KÜNNETH FORMULA IN CYCLIC HOMOLOGY

by

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SFB/MPI 85-49

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0. INTRODUCTION:

The cyclic homology $HC_*(A)$ of an associative algebra with identity (unital algebra) A over a field of characteristic zero k was introduced by A. Connes [C] (see also [L,Q]). It comes equipped with a natural degree (-2) k-linear map $S : HC_*(A) \to HC_{*-2}(A)$, which provides $HC_*(A)$ with a $k[u]$ co-module structure, where $HC_*(k) = k[u]$ is the polynomial algebra in the variable u of degree 2 regarded as a coalgebra (see section 3). The purpose of this paper is to prove the following theorem

<u>THEOREM A</u>: 1) <u>Given two unital k-algebras A and B, there exists a short exact sequence</u>

$$0 \to \Sigma\, \mathrm{Cotor}(HC_*(A), HC_*(B)) \xrightarrow{\psi} HC_*(A \otimes B) \xrightarrow{\phi} HC_*(A) \underset{k[u]}{\square} HC_*(B) \to 0$$

<u>natural in A and B, where $\square$ denotes the cotensor product.</u>

2) *If* $HC_*(B)$ *is a quasi-free comodule* [+] *i.e.* $HC_*(B) = k[u] \otimes V_* + W_*$, *then* $HC_*(A \otimes B) = HC_*(A) \otimes W_* + HH_*(A) \otimes V_*$.

As an application we have the following calculation of the cyclic homology of the polynomial algebra $A[t]$ resp. Laurent polynomial algebra $A[t,t^{-1}]$.

COROLLARY B: 1) *If* $A[t]$ *denotes the polynomial algebra with coefficients in* A *then*

$$HC_*(A[t]) = HC_*(A) + \bigoplus_{\alpha \in N} (HH_*(A))_\alpha$$

with N *denoting the natural numbers*, $HH_*(A)_\alpha$ *being a copy of* $HH_*(A)$;

2) *If* $A[t,t^{-1}]$ *denotes the algebra of Laurent polynomials with coefficients in* A *then* $HC_*(A[t,t^{-1}]) = HC_*(A) + HC_{*-1}(A) + \text{Nill}\ HC_*(A)$ *with* $\text{Nill}\ HC_*(A) = \bigoplus_{\alpha \in Z \setminus \{0\}} (HH_*(A))_\alpha$. *This can be re-written as*

3) $HC_*(A[t,t^{-1}]) = HC_*(A[t]) + HC_{*-1}(A) + \text{Nill}\_HC_*(A)$ *with* $\text{Nill}\_HC_*(A) = \bigoplus_{\alpha \in Z \setminus \{0 \cup N\}} (HH_*(A))_\alpha$ *where* $\text{Nill}_+HC_*(A) = \bigoplus_{\alpha \in N} (HH_*(A))_\alpha$, *and* $\text{Nill}\ HC_*(A) = \text{Nill}_+HC_*(A) + \text{Nill}\_HC_*(A)$.

[+] See Section 3 for definition.

The above theorem has a corresponding generalization to differential graded algebras with differential of degree $= 1$. Nill $HC_*(A)$ has interesting geometric applications. Note the above corollary is also verified in $[B]_2$ for $A = k[G]$ a group ring.

In the particular cases of A and B group rings, both Theorem A and Corollary have been verified in $[B]_2$. Theorem A was conjectured by Burghelea and Karoubi in May, 1984 and both of them have provided proofs through different arguments. A subsequent proof was given by C. Ogle [O].(+

The results of this paper have been announced in Oberwolfach, August 1984.

This paper is a substitute for $[B]_3$ and [O], and being shorter than both of them better suitted for publication. The arguments of $[B]_3$ permit stronger conclusions (in particular the fact that ψ resp. ϕ in Theorem A identify to the Loday Quillen product [L,Q] resp. the dual of Connes product in cyclic cohomology), but they are less conceptual and more complicated.

The paper is organized as follows: In section I we review the concept of algebraic S^1-chain complex introduced in $[B]_1$ and describe the "tensor product" of two algebraic S^1-chain complexes. In section II we prove the Künneth formula for the tensor product of two algebraic S^1-chain complexes. In

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(+ C. Kassel K and J. Jones & C. Hood have also announced the Künneth formula for cyclic homology of algebras.

section III we use "acyclic models" to show that Hochschild and cyclic homology of the algebraic S^1-chain complex associated with the tensor product of two cyclic R-modules is the same as of the tensor product of the associated algebraic S^1-chain complexes. In section IV we derive Theorem A and Corollary B.

SECTION 1:

Let R be a commutative ring with unit. An algebraic S^1-chain complex (a chain complex equipped with an algebraic circle action) $\tilde{C} = (C_*, d_*, \beta_*)$ consists of the chain complex of R-modules (C_*, d_*), $d_* : C_n \to C_{n-1}$ satisfying $d_{n+1} d_n = 0$, with the algebraic circle action β_* given by R-linear maps $\beta_* : C_n \to C_{n+1}$ which satisfy $\beta_{n+1}\beta_n = 0$, $d_{n+1}\beta_n + \beta_{n-1}d_n = 0$.

A morphism of algebraic S^1-chain complexes $f_* : (C_*, d_*, \beta_*) \to (C'_*, d'_*, \beta'_*)$ consists of R-linear maps $f_n : C_n \to C'_n$ which commute with the d's and β's.

To an algebraic S^1-chain complex (C_*, d_*, β_*) one can associate the chain complex $({}_\beta C_*, {}_\beta d_*)$ with ${}_\beta C_n = C_n + C_{n-2} + \dots$, ${}_\beta d_n(x_n, x_{n-2}, \dots) = (dx_n + \beta x_{n-2}, dx_{n-2} + \beta x_{n-4}, \dots)$ and the following short exact sequence of chain complexes

$$(*) \qquad 0 \to (C_*, d_*) \xrightarrow{I} ({}_\beta C_*, {}_\beta d_*) \xrightarrow{\pi} \Sigma^2({}_\beta C_*, {}_\beta d_*) \to 0 .$$

Here I is the inclusion $I(x_n) = (x_n, 0, \dots 0)$, Σ denotes the suspension $\Sigma(C_*, d_*) = (B_*, d'_*)$ with $B_{n+1} = C_n, B_0 = 0, d'_{n+1} = d_n$, and π is the projection $\pi(x_n, x_{n-2}, \ldots) = (x_{n-2}, x_{n-4}, \ldots)$

The homology groups $H_*(C_*, d_*)$, resp. $H_*({}_\beta C_*, {}_\beta d_*)$ are by definition the Hochschild resp. cyclic or equivariant homology of $\tilde{C} \equiv (C_*, d_*, \beta_*)$. The long exact homology sequence associated with the short exact sequence (*) becomes, with the above notation:

$$(**) \qquad \longrightarrow HH_*(\tilde{C}) \xrightarrow{I} HC_*(\tilde{C}) \xrightarrow{S} HC_{*-2}(\tilde{C}) \longrightarrow HH_{*-1}(\tilde{C}) \longrightarrow$$

and will be called the Gysin-Connes exact sequence. Obviously a morphism of algebraic S^1-chain complexes $f : \tilde{C} \to \tilde{C}'$ provides a commutative diagram

$$(***) \qquad \begin{array}{ccccccccc} \to & HH_*(\tilde{C}) & \to & HC_*(\tilde{C}) & \to & HC_{*-2}(\tilde{C}) & \to & HH_{*-1}(\tilde{C}) & \to \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \to & HH_*(\tilde{C}') & \to & HC_*(\tilde{C}') & \to & HC_{*-2}(\tilde{C}') & \to & HH_{*-1}(\tilde{C}') & \to \end{array} .$$

Given two algebraic S^1-chain complexes $\tilde{C}'$ and $\tilde{C}''$ one defines the tensor product $\tilde{C}' \otimes \tilde{C}''$ as being the chain complex $(C'_* \otimes C''_*, D_*)$ with

$$(C' \otimes C'')_n = \bigoplus_{k=0}^{n} C'_k \otimes C''_{n-k} , \; D_n(x_k \otimes y_{n-k}) = d'x_k \otimes y_{n-k} + (-1)^k x_k \otimes d''y_{n-k} ,$$

equipped with the algebraic circle action $\overline{\beta}_*$,

$$\overline{\beta}_n(x_k \otimes y_{n-k}) = \beta' x_k \otimes y_{n-k} + (-1)^k x_k \otimes \beta'' y_{n-k} .$$

We denote by chains_R (resp. $S^1\text{-chains}_R$) the category of chain complexes resp. algebraic S^1-chain complexes of R-modules and by F , T : $S^1\text{-chains}_R \rightsquigarrow \text{chains}_R$

the functors which associate with (C_*,d_*,β_*) the chain complexes (C_*,d_*) resp. $({}_\beta C_*,{}_\beta d_*)$.

SECTION II:

Let k be a field of characteristic zero and let $k[u]$ be the graded commutative algebra generated by of degree 2 . $k[u]$ can be also viewed as a co-commutative coalgebra with commultiplication $\Delta : k[u] \to k[u] \otimes k[u]$ given by $\Delta(u^p) = \Sigma u^i \otimes u^{p-i}$ and co-unit given by $\varepsilon(u^i) = \begin{cases} 0 & \text{if } i > 0 \\ 1 & \text{if } i = 0 \end{cases}$. A $k[u]$-comodule is a graded vector space M_* equipped with the k-linear map $\Delta_M : M_* \to k[u] \otimes M_*$ which satisfies the expected axioms. These axioms imply that $\Delta_M(m) = m + u \otimes S(m) + u^2 \otimes S^2(m) + \dots$, where S is a degree -2 k-linear map of M_* . Conversely, any $S : M_* \to M_{*-2}$ provides a $k[u]$-comodule structure on M_* , hence the $k[u]$-comodule structures on a graded vector space M_* identify to the k-linear maps of degree -2 .

EXAMPLE: 1) Suppose V_* is a k-graded vector space. Then $V_* \otimes k[u]$ is equipped with a canonical $k[u]$-comodule structure given by $S(x \otimes u^n) = x \otimes u^{n-1}$ and $S(x) = 0$. This is called the free $k[u]$-comodule of base V_* . A $k[u]$-comodule M_* is free iff $S : M_* \to M_{*-2}$ is surjective in which case a base is provided by $\ker S$.

2) Suppose V_* is a k-graded vector space and $S = 0$. The k[u]-comodule structure given by this S is called the trivial structure.

DEFINITION 2.1: A k[u]-comodule M_* is called quasifree if M_* is the direct sum $M'_* + M''_*$ of two k[u]-comodules ($S_{M_*} = S_{M'_*} + S_{M''_*}$) with M'_* free ($S_{M'_*}$ surjective) and M''_* trivial ($S_{M_*} = 0$).

Given two k[u]-comodules M_* and N_* one defines the graded vector space $M_* \square_{k[u]} N_*$ and $\Sigma^2 \mathrm{Coker}_{k[u]}(M_*, N_*)$ as the kernel resp. cokernel of the linear map $D : M_* \otimes N_* \to \Sigma^2 (M_* \otimes N_*)$ given by $D(m \otimes n) = S_M(m) \otimes n - m \otimes S_N(n)$; S_M and S_N are the degree (-2) - linear maps which define the k[u]-comodule structures of M_* and N_* and $\Sigma^n K_*$ denotes the n-fold suspension of K_*.

If $\tilde{C} = (C_*, d_*, \beta_*)$ is an algebraic S^1-chain complex, then $HC_*(\tilde{C})$ has a k[u]-comodule structure induced by $S : HC_*(\tilde{C}) \to HC_{*-2}(\tilde{C})$.

PROPOSITION 2.2: If $\tilde{C}' = (C'_*, d'_*, \beta'_*)$ and $\tilde{C}'' = (C''_*, d''_*, \beta''_*)$ are two algebraic S^1-chain complexes then there exists a (natural) short exact sequence

$$0 \to \Sigma\, \mathrm{Coker}_{k[u]}(HC_*(\tilde{C}'), HC_*(\tilde{C}'')) \xrightarrow{\psi} HC_*(\tilde{C}' \otimes \tilde{C}'') \xrightarrow{\phi} HC_*(\tilde{C}') \underset{k[u]}{\square} HC_*(\tilde{C}'') \to 0 .$$

<u>If moreover</u> $HC_*(\tilde{C}")$ <u>is quasifree and</u> $HC_*(\tilde{C}") = V_* \otimes k[u] + W_*$ <u>where</u> $V_* \otimes k[u]$ <u>is the free part and</u> W_* <u>the trivial part, then</u>

$$HC_*(\tilde{C}' \otimes \tilde{C}") = HC_*(\tilde{C}') \otimes V_* + H_*(C",d_*") \otimes W_* \quad .$$

<u>PROOF OF PROPOSITION 2.2</u>: Note that if (C_*,d_*,β_*) is an algebraic S^1-chain complex, then the chain complex $({}_\beta C_*,{}_\beta d_*)$ is a chain complex of free k[u]-comodules with ${}_\beta d_*$ being a morphism of k[u]-comodules. If (C_*',d_*',β_*') and $(C_*",d_*",\beta_*")$ are two algebraic S^1-chain complexes, we have the following short exact sequence of chain complexes

$$(*) \qquad 0 \longrightarrow {}_{\tilde{\beta}}(C_*' \otimes C_*") \xrightarrow{I} {}_{\beta'}C_*' \otimes {}_{\beta"}C_*" \longrightarrow \Sigma^2({}_{\beta'}C_* \otimes {}_{\beta"}C_*") \longrightarrow 0 \quad .$$

The differential δ in ${}_\beta C_*' \otimes {}_\beta C_*"$ is given by the tensor product differential, D is defined by $D(\overline{x} \otimes \overline{y}) = S'\overline{x} \otimes \overline{y} - \overline{x} \otimes S"\overline{y}$, $\overline{x} \in {}_\beta C_*', \overline{y} \in {}_\beta C_*"$ with S' resp. $S"$ defining the k[u]-comodule structure of ${}_\beta C_*'$ resp. ${}_\beta C_*"$ and I as follows. We formally write $\overline{x} = (x_n,x_{n-2},x_{n-4},\dots) \in {}_{\beta'}C_n'$ as $\overline{x} = \Sigma x_{n-2k}u^k$, $\overline{y} = (y_r,y_{r-2},y_{r-4},\dots) \in {}_{\beta"}C_r$ as $\overline{y} = \Sigma y_{r-2k}v^k$ and $\overline{z} = (z_s,z_{s-2},z_{2-4},\dots) \in {}_{\tilde{\beta}}(C_*' \otimes C_*")$ as $\overline{z} = \Sigma z_{s-2k}U^k$; then I is given by $I(x_m \otimes y_n U^r) = \sum_{\ell=0}^{r} (x_m u^\ell) \otimes (y_n v^{r-\ell})$. The reader can easily check the exactness of this sequence.

Moreover, if one equippes ${}_{\beta'}C'_* \otimes {}_{\beta''}C''$ with the degree -2 morphism of chain complexes $\mathcal{S} = S \otimes id + id \otimes S$, then ${}_{\beta'}C'_* \otimes {}_{\beta''}C''_*$ is a chain complex of k[u]-comodules and both I and D are morphisms of chain complexes of k[u]-comodules. Since $H_*({}_{\beta'}C'_* \otimes {}_{\beta''}C''_*) = HC_*(\tilde{C}') \otimes HC_*(\tilde{C}'')$ and $H_*(D) = S_{HC_*(\tilde{C}')} \otimes id - id \otimes S_{HC_*(\tilde{C}'')}$ the long exact sequence for homology induced by (*) is

$$\longrightarrow \Sigma^3(HC_*(\tilde{C}'_*) \otimes HC_*(\tilde{C}''_*)) \to HC_*(\tilde{C}'_* \otimes \tilde{C}''_*) \to HC_*(\tilde{C}'_*) \otimes HC_*(\tilde{C}''_*) \xrightarrow{S \otimes id - id \otimes S}$$

$$\longrightarrow \Sigma^2(HC_*(\tilde{C}'_*) \otimes HC_*(\tilde{C}''_*)) \to \dots ,$$

which clearly provides the following short exact sequence

$$0 \to \Sigma \operatorname{Coker}(HC_*(\tilde{C}'_*), HC_*(\tilde{C}''_*)) \to HC_*(\tilde{C}'_* \otimes \tilde{C}''_*) \to HC_*(\tilde{C}'_*) \mathop{\square}_{k[u]} HC_*(\tilde{C}''_*) \to 0$$

or equivalently $HC_*(\tilde{C}'_* \otimes \tilde{C}''_*) = \operatorname{Ker} D + \operatorname{Coker} \Sigma D$.

Suppose now that $HC_*(\tilde{C}'') = k[u] \otimes V_* + W_*$ is quasifree. Then $D : HC_*(\tilde{C}') \otimes HC_*(\tilde{C}'') \to HC_*(\tilde{C}') \otimes HC_*(\tilde{C}'')$ is $D_1 + D_2$ with $D_1 : \tilde{HC}_*(C') \otimes k[u] + V_* \to HC_*(\tilde{C}') \otimes k[u] + V_*$ defined by $D_1(x \otimes u^n \otimes v) = Sx \otimes u^n \otimes v - x \otimes u^{n-1} \otimes v$ and $D_2 : HC_*(\tilde{C}) \otimes W_* \to HC_*(\tilde{C}') \otimes W_*$ defined by $D_2(x \otimes w) = Sx \otimes w$. Clearly $\operatorname{Coker} D_1 = 0$, $\operatorname{Ker} D_1 = HC_*(\tilde{C}') \otimes V_*$. The Gysin Connes exact sequence tensored by W_* gives the exact sequence

$$\longrightarrow \Sigma^{-2} HC_*(\widetilde{C}') \otimes W_*) \xrightarrow{\Sigma^{-1} D_2} H\widetilde{C}_*(C') \otimes W_* \xrightarrow{\Sigma B \otimes id} HH_*(\widetilde{C}') \otimes W_* \xrightarrow{I \otimes id}$$

$$\longrightarrow \Sigma^{-1} HC_*(\widetilde{C}') \otimes W_* \xrightarrow{D_2} \Sigma\, HC_*(\widetilde{C}') \otimes W_* \longrightarrow \dots \quad .$$

This implies $HH_*(\widetilde{C}') \otimes W_* = \text{Coker}\, \Sigma^{-1} D_2 + \text{Ker}\, D_2$, which implies that

$$\text{Ker}\, D + \text{Coker}\, \Sigma D = HC_*(\widetilde{C}') \otimes V_* + HH_*(\widetilde{C}') \otimes W_* \quad .$$

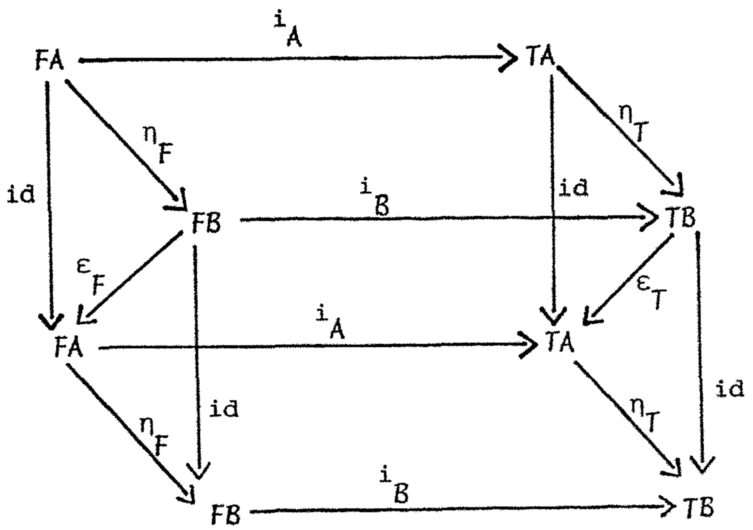
SECTION III:

We recall that a cyclic set (R-module) see [C] or [BF], (X_*, t_*) consists of a simplicial set (R-module) $X_* = (X_n, d_n^i, s_n^i;\ 0 \le i \le n)$ and a cyclic structure $t_* = (t_n : X_n \to X_n)$ which satisfies $t_n^{n+1} = id$, $t_{n-1} d_n^{i-1} = d_n^i t_n$, $t_n s_n^{i-1} = s_n^i t_n$ for $1 \le i \le n$. Let A_R resp. $\widetilde{A}_R$ denote the category of simplicial R-modules resp. cyclic R-modules (when there is no danger of confusion we will write A, $\widetilde{A}$, chains, S^1-chains instead of A_R, $\widetilde{A}_R$, chains$_R$, S^1-chains$_R$) .

As with A , $\widetilde{A}$ is equipped with an internal tensor product $(G_n, d_n^i, s_n^i, t_n) \otimes (G'_n, d'^i_n, s'^i_n, t'_n) = (G_n \otimes G'_n, d_n^i \otimes d'^i_n, s_n^i \otimes s'^i_n, t_n \otimes t'_n)$. With any cyclic R-module (G_n, d_n^i, s_n^i, t_n) one associates the S^1-algebraic chain complex $(G_n, d_n = \sum_{i=0}^{n} (-1)^i d_n^i, \beta_n = (-1)^n (1-(-1)^{n+1} t_{n+1}) s_n^n (1+(-1)^n t_n + \dots (-1)^{n^2} t_n^n)$ denote by $\widetilde{C}(C_*, t_*)$. The purpose of this section is to prove

that Hochschild resp. cyclic homology of $\tilde{C}(G_*,t_*) \otimes \tilde{C}(G'_*,t'_*)$ and $\tilde{C}(G_* \otimes G'_*,\ t_* \otimes t'_*)$ are naturally isomorphic. Prec[illegible]
$A, B : \tilde{A} \times \tilde{A} \rightsquigarrow S^1$-chains are the functors defined by
$A((G_*,t_*),(G'_*,t'_*)) = \tilde{C}(G_* \otimes G'_*, t_* \otimes t'_*)$, $B((G_*,t_*),(G'_*,t'_*)) =$
$= \tilde{C}(G_*,t_*) \otimes \tilde{C}(G'_*,t'_*)$ then we have

<u>THEOREM 3.1</u>: <u>There exists the diagram of functors and natural transformations which is naturally homotopy commutative</u>.



The proof will require the Theorem of acyclic models [M,p.128] which we will review below.

Let $\underline{\underline{C}}$ be a category and $M \subset \text{ob}\,\underline{\underline{C}}$ a set of objects called <u>models</u>. Given a covariant functor $L : \underline{\underline{C}} \longrightarrow Ab$, A = the category of abelian groups one can define a new covariant functor $\underline{L} : \underline{\underline{C}} \longrightarrow As$ and a natural transformation $\eta : \underline{L} \rightsquigarrow L$ by
$\underline{L}(K)$ = the free abelian group generated by
$X(K) = \bigcup_{M \in M}(\text{Hom}(M,K) \times L(M))$ for $K \in \text{ob}\,\underline{\underline{C}}$, $\underline{L}(f)(\alpha,u) = (f \circ \alpha, u)$
for $f \in \text{Hom}(K,L)$, $\alpha \in \text{Hom}(M,K)$ and $u \in L(M)$, with

$\eta^K : \underline{L}(K) \longrightarrow L(K)$ given by $\eta^K(\alpha,u) = A(\alpha)(u)$. The functor L is called <u>representable with respect to</u> M iff η admits a right inverse, i.e. a natural transformation $\phi^L : L \rightsquigarrow \underline{L}$ with $\eta \circ \phi = id$.

<u>THEOREM of acyclic models [M,p.128]</u>: <u>Let</u> $A,B : \underline{\underline{C}} \rightsquigarrow$ <u>chains</u> <u>be two covariant functors</u>, $f = \{f_i : (A)_i \longrightarrow (B)_i \ , \ 0 \le i \le n\}$ <u>a natural transformation of chain complex functors through dimension</u> n <u>and</u> M <u>a set of models in</u> $\underline{\underline{C}}$. <u>If</u> A_i <u>is representable for all</u> i , B(M) <u>is acyclic in dimension</u> $>n$ <u>for all</u> $M \in M$ <u>and</u> $f_n(\mathrm{Im}\, d^A_{n+1}) \subset \mathrm{Im}(d^B_{n+1})$ <u>then there exists a natural transformation</u> $f : A \rightsquigarrow B$ <u>extending</u> $\{f_i\}_{i \le n}$. <u>Moreover the extension</u> f <u>is unique up to all higher homotopies</u>.

<u>PROOF of Theorem 3.1</u>: We take η_F and ε_F as given by the "Alexander Whitney map" resp. "shuffle map"; $\eta_F \circ \varepsilon_F$ resp. $\eta_F \circ \varepsilon_F$ are naturally homotopic to the identity, see [M]pp. §§29. We also take $(\eta_T)_0 = id$ and $(\varepsilon_T)_0 = id$ and we will verify that all functors involving T and F are representable with respect to the class of models $M_\wedge \in ob(\tilde{A} \times \tilde{A})$ defined below. By verifying the acyclicity of TA and TB applied to the models we can use the Theorem of acyclic models as follows:

i) Take $A = TA$ and $B = TB$ resp. $A = TB$ and $B = TA$ to obtain the extensions η_T resp. ε_T .

ii) Take $A = B = TA = TB$ to obtain the natural homotopy between $\varepsilon_T \circ \eta_T$ and id resp. $\eta_T \circ \varepsilon_T$ and id .
iii) Take $A = FA$ and $B = TB$ resp. $A = FB$ and $A = TA$ to obtain the natural homotopy between $\eta_T \circ i_A$ and $i_B \circ \eta_T$ resp. $\varepsilon_T \circ i_B$ and $i_A \circ \varepsilon_F$.

<u>MODELS</u>: In [M]pp. 130, $M^p_* = (M^p_n, d^i_n, s^i_*)$ is defined to be the free simplicial R-module generated by the standard p-simplex $\Delta[p]$ $(\Delta[p]_n = \mathrm{Hom}_\Delta(\underline{n},\underline{p}))$ and $M = \{(M^p_*, M^r_*)\,|\,p,r \geq 0\} \subset \mathrm{ob}\ \mathbb{A} \times \mathbb{A}$ the set of models used to prove the standard Eilenberg Zilber theorem. In analogy let M^p_Λ be the free cyclic R-module generated by the cyclic set $\Lambda[p]$. By $\Lambda[p]$ we denote the "free" cyclic set generated by $\Delta[p]$ (see [B.F] definition 1.3). It follows from [BF] (Proposition 1.4 that the geometric realization of the underlying simplicial set $\Lambda[p]$ is homotopy equivalent to S^1 by an S^1-equivariant map. Let $M_\Lambda = \{(M^p_\Lambda, M^q_\Lambda)|\ p,q \geq 0\} \subset \mathrm{ob}\ \tilde{\mathbb{A}} \times \tilde{\mathbb{A}}$.

<u>REPRESENTABILITY</u>: $(TA)_n$ and $(TB)_n$ are direct sums of functors of type $(FA)_n = \tilde{A}_n$ resp. $(FB)_n = \tilde{B}_n$ so it suffices to check the representability for $\tilde{A}_n$ resp. $\tilde{B}_n$ in order to do it for $(TA)_n$ and $(TB)_n$. This is done as in [M] Lemma 2.9.1 by using the "free-ness" of our models. Precisely if $x_n \in K_n$ $K \in \mathrm{ob}\ \tilde{\mathbb{A}}$ it induces a simplicial map $\Delta[n] \xrightarrow{\hat{x}_n} K$ and then a cyclic map $\Lambda[n] \xrightarrow{\hat{\hat{x}}_n} K$. This induces the homomorphism of cyclic

R-modules $M^n_\wedge \xrightarrow{\overline{x}_n} K$. So if $(K,L) \in \text{ob}\,\tilde{A} \times \tilde{A}$

$\phi^{\tilde{A}_n} : \tilde{A}_n(K,L) = K_n \otimes L_n \longrightarrow \underline{A}_n(K,L)$ and

$\phi^{\tilde{B}_n} : \tilde{B}_n(K,L) = \sum_{r=0}^{r} K_r \otimes L_{n-r} \longrightarrow \underline{B}_n(K,L)$ are defined by the formulas $\phi^{\tilde{A}_n}(x_n \otimes y_n) = (\overline{x}_n \times \overline{y}_n , \lambda_n \otimes \lambda_n) \in \text{Hom}_{\tilde{A}\times\tilde{A}}(M^n_\wedge \times M^n_\wedge, K \times L) \times$ $\times A_n(M^n_\wedge, M^n_\wedge)$, $\phi^{\tilde{B}_n}(x_p \otimes x_{n-p}) = (\overline{x}_p \times \overline{x}_{n-p} , \lambda_p \otimes \lambda_n) \in$ $\in \text{Hom}_{\tilde{A}\times\tilde{A}}(M^p_\wedge \times M^{n-p}_\wedge ; K \times L) \times B_n(M^p_\wedge, M^{n-p}_\wedge)$ with λ_* the prefered generator of $M^n_\wedge$. It is straightforward to verify $\phi^{\tilde{A}_n}$ and $\phi^{\tilde{B}_n}$ are natural transformations inverse to $\eta^{\tilde{A}_n}$ and $\eta^{\tilde{B}_n}$.

<u>ACYCLICITY</u>: By definition $H_*(TA(M^n_\wedge, M^p_\wedge)) = HC_*(\tilde{C}_*(M^n_\wedge) \otimes \tilde{C}_*(M^p_\wedge))$ and $H_*(TB(M^n_\wedge, M^p_\wedge)) = HC_*(\tilde{C}_*(M^n_\wedge \otimes M^p_\wedge))$ ($M^n_\wedge \otimes M^p_\wedge$ in the free cyclic R-module generated by the cyclic set $\Lambda[n] \times \Lambda[p]$) . By [BF] section I $HC_*(\tilde{C}_*(M^n_\wedge)) = H_*(|\Delta[n]| ; R)$ and $HC_*(\tilde{C}_*(M^n_\wedge \otimes M^p_\wedge) = H_*(\Delta[n] \times \Delta[p] ; R)$ and $HH_*(\tilde{C}_*(M^n_\wedge)) = H_*(S^1;R)$. Combined with Proposition 2.2 one concludes that

$$H_*(TA(M^n_\wedge, M^p_\wedge)) = H_*(TB(M^n_\wedge, M^p_\wedge)) = \begin{cases} 0 & \text{if } * > 0 \\ R & \text{if } * = 0 \end{cases} .$$

Q.E.D.

SECTION IV:

PROOF of Theorem A: Given an R-algebra A the Hochschild resp. cyclic homology of A are calculated by the algebraic S^1-chain complex $(T_n(A), d_n^i, s_n^i, t_n)$ with $T_n(A) = \underbrace{A \otimes \ldots \otimes A}_{n+1}$

$$d_n^i(a_0 \otimes \ldots \otimes a_n) = \begin{cases} a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n & \text{if } i \leq n-1 \\ a_n a_0 \otimes \ldots \otimes a_i \otimes \ldots \otimes a_{n-1} & \text{if } i = n \end{cases}$$

$$s_n^i(a_0 \otimes \ldots \otimes a_n) = a_0 \otimes \ldots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \ldots \otimes a_n$$

$$t_n^A(a_0 \otimes \ldots \otimes a_n) = a_n \otimes a_0 \otimes \ldots \otimes a_{n-1} \quad .$$

Theorem 3.1 implies that Hochschild resp. cyclic homology of $\tilde{C}(T_*(A \otimes B), t_*^A)$, and of $\tilde{C}(T_*(A), t_*^A) \otimes \tilde{C}(T_*(B), t_*^B)$ are naturaly isomorphic. Theorem A follows then from Proposition 2.2.

PROOF of Corollary B: This follows from the calculation of the Hochschild resp. cyclic homology of $k[t]$ and $k[t,t^{-1}]$ given in [LQ] section 2. In both cases the cyclic homology is quasifree $k[u]$-comodule with the free part isomorphic to $k[t]$ resp. $k[t,t^{-1}]$ regarded as graded vector spaces concentrated in degree zero.

Q.E.D.

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