3.1.

b) Left $G_{J_1}-$ and right $G_{J_2}-$ invariant functions commute in $\{\,,\,\}_{(\,J_{1_\delta},\,J_{2_\delta})}$.

Proof:

For $\varphi,\, \psi \in C^\infty(G)$, let $\widehat{\varphi} = \varphi \circ \pi_r$, $\widehat{\psi} = \psi \circ \pi_r \in C^\infty(D)$. By definition, the right reduced bracket is given by

$$\{\varphi,\, \psi\}_{red.}(x) = \frac{1}{2}\langle J_{1_\delta}(D\widehat{\varphi}(x,\, x)),\, D\widehat{\psi}(x,\, x)\rangle + \frac{1}{2}\langle J_{2_\delta}(D'\widehat{\varphi}(x,\, x)),\, D'\widehat{\psi}(x,\, x)\,\rangle\,.$$

The claim then follows easily upon inserting the gradients

$$D'\widehat{\varphi}(x,\, x) = \big(\Psi(D\varphi(x)) + J_{2_+}(D'\varphi(x)),\, \Psi(D\varphi(x)) + J_{2_-}(D'\varphi(x))\big)$$

$$D\widehat{\varphi}(x,\,x) = Ad_{(x,\,x)}\, D'\widehat{\varphi}(x,\,x),$$

in the expression above.
b) Let $\widehat{\varphi} \in C^\infty(D)$ be left G_{J_1}–invariant and $\widehat{\psi} \in C^\infty(D)$ be right G_{J_2}–invariant.Set $D\widehat{\varphi}(x,\,y) = (X_1,\, X_2)$, $D\widehat{\psi}(x,\,y) = (Y_1,\, Y_2)$, $D'\widehat{\varphi}(x,\,y) = (X_1',\, X_2')$, $D'\widehat{\psi}(x,\,y) = (Y_1',\, Y_2')$.
The left invariance of $\widehat{\varphi}$ implies

$$J_{1_+}(X_2) - J_{1_-}(X_1) + \Psi^\star(X_2' - X_1') = 0\,,$$

while the right invariance of $\widehat{\psi}$ implies

$$J_{2_+}(Y_2') - J_{2_-}(Y_1') + \Psi(Y_2 - Y_1) = 0\,.$$

Inserting these conditions into the Poisson bracket

$$\{\widehat{\varphi}, \widehat{\phi}\}_{(J_{1_\delta},\, J_{2_\delta})} = \frac{1}{2}\langle\, \Pi_{(\delta_{\mathcal{G}})_1}(X_1,\, X_2),\, (Y_1,\, Y_2)\,\rangle - \frac{1}{2}\langle\, (X_1',\, X_2'),\, \Pi_{(\delta_{\mathcal{G}})_2}(Y_1',\, Y_2')\,\rangle$$
$$-\frac{1}{2}\langle\, \Pi_{\mathcal{G}_{J_1}}(X_1,\, X_2),\, (Y_1,\, Y_2)\,\rangle + \frac{1}{2}\langle\, (X_1',\, X_2'),\, \Pi_{\mathcal{G}_{J_2}}(Y_1',\, Y_2')\,\rangle\,,$$

the two first and two last terms are seen to cancel out separately. Here, $\Pi_{(\delta_{\mathcal{G}})_i}$ refers to the decompositions $\delta_{J_{i_\delta}} =^\delta \mathcal{G} \ominus \mathcal{G}_{J_i}$, $i = 1, 2$. ♠

Now, noting that π_r and π_l are both submersions onto G, we may collect the above results in the following theorem.

Theorem 3.3
The diagram $G \overset{\pi_l}{\longleftarrow} (G \times G)_{(J_{1_\delta},\, J_{2_\delta})} \overset{\pi_r}{\longrightarrow} G$ *provides a full symplectic dual pair for the twisted affine structure of Theorem 3.1.*♠

Theorem 3.3 has the immediate corollary:

Corollary 3.1
The connected component of the symplectic leaf of the twisted affine Poisson structure of Theorem 3.1 passing through $x \in G$ *is given by*

$$\{\, \big(\, \widehat{\Psi^\star} \circ m_2^{-1}(\widehat{\Psi}(g)\, x^{-1} m_1(g^{-1})\, x \widehat{\Psi}(g^{-1}))\big)^{-1} \cdot$$
$$\widehat{J_{1_+}}(g)\, x \widehat{\Psi}(g^{-1}) \cdot \big(\widehat{\Psi}(g)\, x^{-1} m_1(g^{-1})\, x\, \widehat{\Psi}(g^{-1})\big)_{+_2} \quad |\, g \in G_{J_1} \}$$

Proof:

This follows from a general result of A.Weinstein [W 83] according to which , for a full symplectic dual pair, the connected components of the symplectic leaves are given by $\pi_r \circ \pi_l^{-1}(x)$.

Now, $\pi_l^{-1}(x) = G_{J_1} \cdot (x, x) = \{(\widehat{J_{1+}}(g)\, x(\widehat{\Psi}(g))^{-1}, \widehat{J_{1-}}(g)\, x(\widehat{\Psi}(g))^{-1}) \quad | \; g \in G_{J_1}\}$, from which the statement is clear.♠

We close this section with two comments.

Comments

a) The Poisson manifold $(G, \{\,,\,\}_{(\Delta, 0, 1)})$ (see Theorem 3.2) coincides (see Remark 3.3 a) with the Poisson Lie group dual to $(G, \{\,,\,\}_{(J^+, -J^+)})$. In [STS 85] it is shown that the diagram

$$G \overset{\pi_l'}{\longleftarrow} D_{(J_\delta^+,\, J_\delta^+)} \overset{\pi_r'}{\longrightarrow} G,$$

$$\pi_l'(x, y) = y^{-1} x, \quad \pi_r'(x, y) = x\, y^{-1}, \quad x, y \in G,$$

provides a full symplectic dual pair for $\{\,,\,\}_{(\Delta, 0, 1)}$. In particular, its symplectic leaves are the conjugacy classes of G.
For $z \in G$, $z_\pm$ denote the factors of the decomposition (2.2) relative to J^+.
Setting $J_1 = -J^+$, $J_2 = J^+$, $\Psi = -(J^+)_-$, and hence $\Psi^\star = (J^+)_+$ (see Remark 3.1 and (3.16)), we find
$(z)_{+_1} = (z^{-1})_- = \widehat{\Psi} \circ m_1^{-1}(z)$, $\widehat{\Psi^\star} \circ m_2^{-1}(z) = z_+$. Theorem 3.3 then asserts that the diagram

$$G \overset{\pi_l}{\longleftarrow} D_{((-J^+)_\delta,\, J_\delta^+)} \overset{\pi_r}{\longrightarrow} G,$$

$$\pi_l(x, y) = (y\, x^{-1})_-^{-1}\, x(y\, x^{-1})_-, \quad \pi_r(x, y) = (x^{-1} y)_+^{-1}\, x(x^{-1} y)_+,$$

also provides a full symplectic dual pair for $\{\,,\,\}_{(\Delta, 0, 1)}$. The symplectic leaf $\mathcal{L}_x \subset G$ passing through x is now given by (see Corollary 3.1)

$$\mathcal{L}_x = \pi_r \circ \pi_l^{-1}(x) = \{(kx^{-1}k^{-1}x)_+^{-1} x\, (kx^{-1}k^{-1}x)_- \;|\; k \in G\},$$

where we have used $g_\pm z_\pm = (g_+ z\, g_-^{-1})_\pm$ for all $g \in G_{J^+}$ and $z \in G$ and have set $m(g) = k$. The leaf $\mathcal{L}_x$ is (as it should be) diffeomorphic to the G-conjugacy class of x. Indeed, if we let $\iota_G(z) = z^{-1}$, $\iota_{G_{J^+}}(z) = z_+^{-1} z_-$, and $i_y(z) = y\, z\, y^{-1}$, one easily verifies that

$$\iota_G \circ i_{x_+} \circ \iota_{G_{J^+}}(\mathcal{L}_x) = \{\, k\, x\, k^{-1} \;|\; k \in G\}.$$

b) J-H Lu, in her Doctoral Thesis [Lu 90] (see also the remark at the end of [Lu 91]), has described the symplectic transformation groupoid for affine Poisson structures. Her description makes use of a (nondegenerate) semi-direct Poisson structure $\{\,,\,\}_{\times_{\frac{1}{2}}}$ which, for the bracket $\{\,,\,\}_{(-J_1, J_2)}$ of Theorem 2.1, is constructed as follows.
Let $\{\,,\,\}_+$ be the (nondegenerate) Poisson structure on $G \times G_{J_2}$ induced by the isomorphism (see (2.2))

$$M : G \times G_{J_2} \xrightarrow{\sim} D_{(J_{2_\delta}, J_{2_\delta})},\, (g, h) \longmapsto (g, g) \cdot (\widehat{J_{2+}}(h), \widehat{J_{2-}}(h)).$$

The restriction of the right Poisson action

$$\rho: \left(G_{(-J_1,J_2)} \times (G \times G_{J_2})_{\{\,,\,\}_+}\right) \times \left(G_{(-J_2,J_2)} \times G_{(J_2,-J_2)}\right) \longrightarrow G_{(-J_1,J_2)} \times (G \times G_{J_2})_{\{\,,\,\}_+}$$

$$((p,(x,\,y)),\,(h_1,\,h_2)) \longmapsto (p\,h_1,\,h_2^{-1}x,\,y)\,,$$

to ${}^\delta G \subset D$ is admissible. $\{\,,\,\}_{\times_{\frac{1}{2}}}$ is then defined as the corresponding induced Poisson bracket on the quotient $(G \times G \times G_{J_2})/{}^\delta G \simeq G \times G_{J_2}$.
It is isomorphic to $\{\,,\,\}_{(J_{1_\delta},J_{2_\delta})}$ which we used in Theorem 3.3. Indeed, a direct calculation shows that

$$M : (G \times G_{J_2})_{\{\,,\,\}_{\times_{\frac{1}{2}}}} \xrightarrow{\sim} D_{(J_{1_\delta},J_{2_\delta})}$$

is a symplectomorphism.

We shall not pursue further the study of the symplectic leaves of the Poisson brackets of Theorem 3.2 and hope to return to this elsewhere.

4. A class of morphisms of the Lie bialgebras $(\mathcal{G}_{J^\pm},\mathcal{G})$ and solutions of (MCYB).

$(\mathcal{G},[\,,\,])$ denotes either a complex semi-simple Lie algebra or its split real form (see below), and $(\,,\,)$ is the Killing form. We retain the notations of section 3.1.
Our main purpose in this section is to provide an explicit characterization of the set of symmetric (w.r.t.$(\,,\,)$) Lie algebra homomorphisms $\mathrm{Hom}^+_{sym}(\mathcal{G}_{J^+},\mathcal{G})$ subject to a (natural) additional constraint (see (4.1)).
Let (J_1, J_2, Ψ) be a triple satisfying assumption (A). Recall (see Remark 3.1 b) that if $J_2 = J_1 + 2(\Psi^\star - \Psi)$, the bracket $\{\,,\,\}_{(-J_1,J_2,\Psi)}$ of Theorem 3.1 is linearizable at the identity of G (with associated R-matrix $R = J_2 + 2\Psi \equiv J_1 + 2\Psi^\star$). When $J_1 = J_2 = J^+$, $\mathrm{Hom}^+_{sym}(\mathcal{G}_{J^+},\mathcal{G})$ thus parametrizes the class of linearizable twisted Poisson Lie groups $\{\,,\,\}_{(-J^+,J^+,\Psi)}$ introduced in [LP 89]. It is a simple fact (see Lemma 4.2) that such homomorphisms also belong to $\mathrm{Hom}^-(\mathcal{G}_{J^-},\mathcal{G})$ for a suitable choice of $\Gamma \subset \Delta$ (see (3.13)) and that they may further be composed with a natural involution of $\mathcal{G}$ (see Lemma 4.3 and Corollary 4.1). Therefore, the knowledge of $\mathrm{Hom}^+_{sym}(\mathcal{G}_{J^+},\mathcal{G})$ also provides us with a large class of natural examples of twisted affine Poisson algebras associated with the pair (J^+, J^-) of skew-symmetric solutions of (MCYB). We have used [B 75] and [H 80] as references on Lie algebras.

Before stating the main proposition of this section, we need a preparatory lemma and some definitions.

Lemma 4.1

The Lie algebra $\mathcal{G}_{J^+}$ is solvable.

Proof:

We have to show that the derived sequence

$$\mathcal{G}^{(0)}_{J^+} = \mathcal{G}_{J^+},\, \mathcal{G}^{(1)}_{J^+} = [\mathcal{G}_{J^+},\,\mathcal{G}_{J^+}]_{J^+},\,\ldots,\,\mathcal{G}^{(k)}_{J^+} = [\mathcal{G}^{(k-1)}_{J^+},\,\mathcal{G}^{(k-1)}_{J^+}]_{J^+}\,,\,\ldots$$

terminates. Now, as a Lie algebra,

$$\mathcal{G}_{J^+} = \mathcal{H} \oplus_{\frac{1}{2}} (\mathcal{N}^+ \ominus \mathcal{N}^-),$$

where the semi-direct sum is defined by $\sigma : \mathcal{H} \longrightarrow \text{Der}\,(\mathcal{N}^+ \ominus \mathcal{N}^-)$, $\sigma(H)(\zeta^+ + \zeta^-) = \frac{1}{2}[H, \zeta^+ - \zeta^-]$, $H \in \mathcal{H}, \zeta^\pm \in \mathcal{N}^\pm$. Therefore,

$$\mathcal{G}_{J^+}^{(k+1)} \subset (\mathcal{N}^+)^{(k)} \ominus (\mathcal{N}^-)^{(k)},$$

which terminates since $\mathcal{N}^\pm$ are nilpotent.♠

Let $\phi \in \text{Hom}^+_{sym}\,(\mathcal{G}_{J^+}, \mathcal{G})$. Since solvability is preserved by homomorphisms, $\phi(\mathcal{G}_{J^+}) \subset \mathcal{G}$ is a solvable Lie subalgebra. As such it must lie in a Borel subalgebra $\mathcal{B}' \subset \mathcal{G}$. For the rest of the section we shall assume that $\mathcal{B}'$ is standard w.r.t. $\mathcal{H}$. Thus, we shall impose that

$$\phi(\mathcal{G}_{J^+}) \subset \mathcal{B}' = \mathcal{H} \oplus \amalg_{\alpha \in \Phi'^+} \mathcal{G}^\alpha, \tag{4.1}$$

for some base $\Delta' \subset \Phi$ with associated positive/negative root subsets $\Phi'^\pm \subset \Phi$.

Remark 4.1

In general, $\Delta' \neq \Delta$.

Notations

As in Section 3, if $\Lambda \subset \Delta$, we denote by $\widehat{\Lambda} \subset \Phi$ the set of roots supported by Λ.

Choose two subsets $\Gamma^\pm \subset \Delta$ with $\Gamma^+ \cap \Gamma^- = \emptyset$, and let $\sigma^\pm : \Gamma^\pm \longrightarrow \Gamma^\pm$ be bijections.

Definition 4.1

We shall say that the quadruple $(\Gamma^+, \Gamma^-, \sigma^+, \sigma^-)$ is admissible if $(\alpha_+, \beta_-) = 0$ and $(\sigma^\pm(\alpha_\pm), \sigma^\pm(\beta_\pm)) = (\alpha_\pm, \beta_\pm)$ for all $\alpha_\pm, \beta_\pm \in \Gamma^\pm$.

Remark 4.2

The quadruple $(\Gamma^+, \emptyset, id_{\Gamma^+}, 0)$ is always admissible. If Γ^+ and Γ^- are mutually orthogonal, the quadruple $(\Gamma^+, \Gamma^-, id_{\Gamma^+}, id_{\Gamma^-})$ is admissible.

Example 4.1

Let $\mathcal{G} = sl(n, C)$ with Cartan subalgebra $\mathcal{H} = \{$ (traceless) diagonal matrices $\}$, and root system $\Phi = \{\alpha_{i\,j}, 1 \leq i \neq j \leq n\}$, where $\alpha_{ij}(H) = H_{ii} - H_{jj}$, $H \in \mathcal{H}$. We shall take the base $\Delta = \{\alpha_{i\,i+1}, 1 \leq i \leq n-1\}$.
Choose integers $l_\pm, u_\pm$ such that $1 \leq l_+ \leq u_+ < l_- - 1 \leq u_- - 1 \leq n-2$, set

$$\Gamma^\pm = \{\alpha_{i,\,i+1} \mid l_\pm \leq i \leq u_\pm\},$$

and let the bijections $\sigma^\pm : \Gamma^\pm \longrightarrow \Gamma^\pm$ be given by

$$\sigma^\pm(\alpha_{l_\pm + j\; l_\pm + j+1}) = \alpha_{u_\pm - j\; u_\pm - j+1}, \quad 0 \leq j \leq u_\pm - l_\pm.$$

The quadruple $(\Gamma^+, \Gamma^-, \sigma^+, \sigma^-)$ is then admissible (this is clear from $(\alpha_{i\,i+1}, \alpha_{j\,j+1}) = 2n(2\delta_{i\,j} - \delta_{i\,j-1} - \delta_{i\,j+1})$).

Recall [H 80, sect.25.2 p.146] that a complex semi-simple Lie algebra $\mathcal{L}$ admits a Weyl – Chevalley basis $\{ Z_\beta,\, H_\alpha \quad | \, \beta \in \Phi,\, \alpha \in \Delta \}$: $Z_\beta \in \mathcal{G}^\beta$, $H_\beta = \frac{2\beta^*}{(\beta,\beta)}$, $[Z_\beta, Z_{-\beta}] = H_\beta$, and if $\alpha + \beta \in \Phi$, $[Z_\alpha, Z_\beta] = c_{\alpha,\beta}\, Z_{\alpha+\beta}$, then $c_{\alpha,\beta} = -c_{-\alpha,-\beta}$. Moreover, the structure constants $c_{\alpha,\beta}$ are real (in fact integer). The split real form of $\mathcal{L}$ is defined as the real span of such a basis.

Let $(\Gamma^+, \Gamma^-, \sigma^+, \sigma^-)$ be admissible, and define the isomorphisms of Lie subalgebras

$$\phi^+ : \amalg_{\alpha\in\widehat{\Gamma^+}} \mathcal{G}^\alpha \cap \mathcal{N}^+ \longrightarrow \amalg_{\alpha\in\widehat{\Gamma^+}} \mathcal{G}^\alpha \cap \mathcal{N}^-,$$

$$\phi^- : \amalg_{\alpha\in\widehat{\Gamma^-}} \mathcal{G}^\alpha \cap \mathcal{N}^- \longrightarrow \amalg_{\alpha\in\widehat{\Gamma^-}} \mathcal{G}^\alpha \cap \mathcal{N}^+,$$

by the assignment of generators

$$\phi^+(Z_{\alpha_+}) = \lambda_{\alpha_+} Z_{-\sigma^+(\alpha_+)}, \quad \phi^-(Z_{-\alpha_-}) = \lambda_{\alpha_-} Z_{\sigma^-(\alpha_-)} \quad \text{for all } \alpha_\pm \in \Gamma^\pm, \tag{4.2}$$

where $\lambda_{\alpha_\pm} \in R^\times$ or $C^\times$. Set $\Gamma = \Gamma^+ \cup \Gamma^-$. Further, let the map

$$\phi^0 : \mathcal{H} \longrightarrow \mathcal{H}$$

be subject to

$$\phi^0(H_{\alpha_+}) = -\frac{1}{2} H_{\sigma^+(\alpha_+)}, \quad \phi^0(H_{\alpha_-}) = \frac{1}{2} H_{\sigma^-(\alpha_-)} \quad \text{for all } \alpha_\pm \in \Gamma^\pm, \tag{4.3}$$

and

$$\phi^0(\mathcal{H}_\Gamma^\perp) \subset \mathcal{H}_\Gamma^\perp. \tag{4.4}$$

Remark 4.3

That the maps $\phi^\pm$ may be defined as above follows from the fact that if $\sigma^\pm$ are orthogonal then, for $\alpha_{i\pm} \in \Gamma^\pm$, $\sum_i \alpha_{i\pm} \in \Phi$ implies $\sum_i \sigma^\pm(\alpha_{i\pm}) \in \Phi$.

Finally, let $\phi \in \text{End}\,(\mathcal{G}_{J^+}, \mathcal{G})$ be given by

$$\phi(\zeta) = \begin{cases} (\phi^+ - \phi^- + \phi^0)(\zeta), & \text{if } \zeta \text{ lies in the domain of these maps,} \\ 0, & \text{otherwise.} \end{cases} \tag{4.5}$$

Proposition 4.1

a) If ϕ is defined by (4.5) for some admissible quadruple $(\Gamma^+, \Gamma^-, \sigma^+, \sigma^-)$, then $\phi \in$ $\text{Hom}^+(\mathcal{G}_{J^+}, \mathcal{G})$.

b) If $\Psi \in \text{Hom}^+_{sym}(\mathcal{G}_{J^+}, \mathcal{G})$ satisfies (4.1), then there exist an admissible quadruple $(\Gamma^+, \Gamma^-, \sigma^+, \sigma^-)$ with $(\sigma^\pm)^2 = Id_{\Gamma^\pm}$ and maps $\phi^\pm$, ϕ^0 (as above) such that Ψ is given by (4.5) where $\lambda_{\alpha_\pm} = \lambda_{\sigma^\pm(\alpha_\pm)}$ for all $\alpha_\pm \in \Gamma^\pm$, and ϕ^0 is symmetric w.r.t. $(\,,\,)\,|_{\mathcal{H}\times\mathcal{H}}$.

Proposition 4.1 b, Theorem 3.1, and Remark 3.1 b yield the following proposition.

Proposition 4.2

Under the assumption (4.1), the linearizable (at e of G) twisted Poisson brackets associated with the Lie bialgebra $(\mathcal{G}_{J^+}, \mathcal{G})$ *are given by* $\{\, ,\, \}_{(-J^+, J^+, \phi^+ - \phi^- + \phi^0)}$, *where the admissible quadruple* $(\Gamma^+, \Gamma^-, \sigma^+, \sigma^-)$ *and the maps* $\phi^\pm$, ϕ^0 *are subject to the conditions in Proposition 4.1 b.*♠

Remark 4.4

The R–matrices which arise in the double Lie algebras tangent to the brackets of Proposition 4.2 (see Remark 3.1 b) read as

$$R = (1 + 2\phi^+) \circ \Pi_{\mathcal{N}^+} - (1 + 2\phi^-) \circ \Pi_{\mathcal{N}^-} + 2\phi^0.$$

Note that $R_\pm(\mathcal{G}_{J^+}) \subset \mathcal{P}^\pm_{\Gamma^\pm}$, with $R_+(\mathcal{G}_{J^+}) \cap R_-(\mathcal{G}_{J^+}) = (\phi^0 + \frac{1}{2})(\mathcal{H}) \cap (\phi^0 - \frac{1}{2})(\mathcal{H})$, and that Ker $R_\pm \supset \mathcal{N}^\mp_{\Gamma^\mp}$. Remark also that these R-matrices are not graded w.r.t. root height.

Before providing a proof of Proposition 4.1 we shall state two elementary lemmas (whose proof we leave to the reader) which relate elements of $\mathrm{Hom}^+(\mathcal{G}_{J^+}, \mathcal{G})$ and of $\mathrm{Hom}^-(\mathcal{G}_{J^-}, \mathcal{G})$ (J^- is as in (3.13) with $\Gamma = \Gamma^+ \cup \Gamma^-$). Let $(\Gamma^+, \Gamma^-, \sigma^+, \sigma^-)$ be admissible, $\phi \in \mathrm{Hom}^+(\mathcal{G}_{J^+}, \mathcal{G})$ be defined by (4.5).

Lemma 4.2

$\phi \in \mathrm{Hom}^-(\mathcal{G}_{J^-}, \mathcal{G})$.♠

Remark 4.5

Let Ψ be as in Proposition 4.1 b. From Lemma 4.2, it follows that $\{\, ,\, \}_{(-J_-, J_-, -\Psi)}$ is a twisted Poisson bracket with associated tangent R-matrix $J^- - 2\Psi$.

Let $\theta \in$ Aut $\mathcal{G}$ be the involution: $\theta(Z_\alpha) = -Z_{-\alpha}$, $\theta(H) = -H$, $\quad \alpha \in \Phi$, $H \in \mathcal{H}$.

Lemma 4.3

$\theta \in \mathrm{Aut}^-(\mathcal{G}_{J^\pm})$.♠

Corollary 4.1

$\theta \circ \phi \in \mathrm{Hom}^+(\mathcal{G}_{J^+}, \mathcal{G}) \cap \mathrm{Hom}^-(\mathcal{G}_{J^-}, \mathcal{G})$,
$\phi \circ \theta \in \mathrm{Hom}^-(\mathcal{G}_{J^+}, \mathcal{G}) \cap \mathrm{Hom}^+(\mathcal{G}_{J^-}, \mathcal{G})$.♠

Remark 4.6

Note that if ϕ is symmetric, $\theta \circ \phi = (\phi \circ \theta)^*$. Therefore, $\{\, ,\, \}_{(-J^+, J^-, \theta\circ\phi)}$ is a twisted affine Poisson bracket.

Example 4.2 (Iwasawa and generalized Gauss class on $sl(n, R)$.)

Let $\mathcal{G} = sl(n, R)$ with Cartan subalgebra $\mathcal{H}$ = { (traceless) diagonal matrices }, root system Φ, and base Δ, as in Example 4.1. If e_{ij} is the matrix whose ij entry is 1 and which has 0 elsewhere, then $\mathcal{G}^{\alpha_{ij}} = R e_{ij}$, and $\{Z_{\alpha_{ij}} = e_{ij}, \alpha_{ij} \in \Phi;\ H_{\alpha_{i\,i+1}} = e_{i\,i} - e_{i+1\,i+1}, \alpha_{i\,i+1} \in \Delta\}$ is a Weyl –Chevalley basis. The involution θ is now given by: $\theta(\zeta) = -\zeta^T$. We consider the simplest class of admissible quadruples: $(\Gamma^+, \emptyset, Id_{\Gamma^+}, 0)$ where Γ^+ ranges from $\emptyset$ to Δ. Set $\lambda_{\alpha_+} = -1$ for all $\alpha_+ \in \Gamma^+$, and take $\phi^0 = -\frac{Id_{\mathcal{H}}}{2}$. With these choices

$\phi = \theta\circ(\Pi_{\mathcal{L}^+_{\Gamma^+}} + \frac{\Pi_{\mathcal{H}}}{2})$ ($\mathcal{L}^+_{\Gamma^+} = \mathcal{L}_{\Gamma^+} \cap \mathcal{N}^+$). By linearizing the corresponding twisted brackets of proposition 4.2, we obtain the family of R−matrices:

$$R = J^+ + 2\theta\circ(\Pi_{\mathcal{L}^+_\Gamma} + \frac{\Pi_{\mathcal{H}}}{2}).$$

When $\Gamma^+ = \Delta$,

$$R = \big((1+\theta)\circ\Pi_{\mathcal{N}^+}\big) - \big(\Pi_{\mathcal{N}^-} - \theta\circ\Pi_{\mathcal{N}^+} + \Pi_{\mathcal{H}}\big),$$

or

$$R = \Pi_{so(n)} - \Pi_{\mathcal{B}^-},$$

where

$$\mathcal{G} = so(n) \oplus \mathcal{B}^-$$

is the Iwasawa decomposition of $sl(n,R)$ relative to the maximal abelian subalgebra $\mathcal{H}$. On the other hand, $\theta\circ\phi = \Psi^+$, and $\phi\circ\theta = \Psi^-$ (cf.(3.14)); the corresponding family of twisted Poisson brackets $\{\,,\,\}_{(-J^+,\, J^-,\, \theta\circ\phi)}$ linearizes to $\mathcal{G}_{J^- + 2\theta\circ\phi} \equiv \mathcal{G}_{R_{(\Gamma^+,\,0)}}$ (see (3.11)).

Proof of Proposition 4.1 a

From the explicit expression of $[\,,\,]_{J^+}$

$$\begin{aligned}[\zeta,\,\eta]_{J^+} &= [\Pi_{\mathcal{N}^+}\zeta,\,\Pi_{\mathcal{N}^+}\eta] - [\Pi_{\mathcal{N}^-}\zeta,\,\Pi_{\mathcal{N}^-}\eta] \\ &+ \frac{1}{2}[\Pi_{\mathcal{H}}\zeta, (\Pi_{\mathcal{N}^+} - \Pi_{\mathcal{N}^-})\eta] + \frac{1}{2}[(\Pi_{\mathcal{N}^+} - \Pi_{\mathcal{N}^-})\zeta,\,\Pi_{\mathcal{H}}\eta],\end{aligned}$$

it follows that $\phi \in \mathrm{Hom}^+(\mathcal{G}_{J^+},\,\mathcal{G})$ if and only if the following six relations

$$\phi([\zeta^\pm,\,\eta^\pm]) = \pm[\phi(\zeta^\pm),\,\phi(\eta^\pm)], \tag{4.6}$$

$$\phi([H,\,\eta^\pm]) = \pm 2[\phi(H), \phi(\eta^\pm)], \tag{4.7}$$

$$[\phi(\zeta^+),\,\phi(\eta^-)] = 0 = [\phi(H),\,\phi(H')], \tag{4.8}$$

hold, $\zeta^\pm,\,\eta^\pm \in \mathcal{N}^\pm$, $H,\,H' \in \mathcal{H}$.

Clearly, the relations (4.6) are satisfied by construction (see (4.2) and (4.5)). Let $\alpha_\pm \in \Gamma^\pm$. For $\eta^+ = Z_{\alpha_+}$ and $\eta^- = Z_{-\alpha_-}$, (4.7) says that

$$\alpha_\pm(H) = \mp 2\sigma^\pm(\alpha_\pm)(\phi(H)) \tag{4.9}$$

For α_+ and $H = H_{\gamma_\pm}$, $\gamma_\pm \in \Gamma^\pm$, the l.h.s. of (4.9) reads as

$$2\frac{(\alpha_+,\,\gamma_\pm)}{(\gamma_\pm,\,\gamma_\pm)},$$

while the r.h.s. is (see (4.3))

$$\pm 2\frac{(\sigma^+(\alpha_+),\,\sigma^\pm(\gamma_\pm))}{(\sigma^\pm(\gamma_\pm),\,\sigma^\pm(\gamma_\pm))}.$$

They clearly are equal since $(\Gamma^+,\Gamma^-,\sigma^+,\sigma^-)$ is admissible. The same is true for α_-. On the other hand, $H\in\mathcal{H}_\Gamma^\perp$ means that $\alpha_\pm(H)=0$ for all $\alpha_\pm\in\Gamma^\pm$. Thus (4.9) also holds in this case by (4.4), and hence it holds for all $H\in\mathcal{H}$ since $\mathcal{H}=\mathcal{H}_\Gamma\oplus\mathcal{H}_\Gamma^\perp\,(\Gamma=\Gamma^+\cup\Gamma^-)$. Finally, note that if (4.7) holds for the generators Z_{α_+} and $Z_{-\alpha_-}$, it also holds for all $\eta^\pm$ by the Jacobi identity. It remains to verify (4.8). Clearly, $[\phi(H),\,\phi(H')]=0$ by construction. Now, if $\gamma_\pm\in\widehat{\Gamma^\pm}\cap\Phi^\mp$, the condition $\Gamma^+\cap\Gamma^-=\emptyset$ implies that $[\mathcal{G}^{\gamma_+},\,\mathcal{G}^{\gamma_-}]=0$. Therefore, $[\phi(\zeta^+),\,\phi(\eta^-)]\equiv-[\phi^+(\zeta^+),\,\phi^-(\eta^-)]\in[\amalg_{\gamma_+\in\widehat{\Gamma^+}\cap\Phi^-}\mathcal{G}^{\gamma_+},\,\amalg_{\gamma_-\in\widehat{\Gamma^-}\cap\Phi^+}\mathcal{G}^{\gamma_-}]=0$, which concludes the proof. ♠

Proof of Proposition 4.1 b

We shall need the orthogonality properties

$$(\mathcal{G}^\alpha,\,\mathcal{G}^\beta)=0,\ \text{unless}\ \alpha+\beta=0,\ \text{and}\ (\mathcal{G}^\alpha,\mathcal{H})=0,\quad \alpha,\beta\in\Phi. \tag{4.10}$$

Let $\Psi\in\mathrm{Hom}^+_{sym}(\mathcal{G}_{J^+},\,\mathcal{G})$ satisfy (4.1). Recall that the sets $\Phi'^\pm\subset\Phi$ satisfy

$$-\Phi'^+=\Phi'^-,\ \Phi'^+\cap\Phi'^-=\emptyset,\ \text{and}\ \Phi'^+\cup\Phi'^-=\Phi, \tag{4.11}$$

and that, if we set $\mathcal{N}'^\pm=\amalg_{\alpha\in\Phi'^\pm}\mathcal{G}^\alpha$,

$$[\mathcal{B}'^+,\,\mathcal{B}'^+]=\mathcal{N}'^+. \tag{4.12}$$

Lemma 4.4

a) $\mathcal{N}'^+\subset Ker(\Psi)$. *b)* $\Psi(\mathcal{N}'^-)\subset\mathcal{N}'^+$. *c)* $\Psi(\mathcal{H})\subset\mathcal{H}$.

Proof:

a) follows from the assumption (4.1), the symmetry of Ψ, and the orthogonality properties (4.10). b) follows from (4.7), (4.1), and (4.12). c) follows from a), b), and (4.10). ♠

Let $\Sigma^\pm\subset\Phi^\pm$ be defined by

$$\Sigma^\pm=\{\,\gamma_0+\gamma_1+\cdots+\gamma_{k^\pm},\ \text{for some}\ k^\pm\geq0,\gamma_0\in\Phi'^+\cap\Phi^\pm\ \text{and}\ \gamma_i\in\Phi^\pm,1\leq i\leq k^\pm\}$$

Lemma 4.5

$\amalg_{\alpha\in\Sigma^+\cup\Sigma^-}\mathcal{G}^\alpha\subset Ker(\Psi)$.

Proof:

Recall (see the proof of Lemma 4.1) that $\mathcal{G}_{J^+}=\mathcal{H}\oplus_{\frac{1}{2}}(\mathcal{N}^+\ominus\mathcal{N}^-)$. From Lemma 4.4, $Ker(\Psi)\supset\mathcal{N}'^+$. Now, since $\Psi\in\mathrm{Hom}\,^+(\mathcal{G}_{J^+},\,\mathcal{G})$, $Ker\,(\Psi)\subset\mathcal{G}_{J^+}$ is a Lie ideal. Thus

$$[\mathcal{N}'^+\cap\mathcal{N}^\pm,\,\mathcal{N}^\pm]_{J^+}\equiv[\mathcal{N}'^+\cap\mathcal{N}^\pm,\,\mathcal{N}^\pm]\subset Ker(\Psi).$$

Iterating gives

$$[\cdots[\mathcal{N}'^+\cap\mathcal{N}^\pm,\,\mathcal{N}^\pm],\cdots\mathcal{N}^\pm]\subset Ker(\Psi).$$

The claim then follows from $[\mathcal{G}^\alpha, \mathcal{G}^\beta] = \mathcal{G}^{\alpha+\beta}$ for all $\alpha, \beta \in \Phi$ such that $\alpha + \beta \in \Phi$.♠

Let $\Delta_+ = \Phi'^+ \cap \Delta$, $\mathcal{R}_\pm = \Phi^\pm \setminus \Sigma^\pm$.

Let us make a parenthetical remark.

Remark 4.7

First, note that $\mathcal{R}_+ \subset \widehat{\Delta \setminus \Delta_+} \cap \Phi^+$ and (using (4.11)) $\mathcal{R}_- \subset \widehat{\Delta_+} \cap \Phi^-$. Further note that (4.11) and $\Delta_+ = \Phi'^+ \cap \Delta$ imply $(\widehat{\Delta \setminus \Delta_+}) \cap (\Phi'^+ \cap \Phi^+) = \emptyset = \widehat{\Delta_+} \cap (\Phi'^+ \cap \Phi^-)$, and hence, $\Sigma^+ \cap (\widehat{\Delta \setminus \Delta_+}) = \emptyset = \Sigma^- \cap \widehat{\Delta_+}$. This shows that $\mathcal{R}_+ = (\widehat{\Delta \setminus \Delta_+}) \cap \Phi^+$, and $\mathcal{R}_- = \widehat{\Delta_+} \cap \Phi^-$.

By Lemma 4.4 and Lemma 4.5, it now remains to characterize the restrictions $\Psi_| : \amalg_{\alpha \in \mathcal{R}_+ \cup \mathcal{R}_-} \mathcal{G}^\alpha \longrightarrow \mathcal{N}'^+$ and $\Psi_| : \mathcal{H} \longrightarrow \mathcal{H}$. For $\alpha \in \mathcal{R}_+ \cup \mathcal{R}_-$, let us write

$$\Psi(Z_\alpha) = \sum_{\mu \in \Phi'^+ \cap \Phi^+} \Psi_{\alpha,\mu} Z_\mu + \sum_{\nu \in \Phi'^+ \cap \Phi^-} \Psi_{\alpha,\nu} Z_\nu. \tag{4.13}$$

Lemma 4.6

Let $\alpha_+ \in \mathcal{R}_+$. If $\mathcal{G}^{\alpha_+} \not\subset Ker\,\Psi$, then $\Psi(\mathcal{G}^{\alpha_+}) = \mathcal{G}^{-\beta_+}$ for some $\beta_+ \in \mathcal{R}_+$ and $\Psi(\mathcal{G}^{\beta_+}) = \mathcal{G}^{-\alpha_+}$. If α_+ is simple (w.r.t. Δ), so is β_+. The same assertion holds if the index $+$ is replaced by $-$.

Proof:

The verification consists of four simple observations.

Observation 1. $\Psi(\amalg_{\alpha \in \mathcal{R}_+ \cup \mathcal{R}_-} \mathcal{G}^\alpha) \subset \amalg_{\alpha \in -(\mathcal{R}_+ \cup \mathcal{R}_-)} \mathcal{G}^\alpha$.

Proof: The symmetry of Ψ, together with the orthogonality (4.10) and Lemma 4.5 imply $\Psi_{\alpha,\mu} = 0$ if $\mu \in -\Sigma^- \cap \Phi'^+$ and $\Psi_{\alpha,\nu} = 0$ if $\nu \in -\Sigma^+ \cap \Phi'^+$. Thus, it suffices to check that $-\mathcal{R}_\mp = (\Phi'^+ \cap \Phi^\pm) \setminus (-\Sigma^\mp \cap \Phi'^+)$. But this follows from the definitions and (4.11).

Observation 2. *Let $\gamma \in \mathcal{R}_+ \cup \mathcal{R}_-$. $\Psi_{\alpha,-\gamma} \neq 0$ for at most one $\alpha \in \mathcal{R}_+ \cup \mathcal{R}_-$.*

Proof: Let $\alpha_\pm \in \mathcal{R}_\pm$ (here, by $\pm$, we mean $+$ or $-$), $H \in \mathcal{H}$. If $\Psi_{\alpha_\pm,-\gamma} \neq 0$, (4.7) implies $\pm\frac{1}{2}\alpha_\pm(H) = -\gamma(\Psi(H))$. Thus, if $\beta_\pm \in \mathcal{R}_\pm$ is such that $\Psi_{\beta_\pm,\gamma} \neq 0$ also, we must have either $\alpha_\pm = \beta_\pm$ or $\alpha_\pm = -\beta_\mp$. The second case cannot hold however since $\mathcal{R}_+ \cap -\mathcal{R}_- = \emptyset$ (see Remark 4.7).

Observation 3. *If $\mathcal{G}^\alpha \not\subset Ker(\Psi)$,* then *$\Psi(\mathcal{G}^\alpha) = \mathcal{G}^{-\beta}$ for some $\beta \in (\mathcal{R}_+ \cup \mathcal{R}_-)$ and $\Psi(\mathcal{G}^\beta) = \mathcal{G}^{-\alpha}$. If α is simple (w.r.t. Δ), so is β.*

Proof: By symmetry $\Psi_{\gamma,-\delta}(Z_{-\delta}, Z_\delta) = \Psi_{\delta,-\gamma}(Z_\gamma, Z_{-\gamma})$ for all $\gamma, \delta \in \mathcal{R}_+ \cup \mathcal{R}_-$. The first assertion is then clear from observation 2 (see 4.13). Now, assume $\Psi(\mathcal{G}^\beta) = \mathcal{G}^{-\alpha}$. If α is simple, so is β, by (4.6). By symmetry, $\Psi(\mathcal{G}^\alpha) = \mathcal{G}^{-\beta}$, from which the claim follows.

Observation 4. $\Psi(\amalg_{\alpha \in \mathcal{R}_\pm} \mathcal{G}^\alpha) \subset \amalg_{\alpha \in -\mathcal{R}_\pm} \mathcal{G}^\alpha$.

Proof: Assume that the assertion is false. Then, by Observation 3., there is a pair of roots $\gamma_+ \in \mathcal{R}_+$, and $\gamma_- \in \mathcal{R}_-$ with $\Psi(\mathcal{G}^{\gamma_\pm}) = \mathcal{G}^{-\gamma_\mp}$. (4.7) then implies $\frac{1}{2}\gamma_\pm(H) = \mp\gamma_\mp(\Psi(H))$ for all $H \in \mathcal{H}$. For $H = \gamma_\pm^\star$, the latter relation reads as $\frac{1}{2}(\gamma_\pm, \gamma_\pm) = \mp(\gamma_\mp^\star, \Psi(\gamma_\pm^\star))$. The symmetry of $\Psi|_{\mathcal{H}}$ then implies $(\gamma_+, \gamma_+) = -(\gamma_-, \gamma_-)$. But this contradicts the positive definiteness of $(,)|_{(\text{real span of roots})}$.♠

Let $\pm\Gamma^{\pm} \subset \mathcal{R}_{\pm} \cap (\pm\Delta)$ be the subsets of simple roots such that $\Psi(\mathcal{G}^{\alpha}) \neq 0$ if $\alpha \in \pm\Gamma^{\pm}$. Note that $\Gamma^{+} \cap \Gamma^{-} = \emptyset$. Lemma 4.6 then implies that there exist bijections $\sigma^{\pm} : \Gamma^{\pm} \longrightarrow \Gamma^{\pm}$, with $(\sigma^{\pm})^2 = Id_{\Gamma^{\pm}}$, such that the restriction of Ψ to $\mathcal{N}^{+} \oplus \mathcal{N}^{-}$ is given by (4.2) and (4.5), where, by symmetry (use $(Z_{\gamma}, Z_{-\gamma}) = \frac{2}{(\gamma,\gamma)}$), $\lambda_{\alpha_{\pm}} \cdot (\alpha_{\pm}, \alpha_{\pm}) = \lambda_{\sigma^{\pm}(\alpha_{\pm})} \cdot (\sigma^{\pm}(\alpha_{\pm}), \sigma^{\pm}(\alpha_{\pm}))$, $\alpha_{\pm} \in \Gamma^{\pm}$. Furthermore, since $\Psi(\mathcal{H}) \subset \mathcal{H}$ and $\Gamma^{+} \cap \Gamma^{-} = \emptyset$ (4.8) imposes no further restrictions.

Let us now consider the compatibility of the remaining conditions (4.7) with the symmetry

$$(\Psi(H), H') = (H, \Psi(H')), \quad H, H' \in \mathcal{H}. \tag{4.14}$$

Lemma 4.7

The system of equations (4.7) and (4.14) is compatible if and only if $(\Gamma^{+}, \Gamma^{-}, \sigma^{+}, \sigma^{-})$ is an admissible quadruple.

Proof:

Set $r_{\alpha,\beta} = (\alpha^{\star}, \Psi(\beta^{\star}))$, $\alpha, \beta \in \Delta$. (4.7) and (4.14) are then equivalent to the relations (use $(\sigma^{\pm})^2 = Id_{\Gamma^{\pm}}$)

1. $r_{\alpha_+,\gamma} = -\frac{1}{2}(\sigma^{+}(\alpha_+), \gamma), \quad \alpha_+ \in \Gamma^{+}, \gamma \in \Delta,$
2. $r_{\alpha_-,\gamma} = \frac{1}{2}(\sigma^{-}(\alpha_-), \gamma), \quad \alpha_- \in \Gamma^{-}, \gamma \in \Delta,$
3. $r_{\alpha,\beta} = r_{\beta,\alpha}, \quad \alpha, \beta \in \Delta.$

If $\alpha_+, \gamma_+ \in \Gamma^{+}$ (respectively $\alpha_-, \gamma_- \in \Gamma^{-}$), 1. (resp. 2.) is compatible with 3. iff σ^{+} (resp. σ^{-}) is orthogonal. Now, if $\alpha_+ \in \Gamma^{+}$ and $\gamma_- \in \Gamma^{-}$, the compatibility of 1., 2., and 3. is equivalent to $(\sigma^{+}(\alpha_+), \sigma^{-}(\gamma_-)) = -(\alpha_+, \gamma_-)$. But it is known [H 80, Sect.10.1 p.47] that if $\alpha, \beta \in \Delta$, then $(\alpha, \beta) \leq 0$ for $\alpha \neq \beta$. Thus $(\alpha_+, \gamma_-) = 0$.♠

To conclude the proof of Proposition 4.1 b it suffices to note that for $\sigma^{\pm}$ orthogonal, 1. and 2. in the latter proof are equivalent to (4.3), while (4.9) implies (4.4) where $\Psi|_{\mathcal{H}^{\pm}_{\Gamma}}$ is symmetric.♠

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Appendix.

In this appendix, we sketch how one may adapt an argument of Semenov-Tian-Shansky in [S-T-S 85] to provide a geometric description of the statement c of Theorem 3.2.

Let J, $R \in \operatorname{End} \mathcal{G}$ be two solutions of (**MCYB**) with J skew-symmetric w.r.t. $(\,,\,)$. Let $\tau \in \operatorname{Aut} G$ be an automorphism of G whose induced map on $\mathcal{G}$ (denoted by the same letter) commutes with J and R. Let $\delta = \mathcal{G} \oplus \mathcal{G}$, and define $\widehat{\tau} \in \operatorname{Aut} \delta$ by $\widehat{\tau}(\zeta, \eta) = (\zeta, \tau(\eta))$, $\zeta, \eta \in \mathcal{G}$. The map ${}^{\tau}J_\delta := \widehat{\tau} \circ J_\delta \circ \widehat{\tau}^{-1}$ is then a solution of (**MCYB**) which is skew-symmetric w.r.t. the pairing $\langle\,,\,\rangle$ (see (2.3)). The Lie bialgebra $(\delta_{{}^{\tau}J_\delta}, \delta)$ is called the twisted square of $(\mathcal{G}_J, \mathcal{G})$. Let $D = G \times G$. We embed G_R into $D \times D$ by

$$i_l : G_R \hookrightarrow D \times D,\ g \longmapsto (g_+, \tau(g_+), g_-, g_-), \quad g_\pm = \widehat{R_\pm}(g),$$

and embed $\mathcal{G}_R$ into $\delta \oplus \delta$ via $T_e i_l$. We denote by $\mathcal{G}_R^\perp \subset \delta \oplus \delta$ the orthogonal complement of $\mathcal{G}_R$ w.r.t. the pairing $\langle\langle\,,\,\rangle\rangle$ (see the proof of Lemma 3.3).

Lemma A.1

$\mathcal{G}_R^\perp \subset \delta_{{}^{\tau}J_\delta} \oplus \delta_{J_\delta}$ is a Lie subalgebra if and only if

$$R_+^\star\big([R_-^\star(\zeta), R_-^\star(\eta)]_J\big) = -R_-^\star\big([R_+^\star(\zeta), R_+^\star(\eta)]_J\big).$$

The proof of this lemma is similar to that of Lemma 3.3; we shall leave it to the reader.♠

Remark

Note that a sign mistake was made in a similar statement in [LP 89, p.553, Remark b].

Now, consider the left Poisson action

$$L : \big(D_{({}^{\tau}J_\delta, -{}^{\tau}J_\delta)} \times D_{J_\delta, -J_\delta)}\big) \times D_{({}^{\tau}J_\delta, J_\delta)} \longrightarrow D_{({}^{\tau}J_\delta, J_\delta)}$$

$$\big((\widehat{g_1}, \widehat{g_2}), \widehat{x}\big) \longmapsto \widehat{g_1} \cdot \widehat{x} \cdot \widehat{g_1}^{-1},$$

and let l be the action of G_R on $D_{({}^{\tau}J_\delta, J_\delta)}$ obtained by restricting L to $i_l(G_R)$. By Proposition 2.1, if the condition of Lemma A.1 is satisfied, the action l is admissible so that there exists a unique Poisson structure $\{\,,\,\}_{red.}$ on the quotient $G_R \backslash D$ such that the projection $\pi : D_{({}^{\tau}J_\delta, J_\delta)} \longrightarrow G_R \backslash D \simeq G$, $(x, y) \longmapsto \tau^{-1}(y_+^{-1}) x\, y_-$, is a Poisson map (here $y_\pm$ are the factors of the decomposition (2.2) relative to R). To specialize to Theorem 3.2, we take $R = R_{(\Gamma, C)}$ and $J = J^- + A_C$. It is then easy to see that, for this choice, the l.h.s. and the r.h.s. of the condition in lemma 4.1 are both zero. A somewhat lengthy computation then yields

$$\{\,,\,\}_{red.} = \{\,,\,\}_{(\Gamma, C, \tau)} + \{\,,\,\}_{\mathcal{H}_\Gamma^\perp},$$

where the perturbation reads as

$$\begin{aligned}\{\varphi, \psi\}_{\mathcal{H}_\Gamma^\perp} = &(P(C)(D\varphi), D\psi) + (P(C)(D'\varphi), D'\psi) \\ &- (P(C) \circ \tau(D\varphi), D'\psi) - (P(C) \circ \tau^{-1}(D'\varphi), D\psi).\end{aligned}$$

Here, $P(C) \in \operatorname{End} \mathcal{H}_\Gamma^\perp$ is a cubic polynomial in C which vanishes when C is symmetric. Note that such a perturbation of the Poisson bracket does not change the Hamiltonian vector fields of twisted invariant functions. The rest of the argument of Semenov-Tian-Shansky in [S-T-S 85] remains unchanged and we shall not repeat it here.

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