ANTICANONICAL MODELS OF FANO 3-FOLDS OF DEGREE FOUR

Ivan Cheltsov

Abstract. We study birational transformations of so-called birationally rigid Fano 3-folds into Fano 3-folds with canonical singularities, elliptic fibrations and fibrations on surfaces of Kodaira dimension zero. We descirbe all such maps for smooth quartic 3-folds, general quartic 3-folds with one simple double point, smooth double covers of quadric 3-fold ramified in octic.

§1. Introduction.

Let X be one of the following 3-folds: smooth quartic 3-fold in $\mathbb{P}^4$; general enough quartic 3-fold in $\mathbb{P}^4$ having one simple double point $O \in X$; double cover of a smooth quadric $Q \subset \mathbb{P}^4$ ramified in a smooth octic surface $S \subset Q$. Then X is a Fano 3-fold with terminal $\mathbb{Q}$-factorial singularities, $Pic(X) = \mathbb{Z}$ and $-K_X^3 = 4$. Moreover, anticanonical linear system $|-K_X|$ is free and morphism $\phi_{|-K_X|}$ is an embedding in $\mathbb{P}^4$ in case when X is a quartic 3-fold and a double cover of smooth quadric in $\mathbb{P}^4$ otherwise.

Definition 1.1. Fano variety V with $Pic(V) = \mathbb{Z}$ and terminal $\mathbb{Q}$-factorial singularities is called birationally rigid if V is not birationally equivalent to any fibration, whose fibers have Kodaira dimension $-\infty$, and to any other Fano variety with terminal $\mathbb{Q}$-factorial singularities and Picard group $\mathbb{Z}$.

Well known that Fano 3-fold X is birationally rigid (see [1-3]). Thus, X is not birationally isomorphic to any Fano 3-fold with $\mathbb{Q}$-factorial terminal singularities and Picard group $\mathbb{Z}$ different from X. However, apriori X can be birationally isomorphic to a Fano 3-fold with canonical and even with terminal singularities different from 3-fold X.

Example 1.2. Let X be a quartic 3-fold with a simple double point O and $\rho : V \to X$ be a blow up of O. Then for $n \gg 0$ linear system $|-nK_V|$ is free, map $\psi_O = \phi_{|-nK_V|} \circ \rho^{-1}$ is birational and $X_O = \psi_O(X)$ is a Fano 3-fold with terminal singularities and $-K_{X_O}^3 = 2$.

Author would like to thank V.Chekunov, V.A.Iskovskikh, S.Kasyanov, A.Pankratiev, A.V.Pukhlikov and V.V.Shokurov for useful conversations and Max-Planck-Institut für Mathematik for hospitality.

All varieties are assumed to be projective and defined over $\mathbb{C}$.

Note, that 3-fold X_O in Example 1.2 is a double cover of $\mathbb{P}^3$ ramified in sextic with 24 simple double points. Actually, there are other birational transformations of quartic 3-fold with a simple double point into the Fano 3-fold with terminal singularities.

PROPOSITION 1.3. *Let $X \subset \mathbb{P}^4$ be a general enough*[1] *quartic 3-fold with a simple double point O. Then for every line C on quartic X passing through singular point O there is birational map $\psi_C : X \dashrightarrow X_C$, such that X_C is a Fano 3-fold with terminal singularities and $-K^3_{X_C} = \frac{1}{2}$.*

PROPOSITION 1.4. *Let X be a double cover of a smooth quadric Q in $\mathbb{P}^4$ ramified in a smooth octic surface $S \subset Q$. Then for every curve C on X with $-K_X \cdot C = 1$ there is a birational map $\psi_C : X \dashrightarrow X_C$, where X_C is a Fano 3-fold with $-K^3_{X_C} = \frac{1}{2}$ having terminal singularities in case when the image of the curve C on the quadric Q is not contained in the ramification surface S and canonical singularities otherwise.*

We will explicitly prove Propositions 1.3 and 1.4 in chapter 4. The main purpose of this paper is to prove the following result.

MAIN THEOREM. *Let Y be a Fano 3-fold with canonical singularities which is birationally equivalent to X. Then $Y \cong X$ in case when X is a smooth quartic 3-fold,*

$$Y \cong \begin{cases} X, \\ X_O, \\ X_C \text{ for a line } C \text{ on } X \text{ containing point } O. \end{cases}$$

in case when X is a general enough quartic 3-fold in $\mathbb{P}^4$ with a simple double point O,

$$Y \cong \begin{cases} X, \\ X_C \text{ for a curve } C \text{ on } X \text{ with } -K_X \cdot C = 1. \end{cases}$$

in case when X is a double cover of a smooth quadric.

The group of birational automorphisms of 3-fold X was described in [1-3]. In particular, it was proved that there is a set $\mathcal{S}_X$ of non-biregular birational involutions of X, such that the sequence of groups

$$1 \to \prod_{\tau \in \mathcal{S}_X} \tau \to Bir(X) \to Aut(X) \to 1$$

is exact, the set $\mathcal{S}_X$ is empty in case when X is a smooth quartic 3-fold, the set $\mathcal{S}_X$ has exactly 25 elements in case when X is a general enough quartic 3-fold with a simple double point – one involution is induced by the singular point and the rest are induced by lines

[1] We assume that there are exactly 24 lines on X passing through singular point O.

on quartic X passing through the singular point, in case when X is a double cover of a smooth 3-fold quadric Q ramified in a smooth octic surface $S \subset Q$ all involutions in the set $\mathcal{S}_X$ are induced by curves C on X, such that their images on quadric Q are not contained in surface S and $-K_X \cdot C = 1$.

COROLLARY 1.5. *Involutions in set $\mathcal{S}_X$ is in one-to-one correspondance with the Fano 3-folds with terminal singularities, which are non-biregular but birational to X.*

PROPOSITION 1.6. *Let V be any birationally rigid Fano 3-fold and G be any finite subgroup in $Bir(V)$. Then there is a birational map $\gamma : V \dashrightarrow Y$, such that Y is a Fano 3-fold with terminal singularities and $\gamma \circ G \circ \gamma^{-1} \subset Aut(Y)$.*

Proof. Let $K(V)$ be a field of rational function on V, W be any normal projective model of field $K(V)^G$ and Y be a normalization of 3-fold W in $K(V)$. Then Y is birationally equivalent to V and group G acts biregularly on Y. The claim is follows now from the the equivariant resolution of singularities and Minimal Model Program. □

Thus, for every involution τ in $\mathcal{S}_X$ there is a birational map $\gamma : X \dashrightarrow Y$, such that Y is a Fano 3-fold with terminal singularities and $\gamma \circ \tau \circ \gamma^{-1} \subset Aut(Y)$. Vice versa, for every Fano 3-fold Y with terminal singularities birationally and not biregularly equivalent to 3-fold X there is an involution $\tau_Y \in Aut(Y)$ inducing some birational involution in $\mathcal{S}_X$.

Birational rigidity implies that X is not birationally equivalent to any fibration on varieties of Kodaira dimension $-\infty$. However, X is always birationally equivalent to the fibrations on varieties of Kodaira dimension zero.

Example 1.7. Let $L \subset \phi_{|-K_X|}(X)$ be a line and $\gamma : X \dashrightarrow \mathbb{P}^2$ be a composition of $\phi_{|-K_X|}$ and projection from L. Then the general fiber of γ is an elliptic curve and the resolution of indeterminacy of γ birationally transforms X into the elliptic fibration.

Example 1.8. Let $\gamma : X \dashrightarrow \mathbb{P}^1$ be a rational map given by some pencil in $|-K_X|$. Then the general fiber of γ is a K3 surface and the resolution of indeterminacy of γ gives birational transformation of X into the fibration on K3 surfaces.

Fibrations described in Examples 1.7 and 1.8 can be considered from one point of view: they are given by projections from linear subspaces in $\mathbb{P}^4$ restricted to the anticanonical image of X. So, we will simply call them "fibrations given by projection".

Results of papers [4-6] lead to the natural suggestion that up to the equivalence of fibrations and up to the action of group $Bir(X)$ the fibrations given by projections are the only fibrations on varieties with Kodaira dimension zero which are birationally equivalent to Fano 3-fold X. Actually, this statement was "proved" in paper [6] in case of smooth X, but the proof contains one gap that leads to the missing type of fibrations as was pointed to the author by V.A.Iskovskikh.

Definition 1.9. Smooth point $P \in X$ is called an Iskovskikh point if for some natural m linear system $|-mK_V|$ does not have fixed components, where V is a blow up of P.

PROPOSITION 1.10. *Let P be an Iskovskikh point on 3-fold X, $\rho : V \to X$ be a blow up of point P and m be a natural number, such that linear system $|-mK_V|$ does not have fixed components. Then linear system $|-mK_V|$ is a pencil and its general element is a union of surfaces of Kodaira dimension zero.*

Proof. Consider general element H in linear system $|-K_X|$ passing through point P and consider its proper transforms $\tilde{H}$ on V. Then for every $n \in \mathbb{N}$ exact sequence of sheaves

$$0 \to \mathcal{O}_V(-nK_V - \tilde{H}) \to \mathcal{O}_V(-nK_V) \to \mathcal{O}_{\tilde{H}}(-nK_V) \to 0$$

implies exact sequence of groups

$$H^0(\mathcal{O}_V(-nK_V - \tilde{H})) \to H^0(\mathcal{O}_V(-nK_V)) \to H^0(\mathcal{O}_{\tilde{H}}(-nK_V)).$$

From the equality $K_V^2 \cdot \tilde{H} = 0$ it follows that dimension of $H^0(\mathcal{O}_{\tilde{H}}(-nK_V))$ is bounded by some linear function of n for big enough natural number n. On the other hand, linear system $|-nK_V - \tilde{H}|$ is emtpy for every n, because otherwise we get contradiction with the assumption that linear system $|-mK_V|$ does not have fixed components. Thus, injection

$$0 \to H^0(\mathcal{O}_V(-nK_V)) \to H^0(\mathcal{O}_{\tilde{H}}(-nK_V))$$

implies that linears system $|-mK_V|$ is a pencil.

Suppose that m is the smallest natural number, such that linear system $|-mK_V|$ does not have fixed components. Let S_1 and S_2 be two general enough surfaces in $|-mK_V|$ and $\bar{S}_1$ and $\bar{S}_2$ be their images on X correspondingly. Then surfaces $\bar{S}_1$ and $\bar{S}_2$ are irreducible reduced and have multiplicity $2m$ at point P. To conclude the proof we must show that surfaces S_1 and S_2 have Kodaira dimension zero. Let $\mathcal{H}$ be a linear system of surfaces in anticanonical linear system $|-K_X|$ passing through point P and H be a general enough element of $\mathcal{H}$. Then from inequalities

$$4m^2 = H \cdot \bar{S}_1 \cdot \bar{S}_2 \geq mult_P(\bar{S}_1 \cdot \bar{S}_2) \geq mult_P(\bar{S}_1) mult_P(\bar{S}_2) = 4m^2$$

it follows that intersection $S_1 \cap S_2$ does not have curves in the exceptional divisor of ρ and all curves in $\bar{S}_1 \cap \bar{S}_2$ are contracted by rational map $\phi_{\mathcal{H}}$. Thus, in case when X is a quartic the intersection $\bar{S}_1 \cap \bar{S}_2$ is a union of all lines passing through P. In case when X is a double cover of quadric Q in $\mathbb{P}^4$ equality $H \cdot \bar{S}_1 \cdot \bar{S}_2 = mult_P(\bar{S}_1 \cdot \bar{S}_2)$ implies that P is the only base point of linear system $\phi_{\mathcal{H}}$ contained in $\bar{S}_1 \cap \bar{S}_2$. Therefore, in both cases strict transforms of all curves in intersection $\bar{S}_1 \cap \bar{S}_2$ on surfaces S_1 and S_2 are simultaneously regularly contractible. The subadjunction formula implies that canonical divisors of resolutions of singularities of surfaces S_1 and S_2 are regularly contractible. Therefore, Kodaira dimension of S_1 and S_2 is either zero or $-\infty$, but X is birationally rigid. Thus, the general surface in pencil $|-mK_V|$ has Kodaira dimension zero. □

Corollary 1.11. *Every Iskovskikh point on X gives a birational transformation of X into a fibration on surfaces of Kodaira dimension zero and from the classification theory of algebraic surfaces it follows that general fiber of a fibration given by an Iskovskikh point is either a K3 surface or Enriques surface.*

Example 1.12. Let X be a quartic in $\mathbb{P}^4$, such that the local equation of X in the affine neighbourhood of point P is $q_1+q_2+q_1q_2+q_4=0$, where q_i is a homogeneous polynomial of degree i and the system of equations $q_1 = q_2 = q_4 = 0$ has only finite number of homogeneous solutions. Then P is an Iskovskikh point and P gives a birational transform of quartic X into a fibration on K3 surfaces with one multiple fiber of multiplicity 2.

In this paper we will prove the following result.

Auxiliary Theorem. *Up to the natural equivalence of fibrations and up to the action of group $Bir(X)$ fibrations given by projections and Iskovskikh points are the only fibrations on varieties with Kodaira dimension zero, which are birationally equivalent to X.*

However, in case when X is a double cover of quadric fibrations given by Iskovskikh points are given by projections as well.

Lemma 1.13. *Let X be a smooth double cover of quadric 3-fold Q ramified in octic surface $S \subset Q$ and let P be a point on X. Then point P is an Iskovskikh point if and only if image of P on quadric Q is contained in the ramification surface S.*

Proof. Suppose, that the image of P on Q belongs to surface S. Let $\mathcal{H}_P$ be a linear system of surfaces in $|-K_X|$, whose image on quadric Q tangent to surface S at the image of point P. Then every surface in $\mathcal{H}_P$ is singular in P and $\mathcal{H}_P$ is a pencil, which does not have fixed components. Thereofore, P is an Iskovskikh point.

Suppose now that P is an Iskovkikh point. Let $\rho : V \to X$ be a blow up of P, surfaces S_1 and S_2 be general enough elements of linear system $|-mK_V|$ and $\bar{S}_1$ and $\bar{S}_2$ be their images on X correspondingly, where m is a smallest natural number, such that linear system $|-mK_V|$ does not have fixed components. Then surfaces $\bar{S}_1$ and $\bar{S}_2$ are irreducible reduced and have multiplicity $2m$ at point P.

Suppose that image of point P on quadric Q does not belong to surface S. Let $\mathcal{H}$ be a linear system consisting of all divisors in anticanonical linear system $|-K_X|$ passing through point P and H be a general enough element of $\mathcal{H}$. Then base locus of linear system $\phi_{\mathcal{H}}$ contains some point $\hat{P}$ different from point P and inequalities

$$4m^2 = H \cdot \bar{S}_1 \cdot \bar{S}_2 \geq mult_P(\bar{S}_1 \cdot \bar{S}_2) \geq mult_P(\bar{S}_1) mult_P(\bar{S}_2) = 4m^2$$

implies that $\hat{P} \notin \bar{S}_1 \cap \bar{S}_2$ and all curves in $\bar{S}_1 \cap \bar{S}_2$ are contracted by map $\phi_{\mathcal{H}}$. Thus, intersection $\bar{S}_1 \cap \bar{S}_2$ consists of finite number of curves C_λ, such that all curves C_λ contains point P and $-K_X \cdot C_\lambda = 1$. Let $C \subset \bar{S}_1 \cap \bar{S}_2$ be one of curves C_λ and $\hat{C}$ be a curve on 3-fold X passing through point $\hat{P}$ whose image on quadric Q coincide with the image of curve C. Then $-K_X \cdot C = -K_X \cdot \hat{C} = 1$. Now consider restriction of surface S_1 to the general smooth K3 surface D in $|-K_X|$ containing curves C and $\hat{C}$. We have

$$S_1|_D = mult_C(S_1)C + B,$$

where support of curve B does not contain curves C and $\hat{C}$, because otherwise S_1 contains curve $\hat{C}$ and surface S_2 contains curve $\hat{C}$ as well due to generality of surfaces S_1 and S_2, but $\hat{P} \notin \bar{S}_1 \cap \bar{S}_2$. On the other hand, $C \cdot \hat{C} = 4$ and $C^2 = -2$ on surface D and

$$mult_P(B) \geq mult_P(S_1|_D) - mult_C(S_1) \geq 2m - mult_C(S_1).$$

Therefore, intersecting curve B with curves C and $\hat{C}$ on surface D we get two inconsistent inequalities $m \geq 4mult_C(S_1)$ and $3mult_C(S_1) \geq m$. $\square$

COROLLARY 1.14. *Let X be a smooth double cover of quadric 3-fold Q ramified in octic surface $S \subset Q$, P be a Iskovskikh point on 3-fold X and $\hat{P} \in S$ be an image of point P on quadric Q. Then fibration given by point P is a fibration on K3 surfaces given by projection from two-dimensional linear subspace $T \subset \mathbb{P}^4$ tangent to surface S at point $\hat{P}$.*

§2. Movable log pairs.

Definition 2.1. Movable log pair $(X, M_X) = (X, \sum_{i=1}^n b_i \mathcal{M}_i)$ is a variety X together with a formal finite linear combination of linear systems $\mathcal{M}_i$ without fixed components, such that all $b_i \in \mathbb{Q}_{\geq 0}$.

Every movable log pair can be considered as a usual log pair. So, for a movable log pair (X, M_X) we will say that $K_X + M_X$ is a log canonical divisor and M_X is a boundary of movable log pair (X, M_X). Moreover, for any birational map $\rho : X \dashrightarrow Y$ the strict transform $M_Y = \rho(M_X)$ is well defined – we will say that (X, M_X) and (Y, M_Y) are birationally equivalent. We will assume that log canonical divisors of all considered log pairs are $\mathbb{Q}$-Cartier divisors. Therefore, discrepancies, terminality, canonicity, log terminality and log canonicity can be defined for movable log pairs in the similar way as for usual log pair (see [7]). Note, that every movable log pair is birationally equivalent to a log pair with canonical singularities.

Remark 2.2. Singularities of a movable log pair coincide with singularities of the variety outside of the base locuses of the components of the boundary.

Definition 2.3. Movable log pair (X, M_X) has semiterminal singularities if for some rational number $\lambda > 1$ movable log pair $(X, \lambda M_X)$ has canonical singularities.

Note, that semiterminality of singularities of movable log pair implies canonicity of singularities of variety. Moreover, for a variety with terminal singularities movable log pair on this variety has semiterminal singularities if and only if it has terminal singularities.

Definition 2.4. Proper irreducible subvariety $Y \subset X$ is called a center of canonical singularities of log pair (X, B_X), if there is a birational morphism $f : W \to X$ and such divisor $E \subset W$, that $a(X, B_X, E) \leq 0$ and $f(E) = Y$.

Definition 2.5. $CS(X, B_X)$ will denote a set of all centers of canonical singularities of log pair (X, B_X) and at the same time it will denote the locus of all centers of canonical singularities of log pair (X, B_X).

Definition 2.6. Number

$$\kappa(X, M_X) = \begin{cases} dim(\phi_{|n(K_W+M_W)|}(W)) \text{ for such } n \gg 0, \text{ that } |n(K_W + M_W)| \neq \emptyset, \\ -\infty \text{ when } |n(K_W + M_W)| = \emptyset \text{ for all natural numbers } n. \end{cases}$$

is called Kodaira dimension of movable log pair (X, M_X), where (W, M_W) is a movable log pair with canonical singularities, which is birationally equivalent to (X, M_X).

One can prove that Kodaira dimension of a movable log pair is well defined and does not depend on the choice of the log pair with canonical singularities. By definition Kodaira dimension of a movable log pair is a birational invariant. Moreover, the Kodaira dimension of a movable log pair is a non-decreasing function of the coefficients of the boundary.

Definition 2.7. Movable log pair (V, M_V) is called a canonical model of a movable log pair (X, M_X), if there is a birational map $\psi : X \dashrightarrow V$, such that $M_V = \psi(M_X)$, log canonical divisor $K_V + M_V$ is ample and (V, M_V) has canonical singularities.

THEOREM 2.8. *Canonical model of a movable log pair is unique if it exists.*

Proof. Suppose that both movable log pairs (X, M_X) and (V, M_V) are canonical models and $M_X = \rho(M_V)$ for some birational map $\rho : V \dashrightarrow X$. Let W be a variety and $g : W \to X$ and $f : W \to V$ be birational morphisms, such that $\rho = g \circ f^{-1}$. Then

$$K_W + M_W \sim_{\mathbb{Q}} g^*(K_X + M_X) + \Sigma_X \sim_{\mathbb{Q}} f^*(K_V + M_V) + \Sigma_V,$$

where $M_W = g^{-1}(M_X) = f^{-1}(M_V)$ and Σ_X and Σ_V are exceptional divisors of birational morphisms g and f correspondingly. Canonicity of log pairs (X, M_X) and (V, M_V) implies effectivity of divisors Σ_X and Σ_V. Let n be big enough and divisible enough natural number, such that divisors $n(K_W + M_W)$, $n(K_X + M_X)$ and $n(K_V + M_V)$ are Cartier divisors. Then effectivity of divisors Σ_X and Σ_V implies that

$$\phi_{|n(K_W+M_W)|} = \phi_{|g^*(n(K_X+M_X))|} = \phi_{|f^*(n(K_V+M_V))|}$$

and ρ is an isomorphism, because $K_X + M_X$ and $K_V + M_V$ are ample. $\square$

The existence of a canonical model of the movable log pair implies that its Kodaira dimension equals to the dimension of the variety.

Definition 2.9. A movable log pair (V, M_V) is said to be a minimal model of the movable log pair (X, M_X) if they are birationally equivalent, the divisor $K_V + M_V$ is nef and singularities of movable log pair (V, M_V) are terminal and $\mathbb{Q}$-factorial.

CONJECTURE 2.10. *Let (X, M_X) be a movable log pair. Then either log pair (X, M_X) has non-negative Kodaira dimension and (X, M_X) is birationally equivalent to minimal model, or $\kappa(X, M_X) = -\infty$ and the movable log pair (X, M_X) is birationally equivalent to a movable log pair (Y, M_Y) with terminal $\mathbb{Q}$-factorial singularities, such that the variety Y has the structure of a Mori fibration $f : Y \to Z$ and the divisor $-(K_Y + M_Y)$ is f-ample.*

One can check that Log Minimal Model Program implies Conjecture 2.10. In particular, Conjecture 2.10 holds for surfaces and 3-folds. Note, that Conjecture 2.10 implies that the canonical model of a movable log pair exists if and only if Kodaira dimension of this log pair equals to the dimension of the variety (see [7]).

Definition 2.11. Suppose that the movable log pair (X, M_X) is a minimal model. Then morphism $I(X, M_X) : X \to Z$ is called Iitaka morphism of movable log pair (X, M_X) if log canonical divisor $K_X + M_X$ is $\mathbb{Q}$-rationally equivalent to $I(X, M_X)^*(L)$ for some ample $\mathbb{Q}$-Cartier divisor L on the variety Z and $I(X, M_X)_*(\mathcal{O}_X) = \mathcal{O}_Z$.

CONJECTURE 2.12. *The Iitaka morphism for movable log pairs exists.*

Note, that Conjecture 2.12 is implied by Log Abundance Conjecture, which holds for surfaces and 3-folds (see [7]).

Definition 2.13. Suppose that the movable log pair (X, M_X) is birationally equivalent to the minimal model (V, M_V) by means of a birational map $\rho : X \dashrightarrow V$ and that the Iitaka map of the movable log pair (V, M_V) exists. Then the map $I(X, M_X) = I(V, M_V) \circ \rho$ is called the Iitaka map of the movable log pair (X, M_X).

One can check that Iitaka map of movable log pair is correctly defined.

§3. PRELIMINARY RESULTS.

Fix log pair $(X, B_X) = (X, \sum_{i=1}^{k} b_i B_i)$, such that all $b_i \in \mathbb{Q} \cap [0, 1]$ and log canonical divisor $K_X + B_X$ is $\mathbb{Q}$-Cartier.

Definition 3.1. Proper irreducible subvariety $Y \subset X$ is called a center of log canonical singularities of log pair (X, B_X), if there are such birational morphism $f : W \to X$ and divisor $E \subset W$, that $a(X, B_X, E) \leq -1$ and $f(E) = Y$.

Definition 3.2. $LCS(X, B_X)$ will denote a set of all centers of log canonical singularities of log pair (X, B_X).

Consider birational morphism $f : W \to X$, such that W is smooth and union of all strict transforms of divisors B_i on W and f-exceptional divisors forms divisor with simple normal crossing. Consider log pair (W, B^W), such that $K_W + B^W \sim_{\mathbb{Q}} f^*(K_X + B_X)$ and strict transform of B^W on X is B_X. Birational morphism f is usually called log resolution of log pair (X, B_X) and log pair (W, B^W) is usually called log pull back of (X, B_X).

Definition 3.3. Subscheme associated with ideal sheaf $\mathcal{I}(X, B_X) = f_*(\lceil -B^Y \rceil)$ is called log canonical singularities subscheme of log pair (X, B_X) and denoted as $\mathcal{L}(X, B_X)$.

Definition 3.4. Support of subscheme $\mathcal{L}(X, B_X)$ is called locus of log canonical singularities of log pair (X, B_X) and denoted as $LCS(X, B_X)$.

Note, that all centers of log canonical singularities of log pair (X, B_X) are contained in the locus $LCS(X, B_X)$ and there is a little ambiguity in considering $LCS(X, B_X)$ to be a subvariety and a set simultaniously.

Vanishing Theorem of V.V.Shokurov. *Suppose that for nef and big divisor H on variety X divisor $K_X + B_X + H$ is Cartier. Then $H^i(X, \mathcal{I}(X, B_X) \otimes D) = 0$ for $i > 0$.*

Proof. $R^i f_*(f^*(K_X + B_X + H) + \lceil -B^W \rceil) = 0$ for $i > 0$ by relative Kawamata-Viehweg vanishing. Degeneration of local-to-global spectral sequence and

$$R^0 f_*(f^*(K_X + B_X + H) + \lceil -B^W \rceil) = \mathcal{I}(X, B_X) \otimes D$$

imply that for all i

$$H^i(X, \mathcal{I}(X, B_X) \otimes D) = H^i(W, f^*(K_X + B_X + H) + \lceil -B^W \rceil),$$

but $H^i(W, f^*(K_X + B_X + H) + \lceil -B^W \rceil) = 0$ for $i > 0$ by Kawamata-Viehweg vanishing. □

For any Cartier divisor D on X we have the exact sequence

$$0 \to \mathcal{I}(X, B_X) \otimes D \to \mathcal{O}_X(D) \to \mathcal{O}_{\mathcal{L}(X, B_X)}(D) \to 0$$

and Vanishing Theorem of V.V.Shokurov implies the following three results.

Global Connectedness Theorem. *Suppose that divisor $-(K_X + B_X)$ is nef and big. Then locus $LCS(X, B_X)$ is connected.*

Relative Connectedness Theorem. *Suppose that divisor $-(K_X + B_X)$ is g-nef and g-big for some morphism $g : X \to Z$ with connected fibers. Then locus $LCS(X, B_X)$ is connected in the neighbourhood of any fiber of g.*

THEOREM 3.5. *Let $g : X \to Z$ be a morphism of normal varieties, $D_X = \sum_{i \in I} d_i D_i$ be a divisor on X, $h : V \to X$ be a resolution of singularities of X, such that $g_*(\mathcal{O}_X) = \mathcal{O}_Z$, divisor $-(K_X + D_X)$ is g-nef and g-big, codimension of every subvariety $g(D_i) \subset Z$ is greater or equal then 2 in case $d_i < 0$, union of all divisors $h^{-1}(D_i)$ and all h-exceptional divisors forms a divisor with simple normal crossing. Then locus $\cup_{a_E \leq -1} E$ is connected in a neighbourhood of every fiber of morphism $g \circ h$, where rational numbers a_E are given by $\mathbb{Q}$-rational equivalence $K_V \sim_{\mathbb{Q}} f^*(K_X + D_X) + \sum_{E \subset V} a_E E$.*

Consider another application of Vanishing Theorem of Shokurov.

LEMMA 3.6. *Let (V, B_V) be log pair, such that $V \cong \mathbb{P}^1 \times \mathbb{P}^1$ and effective divisor B_V has type (α, β) in $Pic(V) \otimes \mathbb{Q}$ for some α and β in $\mathbb{Q} \cap [0, 1)$. Then $LCS(V, B_V) = \emptyset$.*

Proof. Intersecting B_V with the rulings of V we conclude that set $LCS(V, B_V)$ does not contain curves. Suppose that $LCS(V, B_V)$ contains some point O. Let H be divisor on V, which has type $(1 - \alpha, 1 - \beta)$ in $Pic(V) \otimes \mathbb{Q}$. Then H is ample, divisor $D = K_V + B_V + H$ is Cartier and $H^0(\mathcal{O}_V(D)) = 0$. On the other hand, map

$$H^0(\mathcal{O}_V(D)) \to H^0(\mathcal{O}_{\mathcal{L}(V, B_V)}(D))$$

is surjective by Vanishing Theorem of V.V.Shokurov. □

THEOREM 3.7. *Let Z be a center of canonical singularities of (X, B_X) and H be an effective irreducible smooth Cartier divisor on X, such that $Z \subset H$ and H is not a component of B_X. Then $LCS(H, B_X|_H) \neq \emptyset$.*

Proof. Let $f : W \to X$ be log resolution of $(X, B_X + H)$ and $\hat{H} = f^{-1}(H)$. Then

$$K_W + \hat{H} \sim_{\mathbb{Q}} f^*(K_X + B_X + H) + \sum_{E \neq \hat{H}} a(X, B_X + H, E)E.$$

By assumption $\{Z, H\} \subset LCS(X, B_X + H)$. Therefore, application of Theorem 3.5 to log pulback of log pair $(X, B_X + H)$ on variety W implies that $\hat{H} \cap E \neq \emptyset$ for some divisor $E \neq \hat{H}$ on W, such that $a(X, B_X, E) \leq -1$. Now equivalences

$$K_{\hat{H}} \sim (K_W + \hat{H})|_{\hat{H}} \sim_{\mathbb{Q}} f|_{\hat{H}}^*(K_H + B_X|_H) + \sum_{E \neq \hat{H}} a(X, B_X + H, E)E|_{\hat{H}}$$

imply the claim. □

Next result is Theorem 3.11 of paper [8].

THEOREM 3.8. *Simple double point O of 3-fold X is a center of canonical singularities of log pair (X, B_X) if and only if inequality $mult_O(B_X) \geq 1$ holds.*

Proof. Let $f : W \to X$ be a blow up of O, $Q = f^{-1}(O)$ and $B_W = f^{-1}(B_X)$. Then

$$K_W + B_W \sim_{\mathbb{Q}} f^*(K_X + B_X) + (1 - mult_O(B_X))Q.$$

Therefore, in case when $mult_O(B_X) \geq 1$ point O is a center of canonical singularities of log pair (X, B_X). On the other hand, in case when $mult_O(B_X) < 1$ surface Q contains some element Z of set $CS(W, B_W)$ and Theorem 3.7 implies that $LCS(Q, B_W|_Q) \neq \emptyset$, which contradicts Lemma 3.6. □

Next result is a slightly modified version of Theorem 3.1 of paper [8].

LEMMA 3.9. *Let O be a smooth point of surface H and for some non-negative rational numbers a_1 and a_2 $O \in LCS(H, (1 - a_1)\Delta_1 + (1 - a_2)\Delta_2 + M_H)$, where boundary M_H is movable and irreducible reduced curves Δ_1 and Δ_2 intersect normally in point O. Then*

$$mult_O(M_H^2) \geq \begin{cases} 4a_1a_2 \text{ in case } a_1 \leq 1 \text{ or } a_2 \leq 1, \\ 4(a_1 + a_2 - 1) \text{ in case } a_1 > 1 \text{ and } a_2 > 1. \end{cases}$$

Note, that most applications use the following special case of Lemma 3.9.

LEMMA 3.10. *Let O be a smooth point on surface H and (H, M_H) be a movable log pair, such that $O \in LCS(H, M_H)$. Then $mult_O(M_H^2) \geq 4$ and equality implies $mult(M_H) = 2$.*

Next result is a theorem of V.A.Iskovskikh in the modern form of A.V.Pukhlikov.

THEOREM 3.11. *Let O be a smooth point on 3-fold X, such that O is a center of canonical singularities of movable log pair (X, M_X). Then $mult_O(M_X^2) \geq 4$ and equality implies $mult_O(M_X) = 2$.*

Proof. Let H be a general very ample divisor on X containing O. Then O is a center of log canonical singularities of log pair $(H, M_X|_H)$ by Theorem 3.7. On the other hand,

$$mult_O(M_X) = mult_O(M_X|_H) \text{ and } mult_O(M_X^2) = mult_O((M_X|_H)^2)$$

and the claim follows from Lemma 3.10. □

§4. Construction of two birational maps.

In this chapter we will prove Propositions 1.3 and 1.4.

Let X be a quartic 3-fold in $\mathbb{P}^4$ with one simple double point O, such that there are exactly 24 lines on X passing through point O. Then $Pic(X) = -\mathbb{Z}K_X$ and $-K_X$ is a hyperplane section of X. Fix some line C on X passing through O, blow up $f : W \to X$ point O and put $Q = f^{-1}(O)$. Blow up $g : V \to W$ proper transform of line C on W and put $E = g^{-1}(f^{-1}(C))$ and $Q_V = g^{-1}(Q)$.

Remark 4.1. Linear system $|-K_V|$ is free and morphism $\psi_{|-K_V|} : V \to \mathbb{P}^2$ is an elliptic fibration with a section Q_V.

The restriction of morphism $\psi_{|-K_V|}$ to Q_V is birational and contracts two smooth rational curves – C_1 and C_2. Consider nef and big divisor $(f \circ g)^*(-K_X) - K_V$. One may easily check that it has zero intersection only with curves C_1 and C_2.

COROLLARY 4.2. *There is a flop $\rho : V \dashrightarrow \hat{V}$ in curves C_1 and C_2.*

The restriction of ρ to the general element of anticanonical linear system $|-K_V|$ is an isomorphism. So, linear system $|-K_{\hat{V}}|$ is free and $\psi_{|-K_{\hat{V}}|} = \psi_{|-K_{\hat{V}}|} \circ \rho$. To obtain birational map ρ explicitly one just must blow up curves C_1 and C_2 and contract exceptional divisors in the "other direction". Unfortunately, explicit way of constructing ρ does not guaranty projectivity of $\hat{V}$.

Remark 4.3. $\rho(Q_V) \cong \mathbb{P}^2$ and its normal bundle is $\mathcal{O}_{\mathbb{P}^2}(-2)$.

Therefore, there is birational morphism $\hat{g} : \hat{V} \to \hat{W}$ contracting divisor $\rho(Q_V)$ to the quotient terminal singular point P of type $\frac{1}{2}(1,1,1)$.

PROPOSITION 4.4. $-K_{\hat{W}}^3 = \frac{1}{2}$ *and base locus of* $|-K_{\hat{W}}|$ *consists of point* P.

Proof. Freeness of $|-K_{\hat{V}}|$ implies that P is the only base point of anticanonical linear system $|-K_{\hat{W}}|$. Equality $-K_{\hat{W}}^3 = \frac{1}{2}$ follows from

$$0 = -K_{\hat{V}}^3 = (\hat{g}^*(-K_{\hat{W}}) - \frac{1}{2}\rho(Q_V))^3 = -K_{\hat{W}}^3 - \frac{1}{2}.$$

□

In particular, $-K_{\hat{W}}$ is nef and big. So, linear system $|-nK_{\hat{W}}|$ is free for $n \gg 0$ and gives birational morphism $\phi_{|-nK_{\hat{W}}|} : \hat{W} \to X_C$, where 3-fold X_C is normal.

COROLLARY 4.5. *X_C is Fano 3-fold with terminal singularities and* $-K_{X_C}^3 = \frac{1}{2}$.

So, we just proved Proposition 1.3 and now we will prove Proposition 1.4.

Let $\theta : X \to Q \subset \mathbb{P}^4$ be a double cover of a smooth 3-fold quadric Q, which is ramified in smooth octic surface S on quadric Q. Then $Pic(X) = -\mathbb{Z}K_X$ and divisor $-K_X$ is a pull back of hyperplane section of Q.

Fix irreducible reduced curve C on X, such that $-K_X \cdot C = 1$. Blow up $f : W \to X$ curve C, put $E = f^{-1}(C)$ and denote C_1 the unique base curve of linear system $|-K_W|$.

Remark 4.6. In case $\theta(C) \not\subset S$ we have $\theta(C) = \theta \circ f(C_1)$, otherwise C_1 is an exceptional curve of ruled surface $E \cong \mathbb{F}_5$ (see [2]).

Blow up $g : V \to W$ curve C_1 and put $G = g^{-1}(C_1)$. Linear system $|-K_V|$ is free, morphism $\phi_{|-K_V|}$ is an elliptic fibration and surface G is its section. Thus, $G \cong \mathbb{F}_1$.

LEMMA 4.7. *Let C_2 be an exceptional section of G. Then $N_{C_2/V} \cong \mathcal{O}_{C_2}(-1) \oplus \mathcal{O}_{C_2}(-1)$.*

Proof. Let $N_{C_2/V} \cong \mathcal{O}_{C_2}(m) \oplus \mathcal{O}_{C_2}(n)$ for $m \geq n$. Then $m+n = -2$ and exact sequence

$$0 \to N_{C_2/G} \to N_{C_2/V} \to N_{C_2/V}|_{C_2} \to 0$$

implies $n \geq deg(N_{G/C_2}) = -1$. $\square$

Blow up $r : Y \to V$ curve C_2, put $R = r^{-1}(C_2) \cong \mathbb{P}^1 \times \mathbb{P}^1$ and contract $\hat{r} : Y \to \hat{V}$ surface R to the curve $\hat{C}_2 \subset \hat{V}$, such that $\hat{r} \circ r^{-1}$ is not regular. Birational map $\hat{r} \circ r^{-1}$ is a flop in curve C_2 and $\hat{r}$ contracts section of ruled surface $r^{-1}(G) \cong \mathbb{F}_1$. Therefore, we get surface $\hat{G} = \hat{r} \circ r^{-1}(G) \cong \mathbb{P}^2$ with $N_{\hat{G}/\hat{V}} \cong \mathcal{O}_{\hat{G}}(-2)$. Contract $\hat{g} : \hat{V} \to \hat{W}$ surface $\hat{G}$ to singular point $\hat{P}$.

Remark 4.8. $\hat{W}$ has one singular point – cyclic quotient singularity of type $\frac{1}{2}(1,1,1)$.

Birational map $\rho = \hat{g} \circ \hat{r} \circ r^{-1} \circ g^{-1}$ is an antiflip in curve C_1. Moreover, birational map ρ is a flip for log terminal log pair $(W, (1+\epsilon)|-K_W|)$ for some small enough positive rational number ϵ. Hence, 3-fold $\hat{W}$ is projective and $\mathbb{Q}$-factorial.

PROPOSITION 4.9. $-K_{\hat{W}}^3 = \frac{1}{2}$ *and base locus of* $|-K_{\hat{W}}|$ *consists of point* $\hat{P}$.

Proof. Linear system $|-K_{\hat{V}}|$ is free and $\phi_{|-K_{\hat{V}}|}$ is an elliptic fibration. So, from the construction of map ρ it follows that base locus of $|-K_{\hat{W}}|$ consists of point $\hat{P}$ and

$$0 = -K_{\hat{V}}^3 = (\hat{g}^*(-K_{\hat{W}}) - \frac{1}{2}\hat{G})^3 = -K_{\hat{W}}^3 - \frac{1}{8}\hat{G}^3 = -K_{\hat{W}}^3 - \frac{1}{2}.$$

$\square$

In particular, $-K_{\hat{W}}$ is nef and big. Hence, for $n \gg 0$ linear system $|-nK_{\hat{W}}|$ is free and gives birational morphism $\phi_{|-nK_{\hat{W}}|} : \hat{W} \to X_C$, where 3-fold X_C is normal.

COROLLARY 4.10. *X_C is Fano 3-fold with canonical singularities and $-K_{X_C}^3 = \frac{1}{2}$.*

In case when image of curve C is not contained in ramification surface S birational map from X to Fano 3-fold X_C does not contract divisors on X, which implies that singularities of 3-fold X_C are terminal. In case when image of curve C is contained in surface S birational map from X to 3-fold X_C contracts a surface spanned by the inverse images on X of all lines on quadric Q tangent to ramification surface S in the points of line $\theta(C)$, which implies that 3-fold X_C has canonical singularities of type $\mathbb{A}_1$ along the general point of one curve. This concludes the proof of Proposition 1.4.

§5. Smooth quartic 3-fold.

In this section we will prove Main and Auxiliary Theorems for smooth quartic 3-folds.

Let X be a smooth quartic 3-fold in $\mathbb{P}^4$. Then $Pic(X) = -\mathbb{Z}K_X$ and anticanonical divisor $-K_X$ is a hyperplane section of quartic X. Consider movable log pair (X, M_X), such that $K_X + M_X \sim_{\mathbb{Q}} 0$.

Theorem 5.1. *Singularities of log pair (X, M_X) are canonical and*

$$CS(X, M_X) = \begin{cases} \emptyset, \\ \{P\} \text{ for Iskovskikh point } P \in X, \\ \{L\} \text{ for line } L \subset \mathbb{P}^4, \\ \{X \cap H\} \text{ for linear subspace } H \subset \mathbb{P}^4 \text{ of dimension } 2. \end{cases}$$

and in case when singularities of (X, M_X) are terminal $M_X = \psi^{-1}(M_Y)$, where rational map $\psi : X \dashrightarrow Y$ is either a rational map given by Iskovskikh point P if set $CS(X, M_X)$ consists of this point P or a projection from locus $CS(X, M_X)$ otherwise.

Proposition 5.2. *Theorem 5.1 implies Main Theorem for smooth quartic 3-fold X.*

Proof. Suppose that there is a birational map $\sigma : X \dashrightarrow Y$, such that Y is a Fano 3-fold with canonical singularities. We have to show that $Y \cong X$.

Consider positive rational number γ, big enough integer n and movable log pair

$$(X, M_X) = (X, \gamma\sigma^{-1}(| - nK_Y|)),$$

such that $K_X + M_X \sim_{\mathbb{Q}} 0$. Then by Theorem 5.1 log pair (X, M_X) has terminal singularities and $\kappa(X, M_X) = 0$. Therefore, for some small enough positive rational number λ both log pairs $(X, \lambda M_X)$ and $(Y, \lambda\gamma|-nK_Y|)$ are birationally equivalent canonical models. However, canonical model of a movable log pair is unique by Theorem 2.8. Thus, $Y \cong X$. □

Proposition 5.3. *Theorem 5.1 implies Auxiliary Theorem for smooth quartic X.*

Proof. Suppose that X is birationally transformed by birational map $\theta : X \dashrightarrow Y$ into some fibration $\tau : Y \to Z$, such that the general fiber of τ is either elliptic curve or a surface of Kodaira dimension zero. Consider movable log pair

$$(X, M_X) = (X, \lambda\theta^{-1}(|\tau^*(D)|))$$

for some big enough divisor D on Z and positive rational λ, such that $K_X + M_X \sim_{\mathbb{Q}} 0$.

Suppose that singularities of log pair (X, M_X) are terminal. Then for some positive rational number γ movable log pair $(X, \gamma M_X)$ is a canonical model. On the other hand, by construction $\kappa(X, \gamma M_X) < 3$. Therefore, $CS(X, M_X) \neq \emptyset$ and Theorem 5.1 implies that

$$CS(X, M_X) = \begin{cases} \{P\} \text{ for Iskovskikh point } P \in X, \\ \{L\} \text{ for line } L \subset \mathbb{P}^4, \\ \{X \cap H\} \text{ for linear subspace } H \subset \mathbb{P}^4 \text{ of dimension } 2. \end{cases}$$

In case when $CS(X, M_X) \neq \{L\}$ for any line L on quartic X Theorem 5.1 implies that fibration τ is equivalent to the fibration given either by Iskovskikh point P or by projection from the two-dimensional linear subspace H in $\mathbb{P}^4$. In the case when Y is a surface Theorem 5.1 imply that τ is birationally equivalent as a fibration to a projection from some line on X. Suppose that τ is a fibration on surfaces of Kodaira dimension zero and the set $CS(X, M_X)$ consist of some line L on quartic X. To conclude the proof we have to show that τ is birationally equivalent as a fibration to a projection from some two-dimensional subspace in $\mathbb{P}^4$.

From Theorem 5.1 it follows that $M_X = \psi^{-1}(M_{\mathbb{P}^2})$, where $\psi : X \dashrightarrow \mathbb{P}^2$ is a projection from line L. Let $f : V \to X$ be a blow up of L, $E = f^{-1}(L)$ and $M_V = f^{-1}(M_X)$. Then boundary M_V lies in the fibers of elliptic fibration $\phi_{|-K_V|} : V \to \mathbb{P}^2$ and

$$K_V + M_V \sim_{\mathbb{Q}} \phi^*_{|-K_V|}(\mathcal{O}_{\mathbb{P}^2}(-1) + M_{\mathbb{P}^2}) \sim_{\mathbb{Q}} 0.$$

Singularities of log pair (V, M_V) can not be terminal, otherwise $\kappa(V, \delta M_V) = 2$ for some small enough rational number $\delta > 1$, but explicit construction of movable log pair (X, M_X) implies that $\kappa(V, \delta M_V) \leq 1$. Therefore, $CS(V, M_V) \neq \emptyset$.

Suppose that the set $CS(V, M_V)$ contains some point $P_V \in V$. Then equivalence

$$K_V + M_V \sim_{\mathbb{Q}} f^*(K_X + M_X)$$

implies that $f(P_V)$ is a senter of canonical singularities of log pair (X, M_X), which contradicts our assumption $CS(X, M_X) = \{L\}$.

Thus, the set $CS(V, M_V)$ contains some curve $\hat{L} \subset V$. Then $\hat{L} \subset E$. On the other hand, movable boundary M_V lies in the fibers of morphism $\phi_{|-K_V|}$. So, curve $\hat{L}$ is contracted by morphism $\phi_{|-K_V|}$ to a point.

Let $\mathcal{H}$ be a linear system of surfaces in $|-K_V|$ passing through curve $\hat{L}$ and D be a general enough element in $\mathcal{H}$. Then $\mathcal{H}$ is a pencil, D is a K3 surface with at most canonical singularities, map $\phi_{\mathcal{H}} \circ f^{-1}$ is a projection from some two-dimensional subspace T in $\mathbb{P}^4$, line L is contained in T and L is cutted on X by T with multiplicity 2, 3 or 4. We will show that τ and $\phi_{\mathcal{H}}$ are birationally equivalent as fibrations.

Suppose that L is cutted on X by T with multiplicity 2 and $T \cap X = L \cup C$, where C is an irreducible reduced conic on X. Consider restriction

$$M_V|_D = \hat{L} + mult_C(M_X)\hat{C} + B_D,$$

where $\hat{C} = f^{-1}(C)$ and B_D is a movable boundary. On surface D divisor $M_V|_D$ is $\mathbb{Q}$-rationally equivalent to $\hat{L} + \hat{C}$ and

$$B_D \cdot \hat{C} = \hat{C}^2(1 - mult_C(M_X)) = -2(1 - mult_C(M_X)) \geq 0.$$

Therefore, curve C is a center of canonical singularities of log pair (X, M_X), which is a contradiction with assumtion that $CS(X, M_X) = \{L\}$.

Suppose that line L is cutted on X by two-dimensional linear subspace $T \subset \mathbb{P}^4$ with multiplicity 2 as above, but $T \cap X = L \cup C_1 \cup C_2$, where C_1 and C_2 are lines on X. Then

$$M_V|_D = \hat{L} + mult_{C_1}(M_X)\hat{C}_1 + mult_{C_2}(M_X)\hat{C}_2 + B_D,$$

where $\hat{C}_1 = f^{-1}(C_1)$, $\hat{C}_2 = f^{-1}(C_2)$ and B_D is a movable boundary. On surface D

$$B_D \sim_{\mathbb{Q}} (1 - mult_{C_1}(M_X))\hat{C}_1 + (1 - mult_{C_2}(M_X)\hat{C}_2)$$

and intersection form of curves $\hat{C}_1$ and $\hat{C}_2$ is negative definite. Movability of B_D implies equalities $mult_{C_1}(M_X) = 1$ and $mult_{C_2}(M_X) = 1$. Thus, lines C_1 and C_2 are centers of canonical singularities of log pair (X, M_X), which contradicts $CS(X, M_X) = \{L\}$.

Suppose that L is cutted on X by two-dimensional linear subspace $T \subset \mathbb{P}^4$ with multiplicity 2 and $T \cap X = L \cup C_0$, where curve C_0 is line on quartic 3-fold X different from line L, such that line C_0 is cutted on quartic X by plane T with multiplicity 2 as well. Let $h : U \to V$ be a blow up of curve $\hat{C}_0 = f^{-1}(C_0)$, $M_U = h^{-1}(M_V)$, S be a general surface in linear system $|-K_U| = h^{-1}(\mathcal{H})$, $\bar{L} = h^{-1}(\hat{L})$ and $\bar{C}_0$ are base curves of linear system $|-K_U|$ different from $\bar{L}$. Then on surface S

$$M_U|_S = \bar{L} + mult_{\bar{C}_0}(M_U)\bar{C}_0 + B_S \sim_{\mathbb{Q}} \bar{L} + \bar{C}_0$$

and $\bar{C}_0^2 = -2$, where B_S is a movable boundary. Hence, $mult_{\bar{C}_0}(M_U) = 1$. On the other hand, curve $\bar{C}_0$ dominates C_0. Thus, $mult_{C_0}(M_X) \geq 1$ and C_0 is a center of canonical singularities of log pair (X, M_X), which contradicts assumtion that $CS(X, M_X) = \{L\}$.

We proved that line L is cutted on quartic X by plane T with multiplicity 3 or 4. Blow up $g : W \to V$ a curve $\hat{L}$ and put $G = g^{-1}(\hat{L})$ and $M_W = g^{-1}(M_V)$. Consider general surface N in $|-K_W| = g^{-1}(\mathcal{H})$ and base curve $\tilde{L} \subset G$ of linear system $|-K_W|$.

Suppose that L is cutted on X by T with multiplicity 3. Then $T \cap X = L \cup R$, where curve R is another line on quartic X. For some movable boundary B_N on surface N

$$M_W|_N = mult_{\tilde{L}}(M_W)\tilde{L} + mult_R(M_X)\tilde{R} + B_N,$$

where $\tilde{R}$ is an strict transform of line R on W. On surface N

$$B_N \sim_{\mathbb{Q}} (1 - mult_{\tilde{L}}(M_W))\tilde{L} + (1 - mult_R(M_X))\tilde{R}$$

and intersection form of curves $\tilde{L}$ and $\tilde{R}$ is negative definite. Therefore, movability of boundary B_N implies that $mult_{\tilde{L}}(M_W) = mult_R(M_X) = 1$ and R is a center of canonical singularities of movable log pair (X, M_X), which contradicts $CS(X, M_X) = \{L\}$.

We proved that line L is cutted on X by T with multiplicity 4. Thus,

$$M_W|_N = \alpha\tilde{L} + B_N$$

for some rational $\alpha \geq mult_{\tilde{L}}(M_W)$ and movable boundary B_N. On the other hand, on surface N boundary B_N is numerically equivalent to $(2-\alpha)\tilde{L}$ and $\tilde{L}^2 < 0$. Hence, movable boundary B_N is empty and $M_W|_N = \alpha\tilde{L}$. However, base locus of pencil $|-K_W|$ consists of curve $\tilde{L}$. Thus, boundary M_W is contained in the fibers of rational map $\phi_{|-K_W|}$. Therefore, projection from T is birationally equivalent to τ as a fibration. □

In the rest of this chapter we will prove Theorem 5.1.

Remark 5.4. Canonicity of every movable log pair on X with $\mathbb{Q}$-rationally trivial log canonical divisor is equivalent to birational rigidity of X.

We will assume that log pair (X, M_X) has canonical singularities.

LEMMA 5.5. *Suppose that point $P \in X$ is a center of canonical singularities of movable log pair (X, M_X). Then P is an Iskovskikh point, $CS(X, M_X) = \{P\}$ and $M_X = \psi^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$, where rational map $\psi : X \dashrightarrow \mathbb{P}^1$ is a rational map given by Iskovskikh point P.*

Proof. Let H_P be general enough hyperplane section of X passing through P. Then inequality $H_P \cdot M_X^2 \geq mult_P(M_X^2)$ and Theorem 3.11 implies that $mult_P(M_X) = 2$.

Let $f : W \to X$ be a blow up of P and $M_W = f^{-1}(M_X)$. Then M_W is $\mathbb{Q}$-rationally equivalent to anticanonical divisor $-K_W$. Therefore, for some natural number n linear system $|-nK_W|$ does not have fixed components. In particular, P is an Iskovskikh point and linear system $|-nK_W|$ is a pencil by Proposition 1.10. From inequalities

$$4 = H_P \cdot M_X^2 \geq mult_P(M_X^2) \geq mult_P^2(M_X) = 4$$

it follows that support of M_X^2 is a union of all lines passing through point P and there are only finite number of them. On the other hand, base locus of pencil $f(|-nK_W|)$ consists of lines passing through point P. Therefore, $M_X = \phi^{-1}_{f(|-nK_W|)}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$.

Now we should prove that P is a unique center of canonical singularities of movable log pair (X, M_X). Suppose that set $CS(X, M_X)$ has other elements except P. We already proved that support of M_X^2 is a union of all lines passing through point P. So, there is a line C on X passing through point P, such that either line C is a center of canonical singularities or set $CS(X, M_X)$ contains another Iskovskikh point on line C.

Consider general hyperplane section H_C of 3-fold X passing through C. Then H_C is a smooth K3 surface and $M_X|_{H_C} = mult_C(M_X)C + B_C$ for some movable boundary B_C on surface H_C. We have

$$2 = mult_P(M_X) \leq mult_P(M_X|_{H_C}) = mult_C(M_X) + mult_P(B_C).$$

Thus, $mult_P(B_C) \geq 2 - mult_C(M_X)$ and from the movability of B_C it follows that

$$B_C^2 = 4 - 2mult_C(M_X) - 2mult_C^2(M_X) \geq (2 - mult_C(M_X))^2,$$

which implies $mult_C(M_X) \leq \frac{2}{3}$. In particular, C is not a center of canonical singularities of log pair (X, M_X) and to conclude the proof we have to show that $CS(X, M_X)$ does not contain another Iskovskikh points on line C. However, in case when set $CS(X, M_X)$ has two points on line C movability of B_C implies

$$B_C^2 = 4 - 2mult_C(M_X) - 2mult_C^2(M_X) \geq 2(2 - mult_C(M_X))^2,$$

which is impossible. $\square$

Thus, we will assume that set $CS(X, M_X)$ does not contains points on X and contains an irreducible and reduced curve C. In particular, canonicity of singularities of movable log pair (X, M_X) implies that $mult_C(M_X) = 1$.

LEMMA 5.6. *Degree of $C \subset \mathbb{P}^4$ is not greater than 4.*

Proof. Equality $mult_C(M_X) = 1$ implies $4 = -K_X \cdot M_X^2 \geq -K_X \cdot C$. $\square$

LEMMA 5.7. *Curve C is contained in two-dimensional linear subspace in $\mathbb{P}^4$.*

Proof. Suppose that curve C is not contained in two-dimensional linear subspace in $\mathbb{P}^4$. Then by Lemma 5.6 curve C is either a smooth curve of degree 3 or 4 or a rational curve of degree 4 with one double point P.

Suppose that C is smooth. Let $f : W \to X$ be a blow up of curve C, E be exceptional divisor of f, $M_W = f^{-1}(M_X)$ and $d = -K_X \cdot C$. Then linear system $|f^*(-dK_X) - E|$ does not have base curves. Therefore,

$$0 \leq (f^*(-dK_X) - E) \cdot M_W^2 = 3d - d^2(C) + 2g(C) - 2 < 0.$$

Thus, curve C is a rational curve of degree 4 with double point P. Let $f : W \to X$ be a composition of blow up of point P and blow up of strict transform of curve C, G and E

be two f-exceptional divisors, such that $f(E) = C$ and $f(G) = P$. Then complete linear system $|f^*(-4K_X) - E - 2G|$ does not have base curves. Thus, for $M_W = f^{-1}(M_X)$

$$0 \leq (f^*(-4K_X) - E - 2G) \cdot M_W^2 = 4mult_P(M_X) - 2mult_P^2(M_X) - 6 < 0,$$

because $2 > mult_P(M_X) \geq mult_C(M_X) = 1$. $\square$

LEMMA 5.8. *Suppose that curve C is a line. Then $M_X = \psi^{-1}(M_{\mathbb{P}^2})$ for some movable boundary $M_{\mathbb{P}^2}$ on $\mathbb{P}^2$, where rational map ψ is a projection from line C.*

Proof. Let $f : W \to X$ be a blow up of curve C and $M_W = f^{-1}(M_X)$. Then linear system $|-K_W|$ is free, $\phi_{|-K_W|}$ is an elliptic fibration and $M_W \sim_{\mathbb{Q}} -K_W$. $\square$

LEMMA 5.9. *Suppose that C is a line, but $CS(X, M_X) \neq \{C\}$. Then there is two-dimensional linear subspace $T \subset \mathbb{P}^4$ containing line C, such that set $CS(X, M_X)$ consists of all components of $X \cap T$ and $M_X = \psi^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$, where rational map map ψ is a projection from T.*

Proof. Let $f : V \to X$ be a blow up of line C, $E = f^{-1}(C)$ and $M_V = f^{-1}(M_X)$. Then

$$K_V + M_V \sim_{\mathbb{Q}} f^*(K_X + M_X) \sim_{\mathbb{Q}} 0$$

and Lemma 5.8 implies that movable boundary M_V is contained in the fibers of elliptic fibration $\phi_{|-K_V|} : V \to \mathbb{P}^2$. By assumption set $CS(V, M_V)$ contains some curve $\hat{L}$, such that $\hat{L} \not\subset E$. Movability of M_V implies that curve $\hat{L}$ is contracted by morphism $\phi_{|-K_V|}$ to a point.

Let $\mathcal{H}$ be a linear system of surfaces in $|-K_V|$ passing through curve $\hat{L}$ and D be a general enough element in $\mathcal{H}$. Then $\mathcal{H}$ is a pencil, D is a K3 surface with at most canonical singularities, map

$$\psi = \phi_{\mathcal{H}} \circ f^{-1} : X \dashrightarrow \mathbb{P}^1$$

is a projection from some two-dimensional subspace T in $\mathbb{P}^4$, line C and curve $L = f(\hat{L})$ are both contained in T and C is cutted on X by T with multiplicity at most 3. We will show that $M_V = \phi_{\mathcal{H}}^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$.

Suppose that C is cutted on quartic X by two-dimensional linear subspace $T \subset \mathbb{P}^4$ with multiplicity 3. Then $T \cap X = C \cup L$, where $L = f(\hat{L})$ is a line on X and base locus of linear system $\mathcal{H}$ consists of curve $\hat{L}$ and smooth rational curve $\hat{C} \subset E$. We have

$$M_V|_D = \alpha\hat{C} + \hat{L} + B_D$$

for some rational number $\alpha \geq mult_{\hat{C}}(M_V)$ and some movable boundary B_D. However, on surface D boundary B_D is $\mathbb{Q}$-rationally equivalent to $(2-\alpha)\hat{C}$ and $\hat{C}^2 < 0$. Thus, boundary B_D is empty and from the fact that base locus of linear system $\mathcal{H}$ consists of curves $\hat{C}$ and $\hat{L}$ it follows that $M_V = \phi_{\mathcal{H}}^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$.

Suppose that line C is cutted on X by T with multiplicity 2 and L is an irreducible reduced conic on X. In this case $T \cap X = C \cup L$. Consider restriction

$$M_V|_D = mult_{\hat{C}}(M_V)\hat{C} + \hat{L} + B_D,$$

where $\hat{L}$ and $\hat{C} \subset E$ are base curves of pencil $\mathcal{H}$ and B_D is a movable boundary. On surface D divisor $M_V|_D$ is $\mathbb{Q}$-rationally equivalent to $\hat{C} + \hat{L}$ and $\hat{C}^2 < 0$. So, boundary B_D is empty and $mult_{\hat{C}}(M_V) = 1$. Hence, the fact that base locus of linear system $\mathcal{H}$ consists of curves $\hat{C}$ and $\hat{L}$ implies that $M_V = \phi_{\mathcal{H}}^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$.

Suppose that line C is cutted on quartic X by two-dimensional linear subspace $T \subset \mathbb{P}^4$ with multiplicity 2, but $T \cap X = C \cup L \cup L_1$, where L and L_1 are two different lines on quartic 3-fold X. Consider restriction

$$M_V|_D = mult_{\hat{C}}(M_V)\hat{C} + \hat{L} + mult_{L_1}(M_X)\hat{L}_1 + B_D,$$

where $\hat{L}$, $\hat{L}_1 = f^{-1}(L_1)$ and $\hat{C} \subset E$ are base curves of pencil $\mathcal{H}$ and B_D is a movable boundary. On surface D divisor $M_V|_D$ is $\mathbb{Q}$-rationally equivalent $\hat{C}+\hat{L}+\hat{L}_1$ and intersection form of curves $\hat{C}$ and $\hat{L}_1$ is negative definite. Therefore, boundary B_D is empty and equalities $mult_{\hat{C}}(M_V) = 1$ and $mult_{L_1}(M_X) = 1$ hold. In particular, line L_1 is a center of canonical singularities of movable log pair (X, M_X). Base locus of pencil $\mathcal{H}$ consists of curves $\hat{C}$, $\hat{L}$ and $\hat{L}_1$. Thus, movability of M_V implies that $M_V = \phi_{\mathcal{H}}^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$.

Suppose now that C is cutted on X by T with multiplicity 2 and $T \cap X = C \cup L$, where curve L is a line on X also cutted by T with multiplicity 2. Let $h : U \to V$ be a blow up of $\hat{L}$, $M_U = h^{-1}(M_V)$, S be a general surface in pencil $|-K_U| = h^{-1}(\mathcal{H})$, $\bar{C}$ and $\bar{L}$ be two smooth rational base curves of $|-K_U|$. Then

$$M_U|_S = mult_{\bar{C}}(M_U)\bar{C} + mult_{\bar{L}}(M_U)\bar{L} + B_S,$$

where B_S is a movable boundary. On surface S

$$B_S \sim_{\mathbb{Q}} (1 - mult_{\bar{C}}(M_U))\bar{C} + (1 - mult_{\bar{L}}(M_U))\bar{L}$$

and intersection form of curves $\bar{C}$ and $\bar{L}$ is negative definite. Therefore, boundary B_S is empty and $M_U|_S = \bar{C} + \bar{L}$. So, $M_U = \phi_{|-K_U|}^{-1}(M_{\mathbb{P}^1})$ for some $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$.

So, we may assume that C is cutted on X by two-dimensional subspace $T \subset \mathbb{P}^4$ with multiplicity 1. In case when L is a plane cubic we have $M_V = \phi_{\mathcal{H}}^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$, because $M_V|_D = \bar{L}$ and base locus of pencil $\mathcal{H}$ consists of curve $\hat{L}$. Thus, we will assume that L is either a conic or a line on X.

Suppose that L is a conic on quartic X. Then $T \cap X = C \cup L \cup R$, where R is a line on quartic X different from C. Consider restriction

$$M_V|_D = \hat{L} + mult_R(M_X)\hat{R} + B_D,$$

where $\hat{R}=f^{-1}(R)$. On surface D we have $M_V|_D \sim_{\mathbb{Q}} \hat{L}+\hat{R}$ and $\hat{R}^2<0$. Therefore, boundary B_D is empty and $mult_R(M_X)=1$. Hence, line R is a center of canonical singularities of movable log pair (X, M_X). On the other hand, base locus of pencil $\mathcal{H}$ consists of curves $\hat{L}$ and $\hat{R}$. Thus, from the movability of M_V and $M_V|_D=\hat{L}+\hat{R}$ it follows that $M_V=\phi_{\mathcal{H}}^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$.

So, we may assume that both curves C and L are lines. Moreover, we already considered case when set $CS(X, M_X)$ contains two lines in $X\cap T$, such that one of them is cutted on quartic X by plane T with multiplicity greater or equal than 2. Thus, to conclude the proof we should consider the following two cases:

$$X\cap T=\begin{cases} C\cup L\cup R_0 \text{ for irreducible reduced conic } R_0\subset X,\\ C\cup L\cup R_1\cup R_2 \text{ for two different lines } R_1 \text{ and } R_2 \text{ on } X.\end{cases}$$

Suppose that $X\cap T=C\cup L\cup R_0$, where R_0 is an irreducible reduced conic. Let

$$M_V|_D=\hat{L}+mult_{R_0}(M_X)\hat{R}_0+B_D,$$

where $\hat{R}_0=f^{-1}(R_0)$. Then on surface D we have $M_V|_D\sim_{\mathbb{Q}}\hat{L}+\hat{R}_0$ and $\hat{R}_0^2<0$. Therefore, boundary B_D is empty and $mult_{R_0}(M_X)=1$. Hence, conic R_0 is a center of canonical singularities of log pair (X, M_X). On the other hand, base locus of pencil $\mathcal{H}$ consists of both curves $\hat{L}$ and $\hat{R}_0$. Therefore, $M_V|_D=\hat{L}+\hat{R}_0$ implies that $M_V=\phi_{\mathcal{H}}^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$.

Suppose now that $X\cap T=C\cup L\cup R_1\cup R_2$, where R_1 and R_2 are different lines on quartic 3-fold X. Consider restriction

$$M_V|_D=\hat{L}+mult_{R_1}(M_X)\hat{R}_1+mult_{R_2}(M_X)\hat{R}_2+B_D,$$

where $\hat{R}_1=f^{-1}(R_1)$ and $\hat{R}_2=f^{-1}(R_2)$. On surface D

$$B_D\sim_{\mathbb{Q}}(1-mult_{R_1}(M_X))\hat{R}_1+(1-mult_{R_2}(M_X))\hat{R}_2$$

and intersection form of curves $\hat{R}_1$ and $\hat{R}_2$ is negative definite. Thus, $B_D=\emptyset$ and lines R_1 and R_2 are centers of canonical singularities of (X, M_X). Base locus of pencil $\mathcal{H}$ consists of curves $\hat{L}$, $\hat{R}_1$ and $\hat{R}_2$, but $M_V|_D=\hat{L}+\hat{R}_1+\hat{R}_2$. Therefore, $M_V=\phi_{\mathcal{H}}^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$. $\square$

LEMMA 5.10. *Suppose that degree of curve C is 4. Then $M_X=\psi^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$, where ψ is a projection from the two-dimensional linear subspace of $\mathbb{P}^4$ containing curve C.*

Proof. By Lemma 5.7 curve C is cutted on quartic X by some two-dimensional linear subspace $T\subset\mathbb{P}^4$. Let $\mathcal{H}_T$ be a linear system of hyperplane sections of quartic X passing through T, $f: W\to X$ be a resolution of indeterminacy of rational map $\phi_{\mathcal{H}_T}$, D be a general fiber of morphism $g = \phi_{\mathcal{H}_T} \circ f^{-1}$ and $M_W = f^{-1}(M_X)$, such that 3-fold W contains one exceptional divisor E dominating curve C and f is an isomorphism outside of C. Then

$$D \sim f^*(-K_X) - E - \sum_{i \in I} a_i F_i \text{ and } M_W|_D \sim_{\mathbb{Q}} ((1 - mult_C(M_X))E + \sum_{i \in I} c_i F_i)|_D,$$

where all $f(F_i)$ are points on curve C, a_i are natural numbers and all c_i are rational numbers. Thus, movability of $M_W|_D$ implies $M_W|_D = \emptyset$. Hence, boundary M_W lies in the fibers of g. Therefore, $M_W = g^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$. $\square$

LEMMA 5.11. *Suppose that curve C is either a conic or a plain cubic, which is contained in two-dimensional linear subspace $T \subset \mathbb{P}^4$. Then set $CS(X, M_X)$ consists of all components of $X \cap T$ and $M_X = \psi^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$, where rational map ψ is a projection from T.*

Proof. Let $\mathcal{H}_T$ be a pencil of hyperplane sections of X passing through T and D be a general enough surface in $\mathcal{H}_T$. Then D is a K3 surface with at most canonical singularities.

Suppose that C is a plane cubic. Then $X \cap T = C \cup L$ and

$$M_X|_D = C + mult_L(M_X)L + B_D,$$

where L is a line on quartic X and B_D is a movable boundary. On the other hand, on surface D boundary B_D is $\mathbb{Q}$-rationally equivalent to $(1 - mult_L(M_X))L$ and $L^2 < 0$, which implies that curve L is a center of canonical singularities of (X, M_X). Therefore, the claim follows from Lemma 5.9.

Suppose that C is a conic and $X \cap T = C \cup L_1 \cup L_2$ for two different lines L_1 and L_2 on quartic X. Then

$$M_X|_D = C + mult_{L_1}(M_X)L_1 + mult_{L_2}(M_X)L_2 + B_D$$

for some movable boundary B_D. On the other hand, on surface D

$$B_D \sim_{\mathbb{Q}} (1 - mult_{L_1}(M_X))L_1 + (1 - mult_{L_2}(M_X))L_2$$

and intersection form of lines L_1 and L_2 is negative definite. Therefore, lines L_1 and L_2 are centers of canonical singularities of (X, M_X) and the claim follows from Lemma 5.9.

Suppose that C is a conic and $X \cap T = C \cup R$ for a line R cutted on X by T with multiplicity 2. Then

$$M_X|_D = C + \alpha R + B_D$$

for some rational number $\alpha \geq mult_R(M_X)$ and a movable boundary B_D. However, on surface D boundary B_D is $\mathbb{Q}$-rationally equivalent to $(2 - \alpha)R$ and $R^2 < 0$, which implies that boundary B_D is empty. On the other hand, base locus of pencil $\mathcal{H}_T$ consists of both curves C and R. Thus, $M_X = \phi_{\mathcal{H}_T}^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$, which implies that line R is a center of canonical singularities of log pair (X, M_X).

Suppose now that C is a conic, $X \cap T = C$ and conic C is cutted on X by T with multiplicity 2. Let $f : V \to X$ be a blow up of C, $E = f^{-1}(C)$, $M_V = f^{-1}(M_X)$ and S be a general surface in pencil $|-K_V| = f^{-1}(\mathcal{H}_T)$. Then for some movable boundary B_S

$$M_V|_S = mult_{\hat{C}}(M_V)\hat{C} + B_S,$$

where $\hat{C} \subset E$ is a base curve of pencil $|-K_V|$. On surface S boundary B_S is $\mathbb{Q}$-rationally equivalent to $mult_{\hat{C}}(M_V)\hat{C}$ and inequality $\hat{C}^2 < 0$ holds. Therefore, boundary B_S is empty and $M_V = \phi^{-1}_{|-K_V|}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$, because base locus of pencil $|-K_V|$ consists of curve $\hat{C}$. $\square$

So, we proved Theorem 5.1.

§6. Singular quartic 3-fold.

In this chapter we will prove Main and Auxiliary Theorems for quartic 3-folds with one simple double point.

Let X be a quartic 3-fold in $\mathbb{P}^4$ with one simple double point O, such that there are exactly 24 lines on quartic X passing through point O. Then $Pic(X) = -\mathbb{Z}K_X$ and anticanonical divisor $-K_X$ is a hyperplane section of X. In Example 1.2 we constructed birational map $\psi_O : X \dashrightarrow X_O$, such that X_O is a Fano 3-fold with terminal singularities and $-K^3_{X_O} = 2$. In chapter 4 for every line C on quartic 3-fold X passing through singular point O we constructed birational map $\psi_C : X \dashrightarrow X_C$, such that X_C is a Fano 3-fold with terminal singularities and $-K^3_{X_C} = \frac{1}{2}$.

Proposition 6.1. *Suppose, that there is a birational map $\sigma : X \dashrightarrow Y$, where Y is a Fano 3-fold with canonical singularities. For $n \gg 0$ put $M_Y = \frac{1}{n}|-nK_Y|$ and consider movable log pairs (X, M_X), (X_O, M_{X_O}) and (X_C, M_{X_C}), which are birationally equivalent to movable log pair (Y, M_Y). Then up to the action of group $Bir(X)$ one of movable log pairs (X, M_X), (X_O, M_{X_O}) or (X_C, M_{X_C}) has semiterminal singularities and $\mathbb{Q}$-rationally trivial log canonical divisor.*

Lemma 6.2. *Main Theorem for quartic X is implied by Proposition 6.1.*

Proof. We may assume that $K_X + M_X \sim_{\mathbb{Q}} 0$ and log pair (X, M_X) has semiterminal singularities. Then for some rational number $\lambda > 1$ log pair $(X, \lambda M_X)$ has canonical singularities. On the other hand, divisor $K_X + \lambda M_X$ is ample and both birationally equivalent log pairs $(X, \lambda M_X)$ and $(Y, \lambda M_Y)$ are canonical models. However, canonical model of a movable log pair is unique by Theorem 2.8. Thus, $Y \cong X$. $\square$

Next result is equivalent to birational rigidity of X, which was proved in [3].

PROPOSITION 6.3. *Let (X, M_X) be a movable log pair with $\kappa(X, M_X) = 0$. Then for some $\tau \in Bir(X)$ the singularities of $(X, \tau(M_X))$ are canonical and $K_X + \tau(M_X) \sim_{\mathbb{Q}} 0$.*

To prove Proposition 6.1 we need a description of the set of centers of canonical singularities of log pairs with canonical singularities and Kodaira dimension zero on X.

THEOREM 6.4. *Let (X, M_X) be a log pair with canonical singularities and $\mathbb{Q}$-rationally trivial log canonical divisor. Then*

$$CS(X, M_X) = \begin{cases} \emptyset, \\ \{O\}, \\ \{P\} \text{ for Iskovskikh point } P \text{ on } X, \\ \{C\} \text{ for line } C \subset X,\ O \in C, \\ \{O, C\} \text{ for line } C \subset X,\ O \in C, \\ \{C\} \text{ for line } C \subset X,\ O \notin C, \\ \{O, X \cap H\} \text{ for two-dimensional linear subspace } H \subset \mathbb{P}^4,\ O \in H, \\ \{X \cap H\} \text{ for two-dimensional linear subspace } H \subset \mathbb{P}^4,\ O \notin H. \end{cases}$$

and in case

$$CS(X, M_X) \neq \begin{cases} \emptyset, \\ \{O\}, \\ \{C\} \text{ for line } C \subset X,\ O \in C. \end{cases}$$

$M_X = \psi^{-1}(M_Y)$, where $\psi : X \dashrightarrow Y$ is either a rational map given by Iskovskikh point P in case $CS(X, M_X) = \{P\}$ or a projection from locus $CS(X, M_X)$ otherwise.

PROPOSITION 6.5. *Theorem 6.4 implies Main Theorem for quartic X.*

Proof. Suppose that there is a birational map $\sigma : X \dashrightarrow Y$, where Y is a Fano 3-fold with canonical singularities. Let $M_Y = \frac{1}{n}|-nK_Y|$ for some big enough natural number n and $M_X = \sigma^{-1}(M_Y)$. Then $\kappa(X, M_X) = \kappa(Y, M_Y) = 0$ and by Proposition 6.3 we may compose σ with some birational automorphism of X and assume that singularities of movable log pair (X, M_X) are canonical and $K_X + M_X \sim_{\mathbb{Q}} 0$.

Let $M_{X_O} = \psi_O(M_X)$ and $M_{X_C} = \psi_C(M_X)$ for some line C on X passing through point O, where ψ_O and ψ_C are birational maps from Example 1.2 and Proposition 1.3 correspondingly. Then by Proposition 6.1 to prove Main Theorem it is enough to show that that one of log pairs (X, M_X), (X_O, M_{X_O}) or (X_C, M_{X_C}) has semiterminal singularities and $\mathbb{Q}$-rationally trivial log canonical divisor.

We may assume that singularities of movable log pair (X, M_X) are not terminal. Then Theorem 6.4 and construction of log pair (Y, M_Y) imply that either $CS(X, M_X) = \{O\}$ or for some line C on X containing point O we have $CS(X, M_X) = \{C\}$.

Suppose that $CS(X, M_X) = \{C\}$ for some line C on quartic X passing through singular point O. Blow up $f : W \to X$ point O and blow up $g : V \to W$ curve $f^{-1}(C)$. Let E be exceptional divisor of g, $Q_V = g^{-1}(f^{-1}(O))$ and $M_V = (f \circ g)^{-1}(M_X)$. Then

$$K_V + M_V \sim_{\mathbb{Q}} (f \circ g)^*(K_X + M_X) + a(V, M_V, Q_V)Q_V$$

and $E \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Suppose that singularities of (V, M_V) are not terminal. Then $CS(V, M_V)$ contains such curve $C_V \subset E$, that C_V dominates line C. Last contradicts to

$$M_V \cdot D = a(V, M_V, Q_V) \in \mathbb{Q} \cap (0, 1),$$

where D is a fiber of morphism $\phi_{|-K_V|}$ passing through some point of curve C_V. Thus, singularities of log pair (V, M_V) is terminal and construction of birational map ψ_C implies that log pair (X_C, M_{X_C}) has semiterminal singularities and $K_{X_C} + M_{X_C}$ is $\mathbb{Q}$-rationally trivial.

To conclude the proof we may assume that $CS(X, M_X) = \{O\}$. Let $f : W \to X$ be a blow up of point O, $Q = f^{-1}(O)$ and $M_W = f^{-1}(M_X)$. Then Theorem 3.8 implies

$$K_W + M_W \sim_{\mathbb{Q}} f^*(K_X + M_X).$$

Birational morphism from W to X_O is crepant for log pair (W, M_W). So, to conclude the proof we have to show that log pair (W, M_W) has terminal singularities. Note, that all elements of set $CS(W, M_W)$ are contained in divisor Q.

Suppose that set $CS(W, M_W)$ has some point $O_W \in Q$, which is not contained in strict transform of any line on X. Then $mult_{O_W}(M_W^2) \geq 4$ by Theorem 3.11. Intersecting M_W^2 with general surface in $|-K_W|$ passing through O_W we get contradiction.

Suppose that set $CS(W, M_W)$ contains point $O_W \in Q$, such that $O_W \in f^{-1}(C)$ for some line C on X passing through O. Let $g : V \to W$ be blow up of $f^{-1}(C)$, E be exceptional divisor of g and $M_V = g^{-1}(M_W)$. Then $LCS(E, M_V|_E) \neq \emptyset$ by Theorem 3.7. On the other hand, $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ and we get contradiction with Lemma 3.6. Therefore, set $CS(W, M_W)$ does not contain points.

Suppose that $CS(W, M_W)$ contains curve Z in Q. Intersecting M_W with rulings of Q we conclude that Z is also a ruling. Let $g : V \to W$ be a blow up of Z, H be a general surface in pencil $|-K_V|$ and $M_V = g^{-1}(M_W)$. Then H is a K3 surface with at most canonical singularities and

$$M_V|_H = mult_{\hat{Z}}(M_V)\hat{Z} + \sum_{i \in I} \alpha_i C_i + B_H$$

for some rational numbers $\alpha_i \geq mult_{C_i}(M_V)$ and movable boundary B_H, where $\hat{Z}$ and C_i are base curves of pencil $|-K_V|$, such that $\phi_{|-K_W|}(g(\hat{Z})) = \phi_{|-K_W|}(Z)$ and curves C_i are mapped to lines on quartic X passing through point O. On the other hand, intersection form of curves $\hat{Z}$ and C_i on surface H is negative definite and

$$M_V|_H \sim_{\mathbb{Q}} H|_H \sim \hat{Z} + \sum_{i \in I} a_i C_i$$

for some natural numbers a_i. Therefore, $B_H=\emptyset$. Hence, boundary M_V lies in the fibers of map $\phi_{|-K_V|}$, which implies that $f\circ g(C_i)$ are centers of canonical singularities of movable log pair (X,M_X), which contradicts our assumption $CS(X,M_X=\{O\}$ and the choice of movable boundary M_Y. $\square$

PROPOSITION 6.6. *Theorem 6.4 implies Auxiliary Theorem for quartic X.*

Proof. Suppose that quartic X is birationally transformed by birational map $\theta: X \dashrightarrow Y$ into some fibration $\tau: Y\to Z$, such that the general fiber of τ is either elliptic curve or a surface of Kodaira dimension zero. Let $M_X=\lambda\theta^{-1}(|\tau^*(D_Z)|)$ for some big enough divisor D_Z and some $\lambda\in\mathbb{Q}_{>0}$. Then by Proposition 6.3 we may compose θ with some birational automorphism of X and choose such positive rational λ, that movable log pair (X,M_X) has canonical singularities and $K_X+M_X\sim_{\mathbb{Q}}0$.

Suppose that singularities of log pair (X,M_X) are terminal. Then for some positive rational number γ movable log pair $(X,\gamma M_X)$ is a canonical model. On the other hand, by construction $\kappa(X,\gamma M_X)<3$. Therefore, $CS(X,M_X)\neq\emptyset$. Moreover, same arguments imply that singularities of movable log pairs (X_O,M_{X_O}) and (X_C,M_{X_C}) are not semiterminal, where $M_{X_O}=\psi_O(M_X)$ and $M_{X_C}=\psi_C(M_X)$ for some line C on X passing through point O and ψ_O and ψ_C are birational maps from Example 1.2 and Proposition 1.3 correspondingly. Therefore, the proof of Proposition 6.5 implies that $CS(X,M_X)\neq\{O\}$ and $CS(X,M_X)\neq\{C\}$ for all lines C on X containing point O. Hence, by Theorem 6.4

$$CS(X,M_X)=\begin{cases}\{P\}\text{ for Iskovskikh point }P\text{ on }X,\\ \{O,C\}\text{ for line }C\subset X,\ O\in C,\\ \{C\}\text{ for line }C\subset X,\ O\notin C,\\ \{O,X\cap H\}\text{ for two-dimensional linear subspace }H\subset\mathbb{P}^4,\ O\in H,\\ \{X\cap H\}\text{ for two-dimensional linear subspace }H\subset\mathbb{P}^4,\ O\notin H.\end{cases}$$

and $M_X=\psi^{-1}(M_T)$, where map $\psi: X\dashrightarrow T$ is either a rational map given by Iskovskikh point P in case $CS(X,M_X)=\{P\}$ or a projection from locus $CS(X,M_X)$ otherwise.

In particular, in case when $CS(X,M_X)\neq\{O,C\}$ for line C on quartic X passing through point O and in case when $CS(X,M_X)\neq\{C\}$ for some line C on quartic X not passing through point O it follows from Theorem 6.4 that fibration τ is a fibration on surfaces of Kodaira dimension zero and is equivalent to the fibration given either by Iskovskikh point P or by projection from some two-dimensional linear subspace H in $\mathbb{P}^4$.

In the case when τ is elliptic fibration Theorem 6.4 trivially imply that τ is birationally equivalent as a fibration to a projection from some line on X.

To conclude the proof we may assume that τ is a fibration on surfaces of Kodaira dimension zero,

$$CS(X,M_X)=\begin{cases}\{O,C\}\text{ for line }C\subset X,\ O\in C,\\ \{C\}\text{ for line }C\subset X,\ O\notin C.\end{cases}$$

and $M_X=\psi^{-1}(M_{\mathbb{P}^2})$ for some movable boundary $M_{\mathbb{P}^2}$ on $\mathbb{P}^2$, where map $\psi: X\dashrightarrow\mathbb{P}^2$ is a projection from line C. We will show that τ is birationally equivalent as a fibration to a projection from some two-dimensional subspace in $\mathbb{P}^4$ containing line C.

Case when $CS(X, M_X) = \{C\}$ for some line C on quartic X not passing through singular point O is covered by the proof of Proposition 5.3. Therefore, we will assume that set $CS(X, M_X)$ consists of point O and some line C on quartic X passing through O.

Let $f : W \to X$ be a blow up of point O, $Q = f^{-1}(O)$ and $M_W = f^{-1}(M_X)$. Then it follows from Theorem 3.8 that $mult_O(M_X) = 1$ and $K_W + M_W \sim_{\mathbb{Q}} 0$.

We claim that there are no points of W which are centers of canonical singularities of movable log pair (W, M_W). Suppose that there is a point O_W on W, which is a center of canonical singularities of movable log pair (W, M_W). By our assumtion O_W is contained in Q and $mult_{O_W}(M_W^2) \geq 4$ by Theorem 3.11. Therefore, if O_W is not contained in strict transforms of any line on X passing through point O intersection of M_W^2 with the general surface in $|-K_W|$ passing through point O_W leads to contradiction. Hence, point O_W is contained in $f^{-1}(L)$ for some line L on X. Let $g : V \to W$ be blow up of $f^{-1}(L)$, E be exceptional divisor of g and $M_V = g^{-1}(M_W)$. Then $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ and singularities of log pair $(E, M_V|_E)$ are not log canonical by Theorem 3.7. On the other hand, divisor $M_V|_E$ has type $(mult_L(M_X), mult_L(M_X))$ in $Pic(E) \otimes \mathbb{Q}$ and $mult_L(M_X) < 1$, which is a contradiction with Lemma 3.6.

Suppose that some curve Z in Q is a center of canonical singularities of log pair (W, M_W). Then $f^{-1}(C) \cap Z \neq \emptyset$ and Z is a ruling of $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$. Let $g : V \to W$ be a blow up of curve Z, H be a general surface in pencil $|-K_V|$ and $M_V = g^{-1}(M_W)$. Then H is a K3 surface with at most canonical singularities and

$$M_V|_H = mult_{\hat{Z}}(M_V)\hat{Z} + \sum_{i \in I} \alpha_i C_i + B_H$$

for some rational numbers $\alpha_i \geq mult_{C_i}(M_V)$ and movable boundary B_H, where curves $\hat{Z}$ and C_i are base curves of pencil $|-K_V|$, such that $\phi_{|-K_W|}(g(\hat{Z})) = \phi_{|-K_W|}(Z)$, images of curves C_i on quartic X are lines passing through point O and $f \circ g(C_j) = C$ for some $j \in I$. On the other hand, intersection form of curves $\hat{Z}$ and C_i on surface H is negative definite and divisor $M_V|_H$ is $\mathbb{Q}$-rationally equivalent to divisor $H|_H$, which is rationally equivalent on surface H to $\hat{Z} + \sum_{i \in I} a_i C_i$ for some natural numbers a_i. Hence, $B_H = \emptyset$ and M_V lies in the fibers of $\phi_{|-K_V|}$, which implies the claim.

Therefore, we may assume that the unique center of canonical singularities of movable log pair (W, M_W) is a curve $\hat{C} = f^{-1}(C)$. Let $g : V \to W$ be a blow up of $\hat{C}$, let E be exceptional divisor of g, $Q_V = g^{-1}(Q)$ and $M_V = g^{-1}(M_W)$. Then

$$K_V + M_V \sim_{\mathbb{Q}} g^*(K_W + M_W) \sim_{\mathbb{Q}} (f \circ g)^*(K_X + M_X) \sim_{\mathbb{Q}} 0,$$

linear system $|-K_V|$ is free, $\phi_{|-K_V|} : V \to \mathbb{P}^2$ is an elliptic fibration with a section Q_V, boundary M_V lies in the fibers of $\phi_{|-K_V|}$ and $E \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Singularities of log pair (V, M_V) can not be terminal, otherwise $\kappa(V, \delta M_V) = 2$ for some rational number $\delta > 1$, but construction of M_Y implies that $\kappa(V, \delta M_V) \leq 1$. Therefore, there is a center of canonical singularities of log pair (V, M_V), which is contained in exceptional divisor E. On the other hand, set $CS(V, M_V)$ does not contains points and curves contracted by morphism g, because otherwise they images on W will be centers of canonical singularities of log pair (W, M_W), but we proved that the unique center of canonical singularities of log pair (W, M_W) is a curve $\hat{C}$. Thus, the set $CS(V, M_V)$ contains some curve $\tilde{C} \subset E$, such that curve $g(\tilde{C}) = \hat{C}$. Movable boundary M_V lies in the fibers of morphism $\phi_{|-K_V|}$. Hence, curve $\tilde{C}$ is contracted by morphism $\phi_{|-K_V|}$ to a point. However, double cover $\phi_{|-K_V|}|_E : E \to \mathbb{P}^2$ does not contract any curves. $\square$

In the rest of this chapter we will prove Theorem 6.4. Fix movable log pair (X, M_X), such that singularities of log pair (X, M_X) are canonical, $K_X + M_X \sim_{\mathbb{Q}} 0$ and $CS(X, M_X) \neq \emptyset$.

LEMMA 6.7. *Suppose that smooth point $P \in X$ is a center of canonical singularities of movable log pair (X, M_X). Then P is an Iskovskikh point, set $CS(X, M_X)$ consists of point P and $M_X = \psi^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$, where $\psi : X \dashrightarrow \mathbb{P}^1$ is a rational map given by Iskovskikh point P.*

Proof. Arguments of the proof of Lemma 5.5 implies that point P is an Iskovskikh point, equality $mult_P(M_X) = 2$ holds, $M_X = \psi^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$, where rational map $\psi : X \dashrightarrow \mathbb{P}^1$ is a rational map given by Iskovskikh point P and either P is a unique center of canonical singularities of log pair (X, M_X) or set $CS(X, M_X)$ contains a center of canonical singularities of log pair (X, M_X), which is a subset of some line C on quartic X passing through points P and O. Suppose, that $CS(X, M_X) \neq \{P\}$.

Consider general hyperplane section H_C of X passing through C. Then H_C is a K3 surface with one simple double point O and for some movable boundary B_C

$$M_X|_{H_C} = mult_C(M_X)C + B_C.$$

Moreover, we have

$$2 = mult_P(M_X) \leq mult_P(M_X|_{H_C}) = mult_C(M_X) + mult_P(B_C).$$

Thus, $mult_P(B_C) \geq 2 - mult_C(M_X)$ and on surface H_C

$$B_C^2 = 4 - 2mult_C(M_X) - \frac{3}{2}mult_C^2(M_X) \geq (2 - mult_C(M_X))^2,$$

which implies $mult_C(M_X) \leq \frac{4}{5}$. In particular, line C is not a center of canonical singularities of movable log pair (X, M_X). Therefore, either set $CS(X, M_X)$ contains another Iskovskikh point on line C or $CS(X, M_X)$ contains singular point O.

Suppose that another Iskovskikh points on line C is a center of canonical singularities of log pair (X, M_X). Then movability of B_C implies inequality

$$B_C^2 = 4 - 2mult_C(M_X) - \frac{3}{2}mult_C^2(M_X) \geq 2(2 - mult_C(M_X))^2,$$

which is impossible.

Hence, set $CS(X, M_X)$ contains singular point O. Therefore, Theorem 3.8 implies that equality $mult_O(M_X) = 1$ holds. Thus, $mult_O(B_C) \geq 1 - \frac{1}{2}mult_C(M_X)$ and

$$B_C^2 = 4 - 2mult_C(M_X) - \frac{3}{2}mult_C^2(M_X) \geq (2 - mult_C(M_X))^2 + 2(1 - \frac{1}{2}mult_C(M_X))^2,$$

which is impossible. □

We will assume that set $CS(X, M_X)$ does not contains smooth points on X and contains an irreducible and reduced curve C. In particular, canonicity of singularities of movable log pair (X, M_X) implies that $mult_C(M_X) = 1$. Moreover, it follows from the arguments of the proof of Lemma 5.6 that degree of curve $C \subset \mathbb{P}^4$ is not greater than 4.

LEMMA 6.8. *Curve C is contained in two-dimensional linear subspace in $\mathbb{P}^4$.*

Proof. Suppose that the claim is false. Then C is either a smooth curve or a rational curve of degree 4 with one double point. Moreover, in case $O \notin C$ the proof of Lemma 5.7 leads to a contradiction. So, we will assume that curve C contains point O. We will show that this assumption leads to a contradiction.

Suppose that C is smooth. Then degree of C is either 3 or 4. Let $f : W \to X$ be a blow up of point O, $g : V \to W$ be a blow up of curve $f^{-1}(C)$, $Q_V = (f \circ g)^{-1}(O)$, E be an exceptional divisor of g, $M_V = (f \circ g)^{-1}(M_X)$ and $d = -K_X \cdot C$. Then

$$M_V \sim_{\mathbb{Q}} (f \circ g)^*(-K_X) - mult_O(M_X)Q_V - E$$

and divisor $(f \circ g)^*(-dK_X) - Q_V - E$ is nef. On the other hand, we have

$$((f \circ g)^*(-dK_X) - Q_V - E) \cdot M_V^2 = 3d - d^2 + 2mult_O(M_X) - 2mult_O^2(M_X) + 2g(C) - 2 < 0.$$

Suppose now that curve C is a rational curve of degree 4 with one double point P, such that $P \neq O$. Let $h : U \to X$ be a composition of blow ups of points P and O with a blow up of strict transform of curve C, $M_U = h^{-1}(M_X)$, $Q_U = h^{-1}(O)$, $F = h^{-1}(P)$ and G be an irreducible exceptional divisor of h, such that $h(G) = C$. Then

$$M_U \sim_{\mathbb{Q}} h^*(-K_X) - mult_O(M_X)Q_U - mult_P(M_X)F - G$$

and linear system $|h^*(-4K_X) - Q_U - 2F - G|$ does not have base curves. Therefore,

$$(h^*(-4K_X) - Q_U - 2F - G) \cdot M_U^2 \geq 0.$$

On the other hand, $(h^*(-4K_X) - Q_U - 2F - G) \cdot M_U^2$ equals to

$$2mult_O(M_X) - 2mult_O^2(M_X) + 4mult_P(M_X) - 2mult_P^2(M_X) - 6 < 0.$$

Suppose now that curve C is a rational curve of degree 4 and point $O \in C$ is a double point on C. Let $f : W \to X$ be blow up of singular point O, $g : V \to W$ be blow up of curve $f^{-1}(C)$, $Q_V = (f \circ g)^{-1}(O)$, $E = g^{-1}(f^{-1}(C))$ and $M_V = (f \circ g)^{-1}(M_X)$. Then

$$M_V \sim_{\mathbb{Q}} (f \circ g)^*(-K_X) - mult_O(M_X)Q_V - E$$

and direct calculations imply

$$((f \circ g)^*(-4K_X) - Q_V - E) \cdot M_V^2 = 4mult_O(M_X) - 2mult_O^2(M_X) - 6 < 0,$$

but linear system $|(f \circ g)^*(-4K_X) - Q_V - E|$ does not have base curves. $\square$

LEMMA 6.9. *Suppose that C is a line and in case $O \in C$ suppose that point O is a center of canonical singularities of log pair (X, M_X). Then $M_X = \psi^{-1}(M_{\mathbb{P}^2})$ for some movable boundary $M_{\mathbb{P}^2}$ on $\mathbb{P}^2$, where rational map ψ is a projection from line C.*

Proof. In case when point O is not contained in curve C let $f : W \to X$ be a blow up of curve C, otherwise let $f : W \to X$ be a composition of a blow up of point O with a blow up of strict transform of curve C. Then linear system $|-K_W|$ is free and $\phi_{|-K_W|}$ is an elliptic fibration. On the other hand, $M_W \sim_{\mathbb{Q}} -K_W$ and $M_W \cdot L = 0$ for general enough fiber L of morphism $\phi_{|-K_W|}$. Therefore, M_W lies in fibers of elliptic fibration $\phi_{|-K_W|}$. $\square$

LEMMA 6.10. *Suppose that C is a line, $O \notin C$ and $CS(X, M_X)$ contains irreducible reduced curve L different from C. Then there is two-dimensional linear subspace $T \subset \mathbb{P}^4$ containing both curves C and L, such that*

$$CS(X, M_X) = \begin{cases} \{O, X \cap T\} & \text{in case } O \in T, \\ \{X \cap T\} & \text{in case } O \notin T. \end{cases}$$

Moreover, $M_X = \psi^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$, where rational map ψ is a projection from T.

Proof. Let $f : V \to X$ be a blow up of C, $E = f^{-1}(C)$ and $M_V = f^{-1}(M_X)$. Then linear system $|-K_V|$ is free, morphihsm $\phi_{|-K_V|} : V \to \mathbb{P}^2$ in an elliptic fibration and if follows from Lemma 6.9 that boundary M_V is contained in the fibers of $\phi_{|-K_V|}$. On the other hand, set $CS(V, M_V)$ contains curve $\hat{L} = f^{-1}(L)$. Thus, movability of M_V implies that curve $\hat{L}$ is contracted by morphism $\phi_{|-K_V|}$ to a point.

Let $\mathcal{H}$ be a pencil of surfaces in $|-K_V|$ passing through curve $\hat{L}$ and D be a general enough surface in $\mathcal{H}$. Then D is a K3 surfaces with at most canonical singularities and rational map $\phi_{\mathcal{H}} \circ f^{-1}$ is a projection from some two-dimensional linear subspace $T \subset \mathbb{P}^4$, such that curves C and L are contained in T.

In case when T does not contain singular point O it follows from the proof of Lemma 5.9 that set $CS(X, M_X)$ consists of all components of intersection $X \cap T$ and $M_V = \phi_{\mathcal{H}}^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$. Therefore, we will assume that T contains O.

Suppose, that $M_V = \phi_{\mathcal{H}}^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$ and $CS(X, M_X)$ contains all curves in $X \cap T$. Consider resolution of indeterminacy $h : U \to V$ of rational map $\phi_{\mathcal{H}}$, take general fiber S of morphism $\phi_{\mathcal{H}} \circ h^{-1}$ and put $M_U = h^{-1}(M_V)$. Then

$$M_U|_S \sim_{\mathbb{Q}} (1 - mult_O(M_X))Q_U|_S,$$

where Q_U is a h-exceptional divisor, whose image on quartic X is point O. Divisor Q_U is not contained in fibers of $\phi_{\mathcal{H}} \circ h^{-1}$. Therefore, $O \in CS(X, M_X)$. Hence, to conclude the proof we must show that all curves in $X \cap T$ are centers of canonical singularities of movable log pair (X, M_X) and $M_V = \phi_{\mathcal{H}}^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$.

Suppose that L is a plane cubic. Then for some movable boundary B_D on surface D

$$M_X|_D = \hat{L} + B_D.$$

However, base locus of pencil $\mathcal{H}$ consists of curve $\hat{L}$ and $M_X|_D \sim_{\mathbb{Q}} \hat{L}$. Therefore, movable boundary B_D is empty and $M_V = \phi_{\mathcal{H}}^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$.

Suppose that C is cutted on X by two-dimensional linear subspace $T \subset \mathbb{P}^4$ with multiplicity 3. Then L is a line, $T \cap X = C \cup L$ and base locus of pencil $\mathcal{H}$ consists of curve $\hat{L}$ and smooth rational curve $\hat{C} \subset E$. We have

$$M_V|_D = \alpha\hat{C} + \hat{L} + B_D$$

for some rational number $\alpha \geq mult_{\hat{C}}(M_V)$ and movable boundary B_D, but on surface D movable boundary B_D is $\mathbb{Q}$-rationally equivalent to $(2 - \alpha)\hat{C}$ and $\hat{C}^2 < 0$. Therefore, boundary B_D is empty. On the other hand, base locus of pencil $\mathcal{H}$ consists of curves $\hat{C}$ and $\hat{L}$. Hence, $M_V = \phi_{\mathcal{H}}^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$.

Suppose that line C is cutted on X by T with multiplicity 2 and L is an irreducible reduced conic on X. In this case $T \cap X = C \cup L$. Consider restriction

$$M_V|_D = mult_{\hat{C}}(M_V)\hat{C} + \hat{L} + B_D,$$

where $\hat{L}$ and $\hat{C} \subset E$ are base curves of pencil $\mathcal{H}$ and B_D is a movable boundary. On surface D restriction $M_V|_D$ is $\mathbb{Q}$-rationally equivalent to $\hat{C} + \hat{L}$ and $\hat{C}^2 < 0$. So, boundary B_D is empty and $mult_{\hat{C}}(M_V) = 1$. Hence, the fact that base locus of linear system $\mathcal{H}$ consists of curves $\hat{C}$ and $\hat{L}$ implies that $M_V = \phi_{\mathcal{H}}^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$.

Suppose that line C is cutted on quartic X by two-dimensional linear subspace $T \subset \mathbb{P}^4$ with multiplicity 2, but $T \cap X = C \cup L \cup L_1$, where L and L_1 are two different lines on quartic 3-fold X. Consider restriction

$$M_V|_D = mult_{\hat{C}}(M_V)\hat{C} + \hat{L} + mult_{L_1}(M_X)\hat{L}_1 + B_D,$$

where $\hat{L}$, $\hat{L}_1 = f^{-1}(L_1)$ and $\hat{C} \subset E$ are base curves of pencil $\mathcal{H}$ and B_D is a movable boundary. On surface D restriction $M_V|_D$ is $\mathbb{Q}$-rationally equivalent to $\hat{C} + \hat{L} + \hat{L}_1$ and intersection form of curves $\hat{C}$ and $\hat{L}_1$ is negative definite. Therefore, boundary B_D is empty and $mult_{\hat{C}}(M_V) = mult_{L_1}(M_X) = 1$. In particular, line L_1 is a center of canonical singularities of movable log pair (X, M_X). Base locus of pencil $\mathcal{H}$ consists of curves $\hat{C}$, $\hat{L}$ and $\hat{L}_1$. Thus, movability of M_V and $M_V|_D = \hat{C} + \hat{L} + \hat{L}_1$ imply that $M_V = \phi_{\mathcal{H}}^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$.

Suppose that C is cutted on X by T with multiplicity 2, L is a line containing point O and $T \cap X = C \cup L$. Let $g : W \to X$ be a composition of blow up of point O and blow up of strict transform of line L, $\mathcal{H}_W = g^{-1}(\mathcal{H}_T)$, $Q_W = g^{-1}(O)$ and E_W be an exceptional divisor of g dominating L. Then linear system $|-K_W|$ is free and morphism $\phi_{|-K_W|}$ is an elliptic fibration with a section Q_W. Assumption that line L is cutted on quartic X by T with multiplicity 2 implies that pencil $\mathcal{H}_W$ has fixed curve $\hat{L}$ contained in E. On the other hand, $\mathcal{H}_W \subset |-K_W|$. Hence, curve $\hat{L}$ is contracted by morphism $\phi_{|-K_W|}$ to a point, but double cover $\phi_{|-K_V|}|_E : E \to \mathbb{P}^2$ does not contract any curves.

So we may assume that C is cutted on X by two-dimensional subspace $T \subset \mathbb{P}^4$ with multiplicity 1. Suppose that L is a conic. Then $T \cap X = C \cup L \cup R$, where R is a line on quartic X different from C. Consider restriction

$$M_V|_D = \hat{L} + mult_R(M_X)\hat{R} + B_D,$$

where $\hat{R} = f^{-1}(R)$. On surface D we have $M_V|_D \sim_{\mathbb{Q}} \hat{L} + \hat{R}$ and $\hat{R}^2 < 0$. Therefore, boundary B_D is empty and $mult_R(M_X) = 1$. Hence, line R is a center of canonical singularities of movable log pair (X, M_X). On the other hand, base locus of pencil $\mathcal{H}$ consists of curves $\hat{L}$ and $\hat{R}$. Thus, from the movability of M_V and $M_V|_D = \hat{L} + \hat{R}$ it follows that $M_V = \phi_{\mathcal{H}}^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$.

So, we may assume that both C and L are lines. Moreover, we already proved that in case when line L pass through singular point O line L is cutted on quartic X by T with multiplicity 1. On the other hand, we already considered case when set $CS(X, M_X)$ has two lines in $X \cap T$, such that both of them do not contain point O and one of them is cutted on quartic X by plane T with multiplicity greater or equal than 2. Thus, we will assume that C and L are cutted on quartic X by T with multiplicity 1.

Let $M_V|_D = \hat{L} + \sum_{i=1}^{k} \alpha_i C_i + B_D$, where C_i are base curves of $\mathcal{H}$, α_i are non-negative rational numbers and B_D is a movable boundary. Then $X \cap T = L \cup C \cup_{i=1}^{k} f(C_i)$ and on surface D we have $M_V|_D \sim_{\mathbb{Q}} D|_D \sim \hat{L} + Z$, where

$$Z = \begin{cases} C_1 + C_2, \text{ where } f(C_1) \text{ and } f(C_2) \text{ are different lines on } X, \\ C_1, \text{ where } f(C_1) \text{ is a reduced irreducible conic on } X, \\ 2C_1, \text{ where } f(C_1) \text{ is a line on } X \text{ and } O \notin f(C_1). \end{cases}$$

However, intersection form of curves C_i on surface D is negative definite. Therefore, boundary B_D is empty, $M_V = \phi_{\mathcal{H}}^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$ and explicit structure of M_V implies that curves $f(C_i)$ are centers of canonical singularities of log pair (X, M_X). $\square$

LEMMA 6.11. *Suppose that C is a line passing through singular point O and some irreducible reduced curve $L \neq C$ on X is a center of canonical singularities of movable* *log pair (X, M_X). Then there is two-dimensional linear subspace $T \subset \mathbb{P}^4$ containing both curves C and L, such that $CS(X, M_X) = \{O, X \cap T\}$ and $M_X = \psi^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$, where ψ is a projection from T.*

Proof. Take general hyperplane section S of X passing through C. Surface S is a K3 surface with one simple double point O. Consider restriction $M_X|_S = C + B_S$, where B_S is a movable boundary on surface S. Suppose that curves C and L are not contained in any two-dimensional linear subspace of $\mathbb{P}^4$. Then intersection $S \cap L$ contains point P, which does not belong to line C. Therefore, on surface S

$$1 > B_S^2 \geq mult_P^2(B_S) \geq mult_P^2(M_X) = 1.$$

Hence, there is two-dimensional linear subspace $T \subset \mathbb{P}^4$ containing curves C and L.

Suppose, that $CS(X, M_X)$ contains all curves in $X \cap T$ and $M_X = \psi^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$ and projection ψ from T. Let $h : U \to V$ be a resolution of indeterminacy of ψ, S_U be a general fiber of $\psi \circ h^{-1}$ and $M_U = h^{-1}(M_X)$. Then

$$M_U|_{S_U} \sim_{\mathbb{Q}} (1 - mult_O(M_X))Q_U|_{S_U},$$

where Q_U is a h-exceptional divisor, such that $h(Q_U) = O$. However, divisor Q_U is not contained in fibers of $\psi \circ h^{-1}$. Therefore, $mult_O(M_X) = 1$ and singular point O is a center of canonical singularities of log pair (X, M_X). Hence, to conclude the proof we must show that set $CS(X, M_X)$ contains all components of $X \cap T$ and $M_X = \psi^{-1}(M_{\mathbb{P}^1})$, where ψ is a projection from T and $M_{\mathbb{P}^1}$ is a boundary on $\mathbb{P}^1$.

Let $\mathcal{H}_T$ be a pencil of all hyperplane sections of quartic X passing through $X \cap T$ and let D_T be a general enough surface in $\mathcal{H}_T$. Then D_T is a K3 surface with at most canonical singularities.

Suppose that L is a plane cubic. For some movable boundary B_{D_T} on surface D_T

$$M_X|_{D_T} = L + C + B_{D_T}.$$

However, base locus of $\mathcal{H}_T$ consists of curves C and L and $M_X|_{D_T} \sim_{\mathbb{Q}} L + C$. Therefore, boundary B_{D_T} is empty and $M_X = \phi_{\mathcal{H}_T}^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$.

Suppose that L is an irreducible reduced conic. Then

$$M_X|_{D_T} = L + C + mult_R(M_X)R + B_{D_T},$$

where R is a line in $X \cap T$ and B_{D_T} is a movable boundary on surface D_T. On the other hand, on surface D_T boundary B_{D_T} is $\mathbb{Q}$-rationally equivalent to $(1 - mult_R(M_X))R$ and inequality $R^2 < 0$ holds. Therefore, boundary B_{D_T} is empty and line R is a center of canonical singularities of log pair (X, M_X). Moreover, base locus of pencil $\mathcal{H}_T$ consists of curves C, L and R. So, $M_X = \phi_{\mathcal{H}_T}^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$.

Thus, we may assume that L is a line on X. However, in case when point O is not contained in L the claim is follows from Lemma 6.10. Thus, we will assume that line L

contains point O. Moreover, from the proof of Lemma 6.10 it follows that both lines C and L are cutted on X by T with multiplicity 1.

Let $X \cap T = L \cup C \cup_{i=1}^{k} C_i$. Then on surface D_T

$$M_X|_{D_T} = C + L + \sum_{i=1}^{k} \alpha_i C_i + B_{D_T},$$

where α_i are non-negative rational numbers and B_{D_T} is a movable boundary. However, on surface D restriction $M_X|_{D_T}$ is $\mathbb{Q}$-rationally equivalent to $C + L + Z$, where

$$Z = \begin{cases} C_1 + C_2, \text{ where } C_1 \text{ and } C_2 \text{ are different lines on } X, \\ C_1, \text{ where } C_1 \text{ is a reduced irreducible conic on } X, \\ 2C_1, \text{ where } C_1 \text{ is a line on } X \text{ and } O \notin C_1. \end{cases}$$

In every case the intersection form of curves C_i on surface D_T is negative definite, which implies that boundary B_{D_T} is empty. On the other hand, curves C, L and C_i are base curves of pencil $\mathcal{H}_T$. Thus, $M_V = \phi_{\mathcal{H}}^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$ and it follows from the explicit structure of boundary M_V that curves C_i are centers of canonical singularities of log pair (X, M_X). □

LEMMA 6.12. *Suppose C is a curve of degree 4, which is contained in two-dimensional linear subspace $T \subset \mathbb{P}^4$. Then $M_X = \psi^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$, where ψ is a projection from T. Moreover, point O is a center of canonical singularities of log pair (X, M_X) if T contains O.*

Proof. Consider a pencil $\mathcal{H}_T$ of hyperplane sections of X passing through T and resolve indeterminacy of map $\phi_{\mathcal{H}_T}$ by means of birational morphism $f : W \to X$, such that 3-fold W contains one exceptional divisor E dominating curve C and f is an isomorphism outside of C. Let D be a general fiber of morphism $g = \phi_{\mathcal{H}_T} \circ f^{-1}$ and $M_W = f^{-1}(M_X)$. Then

$$D \sim f^*(-K_X) - E - \sum_{i \in I} a_i F_i,$$

where all $f(F_i)$ are points on curve C and a_i are natural numbers. On the other hand,

$$M_W|_D \sim_{\mathbb{Q}} \sum_{i \in I} c_i F_i|_D$$

for some rational numbers c_i. Now movability of $M_W|_D$ implies $M_W|_D = \emptyset$. Therefore, movable boundary M_W lies in the fibers of morphism g. Hence, $M_W = g^{-1}(M_{\mathbb{P}^1})$ for some boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$.

Suppose now that curve C contains point O. In this case we may choose morphism f in such a way that $f = g \circ h$, where $g : V \to X$ is a blow up of O and morphism h is birational. Let Q be an exceptional divisor of g and $Q_W = h^{-1}(Q)$. Then $F_j = Q_W$ for some $j \in I$ and restriction $F_j|_D$ is non-trivial. However, $c_j = 1 - mult_O(M_X)$. Therefore, point O is a center of canonical singularities of log pair (X, M_X). $\square$

Lemma 6.13. *Suppose that curve C is either a conic or a plain cubic, which is contained in two-dimensional linear subspace $T \subset \mathbb{P}^4$. Then*

$$CS(X, M_X) = \begin{cases} \{O, X \cap T\} \text{ in case } O \in T, \\ \{X \cap T\} \text{ in case } O \notin T. \end{cases}$$

and $M_X = \psi^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$, where rational map map ψ is a projection from T.

Proof. Let $\mathcal{H}_T$ be a pencil of hyperplane sections of X passing through T and D be a general enough surface in $\mathcal{H}_T$. Then D is a K3 surface with at most canonical singularities.

Suppose that C is a plane cubic. Then $X \cap T = C \cup L$ and

$$M_X|_D = C + mult_L(M_X)L + B_D,$$

where L is a line on X and B_D is a movable boundary. However, on surface D boundary B_D is $\mathbb{Q}$-rationally equivalent to $(1 - mult_L(M_X))L$ and $L^2 < 0$. Therefore, line L is a center of canonical singularities of log pair (X, M_X) and Lemmas 6.10 and 6.11 imply the claim.

Suppose that C is a conic and $X \cap T = C \cup L_1 \cup L_2$ for two different lines L_1 and L_2 on quartic X. Then

$$M_X|_D = C + mult_{L_1}(M_X)L_1 + mult_{L_2}(M_X)L_2 + B_D$$

for some movable boundary B_D. On the other hand, on surface D

$$B_D \sim_{\mathbb{Q}} (1 - mult_{L_1}(M_X))L_1 + (1 - mult_{L_2}(M_X))L_2$$

and intersection form of lines L_1 and L_2 is negative definite. Therefore, lines L_1 and L_2 are centers of canonical singularities of (X, M_X) and Lemmas 6.10 and 6.11 imply the claim.

Suppose that C is a conic and $X \cap T = C \cup R$ for a line R cutted on X by two-dimensional linear subspace $T \subset \mathbb{P}^4$ with multiplicity 2. Then

$$M_X|_D = C + \alpha R + B_D$$

for some rational number $\alpha \geq mult_R(M_X)$ and a movable boundary B_D. On surface D boundary B_D is $\mathbb{Q}$-rationally equivalent to $(2 - \alpha)R$ and $R^2 < 0$. Therefore, boundary B_D is empty. Thus, $M_X = \phi^{-1}_{\mathcal{H}_T}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$, because base locus of pencil $\mathcal{H}_T$ consists of curves C and R. Moreover, explicit structure of boundary M_X implies that $mult_R(M_X) = 1$. Hence, line R is a center of canonical singularities of log pair (X, M_X).

Suppose now that C is a conic, $X \cap T = C$ and conic C is cutted on X by T with multiplicity 2. The claim is implied by Lemma 5.11 in case when conic C does not pass through O. Therefore, we will assume that C contains O. Let $f : V \to X$ be a composition of blow up of point O with a blow up of strict transform of C, $Q_V = f^{-1}(O)$, E be an exceptional divisor of f dominating conic C, $M_V = f^{-1}(M_X)$ and S be a general enough surface in pencil $|-K_V| = f^{-1}(\mathcal{H}_T)$. Then for some movable boundary B_S

$$M_V|_S = mult_{\hat{C}}(M_V)\hat{C} + \sum_{i=1}^{k} \alpha_i Z_i + B_S,$$

where $\hat{C} \subset E$ and $Z_i \subset Q_V$ are a base curves of pencil $|-K_V|$. However, on surface S boundary B_S is $\mathbb{Q}$-rationally equivalent to $mult_{\hat{C}}(M_V)\hat{C} + \sum_{i=1}^{k} \beta_i Z_i$ for some rational numbers β_i. On the other hand, intersection form of curves $\hat{C}$ and Z_i on surface S is negative definited. Therefore, B_S is empty and $M_V = \phi_{|-K_V|}^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$. Now the explicit structure of M_X imples that O is a center of canonical singularities of log pair (X, M_X). $\square$

Thus, Theorem 6.4 is proved.

§7. Double cover of quadric.

In this section we will prove Main and Auxiliary Theorems for smooth Fano double cover of quadric 3-fold.

Let $\theta : X \to Q \subset \mathbb{P}^4$ be a double cover of a smooth 3-fold quadric Q ramified in smooth octic surface $S \subset Q$. Then $Pic(X) = -\mathbb{Z}K_X$ and anticanonical divisor $-K_X$ is a pull back of a hyperplane section of quadric Q. In chapter 4 for every irreducible reduced curve C on X with $-K_X \cdot C = 1$ we explicitly constructed birational map $\psi_C : X \dashrightarrow X_C$, such that X_C is a Fano 3-fold with terminal singularities and $-K_{X_C}^3 = \frac{1}{2}$.

Proposition 7.1. *Suppose, that there is a birational map $\sigma : X \dashrightarrow Y$, where Y is a Fano 3-fold with canonical singularities. For $n \gg 0$ put $M_Y = \frac{1}{n}|-nK_Y|$ and consider log pairs (X, M_X) and (X_C, M_{X_C}) birationally equivalent to (Y, M_Y). Then up to the action of group $Bir(X)$ either log pair (X, M_X) or one of log pairs (X_C, M_{X_C}) has semiterminal singularities and $\mathbb{Q}$-rationally trivial log canonical divisor.*

Lemma 7.2. *Main Theorem for 3-fold X follows from the Proposition 7.1.*

Proof. See proof of Lemma 6.2. $\square$

The following result proved in [2] is equivalent to birational rigidity of X.

PROPOSITION 7.3. *Let (X, M_X) be a movable log pair with $\kappa(X, M_X) = 0$. Then for some $\tau \in Bir(X)$ the singularities of $(X, \tau(M_X))$ are canonical and $K_X + \tau(M_X) \sim_{\mathbb{Q}} 0$.*

To prove Proposition 7.1 we will need the description of the locus of canonical singularities of log pairs with Kodaira dimension zero on 3-fold quartic X.

THEOREM 7.4. *Let (X, M_X) be a log pair with canonical singularities and $\mathbb{Q}$-rationally trivial log canonical divisor. Then*

$$CS(X, M_X) = \begin{cases} \emptyset, \\ \{C\} \text{ for a curve } C \subset X \text{ with } -K_X \cdot C = 1, \\ \{C\} \text{ for a curve } C \subset X, \text{ such that } \theta(C) \text{ is a line and } -K_X \cdot C = 2, \\ \{C_1 \cup C_2\}, \text{ where } \theta(C_1) = \theta(C_2) \text{ is a line and } -K_X \cdot (C_1 + C_2) = 2, \\ \{\cup_{i \in I} C_i, \cup_{j \in J} P_j\}, \text{ where } C_i \text{ are all base curves of pencil } \mathcal{H} \subset |-K_X| \\ \qquad\qquad \text{and } P_j \text{ are singular points of all surfaces in } \mathcal{H}. \end{cases}$$

and one of the following holds: log pair (X, M_X) is terminal; set $CS(X, M_X)$ consists of some curve C with $-K_X \cdot C = 1$, $(\psi_C(X), \psi_C(M_X))$ has semiterminal singularities and $\mathbb{Q}$-rationally trivial log canonical divisor, where ψ_C is a map from Proposition 1.4; set $CS(X, M_X)$ contains some point P, $\theta(P) \in S$ and $M_X = \psi_P^{-1}(M_{\mathbb{P}^1})$ for a movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$, where ψ_P is a composition of θ with a projection from tangent space to surface S at point $\theta(P)$; $M_X = \psi^{-1}(M_Y)$, where $\psi : X \dashrightarrow Y$ is a composition of θ with a projection from locus $\theta(CS(X, M_X))$.

PROPOSITION 7.5. *Theorem 7.4 implies Main Theorem for 3-fold X.*

Proof. Suppose that there is a birational map $\sigma : X \dashrightarrow Y$, where Y is a Fano 3-fold with canonical singularities. Let $M_Y = \frac{1}{n}|-nK_Y|$ for some $n \gg 0$ and $M_X = \sigma^{-1}(M_Y)$. Then $\kappa(X, M_X) = \kappa(Y, M_Y) = 0$ and by Proposition 7.3 we may compose σ with some birational automorphism of X and assume that movable log pair (X, M_X) has canonical singularities and $K_X + M_X \sim_{\mathbb{Q}} 0$. Therefore, Theorem 7.4 and structure of M_Y imply that either singularities of (X, M_X) are terminal or $CS(X, M_X) = \{C\}$ for some curve C on 3-fold X with $-K_X \cdot C = 1$, singularities of movable log pair (X_C, M_{X_C}) are semiterminal and $K_{X_C} + M_{X_C} \sim_{\mathbb{Q}} 0$, where $M_{X_C} = \psi_C(M_X)$ and $\psi_C : X \dashrightarrow X_C$ is a birational map from Proposition 1.4. Now Proposition 7.1 implies the claim. □

PROPOSITION 7.6. *Theorem 7.4 implies Auxiliary Theorem for 3-fold X.*

Proof. Suppose that X is birationally transformed by birational map $\sigma : X \dashrightarrow Y$ into some fibration $\tau : Y \to Z$, such that the general enough fiber of fibration τ is either an elliptic curve or a surface of Kodaira dimension zero. Let $M_X = \lambda\sigma^{-1}(|\tau^*(D_Z)|)$ for some big enough divisor D_Z and some $\lambda \in \mathbb{Q}_{>0}$. Then by Proposition 7.3 we may compose σ

with some birational automorphism of X and choose such positive rational λ, that movable log pair (X, M_X) has canonical singularities and $K_X + M_X \sim_{\mathbb{Q}} 0$.

Suppose that singularities of log pair (X, M_X) are terminal. For some positive rational number γ movable log pair $(X, \gamma M_X)$ is a canonical model. On the other hand, construction of M_Y implies that $\kappa(Y, \gamma M_Y) < 3$. Therefore, $CS(X, M_X) \neq \emptyset$.

Let L be a curve on 3-fold X with $-K_X \cdot L = 1$. Consider movable log pair (X_L, M_{X_L}), where $M_{X_L} = \psi_L(M_X)$ and $\psi_L : X \dashrightarrow X_L$ is a birational map from Proposition 1.4. Suppose that singularities of log pair (X_L, M_{X_L}) are semiterminal and its log canonical divisor is $\mathbb{Q}$-rationally trivial. Then movable log pair $(X_L, \gamma M_{X_L})$ is a canonical model for some positive rational number γ, which contradicts inequality $\kappa(Y, \gamma M_Y) < 3$.

Thus, by Theorem 7.4 either set $CS(X, M_X)$ contains some point P and $\theta(P) \in S$ or

$$CS(X, M_X) = \begin{cases} \{L\} \text{ for a curve } L \subset X \text{ with } -K_X \cdot L = 1, \\ \{C\} \text{ for a curve } C \subset X, \text{ such that } \theta(C) \text{ is a line and } -K_X \cdot C = 2, \\ \{L_1 \cup L_2\}, \text{ where } \theta(L_1) = \theta(L_2) \text{ is a line and } -K_X \cdot (L_1 + L_2) = 2, \\ \{\cup_{i \in I} C_i\}, \text{ where } C_i \text{ are all base curves of some pencil } \mathcal{H} \subset |-K_X|. \end{cases}$$

and in both cases $M_X = \psi^{-1}(M_T)$, where $\psi : X \dashrightarrow T$ is either a composition of θ with a projection from locus $\theta(CS(X, M_X))$ or a composition of double cover θ and a projection from tangent space to surface S at $\theta(P)$ in case $P \in CS(X, M_X)$.

In particular, in case when set $CS(X, M_X)$ contains some point P on 3-fold X it follows from Theoren 7.4 that τ is a fibration on K3 surfaces and τ is equivalent to the fibration given projection from the tangent space to surface S at $\theta(P)$. In case when set $CS(X, M_X)$ consists of all base curves of some pencil $\mathcal{H}$ in anticanonical linear system $|-K_X|$ and image of locus $CS(X, M_X)$ on Q is not a line Theorem 7.4 implies that τ is a fibration on K3 surfaces, which is equivalent to the fibration given by rational map $\phi_{\mathcal{H}}$. In case when τ is elliptic fibration Theorem 7.4 implies that τ is birationally equivalent as a fibration to a fibration given by composition of double cover θ with a projection from some line on Q.

Thus, we may assume that τ is a fibration on surfaces of Kodaira dimension zero,

$$CS(X, M_X) = \begin{cases} \{L\} \text{ for a curve } C \subset X \text{ with } -K_X \cdot L = 1, \\ \{C\} \text{ for a curve } C \subset X, \text{ such that } \theta(C) \text{ is a line and } -K_X \cdot C = 2, \\ \{L_1 \cup L_2\}, \text{ where } \theta(L_1) = \theta(L_2) \text{ is a line and } -K_X \cdot (L_1 + L_2) = 2, \end{cases}$$

and $M_X = \psi^{-1}(M_{\mathbb{P}^2})$ for some movable boundary $M_{\mathbb{P}^2}$ on $\mathbb{P}^2$, where map $\psi : X \dashrightarrow \mathbb{P}^2$ is a composition of double cover θ with a projection from line $\theta(CS(X, M_X))$. To conclude the proof we must show that $M_X = \phi_{\mathcal{H}}^{-1}(M_{\mathbb{P}^1})$ for some pencil $\mathcal{H}$ in anticanonical linear system $|-K_X|$ and some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$.

Suppose that the unique center of canonical singularities of log pair (X, M_X) is a smooth rational curve L and $-K_X \cdot L = 1$. Let $f : W \to X$ be a blow up of L, E be an exceptional divisor of f, $M_W = f^{-1}(M_X)$ and $\hat{L}$ be a base curve of linear system $|-K_W|$. Then $\hat{L}$ is a smooth rational curve and construction of birational map ψ_L from Proposition 1.4 implies that for some integer $n \gg 0$

$$\psi_L = \phi_{|-nK_{\hat{W}}|} \circ \rho,$$

where $\rho : W \dashrightarrow \hat{W}$ is an antiflip in curve $\hat{L}$.

Suppose, that singularities of log pair (W, M_W) are terminal. Take $\zeta \in \mathbb{Q}_{>1}$, such that singularities of log pair $(W, \zeta M_W)$ are still terminal. Let $M_{\hat{W}} = \rho(M_W)$. Then singularities of movable log pair $(\hat{W}, \zeta M_{\hat{W}})$ are terminal as well, because ρ is a log flip for movable log pair $(W, \zeta M_W)$. On the other hand, log canononical divisor $K_{\hat{W}} + M_{\hat{W}}$ is $\mathbb{Q}$-rationally trivial. In particular, morphism $\phi_{|-nK_{\hat{W}}|}$ is crepant for terminal log pair $(\hat{W}, \zeta M_{\hat{W}})$. Hence, log pair $(X_L, \psi_L(M_X))$ has semiterminal singularities and $\mathbb{Q}$-rationally trivial canonical divisor, but we already proved that this is impossible. Therefore, singularities of movable log pair (W, M_W) are not terminal and equivalence

$$K_W + M_W \sim_{\mathbb{Q}} f^*(K_X + M_X)$$

implies that set $CS(W, M_W)$ contains an irrreducible curve $T \subset E$, such that $f(T) = C$.

Suppose that $T \neq \hat{L}$ and $\hat{L} \subset E$. Then intersecting M_W with a general enough fiber of morphism $f|_E : E \to C$ we get $E \subset M_W$, which contradicts the movability of M_W.

Suppose that $T \neq \hat{L}$ and $\hat{L} \not\subset E$. Let D_W be a general enough surface in anticanonical linear system $|-K_W|$. Then D_W is a smooth K3 surface, $T \cap D_W \neq \emptyset$ and

$$M_W|_{D_W} = mult_{\hat{L}}(M_W)\hat{L} + R_{D_W} \sim_{\mathbb{Q}} \hat{L} + F$$

for some movable boundary R_{D_W} and elliptic curve F, such that $F \cdot \hat{L} = 1$. On the other hand, $M_W \cdot \hat{L} = -1$. Therefore, $mult_{\hat{L}}(M_W) > 0$. Hence, intersecting $M_W|_{D_W}$ with curve in pencil $|F|$ passing through a point of intersection $T \cap D_W$ we get a contradiction.

Therefore, $T = \hat{L}$, curve $\hat{L}$ is contained in surface E, image of L on quadric Q is contained in S and $\hat{L}$ is an exceptional section of ruled surface E. Let $g : V \to W$ be a blow up of curve $\hat{L}$, $G_V = f^{-1}(\hat{L})$ and $M_V = g^{-1}(M_W)$. Then linear system $|-K_V|$ is free, morphism $\phi_{|-K_V|} : V \to \mathbb{P}^2$ is an elliptic fibration with a section $G_V \cong \mathbb{F}_1$ and $M_V \cdot K_V^2 = 0$. Hence, boundary M_V lies in the fibers of $\phi_{|-K_V|}$.

Singularities of log pair (V, M_V) are not terminal, because in case when singularity of log pair (V, M_V) are terminal $\kappa(V, \gamma M_V) = 2$ for every rational number $\gamma > 1$, but by construction $\kappa(X, \gamma M_X) \leq 1$. On the other hand, image of every center of canonical singularities of (V, M_V) on 3-fold X is a center of canonical singularities of log pair (X, M_X). Thus, set $CS(V, M_V)$ contains some curve $\tilde{L} \subset G_V$. However, we already proved that M_V is contained in fibers of elliptic fibration $\phi_{|-K_V|}$. Therefore, curve $\tilde{L}$ is contracted by morphism $\phi_{|-K_V|}$ to a point. Hence, $\tilde{L}$ is an exceptional section of surface $G_V \cong \mathbb{F}_1$.

Let $h : U \to V$ be a blow up of $\tilde{L}$, $R = h^{-1}(\tilde{L})$, $M_U = h^{-1}(M_V)$, D be a general enough surface in pencil $|-K_U|$. Then D is a K3 surface with at most canonical singularities and on surface D

$$M_U|_D = \sum_{i=1}^{k} \alpha_i \tilde{L}_i + B_D \sim_{\mathbb{Q}} D|_D \sim \sum_{i=1}^{k} \beta_i \tilde{L}_i,$$

where $\tilde{L}_i$ are base curves of pencil $|-K_U|$, B_D is a movable boundary on surfaces D, α_i are non-negative rational numbers and β_i are natural numbers. On the other hand, the intersection form of curves $\tilde{L}_i$ on surface D is negative definite. Therefore, boundary B_D is empty, which implies that $M_U = \phi^{-1}_{|-K_U|}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$.

Suppose that the unique center of canonical singularities of (X, M_X) is a curve C, such that $-K_X \cdot C = 2$ and image of C on quadric Q is a line. Then C is either a smooth elliptic curve or a rational curve with one singular point. Let $\mathcal{H}_C$ be a linear subsystem of $|-K_X|$ consisting of surfaces passing through curve C. Then map $\phi_{\mathcal{H}_C} : X \dashrightarrow \mathbb{P}^2$ is a composition of a double cover θ and a projection from line $\theta(C)$. Let $r : V_C \to X$ be a birational map, such that 3-fold V_C has terminal singularities, morphism r has exactly one exceptional divisor G, surface G dominates curve C, linear system $\mathcal{H}_{V_C} = r^{-1}(\mathcal{H}_C)$ is free and morphism $\phi_{\mathcal{H}_{V_C}}$ is an elliptic fibration. Morphism r is a terminal blow up for canonical log pair $(X, \mathcal{H}_C)$ and in case when C is smooth birational morphism r is a blow up of curve C. Explicit discription of birational morphism r in case when C is singular is the following: blow up the singular point of C, blow up the strict transform of curve C and make an antiflip in the strict transform of a line in exceptional divisor of blow up of singular point of C, which passes through two points of intersection of strict transform of curve C with the exceptional divisor of blow up of singular point of curve C.

Let $M_{V_C} = r^{-1}(M_X)$. Then equality $mult_C(M_X) = 1$ implies $\mathcal{H}^2_{V_C} \cdot M_{V_C} = 0$. Therefore, boundary M_{V_C} lies in the fibers of elliptic fibration $\phi_{\mathcal{H}_{V_C}}$ and boundary M_{V_C} is $\mathbb{Q}$-rationally equivalent to $\mathcal{H}_{V_C}$. Hence, singularities of log pair (V_C, M_{V_C}) are not terminal, because otherwise $\kappa(V_C, \gamma M_{V_C}) = 2$ for every rational number $\gamma > 1$, but construction of movable boundary M_X implies $\kappa(X, \gamma M_X) \leq 1$. Thus, set $CS(V_C, M_{V_C})$ contains curve $\hat{C}$, such that $r(\hat{C}) = C$. However, we already proved that boundary M_{V_C} is contained in the fibers of elliptic fibration $\phi_{|-K_V|}$. Therefore, $\hat{C}$ is contracted by morphism $\phi_{|-K_V|}$ to a point. Thus, curve $\hat{C}$ is an exceptional section of ruled surface G. Let $\mathcal{H}_{\hat{C}}$ be a pencil in linear system $\mathcal{H}_{V_C}$ containing surfaces passing through curve $\hat{C}$ and $D_{\hat{C}}$ be a general enough surface in $\mathcal{H}_{V_C}$. Then on surface $D_{\hat{C}}$

$$M_{V_C}|_{D_{\hat{C}}} = mult_{\hat{C}}(M_{V_C})\hat{C} + B_{D_{\hat{C}}} \sim_{\mathbb{Q}} D_{\hat{C}}|_{D_{\hat{C}}} \sim \hat{C}.$$

Therefore, $M_{V_C}|_{D_{\hat{C}}} = \hat{C}$, but curve $\hat{C}$ is a base locus of pencil $\mathcal{H}_{\hat{C}}$. Thus, $M_{V_C} = \phi^{-1}_{\mathcal{H}_{\hat{C}}}(M_{\mathbb{P}^1})$ for some movable boundary on $\mathbb{P}^1$.

To conclude the proof we may assume that set $CS(X, M_X)$ consists of two different smooth rational curves L_1 and L_2, which are mapped by double cover θ to one line on quadric Q. Let $s : \bar{X} \to X$ be a composition of a blow up of L_1 with a blow up of strict transform of L_2, $M_{\bar{X}} = s^{-1}(M_X)$, E_1 and L_2 be exceptional divisors of morphism s dominating curves L_1 and E_2 correspondingly. Then linear system $|-K_{\bar{X}}|$ is free, morphism $\phi_{|-K_{\bar{X}}|}$ is an elliptic fibration with two sections E_1 and E_2, boundary $M_{\bar{X}}$ lies in the fibers of fibration $\phi_{|-K_{\bar{X}}|}$ and $E_2 \cong \mathbb{F}_1$. Hence, singularities of log pair $(\bar{X}, M_{\bar{X}})$ are not terminal, because otherwise $\kappa(\bar{X}, \gamma M_{\bar{X}}) = 2$ for every rational number $\gamma > 1$, but by construction $\kappa(X, \gamma M_X) \leq 1$. Therefore, set $CS(V_C, M_{V_C})$ contains some curve $\bar{L}_2$, such that image of curve $\bar{L}_2$ on 3-fold X is either a curve L_1 or a curve L_2. Moreover, we may assume that $\bar{L}_2$ dominates curve L_2, because otherwise we may consider birational morphism s to be a composition of blow up of curve L_2 with a composition of blow up of strict transform of curve L_1. Movable boundary $M_{\bar{X}}$ is contained in fibers of elliptic fibration $\phi_{|-K_{\bar{X}}|}$. Therefore, $\bar{L}_2$ is an exeptional section of surface E_2.

Let $\mathcal{H}_{\bar{X}}$ be a pencil of all surfaces in $|-K_{\bar{X}}|$ passing though curve $\bar{L}_2$ and $D_{\bar{X}}$ be a general enough surface in pencil $\mathcal{H}_{\bar{X}}$. Then on surface $D_{\bar{X}}$

$$M_{\bar{X}}|_{D_{\bar{X}}} = \bar{L}_2 + \sum_{i=1}^{k} a_i \bar{Z}_i + B_{D_{\bar{X}}} \sim_{\mathbb{Q}} D_{\bar{X}}|_{D_{\bar{X}}} \sim \bar{L}_2 + \sum_{i=1}^{k} b_i \bar{Z}_i,$$

where $B_{D_{\bar{X}}}$ is a movable boundary on $D_{\bar{X}}$, $\bar{Z}_i$ are base curves of pencil $\mathcal{H}_{\bar{X}}$ different from curve $\bar{L}_2$, all a_i are non-negative rational numbers and all b_i are natural numbers. However, the intersection form of curves $\bar{Z}_i$ on surface $D_{\bar{X}}$ is negative definite. Therefore, movable boundary $B_{D_{\bar{X}}}$ is empty and support of restriction $M_{\bar{X}}|_{D_{\bar{X}}}$ is contained in the base locus of pencil $\mathcal{H}_{\bar{X}}$. Thus, $M_{\bar{X}} = \phi^{-1}_{\mathcal{H}_{\bar{X}}}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$. □

In the rest of the chapter we will prove Theorem 7.4. Consider movable log pair (X, M_X), such that singularities of (X, M_X) are canonical, $K_X + M_X \sim_{\mathbb{Q}} 0$ and $CS(X, M_X) \neq \emptyset$.

LEMMA 7.7. *Suppose that some point P on 3-fold X is a center of canonical singularities of movable log pair (X, M_X). Then image of point P on quadric Q is contained in ramification surface S, $CS(X, M_X) = \{P, \cup_{i\in I} C_i, \cup_{i\in I} P_j\}$ and $M_X = \phi^{-1}_{\mathcal{H}}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$, where $\mathcal{H}$ is a pencil in linear system $|-K_X|$ consisting of surfaces singular at point P, C_i are all base curves of pencil $\mathcal{H}$ and P_j are base points of pencil $\mathcal{H}$, such that $\theta(P_j) \in S$ and tangent spaces of surface S at points $\theta(P_j)$ are coincide with tangent space of surface S at point $\theta(P)$.*

Proof. Let H_P be general enough surface in linear system $|-K_X|$ containing P. Then inequality $H_P \cdot M_X^2 \geq mult_P(M_X^2)$ and Theorem 3.11 imply $mult_P(M_X) = 2$.

Let $f : W \to X$ be a blow up of point P, $E = f^{-1}(P)$ and $M_W = f^{-1}(M_X)$. Then boundary M_W is $\mathbb{Q}$-rationally equivalent to anticanonical divisor $-K_W$. Therefore, for some big enough natural number n linear system $|-nK_W|$ does not have fixed components and P is an Iskovskikh point. Thus, image of point P on quadric Q belongs to the ramification surface S by Lemma 1.13. In particular, linear system $|-K_W|$ is a pencil without fixed components.

Let $C \subset X$ be either a curve with $-K_X \cdot C = 1$ or a curve with $-K_X \cdot C = 2$ having singularity at point P. Then image of curve C on quadric Q is a line, C is contained in the base locus of pencil $f(|-K_W|)$ and C is contained in the support of M_X^2. On the other hand, on 3-fold W

$$M_W^2 \cdot (f^*(-K_X) - E) = -K_W^2 \cdot (f^*(-K_X) - E) = 0,$$

divisor $f^*(-K_X) - E$ is nef and big and for some big enough natural number k complete linear system $|k(f^*(-K_X) - E)|$ is free. Therefore, base locus of of pencil $f(|-K_W|)$ and support of M_X^2 consist of finite number of curves C_λ, such that either curve C_λ is smooth and $-K_X \cdot C_\lambda = 1$ or curve C_λ is singular at point P and $-K_X \cdot C_\lambda = 2$. In particular, for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$ we get $M_X = \phi^{-1}_{f(|-K_W|)}(M_{\mathbb{P}^1})$. Therefore,

$$CS(X, M_X) = CS(X, f(|-K_W|),$$

which implies the claim. $\square$

Thus, we will assume that set $CS(X, M_X)$ does not contain points on X and contains some irreducible and reduced curve C on X.

LEMMA 7.8. *Suppose that $-K_X \cdot C = 1$ and curve C is unique center of canonical singularities of movable log pair (X, M_X). Then either $M_X = \psi^{-1}(M_{\mathbb{P}^2})$ for some movable boundary $M_{\mathbb{P}^2}$ on $\mathbb{P}^2$ or movable log pair $(X_C, \psi_C(M_X))$ has semiterminal singularities and $\mathbb{Q}$-rationally trivial log canonical divisor, where $\psi : X \dashrightarrow \mathbb{P}^2$ is a composition of θ with a projection from $\theta(C)$ and $\psi_C : X \dashrightarrow X_C$ is a birational map from Proposition 1.4.*

Proof. Curve C is smooth and rational and its image on Q is a line. Let $f : W \to X$ be a blow up of curve C, E be exceptional divisor of f, $M_W = f^{-1}(M_X)$ and C_1 be a unique base curve of linear system $|-K_W|$. Then C_1 is a smooth rational curve and construction of birational map ψ_C implies that $\psi_C = \phi_{|-nK_{\hat{W}}|} \circ \rho$ for some big enough natural n, where birational map $\rho : W \dashrightarrow \hat{W}$ is an antiflip in curve C_1.

Suppose, that singularities of log pair (W, M_W) are terminal. Take $\zeta \in \mathbb{Q}_{>1}$, such that singularities of log pair $(W, \zeta M_W)$ are still terminal. Let $M_{\hat{W}} = \rho(M_W)$. Then singularities of log pair $(\hat{W}, \zeta M_{\hat{W}})$ are terminal, because ρ is log flip for log pair $(W, \zeta M_W)$. On the other hand, log canononical divisor $K_{\hat{W}} + M_{\hat{W}}$ is $\mathbb{Q}$-rationally trivial. In particular, birational morphism $\phi_{|-nK_{\hat{W}}|}$ is crepant for log pair $(\hat{W}, \zeta M_{\hat{W}})$. Therefore, log pair (X_C, M_{X_C}) has semiterminal singularities and $K_{X_C} + M_{X_C} \sim_{\mathbb{Q}} 0$, where $M_{X_C} = \psi_C(M_X)$.

Suppose that singularities of (X_C, M_{X_C}) are semiterminal and $K_{X_C} + M_{X_C} \sim_{\mathbb{Q}} 0$. Then singularities of log pair (W, M_W) are semiterminal too, because log pair $(\hat{W}, M_{\hat{W}})$ is a crepant pull back of log pair (X_C, M_{X_C}) and map ρ^{-1} is a flop for log pair $(\hat{W}, M_{\hat{W}})$.

To conclude the proof we may assume that singularities of log pair (W, M_W) are not terminal. Equivalence

$$K_W + M_W \sim_{\mathbb{Q}} f^*(K_X + M_X)$$

implies that set $CS(W, M_W)$ contains an irrreducible curve $T \subset E$, such that $f(T) = C$.

Suppose that $T = C_1$ and C_1 is contained in surface E. Let $g : V \to W$ be a blow up of curve C_1 and $M_V = g^{-1}(M_W)$. Then $|-K_V|$ is free, $\phi_{|-K_V|} : V \to \mathbb{P}^2$ is an elliptic fibration and $M_V \cdot K_V^2 = 0$. Therefore, M_V lies in the fibers of $\phi_{|-K_V|}$ and for some movable boundary $M_{\mathbb{P}^2}$ on $\mathbb{P}^2$ we have $M_V = \phi^{-1}_{|-K_V|}(M_{\mathbb{P}^2})$ and

$$M_X = (\phi_{|-K_V|} \circ g^{-1} \circ f^{-1})^{-1}(M_{\mathbb{P}^2}),$$

but $\phi_{|-K_V|} \circ g^{-1} \circ f^{-1}$ is a composition of θ with a projection from line $\theta(C)$. Thus, inequality $\kappa(V, \gamma M_V) < 3$ holds for every rational number $\gamma > 1$. In particular, movable log pair (X_C, M_{X_C}) does not have semiterminal singularities and $\mathbb{Q}$-rationally trivial log canonical divisor.

Suppose that $T \neq C_1$ and $C_1 \subset E$. Then intersecting M_W with a general enough fiber of morphism $f|_E : E \to C$ we get $E \subset M_W$, which contradicts the movability of M_W.

Suppose now that $T \neq C_1$ and $C_1 \not\subset E$. Let D_W be a general enough divisor in linear system $|-K_W|$. Then D_W is a smooth K3 surface, $T \cap D_W \neq \emptyset$ and

$$M_W|_{D_W} = mult_{C_1}(M_W)C_1 + R_{D_W} \sim_{\mathbb{Q}} C_1 + F$$

for some movable boundary R_{D_W} and elliptic curve F, such that $F \cdot C_2 = 1$. On the other hand, $M_W \cdot C_1 = -1$. Therefore, $mult_{C_1}(M_W) > 0$. Hence, intersecting of $M_W|_{D_W}$ with curve in pencil $|F|$ passing through a point of $T \cap D_W$ we get a contradiction. $\square$

LEMMA 7.9. *Inequality $-K_X \cdot C \leq 4$ holds.*

Proof. Canonicity of log pair (X, M_X) implies $mult_C(M_X) = 1$. So,

$$4 = H \cdot M_X^2 \geq mult_C(M_X^2)H \cdot C \geq mult_C(M_X)^2 H \cdot C = -K_X \cdot C,$$

where H is a general enough divisor in $|-K_X|$. $\square$

LEMMA 7.10. *Image of curve C on quadric Q is either a line or a conic.*

Proof. Suppose that degree of curve $\theta(C)$ is greater than 2. Then Lemma 7.9 implies that restriction $\theta|_C$ is an isomorphism and image of curve C on quadric Q is either a singular rational curve of degree 4 or a smooth curve of degree 3 or 4.

Suppose that curve C is smooth. Then C ie either a rational curve or an elliptic curve and $-K_X \cdot C = 4$. Let $f : W \to X$ be a blow up of curve C, E be an exceptional divisor of f, $M_W = f^{-1}(M_X)$ and $d = -K_X \cdot C$. We claim that divisor $f^*(-dK_X) - E$ is nef.

In case when image of curve C on quadric Q is not contained in ramification surface S let $\tilde{C} \neq C$ be a curve on X, whose image on Q coincide with the image of curve C. Then base locus of linear system $|f^*(-dK_X) - E|$ is contained in curve $\hat{C} = f^{-1}(\tilde{C})$. Therefore, divisor $f^*(-dK_X) - E$ has non-negative intersection with all curves on 3-fold W except may be with $\hat{C}$, but $(f^*(-dK_X) - E) \cdot \hat{C} > 0$.

In case when image of curve C on quadric Q is contained in ramification surface S base locus of linear system $|f^*(-dK_X) - E|$ is contained in exceptional divisor E. We may assume that normal bundle $N_{C/X}$ is decomposable and s_∞ is an exceptional section of ruled surface E. We should prrove that $(f^*(-dK_X) - E)|_E \cdot s_\infty \geq 0$. However,

$$(f^*(-dK_X) - E)|_E \cdot s_\infty = \frac{d(2d-1) + s_\infty^2}{2} + 1 - g(C).$$

Hence, we must show that $s_\infty^2 \geq 2g(C) - 2 - d(2d-1)$. Let $N_{C/X} \cong \mathcal{O}_C(m) \oplus \mathcal{O}_C(n)$ for integer numbers $m \geq n$. Then $m + n = d - 2 + 2g(C)$ and exact sequence

$$0 \to N_{C/\tilde{S}} \to N_{C/X} \to N_{\tilde{S}/X}|_C \to 0$$

implies $n \geq deg(N_{C/\tilde{S}}) = 2g(C) - 2 - d$, where $\tilde{S} = \theta^{-1}(S)$. Therefore,

$$s_\infty^2 = n - m = 2n + 2 - 2g(C) - d \geq 2g(C) - 2 - 3d > 2g(C) - 2 - d(2d - 1).$$

Thus, we proved that divisor $f^*(-dK_X) - E$ is nef. In particular,

$$0 \leq (f^*(-dK_X) - E) \cdot M_W^2 = 3d - d^2 - 2 + 2g(C) < 0.$$

To conclude the proof we may assume that $-K_X \cdot C = 4$ and curve C is a rational curve with one double point P. Let $g : V \to X$ be a composition of blow up of point P and blow up of strict transform of curve C, $M_V = g^{-1}(M_X)$, G and H be exceptional divisors of birational morphism g, where H is an exceptional divisor of blow up of strict transform of curve C. Using the same arguments as in the previous case one can show that on 3-fold V divisor $f^*(-4K_X) - H - 2G$ is nef. Thus,

$$0 \leq (f^*(-4K_X) - H - 2G) \cdot M_W^2 = 4mult_P(M_X) - 2mult_P^2(M_X) - 6 < 0,$$

because $2 > mult_P(M_X) \geq mult_C(M_X) = 1$. $\square$

LEMMA 7.11. *Suppose that $\theta(C)$ is a conic. Then set $CS(X, M_X)$ consists of all base curves of pencil $\mathcal{H}$ and $M_X = \phi_{\mathcal{H}}^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$, where pencil $\mathcal{H}$ consists of surfaces in linear system $|-K_X|$ passing through curve C.*

Proof. Either $-K_X \cdot C = 4$ and $\theta|_C$ is an isomorphism or $-K_X \cdot C = 2$ and $\theta|_C$ is a double cover. Suppose that $-K_X \cdot C = 4$. Let $f : W \to X$ be a resolution of indeterminacy of rational map $\phi_{\mathcal{H}}$, $\tau = \phi_{\mathcal{H}} \circ f^{-1}$ and $M_W = f^{-1}(M_X)$, such that birational morphism f is an isomorphism outside of curve C, W is smooth and W contains exactly one f-exceptional divisor E dominating curve C. Then general enough fiber D of morphism τ is rationally equivalent to divisor $f^*(-K_X) - E - \sum_{i \in I} a_i F_i$, where all $f(F_i)$ are points on C and all a_i are integer numbers. On the other hand, on surface D restriction $M_W|_D$ is $\mathbb{Q}$-rationally equivalent to $\sum_{i \in I} c_i F_i|_D$ for some rational numbers c_i. Hence, $M_W|_D = \emptyset$ and M_W lies in the fibers of τ, which implies the claim.

Suppose now that $-K_X \cdot C = 2$ and image of curve C on quadric Q is not contained in ramification surface S. Consider smooth rational curve $\tilde{C}$ on X, such that $\tilde{C} \neq C$ and images of curves $\tilde{C}$ and C on quadric Q coincide. Let D_C be a general enough divisor in pencil $\mathcal{H}$. Then $\tilde{C} \subset D_C$ and on surface D_C

$$M_X|_{D_C} = C + mult_{\tilde{C}}(M_X)\tilde{C} + R_{D_C} \sim_{\mathbb{Q}} -K_X|_{D_C} \sim C + \tilde{C}$$

for some movable boundary R_{D_C}. On the other hand, $\tilde{C}^2 < 0$ on surface D. Therefore, boundary R_{D_C} is empty, $mult_{\tilde{C}}(M_X) = 1$ and $CS(X, M_X)$ contains curve $\tilde{C}$. However, base locus of pencil $\mathcal{H}$ consists of curves C and $\tilde{C}$, which implies that $M_X = \phi_{\mathcal{H}}^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$.

Thus, we may assume that $-K_X \cdot C = 2$ and $\theta(C) \subset S$. Let $g : V \to X$ be a blow up of curve C, F be an exceptional divisor of morphism g, $M_V = g^{-1}(M_X)$, $\mathcal{H}_V = f^{-1}(\mathcal{H})$

and D_V be a general enough surface in pencil $\mathcal{H}_V$. Then base locus of pencil $\mathcal{H}_V$ consists of section $\hat{C}$ of ruled surface F and on surface D_V

$$M_V|_{D_V} = mult_{\hat{C}}(M_V)\hat{C} + R_{D_V} \sim_{\mathbb{Q}} -K_V|_{D_V} \sim \hat{C},$$

where R_{D_V} is a movable boundary. On the other hand, $\hat{C}^2 < 0$ on surface D_V. Therefore, boundary R_{D_V} is empty. and support of restriction $M_V|_{D_V}$ is contained in the base locus of pencil $\mathcal{H}_V$. Thus, $M_V = \phi_{\mathcal{H}_V}^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$. $\square$

We may assume that image of curve C on quadric Q is a line.

LEMMA 7.12. *Suppose that set $CS(X, M_X)$ contains an irreducible reduced curve L, such that $\theta(L) \neq \theta(C)$. Then image of curve L on quadric Q is a line, $\theta(C) \cap \theta(L) \neq \emptyset$, set $CS(X, M_X)$ consists of all base curves of pencil $\mathcal{H}$ of surfaces in linear system $|-K_X|$ passing through $C \cup L$ and $M_X = \phi_{\mathcal{H}}^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$.*

Proof. From Lemmas 7.10 and 7.11 it follows that image of curve L on quadric Q is a line. Suppose that $-K_X \cdot C = 1$ and $\theta(L) \cap \theta(C) = \emptyset$. Let $f : W \to X$ be a blow up of curve C, E be an exceptional divisor of f, $\hat{L} = f^{-1}(L)$, $M_W = f^{-1}(M_X)$ and D_W be a general enough divisor in linear system $|-K_W|$. Then base locus of linear system $|-K_W|$ consists of one smooth rational curve C_1, D_W is a smooth K3 surface, $\hat{L} \cap D_W \neq \emptyset$ and for some movable boundary R_{D_W} on surface D_W

$$M_W|_{D_W} = mult_{C_1}(M_W)C_1 + R_{D_W} \sim_{\mathbb{Q}} C_1 + F,$$

where F is elliptic curve, such that $F \cdot C_1 = 1$. Moreover, inequality $M_W \cdot C_1 < 0$ implies that $mult_{C_1}(M_W) > 0$. Therefore, intersecting $M_W|_{D_W}$ with any curve in pencil $|F|$ passing through a point of set $\hat{L} \cap D_W$ we get a contradiction.

Suppose that $-K_X \cdot C = 2$ and $\theta(L) \cap \theta(C) = \emptyset$. Then C is either a smooth elliptic curve or a rational curve with one singular point. Let $\mathcal{H}_C$ be a linear system consisting of surfaces in $|-K_X|$ passing through curve C. Then map $\phi_{\mathcal{H}_C} : X \dashrightarrow \mathbb{P}^2$ is a composition of a double cover θ and a projection from line $\theta(C)$. Let $r : V_C \to X$ be a terminal blow up for canonical log pair $(X, \mathcal{H}_C)$ and $M_{V_C} = r^{-1}(M_X)$, such that linear system $\mathcal{H}_{V_C} = r^{-1}(\mathcal{H}_C)$ is free and morphism $\phi_{\mathcal{H}_{V_C}}$ is an elliptic fibration. Then

$$K_{V_C} + M_{V_C} \sim_{\mathbb{Q}} r^*(K_X + M_X) \sim_{\mathbb{Q}} K_{V_C} + \mathcal{H}_{V_C}.$$

Therefore, boundary M_{V_C} lies in the fibers of elliptic fibration $\phi_{\mathcal{H}_{V_C}}$. On the other hand, strict transform of curve L on 3-fold V_C is a center of canonical singularities of movable log pair (V_C, M_{V_C}) and it is not contained in the fibers of elliptic fibration $\phi_{\mathcal{H}_{V_C}}$, which contradicts movability of boundary M_{V_C}.

Therefore, we proved that $\theta(L) \cap \theta(C) \neq \emptyset$. Let $\mathcal{H}$ be a pencil in anticanonical linear system $|-K_X|$ consisting of surfaces passing through both curves C and L. We should prove that set $CS(X, M_X)$ contains all base curves of pencil $\mathcal{H}$ and $M_X = \phi_{\mathcal{H}}^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$.

Suppose that $-K_X\cdot C=-K_X\cdot L=1$. Let $g:V\to X$ be a composition of a blow up of curve C with a blow up of curve L, D_V be a general enough surface in anticanonical linear system $|-K_V|$ and $M_V=g^{-1}(M_X)$. Then on surface D_V

$$M_V|_{D_V}=\sum_{i=1}^{k}\alpha_i Z_i+R_{D_V}\sim_{\mathbb{Q}}-K_V|_{D_V}\sim\sum_{i=1}^{k}\beta_i Z_i,$$

where Z_i are base curves of pencil $|-K_V|$, R_{D_V} is a movable boundary, all α_i are non-negative rational numbers and all β_i are natural numbers. However, intersection form of curves Z_i on surface D_V is negative definite. Therefore, $R_{D_V}=\emptyset$ and $M_V=\phi^{-1}_{|-K_V|}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$, which implies that set $CS(X,M_X)$ contains all base curves of pencil $\mathcal{H}$.

Suppose that $-K_X\cdot C=-K_X\cdot L=2$. Let D be a general enough surface in $\mathcal{H}$. Then

$$M_X|_D=C+L+R_D\sim_{\mathbb{Q}}-K_X|_D\sim C+L$$

for some movable boundary R_D on surface D. Thus, $R_D=\emptyset$ and restriction $M_X|_D$ consists of two base curves of pencil $\mathcal{H}$, which implies that $M_V=\phi^{-1}_{|-K_V|}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$.

Therefore, to conclude the proof of the claim we should consider the following two similar cases: either $-K_X\cdot C=2$ and $-K_X\cdot L=1$ or $-K_X\cdot C=1$ and $-K_X\cdot L=2$. Thus, we may assume that $-K_X\cdot C=2$ and $-K_X\cdot L=1$. Let $h:U\to X$ be a blow up of curve L, $\hat{L}$ be a base curve of linear system $|-K_U|$, $\hat{C}=h^{-1}(C)$, $M_U=h^{-1}(M_X)$, $\mathcal{H}_U=h^{-1}(\mathcal{H})$ and D_U be a general enough surface in pencil $\mathcal{H}_U$. Then curves $\hat{C}$ and $\hat{L}$ is contained in base locus of pencil $\mathcal{H}_U$. Moreover, in case when curve C is singular rational curve and its singular point is contained in curve L base locus of pencil $\mathcal{H}_U$ contains curve Z, which is mapped to singular point of curve C. In case when curve C is smooth or singular point of curve C does not belong to curve L base locus of pencil $\mathcal{H}_U$ consists of curves $\hat{L}$ and $\hat{C}$ and we will put in this case $Z=\emptyset$. On surface D_U

$$M_U|_{D_U}=\hat{C}+mult_{\hat{L}}(M_U)\hat{L}+\alpha Z+R_{D_U}\sim_{\mathbb{Q}}-K_U|_{D_U}\sim\hat{C}+\hat{L}+Z,$$

where R_{D_V} is a movable boundary and $\alpha=mult_Z(M_U)$ in case when $Z\neq\emptyset$. However, on surface D_U inequality $\hat{L}^2<0$ holds and in case $Z\neq\emptyset$ intersection form of curves Z and $\hat{L}$ on surface D_U is negative definite. Therefore, boundary R_{D_U} is empty, $mult_{\hat{L}}(M_U)=1$ and in case when $Z\neq\emptyset$ we have $mult_Z(M_U)=1$. Thus, $M_U=\phi^{-1}_{\mathcal{H}_U}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$ and all base curves of pencil $\mathcal{H}$ are centers of canonical singularities of log pair (X,M_X). Moreover, in case when curve C is a singular rational curve and its singular point is contained in curve L equality $mult_Z(M_U)=1$ implies that singular point of curve C is a center of canonical singularities of log pair (X,M_X) and all surfaces in pencil $\mathcal{H}$ are singular at this point, but we assumed that set $CS(X,M_X)$ does not contain points on 3-fold X. $\square$

Therefore, we may assume that locus $CS(X,M_X)$ is mapped to a line on quadric Q.

LEMMA 7.13. *Suppose that $-K_X \cdot C = 2$. Then curve C is a unique center of canonical singularities in $CS(X, M_X)$ and $M_X = \psi^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$, where $\psi : X \dashrightarrow \mathbb{P}^1$ is a composition of double cover θ with a projection from line $\theta(C)$.*

Proof. Suppose that $-K_X \cdot C = 2$. Consider linear system $\mathcal{H}_C$ of surfaces in linear system $|-K_X|$ containing curve C. Let $f : W \to X$ be a resolution of indeterminacy of rational map $\phi_{\mathcal{H}_C}$, such that there is exactly one f-exceptional divisor $E \subset W$ dominating curve C. Put $\mathcal{H}_W = f^{-1}(\mathcal{H}_C)$ and $M_W = f^{-1}(M_X)$. Then $M_W \cdot \mathcal{H}_W^2 = 0$, which implies the claim. □

Therefore, we may assume that $-K_X \cdot C = 1$ and by Lemmas 7.8 and 7.12 we may assume that set $CS(X, M_X)$ consists of curve C and another curve L on 3-fold X different from curve C, such that $\theta(L) = \theta(C)$. Now arguments of the proof of Lemma 7.13 imply that $M_X = \psi^{-1}(M_{\mathbb{P}^1})$ for some movable boundary $M_{\mathbb{P}^1}$ on $\mathbb{P}^1$, where $\psi : X \dashrightarrow \mathbb{P}^1$ is a composition of θ with a projection from $\theta(C)$. This concludes the proof of Theorem 7.4.

REFERENCES

[1] Iskovskikh V.A., Manin Yu.I., *Three-dimensional quartics and counterexamples to the Lüroth problem*, Mat. Sbornik **15** (1971), 140–166.

[2] Iskovskikh V.A., *Birational automorphisms of three-dimensional algebraic varieties*, Jour. Soviet Math. **13** (1980), 815–868.

[3] Pukhlikov A.V., *Birational automorphisms of a three-dimensional quartic with an elementary singularity*, Mat. Sbornik **135** (1988), 472–496.

[4] Cheltsov I., *Fano 3-fold with unique elliptic structure*, Mat. Sbornik **192** (2001), 145–156.

[5] Cheltsov I., *On smooth quintic 4-fold*, Mat. Sbornik **191** (2000), 139–162.

[6] Cheltsov I., *Log pairs on birationally rigid varieties*, Jour. Math. Sciences **102** (2000), 3843–3875.

[7] Kawamata Y., Matsuda K., Matsuki K., *Introduction to the minimal model problem*, Adv. Stud. Pure Math. **10** (1987), 283–360.

[8] Corti A., *Singularities of linear systems and 3-fold birational geometry*, L.M.S. lecture Note Series **281** (2000), 259–312.

E-mail: cheltsov@yahoo.com