On Q-Fano fiber spaces with two-dimensional base

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Introduction

The aim of this notes is to study extremal contractions from threefolds with only terminal singularities to surfaces. More precisely, we study an analytic analog of such contractions, so called $\mathbb{Q}$-Fano fiberations over two-dimensional base (see (1.1)). We are interesting in the biregular structure of $\mathbb{Q}$-Fano fibrations. For birational structure of fibration on rational curves, constructing standard models, etc see [24]. The study of $\mathbb{Q}$-Fano fibrations may be applied for Sarkisov's program of factorization of birational maps [25], [4] and also for study $\mathbb{Q}$–Fano threefolds with extremal contractions to surfaces (see section 7).

(0.1) Conjecture (special case of Reid's general elephants conjecture). *Let $f : (X, C) \to (S, s)$ be a $\mathbb{Q}$-Fano fiber space with two-dimensional base. Then a general member of the linear system $|-K_X|$ has only Du Val singularities.*

(0.2) Conjecture. *Let $f : (X, C) \to (S, s)$ be a $\mathbb{Q}$-Fano fiber space with two-dimensional base. Then (S, s) is Du Val singularity of type A_n, $n \geq 0$.*

In this paper we shall prove that conjecture (0.1) implies conjecture (0.2) (propositions (6.3) ,(5.5)). We also give detailed analysis of primitive $\mathbb{Q}$-Fano fiber spaces in section 5. In some cases (theorem (5.2)) conjecture (0.1) is proved. Our main tool is Mori's technique of study small extremal contractions [18].

1 Background results and first properties

(1.1) Definition. Let (X, C) be a germ of a three-dimensional complex space along a compact reduced curve C and let (S, s) be a germ of a two-dimensional normal complex space. Suppose that X has at worst terminal singularities. Then we say that proper morphism $f : (X, C) \to (S, s)$ is a *$\mathbb{Q}$-Fano fiber space with two-dimensional base* (or simply $\mathbb{Q}$-Fano fiber space) if

(i) $f^{-1}(s) = C$;

(ii) $f_* \mathcal{O}_X = \mathcal{O}_S$;

(iii) $-K_X$ is f-ample.

A $\mathbb{Q}$-Fano fiber space $f : (X, C) \to (S, s)$ is said to be *minimal* if C is irreducible. A $\mathbb{Q}$-Fano fiber space $f : (X, C) \to (S, s)$ is called *conic bundle* if (S, s) is non-singular and there exists an embedding $i : (X, C) \hookrightarrow \mathbb{P}^2 \times (S, s)$ such that $\mathcal{O}_{\mathbb{P}^2 \times S}(X) = \mathcal{O}_{\mathbb{P}^2 \times S}(2)$ and $i \cdot \mathrm{pr}_2 = f$.

(1.2) Example. Let $\mathbb{P}^1 \times \mathbb{C}^2 \to \mathbb{C}^2$ be the standard projection. Define the action of the group $\mathbb{Z}_n$ on $\mathbb{C}^2_{u,v}$ and $\mathbb{P}^1_{x,y} \times \mathbb{C}^2_{u,v}$:

$$(x, y, u, v) \to (x, \varepsilon^b y, \varepsilon u, \varepsilon^{-1} v),$$

where $\varepsilon = \exp(2\pi i/n)$, $b \in \mathbb{N}$, $(n, b) = 1$. Denote $X = (\mathbb{P}^1 \times \mathbb{C}^2)/\mathbb{Z}_n$, $S = \mathbb{C}^2/\mathbb{Z}_n$. Then the projection $f : X \to S$ is a $\mathbb{Q}$-Fano fiber space. The threefold X has on the fiber $f^{-1}(0)$ exactly two terminal points P_1, P_2 which are cyclic quotients of type $\frac{1}{n}(1, -1, \pm b)$, the surface S has in 0 a Du Val point of type A_{n-1}.

The following is a consequence of the Kawamata-Viehweg vanishing theorem (see [20], §4, [10], 1-2-5).

(1.3) Proposition. *Let $f : (X, C) \to (S, s)$ be a $\mathbb{Q}$-Fano fiber space. Then $R^i f_* \mathcal{O}_X = 0$, $i > 0$.*

(1.3.1) Corollary (cf. [18], (1.2)-(1.3)). (i) *For an arbitrary ideal $\mathcal{I}$ such that $\operatorname{Supp} \mathcal{O}_X/\mathcal{I} \subset C$ we have, $H^1(\mathcal{O}_X/\mathcal{I}) = 0$.*

(ii) *The fiber C is a tree of non-singular rational curves.*

(iii) *If C has ρ irreducible components, then*

$$\mathrm{Pic}(X) \simeq H^2(C, \mathbb{Z}) \simeq \mathbb{Z}^{\oplus \rho}.$$

(1.3.2) Remark. By [7], 4.5 for every threefold X with terminal singularities there exists a projective bimeromorphic morphism $q : X^q \to X$ called *$\mathbb{Q}$-factorialization* of X such that X^q has only terminal (analytically) $\mathbb{Q}$-factorial singularities and q is an isomorphism in codimension 1. If $f : (X, C) \to (S, s)$ is a $\mathbb{Q}$-Fano fiber space, then applying the Minimal Model Program to X^q over (S, s) we obtain a $\mathbb{Q}$-Fano fiber space $f' : (X', C') \to (S, s)$ with analytically $\mathbb{Q}$-factorial singularities, the same base S and $\rho(X', C')/(S, s)$=1. In particular $f' : (X', C') \to (S, s)$ is minimal.

(1.3.3) Remark. Let $f : (X, C) \to (S, s)$ be a $\mathbb{Q}$-Fano fiber space. Since $-K_X$ is f-ample, the Mori cone $\overline{NE}((X, C)/(S, s)) \subset \mathbb{R}^\rho$ is generated by classes of C_i. Thus any C_i generates an extremal ray R_i and $\overline{NE}((X, C)/(S, s)) \subset \mathbb{R}^\rho$ is simplicial. If $\rho \geq 2$, then the contraction of any extremal face of $\overline{NE}((X, C)/(S, s))$ over (S, s) is an extremal neighborhood [18], [13] (not necessary isolated).

(1.4) Proposition [3]. *Let $f : (X, C) \to (S, s)$ be a $\mathbb{Q}$-Fano fiber space. Assume that X has only points of index 1. Then (S, s) is non-singular and $f : (X, C) \to (S, s)$ is a conic bundle (possible singular).*

Note that converse statement is not true (see example (6.2.1)). We only have the following.

(1.4.1) Lemma *Let $f : (X, C) \to (S, s)$ be a $\mathbb{Q}$-Fano fiber space S. Assume that (S, s) is non-singular. Then f is flat.*

PROOF. Since singularities of X are rational, X is Cohen-Macaulay [11]. By [15], 23.1 f is flat. Q.E.D.

(1.5) Du Val singularities. Let (S, s) be a germ of surface log-terminal singularity. By [8], 1.9 $(S, s) \simeq (\mathbb{C}^2, 0)/G$, where $G \subset GL(2, \mathbb{C})$ is a finite group. The projection $(\mathbb{C}^2, 0) \to (S, s)$ is called *topological cover* of (S, s). Order of the group G is called *topological index* of (S, s) and denoted by $I_{top}(S, s)$.

(1.5.1) Is well known that every Du Val singularity is analytically isomorphic one of the following hypersurfaces in $\mathbb{C}^3$:

A_n	$uv + y^{n+1}$ or $z^2 + x^2 + y^{n+1}$	$I_{top}(F, P) = n + 1$
$D_n,\ n \geq 4$	$z^2 + x(y^2 + x^{n-2})$ or for $n = 4$ $z^2 + u^3 + v^3$	$I_{top}(F, P) = 4n - 8$
E_6	$z^2 + x^3 + y^4$	$I_{top}(F, P) = 24$
E_7	$z^2 + x(y^3 + x^2)$	$I_{top}(F, P) = 48$
E_8	$z^2 + x^3 + y^5$	$I_{top}(F, P) = 120$

(1.5.2) Proposition [1]. *Let (F, P) be a germ of Du Val singularity and $\tau : (F, P) \to (F, P)$ be an involution. Then there exists an analytic τ-equivariant embedding $(F, P) \subset (\mathbb{C}^3, 0)$ such that (F, P) can be given by equations (1.5.1). Moreover the action of τ and the quotient $(F, P)/\tau$ are:*

singularity (F,P)	involution τ	quotient $(F,P)/\tau$
A_k, D_k, E_k	$(x,y,z) \to (x,y,-z)$	non-singular
E_6	$(x,y,z) \to (x,-y,z)$	A_2
E_6	$(x,y,z) \to (x,-y,-z)$	E_7
D_k	$(x,y,z) \to (x,-y,z)$	A_1
D_k	$(x,y,z) \to (x,-y,-z)$	D_{2k-2}
A_{2k+1}	$(x,y,z) \to (x,-y,z)$	A_k
A_{2k+1}	$(x,y,z) \to (x,-y,-z)$	D_{k+3}
A_k	$(x,y,z) \to (-x,y,-z)$	A_{2k+1}
A_{2k}	$(u,v,y) \to (-u,v,-y)$	$\frac{1}{2k+1}(k,2k-1)$
A_{2k+1}	$(u,v,y) \to (-u,-v,-y)$	$\frac{1}{4k+4}(2k+1,2k+1)$

(1.5.3) Proposition (see e.g. [23]). *Let (F',P'), (F,P) are two-dimensional singularities and $(F',P') \to (F,P)$ be a finite morphism of degree r. Assume that (F,P) is Du Val and $(F'-\{P'\}) \to (F-\{P\})$ is an étale cover with group $\mathbb{Z}_r$, $r \geq 2$. Then (F',P') is also Du Val and $(F',P') \to (F,P)$ is one of the following:*

r	description	action of $\mathbb{Z}_r$ on (F',P')
any	$A_{k-1} \xrightarrow{r:1} A_{rk-1}$	$(u,v,y) \to (\varepsilon u, \varepsilon^{-1}v, y)$
4	$A_{2k-2} \xrightarrow{4:1} D_{2k+1}$	$(x,y,z) \to (ix,-y,-iz)$
2	$A_{2k-1} \xrightarrow{2:1} D_{k+2}$	$(x,y,z) \to (-x,-y,z)$
3	$D_4 \xrightarrow{3:1} E_6$	$(u,v,z) \to (\varepsilon u, \varepsilon^{-1}v, z)$
2	$D_{k+1} \xrightarrow{2:1} D_{2k}$	$(x,y,z) \to (x,-y,-z)$
2	$E_6 \xrightarrow{2:1} E_7$	$(x,y,z) \to (x,-y,-z)$

where $\varepsilon = \exp(2\pi i/r)$. Moreover except the first case the action $\mathbb{Z}_r$ on the dual graph of the minimal resolution of (F',P') is non-trivial.

(1.6) Terminal singularities. Let (X,P) be a terminal singularity of index $m \geq 1$ and let $\pi : (X^\sharp, P^\sharp) \to (X,P)$ be the canonical cover. Then $(X^\sharp, P^\sharp)$ is a terminal singularity of index 1. It is known [22] that $(X^\sharp, P^\sharp)$ is a hypersurface singularity, i. e. there exist an $\mathbb{Z}_m$-equivariant embedding $(X^\sharp, P^\sharp) \subset (\mathbb{C}^4, 0)$. We fix a generator $\zeta \in \mathbb{Z}_m$ and for $\mathbb{Z}_m$-semi-invariant z define weight $\mathrm{wt}(z) \in \mathbb{Z}$ as

$$\mathrm{wt}(z) \equiv a \bmod m \qquad \text{iff} \qquad \zeta(z) = \varepsilon^a z,$$

where $\varepsilon = \exp 2\pi i/m$. Usually we assume that $0 \leq \mathrm{wt}(z) < m$.

(1.6.1) Theorem [5],[19]. *If $(X^\sharp, P^\sharp)$ is smooth, then it is isomorphic $(\mathbb{C}^3_{x_1,x_2,x_3}, 0)$ such that $\mathrm{wt}(x_1,x_2,x_3) = (a,-a,b)$, where a, b are integer prime to m. Conversely every such singularity is terminal.*

(1.6.2) Theorem [17], [23], [14]. *Assume that $(X^\sharp, P^\sharp)$ is singular and let $\{\phi(x_1,x_2,x_3,x_4) = 0\}$ is an equation of $X^\sharp$ in $(\mathbb{C}^4_{x_1,x_2,x_3,x_4}, 0)$. Then modulo permutation of x_1,x_2,x_3,x_4 we have one of the following:*

(i) *(the main series)* $\mathrm{wt}(x_1, x_2, x_3, x_4; \phi) \equiv (a, -a, b, 0; 0) \bmod m$, *or*

(ii) *(the exceptional case)* $m = 4$, *and* $\mathrm{wt}(x_1, x_2, x_3, x_4; \phi) \equiv (a, -a, b, 2; 2) \bmod 4$, *where* a, b *are integer prime to* m.

(1.6.3) Remark. There is the complete classification of terminal singularities in terms of normal forms of $\phi(x_1, x_2, x_3, x_4)$ and actions of $\mathbb{Z}_m$ [17], see also [23], [14].

(1.6.4) Theorem [23]. *Let* (X, P) *be a germ of terminal singularity. Then a general member* $F \in |-K_X|$ *has only Du Val singularity (at* P*).*

(1.6.5) Definition. Let X be a normal variety and $\mathrm{Cl}(X)$ be its Weil divisor class group. The subgroup of $\mathrm{Cl}(X)$ consisting of Weil divisor classes which are $\mathbb{Q}$-Cartier is called by the semi-Cartier divisor class group. We denote it by $\mathrm{Cl}^{sc}(X)$.

(1.6.6) Theorem [22],[9]. *Let* (X, P) *be a germ of 3-dimensional singularity. Then* $\mathrm{Cl}^{sc}(X, P) \simeq \mathbb{Z}_m$ *and it is generated by the class of* $K_{(X,P)}$.

The following is an easy consequence of (1.6) .

(1.6.7) Lemma. *Let* (X, P) *be a germ of a terminal threefold singularity of index* $m > 1$ *and* $(F, P) \subset (X, P)$ *be a germ of irreducible surface. Assume that* F *is* $\mathbb{Q}$*-Cartier and* (F, P) *is Du Val with topological index* $I_{top}(F, P)$*. Then* $I_{top}(F, P)$ *is divisible by* m*. Moreover if* $I_{top}(F, P) = m$*, then* (X, P) *is a cyclic quotient singularity and* (F, P) *is of type* A_{m-1}.

2 Topological properties of Q-Fano fiber spaces

(2.1) Proposition, [17]. *Let* (X, P) *be a germ of terminal singularity of index* m, $(C, P) \subset (X, P)$ *be a germ of smooth curve.* $\pi : (X^\sharp, P^\sharp) \to (X, P)$ *be the canonical cover and* $C^\sharp := (\pi^{-1}(C))_{\mathrm{red}}$*. Then*

(i) *for arbitrary* $\xi \in \mathrm{Cl}^{sc}(X, P)$*, there exists an effective (Weil) divisor* D *such that* $[D] = \xi$ *and* $D \cap C = \{P\}$.

(ii) $\xi \to (D \cdot C)_P$ *induces a homomorphism*

$$\mathrm{cl}(C, P) : \mathrm{Cl}^{sc}(X, P) \to \frac{1}{m}\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}.$$

(2.2) Definition, [17]. Let things be as in (2.1) . $X \supset C$ is called *primitive* at P if one of the following equivalent conditions is satisfied:

(i) $\mathrm{cl}(C, P) : \mathrm{Cl}^{sc}(X, P) \to \frac{1}{m}\mathbb{Z}/\mathbb{Z}$ is an isomorphism,

(ii) $C^\sharp$ is irreducible,

(iii) $\lim_{U \ni P} \pi_1(U \cap C - P) \simeq \mathbb{Z} \to \lim_{U \ni P} \pi_1(U - P) \simeq \mathbb{Z}_m$ is surjective,

and *imprimitive* otherwise. The order of $\mathrm{Ker}(\mathrm{cl}(C, P))$ is called the *splitting degree* of $X \supset C$ at P and denoted by e.

(2.2.1) Remark. In the situation above $C^\sharp$ has exactly e irreducible components.

(2.3) Now let X be a three-dimensional complex spase with only terminal singularities. and $\mathbb{P}^1 \simeq C \subset X$ be a non-singular rational curve. Assume that C is irreducible and let $P_1, P_2, \ldots, P_n \in X$ be all the points of indices $m_1, m_2, \ldots, m_n > 1$. Then there exists the following exact sequence [18], 1.8:

$$0 \to \mathrm{Pic}(X) \to \mathrm{Cl}^{sc}(X) \to \bigoplus_i \mathrm{Cl}^{sc}(X, P_i) \to 0.$$

(2.4) Corollary, ([18], 1.10) *In notations of (2.3) . The following are equivalent*

(i) $(D \cdot C) = 1/m_1 m_2 \cdots m_n$ *for some* $D \in \mathrm{Cl}^{sc}(X)$;

(ii) $\mathrm{Cl}^{sc}(X) \simeq \mathbb{Z}$;

(iii) $\mathrm{Cl}^{sc}(X)$ *is torsion-free;*

(iv) $X \supset C$ *is locally primitive (i. e. primitive at any point* $P \in C$*) and* $(m_i, m_j) = 1$, *for all* $i \neq j$.

(2.5) Proposition [17]. *Let things be as in (2.3) . Take an effective Cartier divisor* H *such that* $H \cap C$ *is a smooth point of* X *and* $(H \cdot C) = 1$ *and effective Weil* $\mathbb{Q}$*-Cartier divisors* $D_1, \dots, D_l$ *such that* $D_i \cap C = \{P_i\}$ *and* D_i *is a generator of* $\mathrm{Cl}^{sc}(X, P_i)$ *for any* i *(see (2.1)).*

(i) *Assume that* (X, C) *is imprimitive of splitting degree* e *in* P_i. *Then the divisor*

$$D := (m_i/e)D_i - ((m_i D_i \cdot C)/e)H$$

is a e-torsion in $\mathrm{Cl}^{sc}(X, C)$. *It defines a finite Galois* $\mathbb{Z}_e$*-morphism* $g^\flat : X^\flat \to X$ *such that* $P^\flat := {g^\flat}^{-1}(P_i)$ *is one point,* $g^\flat$ *is étale over* $X - \{P_i\}$ *(hence* $X^\flat$ *has only terminal singularities), index of* $(X^\flat, P^\flat)$ *is equal to* m_i/e, $C^\flat := ({g^\flat}^{-1}(C))_{\mathrm{red}}$ *is a union of* e $\mathbb{P}^1$*'s meeting only in* $P^\flat$, *and each irreducible component of* $C^\flat$ *is primitive at* $P^\flat$.

(ii) *Assume that* (X, C) *is locally primitive and for some distinct points* P_i, P_j *we have* $n := (m_i, m_j) > 1$. *Then there are integers* α, β, γ *such that the divisor*

$$D := \alpha D_i + \beta D_j + \gamma H$$

is a n-torsion in $\mathrm{Cl}^{sc}(X, C)$. *It defines a finite Galois* $\mathbb{Z}_n$*-morphism* $g^\natural : X^\natural \to X$ *such that* $P_i^\natural := {g^\natural}^{-1}(P_i)$ *(resp.* $P_j^\natural := {g^\natural}^{-1}(P_j)$*) is one point,* $g^\natural$ *is étale over* $X - \{P_i, P_j\}$ *(hence* $X^\natural$ *has only terminal singularities), index of* $(X^\natural, P_i^\natural)$ *(resp.* $(X^\natural, P_j^\natural)$ *) is equal to* m_i/n *(resp.* m_j/n*), and* $C^\natural := ({g^\natural}^{-1}(C))_{\mathrm{red}} \simeq \mathbb{P}^1$.

(2.6) Let $f : (X, C) \to (S, s)$ be a $\mathbb{Q}$-Fano fiber space. The following easy remark [12], proposition 3.1, (see also [6], proof of 1.6) show that singularities of S are log-terminal: A general hyperplane section $H \subset X$ is non-singular and transversally to C. Hence $H \to S$ is a finite morphism in neighborhood of C. Then by [2], 6.7 (S, s) is log-terminal. We generalize this remark in (2.8) .

(2.7) Construction I. Let $f : (X, C) \to (S, s)$ is a a $\mathbb{Q}$-Fano fiber space. Assume that (S, s) is singular. Then the topological cover $h : (S^\natural, s^\natural) \simeq (\mathbb{C}^2, 0) \to (S, s)$ is non-trivial. Let $X^\natural$ be a normalization of $X \times_S S^\natural$ and $G = \mathrm{Gal}(S^\natural/S)$. Then we have the diagram

$$\begin{array}{ccc} X^\natural & \xrightarrow{g} & X \\ \downarrow f^\natural & & \downarrow f \\ S^\natural & \xrightarrow{h} & S \end{array}$$

The group G acts on $X^\natural$ and clearly $X = X^\natural/G$. Since the action of G on $S^\natural - \{s^\natural\}$ is free, so is the action of G on $X^\natural - C^\natural$, where $C^\natural := ({f^\natural}^{-1}(s^\natural))_{\mathrm{red}}$. Therefore $X^\natural$ has only terminal singularities and the induced action of G on $X^\natural$ is free outside of a finite set of points (see e. g. [2], 6.7). Since $K_{X^\natural} = g^*(K_X)$, we obtain the $\mathbb{Q}$-Fano fiber space $f^\natural : (X^\natural, C^\natural) \to (S^\natural, s^\natural)$ with two dimensional non-singular base.

(2.8) Proposition *Let* $f : (X, C) \to (S, s)$ *be a* $\mathbb{Q}$*-Fano fiber space. Then* (S, s) *is a cyclic quotient singularity.*

PROOF. Because (X,C) is bimeromorphic to the minimal $\mathbb{Q}$-Fano fiber space $f' : (X',C') \to (S,s)$ over (S,s), we consider the case when $f : (X,C) \to (S,s)$ is minimal. It is sufficient to prove only that in (2.7) G is cyclic.

If $C^\natural$ is irreducible, then $C^\natural \simeq \mathbb{P}^1$, so $G \subset PGL(2)$ and therefore G is either cyclic $\mathbb{Z}_n$, dihedral $\mathfrak{D}_n$, $\mathfrak{A}_4$, $\mathfrak{S}_4$ or $\mathfrak{A}_5$. On the other hand, G acts on $(S^\natural, s^\natural) \simeq (\mathbb{C}^2, 0)$. Hence $G \subset GL(2)$. It is easy to check that then G is a cyclic or dihedral. But in the second case the action G on $\mathbb{C}^2$ is not free in codimension 1. Indeed any element in $\mathfrak{D}_n$ of order 2 is either

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{or a reflection} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Therefore G is a cyclic group in this case.

Now we assume that $C^\natural = \cup C_i^\natural$ is reducible (it means that (X,C) contains an imprimitive point). We claim that $\cap C_i^\natural$ is a point. Indeed since the configuration $\cup C_i^\natural$ is tree and the action G on the set $\{C_i^\natural\}$ is transitive, $C_i^\natural \cap (C^\natural - C_i^\natural)$ is a point. Assume that $C_1^\natural \cap \ldots \cap C_k^\natural = \{P^\natural\}$ and let $C_{k+1}^\natural \cap (C_1^\natural \cup \ldots \cup C_k^\natural) \neq \emptyset$. Then $C_{k+1}^\natural \cap (C_1^\natural \cup \ldots \cup C_k^\natural)$ is a point which must be $P^\natural$. The induction proves our claim.

Thus the action G on $X^\natural$ has a fixed point $P^\natural := \cap C_i^\natural$. Let $P = g(P^\natural)$. Take a small neighborhoods $U^\natural \subset X^\natural$ of $P^\natural$ and $U = g(U^\natural) \subset X$ of P. Since $g|_{U^\natural}$ is étale on $U^\natural \backslash \{P^\natural\}$, we have a surjective map $\pi_1(U\backslash\{P\}) \to G$. But on the other hand, $\pi_1(U\backslash\{P\})$ is a cyclic group (see [22], 0.6, [16]). Therefore in this case (S,s) also is a cyclic quotient. This proves the proposition. Q.E.D.

(2.8.1) Corollary. *Let $f : (X,C) \to (S,s)$ be a minimal $\mathbb{Q}$-Fano fiber space and let $P_1, \ldots, P_l$ be all the points of indices $m_1, \ldots, m_l > 1$. Assume that (X,C) is locally primitive and $(m_i, m_j) = 1$ for all $i \neq j$. Then (S,s) is non-singular.*

PROOF. If (S,s) is singular, then by (2.7) the topological cover $(X^\natural, C^\natural)/(S^\natural, s^\natural) \to (X,C)/(S,s)$ is non-trivial. Since it is cyclic Galois cover (2.8) étale over $X - \mathrm{Sing}(X)$, torsion part of $\mathrm{Cl}^{sc}(X)$ is non-trivial, a contradiction with (2.4) . Q.E.D.

(2.9) Construction II. Let $f : (X,C) \to (S,s)$ be a minimal $\mathbb{Q}$-Fano fiber space. Assume that (X,C) has a finite unramified in codimension 2 cover $g : (X^\natural, C^\natural) \to (X,C)$, where $X^\natural$ is normal and $C^\natural$ is connected. Take the Stein factorization

$$\begin{array}{ccc} X^\natural & \xrightarrow{g} & X \\ \downarrow f^\natural & & \downarrow f \\ S^\natural & \xrightarrow{h} & S \end{array}$$

Then h is étale over $S^\natural - \{s^\natural\}$, hence by (2.8) $h : S^\natural \to S$ is a cyclic cover. Therefore $f^\natural : (X^\natural, C^\natural) \to (S^\natural, s^\natural)$ is a $\mathbb{Q}$-Fano fiber space. It is easy to see that $X^\natural$ is the normalization of $S^\natural \times_S X$.

(2.10) Lemma (cf. [18], 1.13). *Let $f : (X,C) \to (S,s)$ be a $\mathbb{Q}$-Fano fiber space. Then any component $C_i \subset C$ contains at most one imprimitive point.*

PROOF. If C is reducible, then our assertion follows from [18], 1.13. Assume that C is irreducible and $P_1, P_2 \in C$ are imprimitive points of splitting degree e_1 and e_2. Let $(X^\flat, C^\flat) \to (X,C)$ and $(X^\natural, C^\natural) \to (X,C)$ be splitting covers corresponding to P_1 and P_2, respectively (see (2.5)). By (2.9) , we can construct two $\mathbb{Q}$-Fano fiber space $(X^\flat, C^\flat) \to (S^\flat, s^\flat)$ and $(X^\natural, C^\natural) \to (S^\natural, s^\natural)$. Then $(X^\flat \times_X X^\natural, C^\flat \times_X C^\natural) \to (S^\flat \times_S S^\natural, s^\flat \times_S s^\natural)$ also is a $\mathbb{Q}$-Fano fiber space. By construction, $C^\flat \times_X C^\natural$ is a Galois $\mathbb{Z}_{e_1 e_2}$ cover of $C \simeq \mathbb{P}^1$ such that each components meets e_1-1 (resp. e_2-1) other components at every point over P_1 (resp. over P_2). Therefore $C^\flat \times_X C^\natural$ contains a cycle of $\mathbb{P}^1$'s, a contradiction with (1.3.1). Q.E.D.

(2.11) Proposition (cf. [18], 0.4.13.3, 6.2). *Let $f:(X,C)\to(S,s)$ be a $\mathbb{Q}$-Fano fiber space. Then any component $C_i\subset C$ cannot contain three points of index >1*

PROOF. As in (2.10) we consider only the case when C is irreducible and locally primitive, because general case can be reduced to this case and [18], 0.4.13.3 by (1.3.3), (2.5) . Assume that $P_1,P_2,P_3\in C$ are points of indices $m_1,m_2,m_3>1$. Using Van Kampen's theorem it is easy to compute the fundamental group of $X-\{P_1,P_2,P_3\}$:

$$\pi_1(X-\{P_1,P_2,P_3\})=\langle\sigma_1,\sigma_2,\sigma_3\rangle/\{\sigma_1^{m_1}=1,\sigma_2^{m_2}=1,\sigma_3^{m_3}=1,\sigma_1\sigma_2\sigma_3=1\}.$$

This group has a finite quotient group G in which the image of σ_i is exactly of order m_i. By (2.9) we obtain a $\mathbb{Q}$-Fano fiber space $f^\natural:(X^\natural,C^\natural)\to(S^\natural,s^\natural)$ with irreducible $C^\natural$. By (2.8) G is cyclic. A contradiction with the fact that any action of cyclic group on $\mathbb{P}^1$ has exactly two fixed points. Q.E.D.

3 Numerical invariants w_P and i_P according to Mori

(3.1) Let X be a normal three-dimensional complex space with only terminal singularities and let $C\subset X$ be a reduced non-singular curve. Denote by $\mathcal{I}_C$ the ideal sheaf of C. As in [18], we consider the following sheafs on C:

$$\operatorname{gr}^0_C\omega := \text{torsion-free part of } \omega_X/(\mathcal{I}_C\omega_X),$$

$$\operatorname{gr}^1_C\mathcal{O} := \text{torsion-free part of } \mathcal{I}_C/\mathcal{I}_C^2.$$

If $C\simeq\mathbb{P}^1$, then we have

$$\omega_X/(\mathcal{I}_C\omega_X)=\operatorname{gr}^0_C\omega+\text{Tors},\qquad \mathcal{I}_C/\mathcal{I}_C^2=\operatorname{gr}^1_C\mathcal{O}+\text{Tors}\,.$$

(3.2) Let m be index of X. The natural map

$$(\omega_X\otimes\mathcal{O}_C)^{\otimes m}\to\mathcal{O}_C(mK_X)$$

induces an injection

$$\beta:(\operatorname{gr}^0_C\omega)^{\otimes m}\to\mathcal{O}_C(mK_X).$$

Denote

$$w_P:=(\operatorname{length}_P\operatorname{Coker}\beta)/m.$$

(3.2.1) Remark. $\deg\operatorname{gr}^0_C\omega<0$, (because $\deg\mathcal{O}_C(mK_X)<0$).

(3.3) We have the natural map

$$\operatorname{gr}^1_C\mathcal{O}\times\operatorname{gr}^1_C\mathcal{O}\times\omega_C\to\omega_X\otimes\mathcal{O}_C\to\operatorname{gr}^0_C\omega,$$

$$x\times y\times zdu\to zdx\wedge dy\wedge du$$

which induces a map

$$\alpha:\wedge^2(\operatorname{gr}^1_C\mathcal{O})\otimes\omega_C\to\operatorname{gr}^0_C\omega\quad.$$

Let

$$i_P := \operatorname{length}_P \operatorname{Coker}(\alpha).$$

Note that $i_P = 0$ if X is smooth in P.

(3.3.1) Lemma ([18], 2.15). *If (X, P) is singular, then $i_P \geq 1$.*

(3.4) Example (cf. [18], 0.4.12.4). Let $\mathbb{Z}_m$ acts on $(\mathbb{C}^3, 0)$ by

$$(x, y, z) \to (\varepsilon^a x, \varepsilon^{-a} y, \varepsilon z),$$

where $\varepsilon = \exp(2\pi i/m)$ and a is an integer prime to m such that $0 < a < m$. Let $C^\sharp \subset \mathbb{C}^3$ be the z-axis. Then $(X, P) := (\mathbb{C}^3, 0)/\mathbb{Z}_m$ is terminal and $C := C^\sharp/\mathbb{Z}_m \subset X$ is a smooth curve. We have the following

(i) $\mathcal{O}_{C,P} = \mathbb{C}\{z^m\}$;
(ii) $\operatorname{gr}_C^0 \omega = \mathcal{O}_C(z^{m-1} dx \wedge dy \wedge dz)$, $\quad \mathcal{O}_C(mK_X) = \mathcal{O}_C(dx \wedge dy \wedge dz)^m$ near P;
(iii) $w_P = (m-1)/m$;
(iv) $\operatorname{gr}_C^1 \mathcal{O} = \mathcal{O}_C(z^{m-a} x) \oplus \mathcal{O}_C(z^a y)$ near P;
(v) $i_P = 1$.

From definitions we have

(3.5) Proposition. If $C \simeq \mathbb{P}^1$, then

$$\deg \operatorname{gr}_C^1 \mathcal{O} = 2 + \deg \operatorname{gr}_C^0 \omega - \sum_P i_P,$$

$$(K_X \cdot C) = \deg \operatorname{gr}_C^0 \omega + \sum_P w_P,$$

(3.6) Proposition. *Let $f : (X, C) \to (S, s)$ be a minimal $\mathbb{Q}$-Fano fiber space. Then*

$$\deg \operatorname{gr}_C^1 \mathcal{O} \geq -2.$$

PROOF. Consider the exact sequence

$$0 \to \mathcal{I}_C/\mathcal{I}_C^2 \to \mathcal{O}_X/\mathcal{I}_C^2 \to \mathcal{O}_C \to 0.$$

By (1.3.1) $H^1(\mathcal{O}_X/\mathcal{I}_C^2) = 0$ and since $H^0(\mathcal{O}_X/\mathcal{I}_C^2) \to H^0(\mathcal{O}_C)$ is onto, we have $H^1(\mathcal{I}_C/\mathcal{I}_C^2) = 0$. Hence $H^1(\operatorname{gr}_C^1 \mathcal{O}) = 0$. It gives us our assertion. Q.E.D.

(3.6.1) Corollary.

$$\sum_P i_P \leq 4 + \deg \operatorname{gr}_C^0 \omega \leq 3,$$

$$\sum_P w_P < -\deg \operatorname{gr}_C^0 \omega \leq 4 - \sum_P i_P(1),$$

$$\sum w_P(0) + \sum i_P(1) < 4.$$

(3.6.2) Example. Let $f : (X, C) \to (S, s)$ be a $\mathbb{Q}$-Fano fiber space as in (1.2) . Then from (3.4) it is easy to compute

(i) $i_{P_1} = i_{P_2} = 1$, $\quad w_{P_1} = w_{P_2} = (n-1)/n$,
(iii) $\deg \operatorname{gr}_C^0 \omega = -2$, $\quad \deg \operatorname{gr}_C^1 \mathcal{O} = -2$,
(iii) $(K_X \cdot C) = -2/n$.

(3.7) Corollary. *Let $f : (X, C) \to (S, s)$ be a minimal $\mathbb{Q}$-Fano fiber space. Then (X, C) contains at most three singular points.*

PROOF. It follows from (3.6.1) and (3.3.1). Q.E.D.

(3.7.1) Remark. If $f : (X, C) \to (S, s)$ is non-minimal, then for every irreducible component $C_i \subset C$ germ (X, C_i) is an extremal neighborhood. By [18], results (3.6), (3.6.1), (3.7) are true for (X, C_i).

4 Computations of i_P and w_P

(4.1) In this section we fix the following notations. Let (X, P) be a germ of three-dimensional terminal singularity of index m and let $(C, P) \subset (X, P)$ be a germ of smooth curve. We assume that P is primitive. Consider the canonical $\mathbb{Z}_m$-cover $(X^\sharp, P^\sharp) \to (X, P)$ and let $C^\sharp = (C \times_X X^\sharp)_{\mathrm{red}}$. Since P is primitive, $C^\sharp$ is irreducible. Then $\mathbb{Z}_m$ naturally acts on $X^\sharp$, $C^\sharp$ and on the normalization of $C^\sharp$. There exists an $\mathbb{Z}_m$-equivariant embedding $(X^\sharp, P^\sharp) \subset (\mathbb{C}^4_{x_1,x_2,x_3,x_4}, 0)$. Let $\phi = \phi(x_1, x_2, x_3, x_4)$ be the equation of $(X^\sharp, P^\sharp) \subset (\mathbb{C}^4_{x_1,x_2,x_3,x_4}, 0)$. Recall that we assume that $\mathrm{wt}(x, \phi) \equiv (a, -a, b, c; c) \bmod m$, where $(a, m) = 1$, $(b, m) = 1$, $1 \le a, b \le m - 1$ and $c = 0$ or $c = 2$, $m = 4$ (exceptional case). For any regular function z on $X^\sharp$ such that $z(0,0,0,0) = 0$ by

$$\mathrm{ord}(z) \in \mathbb{N} \cup \{\infty\}$$

we denote the order of vanishing of z on the normalization of $C^\sharp$. All the numbers $\mathrm{ord}(z) < \infty$ form a simigroup, which is denoted by

$$\mathrm{ord}(C^\sharp).$$

Let $\mathrm{ord}(x_i) = a_i$. Then $\mathrm{ord}(C^\sharp)$ is generated by a_i's. We choose the generator of $\mathbb{Z}_m$ such that

$$a_i = \mathrm{ord}(x_i) \equiv \mathrm{wt}(x_i) \bmod m, \text{ if } a_i \ne \infty.$$

(4.1.1) Lemma (see e. g. [2], 15.5). *In notations above there exists $\mathbb{Z}_m$-invariant coordinate system in $\mathbb{C}^4$ such that $C^\sharp$ is monomial. More precisely, $C^\sharp$ is the image of*

$$t \longrightarrow (t^{a_1}, t^{a_2}, t^{a_3}, t^{a_4}),$$

where $t^{a_i} = 0$ if $a_i = \infty$.

(4.1.2) Lemma–Definition [18], 2.6. In notations (4.1) there exists $\mathbb{Z}_m$-invariant coordinate system in $\mathbb{C}^4$ which satisfies the following conditions:

(i) $a_i < \infty$ and $(a_i - m) \notin \mathrm{ord}(C^\sharp)$ for all $i = 1, 2, 3, 4$.

(ii) $a_i \equiv \mathrm{wt}(x_i) \bmod m$ for all $i = 1, 2, 3, 4$.

Such coordinate system is called by *normalized* coordinate system.

(4.2) Let things be as in (4.1). A local generator of $\omega_{X^\sharp}$ is

$$\Omega := \mathrm{Res}(\phi^{-1} dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4),$$

where Res is the Poincaré residue map. Then we can write a local generator of $\mathrm{gr}^0_C\, \omega$ as $\mathbb{Z}_m$-invariant $\psi\Omega$, where $\mathrm{wt}(\psi) \equiv -\mathrm{wt}(\Omega) = m - b$. Therefore

$$m w_P = \dim(\mathcal{O}_C(mK_X)/(\psi\Omega)^m \mathcal{O}_C(mK_X)).$$

Finally, we have

(4.2.1) Proposition, ([17], 2.10).

$$mw_P(0) = \min\{\mathrm{ord}(\psi) \mid \psi = \psi(x_1, x_2, x_3, x_4),\ \mathrm{wt}(\psi) = -\mathrm{wt}(x_1x_2x_3x_4/\phi) = m - b\}.$$

(4.3) Let t be a local parameter on the normalization of $C^\sharp$ (cf. (4.1.1)). Then dt^m is a local generator of ω_C. Denote by $\mathcal{I}_{C^\sharp}$ (or simply $\mathcal{I}$) the ideal sheaf of $C^\sharp$ in $\mathbb{C}^4$ and by $\mathcal{I}^{\{0\}}$ the invariant part of $\mathcal{I}$. Local generators of $\mathrm{gr}_C^1 \mathcal{O}$ lift back to $\zeta_1, \zeta_2 \in \mathcal{I}^{\{0\}}$. Therefore $\phi_1 \wedge \phi_2 \wedge dt^m$ is a local generator of $\wedge^2(\mathrm{gr}_C^1 \mathcal{O}) \otimes \omega_C$. Computations gives as

$$\phi_1 \wedge \phi_2 \wedge dt^m = t^{m-a_4}\phi_1 \wedge \phi_2 \wedge dx_4 = t^{m-a_4}\psi^{-1}\partial(\phi, \phi_1, \phi_2)/\partial(x_1, x_2, x_3)\psi\Omega.$$

Therefore

$$mi_P = m - a_4 - \mathrm{ord}(\psi) + \mathrm{ord}(\partial(\phi, \phi_1, \phi_2)/\partial(x_1, x_2, x_3)).$$

Let

$$[\zeta_1, \zeta_2, \zeta_3] := \mathrm{ord}(\partial(\zeta_1, \zeta_2, \zeta_3)/\partial(x_1, x_2, x_3)).$$

Finally, we have

(4.3.1) Proposition, [17]. *If* $\mathrm{ord}(x_4) < \infty$, *then*

$$m(i_P(1) + w_P(0)) = m - \mathrm{ord}(x_4) + \min_{\phi_1, \phi_2 \in \mathcal{I}^{\{0\}}}\{[\phi, \phi_1, \phi_2]\}.$$

(4.3.2) Remark. It is easy to see that for any $\zeta_1, \zeta_2, \zeta_3$ one has

$$[\zeta_1, \zeta_2, \zeta_3 + \zeta_3'] \geq \min\{[\zeta_1, \zeta_2, \zeta_3], [\zeta_1, \zeta_2, \zeta_3']\}.$$

Note that if (X, P) is not exceptional, then $\phi \in \mathcal{I}^{\{0\}}$. If (X, P) is exceptional, then we may assume that $\mathrm{ord}(x_4) = 2$ (cf. proof of (4.6)) hence $x_4\phi \in \mathcal{I}^{\{0\}}$. In any case we have

(4.3.3) Corollary. *If* $\mathrm{ord}(x_4) < \infty$, *then*

$$m(i_P + w_P) \geq [\phi_1, \phi_2, \phi_3]$$

for some $\phi_1, \phi_2, \phi_3 \in \mathcal{I}^{\{0\}}$.

(4.3.4) Remark. By (4.1.1) we can take $C^\sharp$ as monomial curve. Using (4.3.2) ϕ_i's may be chosen from

$$x_1x_2 - x_4^{(a_1+a_2)/m}, \quad x_1^p x_3^q - x_4^{(a_1p+a_3q)/m}, \quad x_2^r x_3^s - x_4^{(a_2r+a_3s)/m}, \quad x_j^m - x_4^{a_j},\ j = 1, 2, 3$$

where $p, q, r, s \in \mathbb{N}$, $ap + bq \equiv 0 \bmod m$, $(m-a)r + sb \equiv 0 \bmod m$.

(4.4) Lemma. *For any three-dimensional terminal singularity* (X, P) *a general member of* $F \in |-K_{(X,P)}|$ *is given by a section*

$$\psi\Omega^{-1} \in \mathcal{O}_{X^\sharp}(-K_{X^\sharp}),$$

where $\mathrm{wt}(\psi) = \mathrm{wt}(x_1x_2x_3x_4/\phi) = b$ *and* $\Omega = \mathrm{Res}(\phi^{-1}dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4)$. *Moreover*

$$(F \cdot C)_P = \min\{\mathrm{ord}(\psi) \mid \psi = \psi(x_1, x_2, x_3, x_4),\ \mathrm{wt}(\psi) = b\}.$$

PROOF. It is clear that $F \in |-K_X|$ is given by an invariant section of $|-K_{X^\sharp}|$. By the residue formula this section has form $\psi\Omega^{-1}$, where $\Omega = \mathrm{Res}(\phi^{-1}dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4)$. The rest is obvious. Q.E.D.

(4.5) Lemma. *Let $m = 2$ and assume that $w_P(0) + i_P(1) < 4$. Then $\mathrm{ord}(C^\sharp) = \mathbb{N}$ (i. e. $C^\sharp$ is smooth).*

PROOF. Take a normalized coordinate system for $X^\sharp$ in $\mathbb{C}^4_{x_1,\dots,x_4}$ such that $\mathrm{wt}(x) = (1,1,1,0)$, $\mathrm{ord}(x) = (a_i)$. Since C is smooth, $a_4 = 2$. But then $\mathrm{ord}(C^\sharp)$ is generated by $a_4 = 2$ and the smallest a_i, for example a_3. It is sufficient to show only $1 \in \mathrm{ord}(C^\sharp)$. By (4.3.3) we have

$$8 > 2(w_P + i_P) \geq 2 - \mathrm{ord}(x_4) + [\phi_1, \phi_2, \phi_3] = [\phi_1, \phi_2, \phi_3] \geq \sum \mathrm{ord}(z_i)$$

for some $\phi_i \in \mathcal{I}^{\{0\}}_{C^\sharp}$, where $z_i = \partial\phi_i/\partial x_i$. Hence $\mathrm{ord}(z_i) \leq 2$ for some $i = 1,2,3$. But since $\mathrm{wt}(z_i) = m - \mathrm{wt}(x_i) = 1$, $\mathrm{ord}(z_i) = 1$. Thus $1 \in \mathrm{ord}(C^\sharp)$. Q.E.D.

(4.6) Lemma. *Let things are as in (4.1) . Assume that (X, P) is a singularity of index 4 of exceptional series (see (1.6.2), (ii)) and $w_P(0) + i_P(1) < 4$. Then we have one of the following*

(i) $\mathrm{ord}(C^\sharp) = \mathbb{N}$ *(i. e. $C^\sharp$ is non-singular), or*

(ii) $\mathrm{ord}(C^\sharp) = \langle 2, 3\rangle$.

Moreover $w_P = (4 - b)/4$, except the case

(ii*) $\quad \mathrm{ord}(C^\sharp) = \langle 2, 3\rangle$, $b = 3$, $w_P = 5/4$, $i_P = 2$.

PROOF. Suppose that $\mathrm{ord}(C^\sharp) \neq \mathbb{N}$, then $1 \notin \mathrm{ord}(C^\sharp)$. Since $C^\sharp/\mathbb{Z}_4$ is non-singular, $4 \in \mathrm{ord}(C^\sharp)$. But $\mathrm{ord}(x_i) \neq 4$, by $\mathrm{ord}(x_i) \equiv \mathrm{wt}(x_i) \bmod 4$. Hence $\mathrm{ord}(x_i) + \mathrm{ord}(x_j) = 4$ for some $i, j \in \{1,2,3,4\}$. It is possible only if $\mathrm{ord}(x_4) = 2$. Then $\mathrm{ord}(C^\sharp)$ is generated by 2 and the smallest odd $k \in \mathrm{ord}(C^\sharp)$. Therefore $C^\sharp$ is planar and it is sufficient to show only $3 \in \mathrm{ord}(C^\sharp)$. By (4.3.3) we have

$$16 > 4(i_P + w_P) \geq [\phi_1, \phi_2, \phi_3] \geq \sum \mathrm{ord}(z_i)$$

for some $\phi_i \in \mathcal{I}^{\{0\}}_{C^\sharp}$, where $z_i = \partial\phi_i/\partial x_i$. Moreover z_i are semi-invariants with $\mathrm{ord}(z_i) < \infty$ and $\mathrm{wt}(z_i) = 4 - \mathrm{wt}(x_i)$. Thus $\mathrm{ord}(z_i)$, $i = 1,2,3$ are odd. If $3 \notin \mathrm{ord}(C^\sharp)$, then $\mathrm{ord}(z_i) \geq 5$ for $i = 1,2,3$. It gives as $\mathrm{ord}(z_i) = 5$ for $i = 1,2,3$, which contradicts $\mathrm{wt}(z_i) = 4 - \mathrm{wt}(x_i)$. Therefore $\mathrm{ord}(C^\sharp) = \langle 2, 3\rangle$.

Assume that $w_P \neq (4 - b)/4$. Then by (4.2.1) $4 - b \notin \mathrm{ord}(C^\sharp)$. It is possible only we have the case (2) and $b = 3$. In this case $2 \cdot 4 - b = 5 \in \mathrm{ord}(C^\sharp)$, so $w_P = 5/4$, $i_P \leq 2$. If $i_P = 1$, then $12 > 4(i_P + w_P) \geq \sum \mathrm{ord}(z_i)$, where $\mathrm{ord}(z_1) + \mathrm{ord}(z_2) \equiv 0 \bmod 4$, $\mathrm{ord}(z_1) + \mathrm{ord}(z_2) \geq 8$, $\mathrm{ord}(z_3) \equiv 1 \bmod 4$. It gives us $\mathrm{ord}(z_3) = 1$, a contradiction. This proves the lemma. Q.E.D.

(4.7) Lemma (cf. [18], 3.1). *Assume that (X, P) is not exceptional and $w_P + i_P < 3$. Then up to permutation x_1, x_2, we have one of the following*

(i) $\mathrm{ord}(C^\sharp) = \mathbb{N}$,

(ii) $a_1 = a = 2 \in \mathrm{ord}(C^\sharp)$, *$b$ is odd,*

(iii) $\mathrm{ord}(C^\sharp)$ *is generated by $a_1 = a$ and $a_3 = b$, in this case $m - b \in \mathrm{ord}(C^\sharp)$.*

PROOF. If $m = 2$, then by (4.5) , $\mathrm{ord}(C^\sharp) = \mathbb{N}$. Thus we suppose that (X, P) is in the main series, $m \geq 3$ and $a_i = \mathrm{ord}(x_i) \geq 2$ for $i = 1,2,3,4$. Using (4.3.3), we get

$$3m > m(i_P + w_P) \geq [\phi_1, \phi_2, \phi_3] \geq \sum \mathrm{ord}(z_i)$$

for some $\phi_i \in \mathcal{I}_{C^\sharp}^{\{0\}}$, where $z_i = \partial\phi_i/\partial x_i$. Since $\mathrm{ord}(z_i) \equiv -\mathrm{ord}(x_i) \bmod m$, we have

$$\sum \mathrm{ord}(z_i) \leq 3m - b.$$

By (4.3.4), ϕ_i $(i = 1, 2, 3)$ ahs the form

$$x_1x_2 - x_4^{(a_1+a_2)/m}, \quad x_1^p x_3^q - x_4^{(a_1p+a_3q)/m}, \quad x_2^r x_3^s - x_4^{(a_2r+a_3s)/m}, \quad \text{or} \quad x_j^m - x_4^{a_j},\ j = 1, 2, 3,$$

where $p, q, r, s \in \mathbb{N}$, $ap + bq \equiv 0 \bmod m$, $(m - a)r + sb \equiv 0 \bmod m$.

First we assume that $\phi_i \neq x_j^m - x_4^{a_j}$, $\forall i, j \leq 3$. Then

$$\{\phi_1, \phi_2, \phi_3\} = \{x_1x_2 - x_4^{(a_1+a_2)/m}, \quad x_1^p x_3^q - x_4^{(a_1p+a_3q)/m}, \quad x_2^r x_3^s - x_4^{(a_2r+a_3s)/m}\}, \quad i = 1, 2, 3$$

hence $3m - b \geq \sum \mathrm{ord}(z_i) = a_1p + a_3q + a_2r + a_3s - a_3$. Since $a_1p + a_3q \equiv 0 \bmod m$, $a_2r + a_3s \equiv 0 \bmod m$, we have $a_1p + a_3q \leq 2m$ and $a_2r + a_3c \leq 2m$. Note that we still may permute x_1, x_2, so we assume $a_1p + a_3q \leq a_2r + a_3s$ and if $a_1p + a_3q = a_2r + a_3s$, then $p \geq r$. Consider the following cases:

(4.7.1) $a_1p + a_3q = 2m$, $a_2r + a_3s = 2m$, hence $a_3 > m$. But then $q = s = 1$, $a_1 < m$, $a_2 < m$, $a_1 + a_2 = m$, $(a_1, a_2) = 1$. So $a_1p = a_2r$. It gives us $p = a_2k$, $q = a_1k$ for some $k \in \mathbb{N}$. Thus $a_1a_2k < m = a_1 + a_2$. Therefore $a_1 = 1$ or $a_2 = 1$, a contradiction.

(4.7.2) $a_1p + a_3q = m$, $a_2r + a_3s = 2m$, hence $a_1p < m$, $a_3 < m$, $(a_1, a_3) = 1$. If $a_1 + a_2 \geq 2m$, then $a_2 > m$, $r = 1$, $a_1 + a_2 = 2m$. We obtain $a_1 = a_3s$, a contradiction with $(a_1, a_3) = 1$. Thus $a_1 + a_2 = m$ and we have $a_4 = m = a_1p + a_3q$, $a_2 = m - a_1 = a_1(p-1) + a_3q$. Therefore $\mathrm{ord}(C^\sharp)$ is generated by a_1, a_3 and $m - a_3 = a_1p + a_3(q-1) \in \mathrm{ord}(C^\sharp)$. This is case (iii).

(4.7.3) $a_1p + a_3q = m$, $a_2r + a_3s = m$, then $a_1 + a_2 = m$, $a_1(p-r) + mr + a_3(q+s) = 2m$. It gives us $r = 1$, $a_2 = a_3s$, a contradiction with $(a_3, m) = 1$.

Now we assume that $\phi_i = x_i^m - x_4^{a_i}$ for some i. By $\mathrm{ord}(z_i) = (m-1)a_i \leq 3m - 3$, we have $a_i \leq 3$. If $a_i = 3$, then $\mathrm{ord}(z_j) + \mathrm{ord}(z_k) \leq 3 - b \leq 1$, where $\{i, j, k\} = \{1, 2, 3\}$, a contradiction with $1 \notin \mathrm{ord}(C^\sharp)$. Thus $a_i = 2$ for some $i = 1, 2$ or 3, m is odd and then

$$\mathrm{ord}(z_j) + \mathrm{ord}(z_k) \leq m + 2 - b, \quad \{i, j, k\} = \{1, 2, 3\}.$$

In this situation $\mathrm{ord}(C^\sharp)$ is generated by $a_i = 2$ and the smallest odd integer $\in \mathrm{ord}(C^\sharp)$. We treat the following cases:

(4.7.4) $i = 3$, $a_3 = 2 = b$. Then $\mathrm{ord}(z_1) + \mathrm{ord}(z_2) = m$. By normalizedness of (x), $a_1 = \mathrm{ord}(z_2) < m$, $a_2 = \mathrm{ord}(z_1) < m$. Modulo permutation x_1, x_2 we may assume that a_1 is odd. In this case $\mathrm{ord}(C^\sharp)$ is generated by $a_3 = 2$ and a_1. This is case (iii).

(4.7.5) $i = 1$, $a_1 = a = 2$, $\mathrm{ord}(C^\sharp)$ is generated by 2 and a_3. Then $a_3 < m$, because $m \in \mathrm{ord}(C^\sharp)$. Since $m - a_3$ is even, $m - a_3 \in \mathrm{ord}(C^\sharp)$. We get case (iii).

(4.7.6) $i = 1$, $a_1 = a = 2$, $\mathrm{ord}(C^\sharp)$ is generated by 2 and m. By $\mathrm{ord}(z_2) + \mathrm{ord}(z_3) \leq m + 2 - b$, we have $\mathrm{ord}(z_2), \mathrm{ord}(z_3) < m$. Thus $\mathrm{ord}(z_2), \mathrm{ord}(z_3)$ are even and $z_2 = x_1^{\mathrm{ord}(z_2)/2}$, $z_3 = x_1^{\mathrm{ord}(z_3)/2}$. It is possible only if $\phi_2 = x_1x_2 - x_4^{(2+a_2)/m}$, $\phi_3 = x_1^p x_3 - x_4^{(2p+a_3q)/m}$, where $p = \mathrm{ord}(z_3)/2$ and $2p + b \equiv 0 \bmod m$. Since $\mathrm{ord}(z_3) < m$, one has $2p + b = m$, so b is odd. This is case (ii).

(4.7.7) $i = 1$, $a_1 = a = 2$, $\mathrm{ord}(C^\sharp)$ is generated by 2 and $a_2 = m - 2$. Then, obviously, $m \geq 5$. We only have to show that b is odd. Assume the opposite. Then $b \in \mathrm{ord}(C^\sharp)$

and by normalizedness of (x) $a_3 = b$. From $\operatorname{ord}(z_2) + \operatorname{ord}(z_3) \le m + 2 - b$ we get $\operatorname{ord}(z_2), \operatorname{ord}(z_3) < m$. If both of $\operatorname{ord}(z_2), \operatorname{ord}(z_3)$ are even, then we obtain (ii) as above. So assume that $\operatorname{ord}(z_j)$ is odd, then $m - 2 \le \operatorname{ord}(z_j) < m + 2 - b$. Thus $a_3 = b = 2$. Permuting x_1, x_2, x_3 we obtain case (iii). Q.E.D.

(4.8) Corollary (from proofs of (4.6) , (4.7)). *Let things be as in (4.1) . If $w_P + i_P < 2$, then $C^\sharp$ is non-singular.*

Results of lemmas (4.6) , (4.7) may be summarized in the following

(4.9) Theorem. *Let things be as in (4.1) . Assume that $w_P + i_P < 3$. Then in some (not normalized) coordinate system (x) such that $\operatorname{wt}(x) = (a, m - a, b, c)$, where $(a, m) = (b, m) = 1$, we have one of the following cases*

$(P1)$ the main series, $c = 0$

	$\operatorname{ord}(C^\sharp)$	m	$C^\sharp$	a	b	i_p	w_p	$(F \cdot C)_P$
$(P1.1)$	$\langle a_1 \rangle$ $a_1 = 1$	≥ 2	$x_1 -$ axis	1		1, 2	$(m - b)/m$	b/m
$(P1.2)$	$\langle a_3 \rangle$ $a_3 = 1$	≥ 3	$x_3 -$ axis		1	1, 2	$(m - 1)/m$	$1/m$
$(P1.3)$	$\langle a_1, m \rangle$ $a_1 = 2$	odd ≥ 3	$x_1^m - x_4^2 =$ $x_2 = x_3 = 0$	2	odd	2	$(m - b)/m$	$(m + b)/m$
$(P1.4)$	$\langle a_1, a_2 \rangle$ $a_1 = 2$ $a_2 = m - 2$	odd ≥ 5	$x_1^{m-2} - x_2^2 =$ $x_3 = x_4 = 0$	2	odd $\ne m - 2$	2	$(m - b)/m$	$(m + b)/m$
$(P1.5)$	$\langle a_1, a_3 \rangle$ $a_1 = a$ $a_3 = b$	≥ 5	$x_1^b - x_3^a =$ $x_2 = x_4 = 0$		$(a, b) = 1$ $m = \alpha a + \beta b$ $\alpha \ge 1, \beta \ge 2$	2	$(m - b)/m$	b/m

$(P2)$ the exceptional case, $m = 4$, $c = 2$, $a = 1$

	$\operatorname{ord}(C^\sharp)$	$C^\sharp$	a	b	i_p	w_p	$(F \cdot C)_P$
$(P2.1)$	$\langle a_1 = 1 \rangle$	$x_1 -$ axis	1	1, 3	1, 2	$(4 - b)/4$	$b/4$
$(P2.2)$	$\langle a_2 = 3, a_4 = 2 \rangle$	$x_2^2 - x_4^3 = x_1 = x_3 = 0$	1	1	2	$3/4$	$5/4$

where F is a general member of $|-K_{(X,P)}|$.

(4.10) Lemma. *Let things be as in (4.1) . Suppose that $i_P = 1$ and $2 < w_P < 3$. Then (X, P) is non exceptional, $b \in \operatorname{ord}(C^\sharp)$, and $b \ge 3$.*

Proof. Assume that $b \notin \operatorname{ord}(C^\sharp)$. By (4.6) , (X, P) is not exceptional. From (4.3.3), we have

$$4m > m(i_P + w_P) \ge [\phi_1, \phi_2, \phi_3] = \sum_{i=1}^{3} \operatorname{ord}(z_i),$$

where $\phi_i \in \mathcal{I}^{\{0\}}$, $z_i := \partial\phi_i/\partial x_i \ne 0$. Using $\operatorname{ord}(x_1) + \operatorname{ord}(x_2) \equiv 0 \bmod m$, $\operatorname{ord}(x_3) \equiv b \bmod m$, we obtain

$$4m - b \ge \sum \operatorname{ord} z_i, \qquad 3m - b \ge \operatorname{ord}(z_3).$$

By (4.2.1), $mw_P = \min\{\operatorname{ord}(\psi) | \operatorname{wt}(\psi) = m - b\} = 3m - b$. Hence $\operatorname{ord}(z_3) \ge 3m - b$. Thus $\operatorname{ord}(z_3) = 3m - b$, $\operatorname{ord}(z_1) + \operatorname{ord}(z_2) = m$.

By normalizedness of (x) we get $\mathrm{ord}(z_1) = \mathrm{ord}(x_2)$, $\mathrm{ord}(z_2) = \mathrm{ord}(x_1)$. By our assumption $\mathrm{ord}(x_3) \geq m + b$. Therefore ϕ_1, ϕ_2 depend only on x_1, x_2, x_4 and obviously $\phi_1 = \phi_2 = (x_1x_2 - x_4)$. But then $[\phi_1, \phi_2, \phi_3] = \infty$, a contradiction. Therefore $b \in \mathrm{ord}(C^\sharp)$. If $b \leq 2$, then $2m - b \in \mathrm{ord}(C^\sharp)$ and $w_P \leq 2 - b/m$, a contradiction. This proves lemma. Q.E.D.

(4.11) Lemma. *Let things be as in (4.1) . Suppose that $i_P \leq 2$, $1 < w_P < 2$, $b \leq 2$. Then $i_P = 2$, m is odd, $b = 2$, $b \in \mathrm{ord}(C^\sharp)$ and $\mathrm{ord}(C^\sharp) = \langle 2, m\rangle$.*

PROOF. Assume that $b \notin \mathrm{ord}(C^\sharp)$. By (4.5) , (4.6) (X, P) is not exceptional and $m \geq 3$. By (4.2.1), $mw_P = \min\{\mathrm{ord}(\psi) | \mathrm{wt}(\psi) = m - b\} = 2m - b$. Hence $2m - b \in \mathrm{ord}(C^\sharp)$ and $m - b \notin \mathrm{ord}(C^\sharp)$. From (4.3.3) we have

$$4m > m(i_P + w_P) \geq [\phi_1, \phi_2, \phi_3] = \sum_{i=1}^{3} \mathrm{ord}(z_i), \quad \phi_i \in \mathcal{I}^{\{0\}}, \quad z_i := \partial\phi_i/\partial x_i \neq 0.$$

Since

$$\mathrm{ord}(z_1) + \mathrm{ord}(z_2) \equiv 0 \bmod m, \quad \mathrm{ord}(z_3) \geq 2m - b,$$

it is easy to see

$$4m - b \geq \sum \mathrm{ord}\, z_i.$$

Hence

$$2m - b = \mathrm{ord}(z_3), \quad \mathrm{ord}(z_1) + \mathrm{ord}(z_2) \leq 2m.$$

We can choose ϕ_i's from the following invariants:

$$x_1x_2 - x_4^{(a_1+a_2)/m}, \quad x_1^p x_3^q - x_4^{(a_1p+a_3q)/m}, \quad x_2^r x_3^s - x_4^{(a_2r+a_3s)/m}, \quad x_j^m - x_4^{a_j},\ j = 1, 2, 3.$$

Consider two cases:

(1) $\mathrm{ord}(z_1) + \mathrm{ord}(z_2) = m$. Then $\mathrm{ord}(z_1) = m - a$, $\mathrm{ord}(z_2) = a$, hence $\mathrm{ord}(z_1) = \mathrm{ord}(x_2) = m - a$, $\mathrm{ord}(z_2) = \mathrm{ord}(x_1) = a$ by normalizedness of (x). By our assumptions $b \notin \mathrm{ord}(C^\sharp)$ and $m - b \notin \mathrm{ord}(C^\sharp)$. So $\mathrm{ord}(C^\sharp)$ is generated by $\mathrm{ord}(z_1) = \mathrm{ord}(x_2) = m - a$ and $\mathrm{ord}(z_2) = \mathrm{ord}(x_1) = a$. Since $(a, m - a) = 1$, we have $z_1 = x_2$, $z_2 = x_1$. It gives as $\phi_1 = \phi_2 = x_1x_2 - x_4$. This is impossible.

(2) $\mathrm{ord}(z_1) + \mathrm{ord}(z_2) = 2m$. Permute x_1, x_2 such that $\mathrm{ord}(z_1) \leq \mathrm{ord}(z_2)$, then $\mathrm{ord}(z_1) = m - a$, $\mathrm{ord}(z_2) = m + a$. So, as above, $a_2 = m - a$, $z_1 = x_2$, $\phi_1 = x_1x_2 - x_4^{(a_1+a_2)/m}$. We have the following possibilities for $z_2 = \partial\phi_2/\partial x_2$:

$$\begin{aligned} z_2 &= x_2^{r-1}x_3^s, \quad \text{where } (r-1)(m-a) + sa_3 = m + a \text{ or} \\ z_2 &= x_2^{m-1}, \quad \text{where } (m-a)(m-1) = m + a. \end{aligned}$$

But if $(r - 1)(m - a) + sa_3 = m + a$, then since $a_3 \geq m + b$, we have $s = 1$, $a_3 = m + b$, $m - b > a - b = (r - 1)(m - 2)$, $m - b = r(m - a) \in \mathrm{ord}(C^\sharp)$, a contradiction with our assumption. Therefore $(m - a)(m - 1) = m + a$, then $a = m - 2$, $m - a = 2 \in \mathrm{ord}(C^\sharp)$ and m is odd. Since $m - b \notin \mathrm{ord}(C^\sharp)$, b also is even. Hence $b \in \mathrm{ord}(C^\sharp)$.

Now it is easy to see that $b \neq 1$. Suppose $b = 2$. Then m is odd and $\mathrm{ord}(C^\sharp)$ is generated by 2 and the smallest odd $k \in \mathrm{ord}(C^\sharp)$. Since $m - 2 \notin \mathrm{ord}(C^\sharp)$, $k = m$. Again from $4m - 2 \geq \sum \mathrm{ord}\, z_i$ we have $\mathrm{ord}\, z_1 + \mathrm{ord}\, z_2 = m$ or $2m$. But if $\mathrm{ord}(z_1) + \mathrm{ord}\, z_2 = m$, then $k < m$. Thus $\mathrm{ord}\, z_1 + \mathrm{ord}\, z_2 = 2m$, $m > \mathrm{ord}\, z_1 = a_2 = m - a$ is even, $\mathrm{ord}\, z_2 = m + a$, a is odd. Hence $a \notin \mathrm{ord}(C^\sharp)$, $a_1 = m + a$. This proves the lemma. Q.E.D.

5 Primitive case

(5.1) Lemma. *Let $f : (X, C) \to (S, s)$ be a minimal $\mathbb{Q}$-Fano fiber space with two-dimensional non-singular base, let $P_1, P_2, \dots, P_k \in X$ be all the points of indices $m_1, m_2, \dots, m_k > 1$ and let $m = \text{l.c.m.}(m_1 m_2 \cdots m_k)$ be the global index of X. Then*

(i) *X has no imprimitive points;*

(ii) *$k \le 2$;*

(iii) *if $k = 2$, then $(m_1, m_2) = 1$;*

(iv) *$(-K_X \cdot C) = \delta/m$, where $\delta = 1$ or 2.*

PROOF. (i) Assume that $P \in X$ is an imprimitive point. Then by (2.5) there exists an étale in codimension cyclic cover $X^\natural \to X$. By (2.9) we obtain an étale in codimension 1 cover $S^\natural \to S$. This is impossible because S is smooth.

(ii) It follows from (2.11) .

(iii) Assume for example that $k = 2$. The same arguments as in (i) shows that $(m_1, m_2) = 1$. Since P_1, P_2 are primitive, by (2.4) , we have $(D \cdot C) = 1/m_1 m_2$ for some $D \in \mathrm{Cl}^{sc}(X) \simeq \mathbb{Z}$. Obviously, $(-K_X \cdot C) = \delta/m_1 m_2$ for some $\delta \in \mathbb{Z}$. Hence $-K_X = \delta D$. Let L be a general fiber of f. Then $(-K_X \cdot L) = 2$, therefore $(D \cdot L) = 2/\delta$. But $(D \cdot L)$ is an integer, so $\delta = 1$ or 2. Q.E.D.

(5.1.1) Remark. If $(-K_X \cdot C) = 2/m$, then $-K_X = 2D$ for some $D \in \mathrm{Cl}^{sc}(X)$. In this case equality $-K_X = 2D$ holds in $\mathrm{Cl}^{sc}(X, P_i)$ for any points P_i of index $m_i > 1$. Since $-K_X$ is a generator of $\mathrm{Cl}^{sc}(X, P_i) \simeq \mathbb{Z}_{m_i}$, we obtain $(2, m_i) = 1$. Therefore $(2, m) = 1$ in this case.

(5.1.2) Corollary. *Let $f : (X, C) \to (S, s)$ be a minimal locally primitive $\mathbb{Q}$-Fano fiber space. Assume that (S, s) is singular and let n be topological index of (S, s). Then X contains exactly two singular points P_1, P_2 of indices > 1 and at most one point of index 1. If index of (X, P_i) is equal to m_i, $(i = 1, 2)$, then*

(i) $(m_1, m_2) = n$,

(ii) $(-K_X \cdot C) = \delta n/m_1 m_2$, where $\delta = 1$ or 2.

Moreover $\delta = 2$ only if both of m_1/n and m_2/n are odd.

(5.2) Theorem. *Let $f : (X, C) \to (S, s)$ be a minimal $\mathbb{Q}$-Fano fiber space. Assume that (X, C) is locally primitive. Then*

(O) *$(K_X \cdot C) = \delta/m$, where m is global index of (X, C), $\delta = 1$ or 2.*

(I) $\deg \mathrm{gr}^0_C \omega \ge -2$;

(II) *If* $\deg \mathrm{gr}^0_C \omega = -2$, *then we have one of the following*

> (IIa) *$f : (X, C) \to (S, s)$ is as in example (1.2) .*
>
> (IIb) *X contains only one singular point P of odd index m, $i_P = 2$, $w_P = 2 - 2/m$. In this case (S, s) is non-singular and a general member of $|-K_X|$ does not contain C and has only Du Val singularity at P.*

(III) *If* $\deg \mathrm{gr}^0_C \omega = -1$, *then we have one of the following*

> (IIIa) *X contains three singular points P_1, P_2, P_3 of indices m_1, m_2 and $m_3 = 1$ with $(m_1, m_2) = 1$. In this case $i_{P_1} = i_{P_2} = i_{P_3} = 1$, $w_{P_1} + w_{P_2} = 1 + (K_X \cdot C) < 1$, and (S, s) is non-singular.*

(IIIb) *X contains three singular points P_1, P_2, P_3 of indices $m_1 \geq m_2 = 2$ and $m_3 = 1$, m_1 is even. In this case $m = m_1$, $\delta = 1$, $i_{P_1} = i_{P_2} = i_{P_3} = 1$, $w_{P_2} = 1/2$, $w_{P_1} = 1/2 - 1/m_1$, $w_{P_3} = 0$, and (S, s) is Du Val of type A_1. Furthermore (X, P_2) is a cyclic quotient singularity of type $\frac{1}{2}(1,1,1)$ and a general member of $|-2K_X|$ does not contain C (and has only log-terminal singularity at P_2).*

(IIIc) *X contains two singular points P_1, P_2 of indices $m_1 \geq m_2 > 1$. In this case $i_{P_1} + i_{P_2} \leq 3$, $w_{P_1} + w_{P_2} = 1 + (K_X \cdot C) < 1$. If $n := (m_1, m_2)$, then (S, s) is a cyclic quotient singularity of index n.*

(IIId) *X contains two singular points P_1, P_2 of indices m_1 and $m_2 = 1$. In this case $i_{P_1} + i_{P_2} \leq 3$, $m = m_1$, $w_{P_1} = 1 - \delta/m_1$, and (S, s) is non-singular.*

(IIIe) *X contains only one singular point P of index m with $i_P \leq 3$, $w_P = 1 - \delta/m$ In this case (S, s) is non-singular.*

PROOF. (O) is the same as (5.1.2). (I) By (3.6) $d \leq 4 - \sum_P i_P \leq 3$. Assume that $d = 3$, then X contains only one singular point P with $i_P = 1$. By (5.1.2), (5.1) (S, s) is non-singular and $(K_X \cdot C) = -\delta/m$, where m is index of (X, P), $\delta = 1$ or 2. Then from (3.5) we have $w_P = 3 - \delta/m < 3$. Therefore $m > 2$ and $b = \delta$ (see (4.2.1)). Lemma (4.10) gives us a contradiction.

(II) First assume that the only singular point of X is P. By (5.1.2), (5.1) (S, s) is non-singular and $(K_X \cdot C) = -\delta/m$, where m is index of (X, P), $\delta = 1$ or 2. Then from (3.5) we have $w_P = 2 - \delta/m < 2$. Therefore $m > 2$ and $b = \delta$ (see (4.2.1)). Moreover from (4.11) we have $b = 2$, m is odd, $i_P = 2$ and $\mathrm{ord}(C^\sharp) = (2, m)$. Let $F \in |-K_{(X,P)}|$ be a general member. Then $F \cap C = \{P\}$ and $F + K_X$ is Cartier. By (4.4), $(F \cdot C)_P = 2/m$. It gives us $((F + K_X) \cdot C) = 0$, hence $F \in |-K_X|$. It follows from (1.6.4), then F has only Du Val singularity at P.

Now we consider the case when X has more then one singular point. Then $\sum i_P \leq 2$, so X contains exactly two singular points P_1, P_2 with $i_{P_1} = i_{P_2} = 1$. Let m_1, m_2 their indices and $m = m_1 m_2/(m_1, m_2)$ be global index of X. Then from (3.5) we have $w_{P_1} + w_{P_2} = 2 + (K_X \cdot C) = 2 - \delta/m < 2$. Hence $(X, P_i) \supset (C, P_i)$, $i = 1, 2$ are such as in (4.9). In particular $w_{P_1}, w_{P_2} < 1$ hence $w_{P_1}, w_{P_2} > 0$ and P_1, P_2 are non-Gorenstein. Take general divisors $F_i \in |-K_{(X,P_i)}|$ $(i = 1, 2)$. We claim that $F_1 + F_2 \in |-K_X|$. Indeed it is sufficient to show only $((F_1 + F_2 + K_X) \cdot C) = 0$. But

$$((F_1 + F_2 + K_X) \cdot C) = ((F_1 \cdot C)_P + w_{P_1} - 1) + ((F_2 \cdot C)_P + w_{P_2} - 1).$$

By (4.9), in the last equation both of terms are zero.

Therefore a general member $F_1 + F_2 \in |-K_X|$ does not contain C and has only Du Val singularities. Let L be a general fiber of $f : X \dashrightarrow S$. Since $(-K_X \cdot L) = 2$, $(F_1 \cdot L) = (F_2 \cdot L) = 1$. Hence $(F_1, P_1) \simeq (F_2, P_2) \simeq (S, s)$ are Du Val of type A_{n-1}. By (5.1.2), (1.6.7) $n \geq m_i \geq n$ Thus $m_1 = m_2 = n$ and the topological cover $X^\natural / S^\natural \to X/S$ gives us a conic bundle $f^\natural : X^\natural \to S^\natural$. Moreover F_i lifts back to $\mathbb{Z}_n$-invariant section $F_i^\natural$ of $f^\natural$. Therefore $f^\natural : X^\natural \to S^\natural$ is $\mathbb{Z}_n$-isomorphic to $\mathbb{P}^1 \times \mathbb{C}^2 \to \mathbb{C}^2$. After change of the coordinate system if necessary we obtain a $\mathbb{Q}$-Fano fiber space as in (1.2).

(III) Let $\deg \mathrm{gr}^0_C\, \omega = -1$. Then $\sum i_P \leq 3$, $\sum w_P = 1 + (K_X \cdot C)$. First suppose that X contains three singular points P_1, P_2, P_3 of indices m_1, m_2, m_3. By (2.11), one of m_i's, say m_3 is equal to 1. Let $n = (m_1, m_2)$ be topological index of (S, s). Consider the topological

$\mathbb{Z}_n$-cover (2.7) . Then the fiber $C^\natural$ is irreducible and the cover $g : C^\sharp \simeq \mathbb{P}^1 \to C \simeq \mathbb{P}^1$ is ramified only over two points P_1, P_2. Hence $g^{-1}(P_i)$ is a point $P_i^\natural$ for $i = 1, 2$ and $g^{-1}(P_3) = \{P_3^\natural(1), \ldots, P_3^\natural(n)\}$. Since all the $P_4^\natural(i)$'s are singular, by (3.7) , we have $n \leq 3$. If $n = 1$, then we have case (IIIa).

Let $n = 2$. Then $(X^\natural, C^\natural)$ contains two singular points $P_3^\natural(1)$, $P_3^\natural(2)$, hence at least one of $P_1^\natural$ or $P_2^\natural$ is non-singular. So we may assume that $(X^\natural, P_2^\natural)$ is non-singular and (X, P_2) is a cyclic quotient singularity. This is case (IIIb). Points P_1, P_2 may be only of types (P1.1), (P1.2) or (P2.1) of theorem (4.9) . In particular, $\mathrm{ord}(C^\sharp) = \mathbb{N}$. As in case (II) take a general divisor $D \in |-2K_{(X,P_2)}|$. Since $w_{P_2} < 1/2$, in notations (4.1) we have $b > m/2$. Similar to (4.4) $(D \cdot C)_{P_2} = (2b_2 - m_2)/m_2 = 1 - 2w_{P_2}$. The divisor $D + 2K_X$ on X is Cartier, because index of P_1 is equal to 2. On the other hand $((D + 2K_X) \cdot C) = 1 - 2w_{P_2} + 2(w_{P_1} + w_{P_2} - 1) = 0$. Hence $D \in |-2K_X|$.

Now consider the case $n = 3$. Then $(X^\natural, C^\natural)$ contains three singular points $P_3^\natural(1)$, $P_3^\natural(2)$, $P_3^\natural(3)$. In this case (X, P_1) and (X, P_2) are cyclic quotient singularities. Hence, by (1.4) , $f^\natural : (X^\natural, C^\natural) \to (S^\natural, s^\natural)$ is a conic bundle with irreducible fiber $C^\natural \simeq \mathbb{P}^1$. This contradicts the following

(5.2.1) Lemma. *Let $f : (X, C) \to (\mathbb{C}^2, 0)$ be a minimal conic bundle with only isolated singularities. Then (X, C) contains at most two singular points.*

PROOF. Assume that (X, C) contains three singular points. Then scheme-theoretical fiber $f^{-1}(0)$ is a double line. Hence in some coordinate system $(x_0, x_1, x_2; u, v)$ in $\mathbb{P}^2 \times (\mathbb{C}^2, 0)$ X is given by the equation

$$x_0^2 + \phi(u,v)x_1^2 + \psi(u,v)x_1x_2 + \zeta(u,v)x_2^2 = 0,$$

where $\phi(0,0) = \psi(0,0) = \zeta(0,0) = 0$. Moreover we may assume that singular points are $(x_0, x_1, x_2; u, v) = (0,1,0;0,0)$, $(0,0,1;0,0)$ and $(0,1,1;0,0)$. But then easy computations gives us

$$\partial\phi(0,0)/\partial u = \partial\phi(0,0)/\partial v = \partial\psi(0,0)/\partial u = \partial\psi(0,0)/\partial v = \partial\zeta(0,0)/\partial u = \partial\zeta(0,0)/\partial v = 0.$$

Therefore $C \subset \mathrm{Sing}(X)$, a contradiction. Q.E.D.

The rest assertions of the theorem is only division into cases. Here we use (5.1.2), (2.8) and (2.8.1). Q.E.D.

(5.3) Proposition. *Let $f : (X, C) \to (S, s)$ be a minimal locally primitive $\mathbb{Q}$-Fano fiber space such as in (5.2) (IIIc). Assume that $i_{P_1} = i_{P_2} = 1$, $w_{P_1} < 1/2$ and $w_{P_2} < 1/2$. Then*

(i) *a general member of $|-2K_X|$ does not contain C (and has only log-terminal singularities),*

(ii) *$f : (X, C) \to (S, s)$ is a quotient of the minimal conic bundle $f^\natural : (X^\natural, C^\natural) \to (S^\natural, s^\natural) \simeq (\mathbb{C}^2, 0)$ by $\mathbb{Z}_n$, where the action $\mathbb{Z}_n$ on $\mathbb{C}^2 - \{0\}$ is free.*

PROOF. (i) As in (5.2) (II) take general divisors $F_i \in |-2K_{(X,P_i)}|$. We claim that $F_1 + F_2 \in |-2K_X|$. It is sufficient to show only $((F_1 + F_2 + 2K_X) \cdot C) = 0$. Note that by (4.9) both of (X, P_1) and (X, P_2) are of type (P1.1), (P1.2) or (P2.1). In particular for corresponding $b_i = \mathrm{wt}(x_3)$ we have $b_i = (1 - w_{P_i})m_i > m_i/2$. Similar to (4.4) , $(F_i \cdot C) = (2b_i - m_i)/m_i = 1 - w_{P_i}$, because $\mathrm{ord}(C_i^\sharp) = \mathbb{N}$. It now follows that

$$((F_1 + F_2 + 2K_X) \cdot C) = ((F_1 \cdot C) + w_{P_1} - 1) + ((F_2 \cdot C) + w_{P_2} - 1) = 0.$$

This proves (i).

(ii) Let $n = (m_1, m_2)$. By (2.5) , (2.9) , it is sufficient to prove only $m_1 = m_2 = n$. Let $F_1 + F_2 \in |-2K_X|$ be a general member, where $F_i \in |-2K_{(X,P_i)}|$ and let L be a general fiber of f. We have $((F_1 + F_2) \cdot L) = (-2K_X \cdot L) = 4$. Hence up to permutation $(F_1 \cdot L) = (F_2 \cdot L) = 2$, or $(F_1 \cdot L) = 3(F_2 \cdot L) = 3$. Let us consider these cases.

CASE (1). Then $(F_1 \cdot C) = (F_2 \cdot C) = (-K_X \cdot C) = \delta n/m_1 m_2$. But $(F_i \cdot C) = k_i/m_i$, where $k_i \in \mathbb{N}$. It gives us $k_1(m_2/n) = k_2(m_1/n) = \delta$. Hese $m_i \neq n$ only if $\delta = 2$ and $m_i/n = 2$, a contradiction with (5.1.2).

CASE (2). In this case $(F_2, P_2) \simeq (S, s)$. In particular $I_{top}(F_2, P_2) = I_{top}(S, s) = n$. Hence, by (1.6.7), (X, P_2) is a cyclic quotient singularity of index $n = m_2$. As in case (1) we have $(F_2 \cdot C) = k/n$, $(F_1 \cdot C) = 3k/n$, $(-K_X \cdot C) = 2k/n = \delta/m_1$. We obtain $2k(m_1/n) = \delta$, i. e. $\delta = 2$, $m_1 = m_2 = n$. This proves the proposition. Q.E.D.

(5.4) Example. Consider the following hypersurface in $\mathbb{P}^2_{x,y,z} \times \mathbb{C}^2_{u,v}$:

$$X^\natural : \qquad \{x^2 + uy^2 + vz^2 = 0\}.$$

Define an action of $\mathbb{Z}_n$ on $X^\natural$ as

$$(x, y, z, u, v) \to (\varepsilon^a x, \varepsilon^{-1} y, z, \varepsilon u, \varepsilon^{-1} v),$$

where $2a + 1 = n$, $\varepsilon = \exp(2\pi i/n)$. Then $f : X^\natural/\mathbb{Z}_n \to \mathbb{C}^2/\mathbb{Z}_n$ is a $\mathbb{Q}$-Fano fiber space. The singular locus of $X^\natural/\mathbb{Z}_n$ consist of two cyclic quotient points of index n. The point (S, s) is Du Val of type A_{n-1}.

(5.4.1) Computations. Consider the open set $\{z \neq 0\}$. The local coordinates are $(t_1 = x/z, t_2 = y/z, u)$. Let $\Omega = (1/t_1)(dt_1 \wedge dt_2 \wedge du)$. Since $\Omega \in \omega_{X^\natural}$ is $\mathbb{Z}_n$-invariant, Ω^{-1} defines a general element $F \in |-K_X|$. It is easy to see that F contains central fiber $C = f^{-1}(0)_{\mathrm{red}}$ and has two singular points of type A_{n-1}. Similar to (3.4) we may compute

(i) $(-K_X \cdot C) = 1/n$,

(ii) $i_{P_1}(1) = i_{P_2}(1) = 1$,

(iii) $w_{P_1}(0) = a/n \qquad w_{P_2}(0) = (n - a - 1)/n$,

(iv) $\deg \mathrm{gr}^0_C \omega = -1$.

Therefore this is an example of $\mathbb{Q}$-Fano fiber space as in (5.2) (IIIc).

Now we shall study locally primitive $\mathbb{Q}$-Fano fiber spaces under the assumption the existence of good member in $|-K_X|$.

(5.5) Proposition. *Let $f : (X, C) \to (S, s)$ be a minimal locally primitive $\mathbb{Q}$-Fano fiber space. Assume that a general member of $|-K_X|$ has only Du Val singularities. Then one of the following hold:*

(i) *(S, s) is non-singular,*

(ii) *(S, s) is of type A_1,*

(iii) *$f : (X, C) \to (S, s)$ is quotient of a non-singular conic bundle $f^\natural : (X^\natural, C^\natural) \to (S^\natural, s^\natural)$ with irreducible $C^\natural$ by the group $\mathbb{Z}_m$, where $m \geq 3$ and the action $\mathbb{Z}_m$ on $(S^\natural, s^\natural) \simeq (\mathbb{C}^2, 0)$ is free in codimension 1. Moreover (S, s) has type A_{m-1} in this case.*

PROOF. Let $F \in |-K_X|$ be a general member. If $C \not\subset F$, then we have (5.2) (IIa). So we assume that $F \supset C$ and $n := I_{top}(S, s) \geq 3$. By (5.2) X contains exactly two singular points P_1, P_2 of indices m_1, m_2 with $(m_1, m_2) = n$. Since $-K_X \cdot L = 2$, where L is a general fiber of f, the restriction $f|_F : F \to S$ is generically finite of degree 2. Let

$$f_F : (F, C) \xrightarrow{f_1} (F', P') \xrightarrow{f_2} (S, s)$$

be the Stein factorization, where P' is a point. Then $f_1 : (F, C) \to (F', P')$ is bimeromorphic and $f_2 : (F', P') \to (S, s)$ is finite of degree 2. By the adjunction formula, $K_F = 0$. Therefore the morphism f_1 is crepant and (F', P') is Du Val singularity. Thus there exists the common minimal resolution $(\tilde{F}, \tilde{C} \cup E_1 \cup \ldots \cup E_r) \to (F, C) \to (F', P')$, where $E_1, \ldots, E_r$ are exceptional divisors. Let $\Gamma = \Gamma(F/F')$ be a dual graph for this resolution. Denote vertex corresponding $\tilde{C}$ (resp. E_i) by $\bullet$ (resp. $\circ$). Then white vertices form at least two connected graphs corresponding singular points of (F, C). Note that graphs $\Gamma_i \subset \Gamma$ for points (F, P_1), (F, P_2) has at least $n - 1$ vertices, because $m_i \geq n$ and by (1.6.7). From (1.5.2) keeping in mind that (S, s) is a cyclic quotient singularity we get the following cases for $(F', P') \to (S, s)$:

(1)	$E_6 \xrightarrow{2:1} A_2$,	$n = 3$,
(2)	$A_{2k+1} \xrightarrow{2:1} A_k$,	$n = k + 1$,
(3)	$A_{2k} \xrightarrow{2:1} \frac{1}{2k+1}(k, 2k - 1)$,	$n = 2k + 1$
(4)	$A_k \xrightarrow{2:1} A_{2k+1}$,	$n = 2k + 1$,
(5)	$A_{2k+1} \xrightarrow{2:1} \frac{1}{4k+4}(2k + 1, 2k + 1)$,	$n = 4k + 4$.

Let $\pi^\sharp : (X^\sharp, P_i^\sharp) \to (X, P_i)$, $i = 1, 2$ be the canonical cover and $F_i^\sharp := \pi^{\sharp^{-1}} F$. Then $F_i^\sharp \sim -K_{(X^\sharp, P_i^\sharp)}$ is a Cartier divisor, hence it is normal and $(F_i^\sharp, P_i^\sharp)$ is a Du Val point. Thus we have étale in codimension 1 $\mathbb{Z}_{m_i}$-covers $\pi_i^\sharp : (F_i^\sharp, P_i^\sharp) \to (F, P_i)$ of Du Val singularities, where $(m_1, m_2) = n$.

Consider also the topological cover

$$\begin{array}{ccc} X^\natural & \xrightarrow{g} & X \\ \downarrow f^\natural & & \downarrow f \\ S^\natural & \xrightarrow{h} & S \end{array}$$

It is sufficient to prove that $f^\natural : X^\natural \to S^\natural$ is anon-singular conic bundle.

CASE (1). $(F', P') = E_6$, $(S, s) = A_2$, $n = 3$, $m_i = 3m_i'$. We have only one possibility for Γ.

$$\begin{array}{ccccccccc} & & & & \circ & & & & \\ & & & & | & & & & \\ \circ & - & \circ & - & \bullet & - & \circ & - & \circ \end{array}$$

Then (F, P_i) is type A_2, $m_1 = m_2 = 3$ and $(X^\sharp, P_i^\sharp)$ are non-singular (see (1.5.3)). But then $(X^\natural, P_i^\natural)$ is non-singular too. We obtain case (iii).

CASE (2). $(F', P') = A_{2k+1}$, $(S, s) = A_k$, $n = k + 1$, $m_i = (k + 1)m_i' \geq k + 1$. Then Γ is

$$\underbrace{\circ - \circ - \cdots - \circ}_{l_1} - \bullet - \underbrace{\circ - \cdots - \circ}_{l_2}$$

Whence (F, P_i) is of type A_{l_i}, where $l_i < 2k$. On the other hand $l_i + 1 \geq m_i = (k+1)m'_i$. Hence $l_i = k$, (F, P_i) is type A_k and by (1.5.3) $(X^\natural, P^\natural) = (X^\sharp, P^\sharp)$ is non-singular. As in (1) we get case (iii).

Similarly cases (3), (4), (5), (6) are impossible, by (1.5.3). This proves our proposition. Q.E.D.

6 Some results in imprimitive case

(6.1) Proposition. *Let $f : (X, C) \to (S, s)$ be a minimal $\mathbb{Q}$-Fano fiber space. Assume that (X, C) contains an imprimitive point P of index m and splitting degree e (i. e. $C^\flat$ has exactly e irreducible components). Then*

(i) *The topological cover factors through splitting cover;*

$$\begin{array}{cccccc} g: & X^\natural & \xrightarrow{g^\natural} & X^\flat & \xrightarrow{g^\flat} & X \\ & \downarrow f^\natural & & \downarrow f^\flat & & \downarrow f \\ h: & S^\natural & \xrightarrow{g^\natural} & S^\flat & \xrightarrow{g^\flat} & S \end{array}$$

Hence n (topological index of (S, s)) is divisible by e.

(ii) *X contains no another imprimitive points and at most two primitive points, one of them has index 1.*

(iii) *$g^{-1}(P)$ (resp. ${g^\flat}^{-1}(P)$) is only one point $P^\natural$ (resp. $P^\flat$), all the components of $C^\natural := (g^{-1}(C))_{\text{red}}$ (resp. $({g^\flat}^{-1}(C))_{\text{red}}$) pass through $P^\natural$ (resp. $P^\flat$). In particular m is divisible by n and by e.*

(iv) *If $e \geq 3$, then $(X^\natural, P^\natural)$ and $(X^\flat, P^\flat)$ have index > 1.*

(v) $\deg \operatorname{gr}_C^0 \omega = -1$.

(vi) *$(K_X \cdot C) = (K_{X^\flat} \cdot C^\flat(i)) > -1$, where $C^\flat(i)$ is an irreducible component of $C^\flat$.*

(vii) *$w_P = w_{P^\flat(i)}$, where $P^\flat(i) = P^\flat$ is considered as a point of $(X^\flat, C^\flat(i))$.*

PROOF. (i), (ii), (iii) immediately follows from (2.5) and (2.10). To prove (iv) consider the extremal neighborhood

$$(X^\natural, \cup_{j \neq i} C_j^\natural), \qquad \text{(resp.} \qquad (X^\flat, \cup_{j\neq i} C_j^\flat)),$$

where $C_j^\natural$ (resp. $C_j^\flat$) are irreducible components of $C^\natural$ (resp. $C^\flat$). Since $\cup_{j\neq i} C_j^\natural$ and $\cup_{j \neq i} C_j^\flat$ are reducible, points $(X^\natural, \cap_{j\neq i} C_j^\natural) = (X^\natural, P^\natural)$ and $(X^\flat, \cap_{j \neq i} C_j^\flat) = (X^\flat, P^\flat)$ has indices > 1 by [18], 1.15.

(v) The splitting cover $g^\flat : (X^\flat, C^\flat) \to (X, C)$ induces an isomorphism $C^\flat(i) \xrightarrow{\sim} C \simeq \mathbb{P}^1$, where $C^\flat(i)$ is an irreducible component of $C^\flat$. Hence we have the map

$$\operatorname{gr}^0_{C^\flat(i)} \omega \longrightarrow \operatorname{gr}^0_C \omega.$$

By [18], 2.3.2, $\operatorname{gr}^0_{C^\flat(i)} \omega \simeq \mathcal{O}_{C^\flat(i)}(-1)$. Therefore $\operatorname{gr}^0_C \omega \simeq \mathcal{O}_C(-1)$.

(vi) It follows from $K_{X^\flat} = g^{\flat *}(K_X)$.

(vii) We have

$$w_P + \sum_{Q \neq P} w_Q = 1 + (K_X \cdot C) = 1 + (K_{X^\flat(i)} \cdot C^\flat(i) = w_{P^\flat(i)} + \sum_{Q^\flat(i) \neq P^\flat} w_{Q^\flat(i)}.$$

Since $g^\flat(X^\flat, C^\flat(i)) \to (X, C)$ is an isomorphism outside $P^\flat(i)$, for $Q^\flat(i) = g^\flat(Q)$ one has the equality $w_Q = w_{Q^\flat(i)}$. Whence $w_P = w_{P^\flat(i)}$. This proves the proposition. Q.E.D.

The following is an easy consequence of the classification of extremal neighborhoods of index 2 [13].

(6.2) Proposition. *Let $f : (X, C) \to (S, s)$ be a $\mathbb{Q}$-Fano fiber space. Assume that X has only points of index one and two (we do not assume that C is irreducible). Then we have one of the following:*

(i) *$f : (X, C) \to (S, s) \simeq (\mathbb{C}^2, 0)$ is a conic bundle.*

(ii) *$f : (X, C) \to (S, s)$ is a quotient of a conic bundle $f : (X^\natural, C^\natural) \to (S^\natural, s^\natural) \simeq (\mathbb{C}^2, 0)$ by $\mathbb{Z}_2$, where the action $\mathbb{Z}_2$ on $\mathbb{C}^2 - \{0\}$ is free.*

(iii) *$(S, s) \simeq (\mathbb{C}^2, 0)$, X has a unique point, say P, of index two, $C = \sum C_i$ has at most four components, they all pass through P. Moreover in this case $(-K_X \cdot C_i) = 1/2$ for each irreducible component $C_i \subset C$ and for the scheme-theoretical fiber $Z := f^{-1}(s)$ we have*

(iiia) *$Z \equiv 4C$, C is irreducible,*

(iiib) *$Z \equiv 2C = 2(C_1 + C_2)$,*

(iiic) *$Z \equiv C_1 + 3C_2$, $C = C_1 + C_2$,*

(iiid) *$Z \equiv C_1 + C_2 + 2C_3$, $C = C_1 + C_2 + C_3$, or*

(iiie) *$Z \equiv C = C_1 + C_2 + C_3 + C_4$.*

PROOF. Assume that $(S, s) \simeq (\mathbb{C}^2, 0)$ (is non-singular) and $f : (X, C) \to (S, s)$ is not a conic bundle. Let $Z := f^{-1}(s)$ be the scheme-theoretical fiber of f. Then $Z \equiv \sum \alpha_i C_i$, where $\alpha_i \in \mathbb{N}$ and $C = \sum C_i$.

From lemma (1.4.1) we have $2 = (-K_X \cdot Z) = \sum \alpha(-K_X \cdot C_i)$. Thus the number of components is at most 4. If C is irreducible, then X contains a unique point of index 2 by (5.1.2). If C has 3 or 4 components, then $(X, C_i \cup C_j)$ is an extremal neighborhood for any $C_i, C_j \subset C$ such that $C_i \cap C_j \neq \emptyset$. In this case by [13], 4.7 $C_i \cap C_j$ is the only point of index 2. It gives as case (iii). Consider the case $C = C_1 + C_2$ and let $C_1 \cap C_2 = \{P\}$. Again by [13], 4.7 any of C_1, C_2 contains at most one point of index 2. If P has index 2, then we obtain case (iii), so assume that (X, P) is Gorenstein. Let $P_1 \in C_1$, $P_1 \in C_1$ are points of index 2, $P_i \neq P$. Then by [18], 7.3 general members $F_i \in |-K_{(X,P_i)}|$, $i = 1, 2$ are general members of $|-K_{(X,C_i)}|$ with $(F_i \cdot C_i) = 1/2$. Therefore $F_1 + F_2 \in |-K_{(X,C)}|$. But then $(F_i, P_i) \simeq (S, S)$, a contradiction. The case when only one of (X, C_1), (X, C_2) contains point of index 2 is treated by the similar way.

Now we assume that (S, s) is singular. Consider the topological cover

$$\begin{array}{ccc} X^\natural & \xrightarrow{g} & X \\ \downarrow f^\natural & & \downarrow f \\ S^\natural & \xrightarrow{h} & S \end{array}$$

If $X^\natural$ is Gorenstein, then we have case (ii). In the opposite case $f^\natural : X^\natural \to S^\natural$ is such as in (iii). Then point $P^\natural$ of index 2 is $\mathbb{Z}_2$-invariant. Hence $g(P^\natural)$ has index > 2, a contradiction. Q.E.D.

(6.2.1) Example. Let $V = V_4^3 \subset \mathbb{P}^6$ be a projective cone over the Veronese surface $F = F_4^2 \subset \mathbb{P}^5$ with the vertex O and let $P_1, \dots, P_4 \in V$ be points such that $\langle P_1, \dots, P_4 \rangle = \mathbb{P}^3 \not\ni O$. A unique singular point of V is O, it is cyclic quotient singularity of type $\frac{1}{2}(1,1,1)$. Consider the projection $p : V - - \to \mathbb{P}^2$ from $\langle P_1, \dots, P_4 \rangle = \mathbb{P}^3$. Denote by $s \in \mathbb{P}^2$ the image of O. The resolution of the base locus of p is

$$p : V \xleftarrow{\sigma} X \xrightarrow{f} \mathbb{P}^2,$$

where $X \to V$ is the blow-up of $P_1, \dots, P_4$. Fibers of $X \to \mathbb{P}^2$ are strict transforms of $V \cap \mathbb{P}^4$, hence a general fiber is $\mathbb{P}^1$. The fiber $f^{-1}(s)$ is the union of four $\mathbb{P}^1$'s meeting in the point $O' := \sigma^{-1}(O)$ (of type $\frac{1}{2}(1,1,1)$). Let $f^{-1}(s) = L_1 \cup \ldots \cup L_4$. Easy computations gives us $-2K_V = \mathcal{O}_V(5)$, thus we have $(-K_X \cdot L_i) = (\sigma^*(-K_V) \cdot L_i) - (2E_i \cdot L_i) = 1/2(\mathcal{O}_V(5) \cdot \sigma(L_i)) - 2 = 1/2 > 0$, where E_i is the exceptional divisor over P_i. Therefore $f : (X, L_1 \cup \ldots \cup L_4) \to (\mathbb{P}^2, s)$ is a $\mathbb{Q}$-Fano fiber space with non-singular base and a unique singular point of index 2.

(6.3) Proposition. *Let $f : (X, C) \to (S, s)$ be a minimal $\mathbb{Q}$-Fano fiber space with an imprimitive point P. Assume that a general member $F \in |-K_X|$ has only Du Val singularities. Then we have one of the following*

(i) *(S, s) is Du Val of type A_1, or*

(ii) *(S, s) is Du Val of type A_3, in this case (X, C) has a unique cyclic quotient singularity P of index 8 and has no another points of index > 1, splitting degree of (X, C) is equal to 4.*

PROOF. Let P be an imprimitive point of index m and splitting degree e. It follows from (5.1.2) that (S, s) is singular. Let $n = I_{top}(S, s)$ be topological index of the base. We assume that $n > 2$ (otherwise we have case (i). Remember that m is divisible by n and n is divisible by e (see (6.1)). Consider the topological cover

$$\begin{array}{ccc} X^\natural & \xrightarrow{g} & X \\ \downarrow f^\natural & & \downarrow f \\ S^\natural & \xrightarrow{h} & S \end{array}$$

Then $g^{-1}(P)$ is one point, say $P^\natural$, $C^\natural := (g^{-1}(C))_{\mathrm{red}}$ is reducible, $C^\natural = \sum_{i=1}^{e} C_i^\natural$. Moreover $P^\natural \in C_i^\natural$ for all i.

(6.4) First we assume that $C \not\subset F$. Since $-K_X \cdot L = 2$, where L is a general fiber of f, the restriction $f|_F : F \to S$ is finite of degree 2. If $C \cap F$ is two point P, P_1, then we can assume that $F = F_0 + F_1$, where $F_0 \ni P$, $F_1 \ni P_1$. But then $(F_0, P) \simeq (F_1, P_1) \simeq (S, s)$ are Du Val of type A_{n-1}. Since $F^\natural := g^{-1}(F) \in |-K_X|$, as above, we see that $F^\natural$ has two connected components $F_0^\natural := g^{-1}(F_0)$, $F_1^\natural := g^{-1}(F_1)$. But then $F_1^\natural \not\ni P^\natural$ and hence $F_1^\natural$ intersects only one component of $C^\natural$, because we consider germs (X, C), $(X^\natural, C^\natural)$. This contradicts the fact that $\mathbb{Z}_n$ transitively acts on $\{C_i^\natural\}$.

Now we assume that $C \cap F$ is only one point P. Since P is a unique point of index > 1, the action G on $C^\natural$ is free outside $P^\natural$. In particular, the number of components of $C^\natural$ is divisible by n, so $n = e$. Thus if $n = 2$ and $(X^\natural, P^\natural)$ has index 1, then we have the case (i).

So we assume the opposite. From (1.5.2) we get the following cases for $(F,P) \to (S,s)$ as in (5.5) :

$$
\begin{array}{lll}
(1) & E_6 \xrightarrow{2:1} A_2, & n = 3, \\
(2) & A_{2k+1} \xrightarrow{2:1} A_k, & n = k+1, \\
(3) & A_{2k} \xrightarrow{2:1} \frac{1}{2k+1}(k, 2k-1), & n = 2k+1 \\
(4) & A_k \xrightarrow{2:1} A_{2k+1}, & n = 2k+1, \\
(5) & A_{2k+1} \xrightarrow{2:1} \frac{1}{4k+4}(2k+1, 2k+1), & n = 4k+4.
\end{array}
$$

Let $\pi^\sharp : (X^\sharp, P^\sharp) \to (X,P)$ be the canonical cover and $F^\sharp := \pi^{\sharp -1} F$. Then $F^\sharp \sim -K_{(X^\sharp, P^\sharp)}$ is a Cartier divisor, hence it is normal and $(F^\sharp, P^\sharp)$ is a Du Val point. Thus we have étale in codimension 1 $\mathbb{Z}_m$-cover $\pi^\sharp : (F^\sharp, P^\sharp) \to (F,P)$ of Du Val singularities, where $m \geq n$. By (1.5.3), cases (4), (5) are impossible and (F,P) from (3) admits only cover by non-singular $(F^\sharp, P^\sharp)$ of degree $n = m = 2k+1$. But then $(X^\sharp, P^\sharp) = (X^\natural, P^\natural)$ is a non-singular point. Then $f^\natural : (X^\natural, C^\natural) \to (S^\natural, s^\natural)$ is a conic bundle and $C^\natural$ has only two components. Hence $2k+1 = e = n = 2$, a contradiction. In case (1) $(F,P) = E_6$ admits only cyclic cover $D_4 \xrightarrow{3:1} E_6$. Then $m = n = e = 3$ and $f^\natural : (X^\natural, C^\natural) = (X^\flat, C^\flat) \to (S^\natural, s^\natural)$ is a conic bundle. But in this case $C^\natural = C^\flat$ has only two components, a contradiction.

Finally, consider case (2). If $(X^\natural, P^\natural)$ has index 1, then, as above, $C^\natural$ has only two components, so $n = e = 2$, we get case (i) of our theorem. But if $(X^\natural, P^\natural)$ has index $m > 1$, then $m > n = k+1$ and by (1.5.3) $(F^\sharp, P^\sharp) \xrightarrow{m:1} (F,P)$ is $A_0 \xrightarrow{2k+2:1} A_{2k+1}$, $m = 2k+2$. Then index of $(X^\natural, P^\natural)$ is equal to $m/n = 2$, $(X^\sharp, P^\sharp)$ is non-singular, hence $(X^\natural, P^\natural)$ is a cyclic quotient singularity of type $\frac{1}{2}(1,1,1)$. Therefore $f^\natural : (X^\natural, C^\natural) \to (S^\natural, s^\natural)$ is either as in (6.2) , (iiib), or (6.2) (iiie) (since $\mathbb{Z}_n$ permutes C_i, multiplicities of C_i in Z are the same). Thus $n = e = 2$ or 4. We obtain cases (i) , (ii) of our theorem.

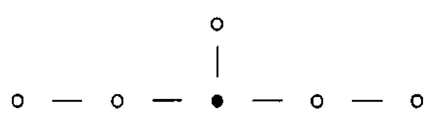
(6.5) Now we assume that $C \subset F$. As in (5.5) consider the Stein factorization

$$f_F : (F, C) \xrightarrow{f_1} (F', P') \xrightarrow{f_2} (S, s),$$

where P' is a point. Then (F', P') is Du Val, $f_1 : (F,C) \to (F',P')$ is bimeromorphic crepant morphism, and $f_2 : (F',P') \to (S,s)$ is finite of degree 2. For $f_2 : (F',P') \to (S,s)$ we have the same possibilities as for $(F,P) \to (S;s)$ in (6.4) . Let us consider these cases. We shall draw graph Γ for $f_1 : (F,C) \to (F',P')$ as in (5.5) .

Case (1). $(F',P') = E_6$, $(S,s) = A_2$, $n = e = 3$, $m = 3m^\natural$, $(X^\natural, P^\natural)$ is not Gorenstein (because $e > 2$)

But points D_5, A_1, and A_4 have no étale in codimension 1 cyclic covers of degree $m = 3m^\natural$.



Then (F,P) is type A_2 and $(X^\sharp, P^\sharp) = (X^\natural, P^\natural)$ is non-singular. But $C^\natural$ has three components, a contradiction with (6.1) .

$$\begin{array}{ccccccccc} & & & & \bullet & & & & \\ & & & & | & & & & \\ \circ & — & \circ & — & \circ & — & \circ & — & \circ \end{array}$$

Then (F,P) is type A_5 and it is a unique singular point on C. As above we obtain that $(X^\natural, P^\natural)$ has index 2, hence $f^\natural : (X^\natural; C^\natural) \to (S^\natural, s^\natural)$ is such as in (6.2) (iiid). This is impossible, since $\mathbb{Z}_3$ permutes $C_i^\natural$.

CASE (2). $(F', P') = A_{2k+1}$, $(S,s) = A_k$, $n = k+1$, $m = (k+1)m^\natural \geq k+1$.

$$\underbrace{\circ — \circ — \cdots — \circ}_{l} — \bullet — \underbrace{\circ — \cdots — \circ}_{r}$$

Then (F,P) is of type A_l, where $l < 2k$. On the other hand $l+1 \geq m = (k+1)m^\natural$. Hence $l = k$, (F,P) is type A_k and $(X^\natural, P^\natural) = (X^\sharp, P^\sharp)$ is non-singular. By (6.1) $C^\natural$ has exactly two components $C_1^\natural, C_2^\natural$. If $k = 1$, then (X,C) contains only points of index 1 or 2, so by (6.2) we have case (i). Thus we assume that $k > 1$. Then $C_1^\natural$ and $C_2^\natural$ are invariant under the action of subgroup $\mathbb{Z}_{k+1/2} \subset \mathbb{Z}_{k+1}$. Therefore there exist fixed points $R_i^\natural \in C_i^\natural$, $R_i^\natural \neq P^\natural$. Thus the point $R := g(R_1^\natural) = g(R_2^\natural) \in X$ has index > 1. Since $l + r = 2k$, we have $r = k$ and (F,R) is of type A_k. Moreover we may assume that $(X^\natural, C^\natural)$ is not Gorenstein, so $(X^\natural, R_1^\natural)$, $(X^\natural, R_2^\natural)$ has (the same) index $m_0 > 1$ and $(X^\natural, C^\natural)$ contains no another points of index > 1. But then index of (X,R) is $m_0(k+1)/2 \leq k+1$. Hence $m_0 = 2$, $(X^\natural, C^\natural)$ has two points of index 2, a contradiction with (6.2) .

As in (6.4) cases (5), (4), (3) are impossible. Q.E.D.

(6.6) Example. Let $X^\natural$ be a hypersurface in $\mathbb{P}^2_{x,y,z} \times \mathbb{C}^2_{u,v}$, defined by the following equation:

$$x^2 + y^2 + z^2\phi(u,v) = 0,$$

where $\{\phi(u,v) = 0\} \subset \mathbb{C}^2$ has an isolated singularity in 0 and $\phi(u,v)$ has only monomials of even degree. Denote by $f^\natural : X^\natural \to \mathbb{C}^2$ the natural projection. Then $X^\natural$ has only one singular point $P^\natural = (x = y = u = v = 0, z = 1)$ on ${f^\natural}^{-1}(0)$. Define the action of $G = \mathbb{Z}_2$ on $X^\natural$ and $\mathbb{C}^2$:

$$(x,y,z,u,v) \to (-x,y,z,-u,-v).$$

Set $X = X^\natural/G$, $S = \mathbb{C}^2/G$. The only fixed point on $X^\natural$ is $P^\natural$. If $(X^\natural, P^\natural)/G$ is terminal, then $f : X \to S$ is a $\mathbb{Q}$-Fano fiber space. The point $P^\natural$ gives us a unique imprimitive point $P \in X$ of index 2. The surface S has Du Val singularity of type A_1 in 0. Consider the following cases for $\phi(u,v)$:

(1) $\phi(u,v) = u^2 + v^{2k}$;

(2) $\phi(u,v) \in \mathfrak{m}^4_{u,v}\mathbb{C}\{u,v\}$.

Then by [14] (X,P) is terminal and has type $cA/2$ and $cAx/2$, respectively. Thus we have examples of $\mathbb{Q}$-Fano fiber spaces as in (6.3) , (i).

(6.7) Example. Let things be as in example (6.2.1). Then the Veronese surface $F_4^2 \subset \mathbb{P}^5$ is the image of

$$q : \mathbb{P}^2 \longrightarrow \mathbb{P}^5, \qquad q : (x,y,z) \longrightarrow (x^2, y^2, z^2, xy, xz, yz).$$

Define the action of $\mathbb{Z}_8$ on $\mathbb{P}^2$ and F_4^2:

$$(x,y,z) \longrightarrow (\varepsilon x, \varepsilon^{-1} y, \varepsilon^3 z),$$

where $\varepsilon := \exp(2\pi i/8)$. Then we can take points $P_i \in V$ as

$$P_1 = q(1,1,1), \qquad P_2 = q(\varepsilon, \varepsilon^{-1}, \varepsilon^3), \qquad P_3 = q(\varepsilon^2, \varepsilon^{-2}, \varepsilon^6), \qquad P_4 = q(\varepsilon^3, \varepsilon^{-3}, \varepsilon^9).$$

Since points $(1,1,1)$, $(\varepsilon, \varepsilon^{-1}, \varepsilon^3)$, $(\varepsilon^2, \varepsilon^{-2}, \varepsilon^6)$, $(\varepsilon^3, \varepsilon^{-3}, \varepsilon^9)$ are in general position, their images $P_1, \ldots, P_4$ generates $\mathbb{P}^3$ such that $\mathbb{P}^3 \cap F_4^2 = \{P_1, \ldots, P_4\}$. Then the induced action $\mathbb{Z}_4$ on V can be lifted on the $\mathbb{Q}$-Fano fiber space $f : (X, L_1 \cup \ldots \cup L_4) \to (\mathbb{P}^2, s)$. It is easy to see that the action $\mathbb{Z}_4$ on $V \subset \mathbb{P}^6$ looks like

$$(x_0, x_1, x_2, x_3, x_4, x_5, x_6) \longrightarrow (x_0, ix_1, -ix_2, -ix_3, x_4, -x_5, ix_6).$$

Projection $p : (V, O) - - \to (\mathbb{P}^2, s)$ gives us the action of $\mathbb{Z}_4$ on $(\mathbb{P}^2, s)$ (in affine neighborhood of s):

$$(y_1, y_2) \longrightarrow (iy_1, -iy_2).$$

Thus we obtain a $\mathbb{Q}$-Fano fiber space $X/\mathbb{Z}_4 \to \mathbb{P}^2/\mathbb{Z}_4$ such as in (6.3) , (ii).

7 Appendix: Q-Fano with extremal contractions to surfaces

(7.1) Definition. A normal projective variety X is called $\mathbb{Q}$-Fano if it has only terminal singularities and, $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor.

In the paper [21] Nikulin obtained some boundedness results for Picard number of $\mathbb{Q}$-Fano threefolds under assumption that there are no small contractions and contractions of extremal faces onto curve or surface. In this direction we discuss the following.

(7.2) Propostion. *Let X be a $\mathbb{Q}$-Fano threefolds with Picard number $\rho(X) \geq 2$. Assume that there exists an contraction of extremal face $f : X \to S$ such that*

(i) $\dim S = 2$,

(ii) *f has only fibers of dimension* 1, (iii) *in small neighborhood of any point $s \in S$ for $f : X \to S$ conjecture (0.2) is true.*

Then S is a rational weak Del Pezzo surface. Furthermore, if $f : X \to S$ as above is a contraction of extremal ray, then $\rho(X) \leq 10$.

PROOF. For a divisible enough m the linear system $-mK_X$ is a very ample system of Cartier divisors. Then the curve $L := f_*((-mK_X)^2)$ is very ample on S. Indeed, L is effective and $(C \cdot f_*((-mK_X)^2)) = f_*(f^*C \cdot ((-mK_X)^2)) > 0$. We have the standard formula $-4K_S \equiv f_*(K_X^2) + \Delta$, where Δ is a reduced Weil divisor on S. Thus

$$-4K_S \equiv \frac{1}{m} L + \Delta. \qquad (*)$$

(7.2.1) Claim *The surface S is rational.*

PROOF. By (*), $-4mK_S$ is effective, hence $\kappa(S) = -\infty$. Since $H^1(S, \mathcal{O}_S) = H^1(X, \mathcal{O}_X) = 0$, S is rational. Q.E.D.

Assume that there exists an irreducible curve $C \subset S$ such that $(-K_S \cdot C) \leq 0$. It follows from (*) that $(-L \cdot C) = (mC \cdot (4K_S + \Delta)) < 0$. Hence $(\Delta \cdot C) < 0$, $C \subset \Delta$,

$((K_S + C)\cdot C) < 0$ and $(C)^2 < 0$. Take a minimal resolution $g : \tilde{S} \to S$. Since S has only Du Val singularities, we have

$$g^*K_S \equiv K_{\tilde{S}}, \qquad g^*C \equiv \tilde{C} + \sum r_i E_i,$$

where $r_i \geq 0$, E_i are exeptional divisors, and $\tilde{C}$ is the proper transform of C. Then

$$0 \leq (K_S \cdot C) = (K_{\tilde{S}} \cdot \tilde{C}),$$

$$0 > (C)^2 = (\tilde{C})^2 + (\tilde{C} \cdot \sum r_i E_i).$$

Since

$$0 > 4(K_S \cdot C) + (C)^2 = 4(K_{\tilde{S}} \cdot \tilde{C}) + (\tilde{C})^2 + (\tilde{C} \cdot \sum r_i E_i),$$

we have $4(K_{\tilde{S}} \cdot \tilde{C}) + (\tilde{C})^2 < 0$ and

$$3(K_{\tilde{S}} \cdot \tilde{C}) + 2p_a(\tilde{C}) - 2 < 0.$$

It follows from $(K_{\tilde{S}} \cdot \tilde{C}) \geq 0$ that $p_a(\tilde{C}) = 0$, $(K_{\tilde{S}} \cdot \tilde{C}) + (\tilde{C})^2 = -2$, and $(3K_{\tilde{S}} \cdot \tilde{C}) - 2 < 0$. Hence $(K_{\tilde{S}} \cdot \tilde{C}) = 0$. It is possible only if $\tilde{C} \simeq \mathbb{P}^1$, $(K_{\tilde{S}} \cdot \tilde{C}) = 0$, $(\tilde{C})^2 = -2$, i. e. $\tilde{C}$ is a (-2)-curve. By definition $\tilde{S}$ and S are weak Del Pezzo surfaces. Q.E.D.

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