ON SOME SYSTEMS OF DIFFERENCE EQUATIONS. Part 9.

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In memory of
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one of the best pupils
of Professor N.M. Korobov.

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§9.0. Foreword.

Let

(1) $$|z| \geq 1, -3\pi/2 < \arg(z) \leq \pi/2, \log(z) = \ln(|z|) + i\arg(z).$$

Then $\log(-z) = \log(z) - i\pi$, if $\Re(z) > 0$ and $\log(z) = \log(-z) - i\pi$, if $\Re(z) < 0$. Let

(2) $$f^{\vee}_{l,1}(z,\nu) = f_{l,1}(z,\nu) = \sum_{k=0}^{\nu}(-1)^{(\nu+k)l}(z)^k\binom{\nu}{k}^{2+l}\binom{\nu+k}{\nu}^{2+l},$$

where $l = 0, 1, 2$, $\nu \in [0, +\infty) \cap \mathbb{Z}$. Let

(3) $$R(t,\nu) = \frac{\prod_{j=1}^{\nu}(t-j)}{\prod_{j=0}^{\nu}(t+j)},$$

where $\nu \in [0, +\infty) \cap \mathbb{Z}$,

(4) $$f^{\vee}_{l,2}(z,\nu)=f_{l,2}(z,\nu)=\sum_{t=1+\nu}^{+\infty}z^{-t}(R(t,\nu))^{2+l},$$

where $l=0,\,1,\,2$ and $\nu\in[0,+\infty)\cap\mathbb{Z}$, and since $(R(t,\nu))^{2+l}$ for $\nu\in\mathbb{N}$ has in the points $t=1,\,\dots\,,\,\nu$, the zeros of the order $2+l$, it follows that

(5) $$f_{l,2}(z,\nu)=\sum_{t=1}^{+\infty}z^{-t}(R(t,\nu))^{2+l},$$

for $l=0,\,1,\,2$ and $\nu\in[0,+\infty)\cap\mathbb{Z}$. Let

(6) $$f^{\vee}_{l,3}(z,\nu)=f_{l,3}(z,\nu)=(\log(z))f_{l,2}(z,\nu)+f_{l,4}(z,\nu),$$

where

(7) $$f_{l,4}(z,\nu)=-\sum_{t=1+\nu}^{+\infty}z^{-t}\left(\frac{\partial}{\partial t}(R^{2+l})\right)(t,\nu),$$

$l=0,\,1,\,2$ and $\nu\in[0,+\infty)\cap\mathbb{Z}$, and since $(R(t,\nu))^{2+l}$ for $\nu\in\mathbb{N}$ has in the points $t=1,\,\dots\,,\,\nu$, the zeros of the order $2+l$, it follows that

(8) $$f_{l,4}(z,\nu)=-\sum_{t=1}^{+\infty}z^{-t}\left(\frac{\partial}{\partial t}(R^{2+l})\right)(t,\nu)$$

for $l=0,\,1,\,2$ and $\nu\in[0,+\infty)\cap\mathbb{Z}$. Let

(9) $$f^{\vee}_{l,5}(z,\nu)=-i\pi f_{l,3}(z,\nu)+f_{l,5}(z,\nu),$$

with $l=1,\,2$, $\nu\in[0,+\infty)\cap\mathbb{Z}$ and

(10) $$f_{l,5}(z,\nu)=$$

$$2^{-1}(\log(z))^2f_{l,2}(z,\nu)+(\log(z))f_{l,4}(z,\nu)+f_{l,6}(z,\nu)=$$
$$=-2^{-1}(\log(z))^2f_{l,2}(z,\nu)+(\log(z))f_{l,3}(z,\nu)+f_{l,6}(z,\nu),$$

where

(11) $$f_{l,6}(z,\nu)=2^{-1}\sum_{t=1+\nu}^{\infty}z^{-t}\left(\left(\frac{\partial}{\partial t}\right)^2(R^{2+l})\right)(t,\nu),$$

and since $(R(t,\nu))^{2+l}$ for $\nu\in\mathbb{N}$ has in the points $t=1,\,\dots\,,\,\nu$, the zeros of the order $2+l$, and $l=1,\,2$ now, it follows that

(12) $$f_{l,6}(z,\nu)=2^{-1}\sum_{t=1+\nu}^{\infty}z^{-t}\left(\left(\frac{\partial}{\partial t}\right)^2(R^{2+l})\right)(t,\nu)$$

for $l=1,\,2$ and $\nu\in[0,+\infty)\cap\mathbb{Z}$. Let

$$f^{\vee}_{l,7}(z,\nu) = f_{l,7}(z,\nu) + (2\pi^2/3)f_{l,3}(z,\nu). \tag{13}$$

with $l = 2$, $\nu \in [0, +\infty) \cap \mathbb{Z}$ and

$$f_{l,7}(z,\nu) = \tag{14}$$

$$-3^{-1}(\log(z))^3 f_{l,2}(z,\nu) + 2^{-1}(\log(z))^2 f_{l,3}(z,\nu) + f_{l,8}(z,\nu)+$$
$$(\log(z))(f_{l,5}(z,\nu) + 2^{-1}(\log(z))^2 f_{l,2}(z,\nu) - (\log(z)) f_{l,3}(z,\nu)) =$$
$$6^{-1}(\log(z))^3 f_{l,2}(z,\nu) - 2^{-1}(\log(z))^2 f_{l,3}(z,\nu) + (\log(z)) f_{l,5}(z,\nu) + f_{l,8}(z,\nu) =$$
$$(1/6)(\log(z))^3 f_{l,2}(z,\nu) + (1/2)(\log(z))^2 f_{l,4}(z,\nu)+$$
$$(\log(z)) f_{l,6}(z,\nu) + f_{l,8}(z,\nu),$$

where

$$f_{l,8}(z,\nu) = -6^{-1} \sum_{t=\nu+1}^{\infty} z^{-t} \left(\left(\frac{\partial}{\partial t} \right)^3 (R^{2+l}) \right)(t,\nu), \tag{15}$$

and, since $(R(t,\nu))^{2+l}$ for $\nu \in \mathbb{N}$ have in the points $t = 1, \ldots, \nu$, the zeros of the order $2 + l$, and $l = 2$ now, it follows that

$$f_{l,8}(z,\nu) = -6^{-1} \sum_{t=1}^{\infty} z^{-t} \left(\left(\frac{\partial}{\partial t} \right)^3 (R^{2+l}) \right)(t,\nu). \tag{16}$$

Let

$$\mathfrak{K}_0 = \{1,\ 2,\ 3\},\ \mathfrak{K}_1 = \{1,\ 2,\ 3,\ 5\},\ \mathfrak{K}_2 = \{1,\ 2,\ 3,\ 5,\ 7\}.$$

Let λ be a variable. We denote by $T_{n,\lambda}$ the diagonal $n \times n$-matrix, i-th diagonal element of which is equal to λ^{i-1} for $i = 1, ..., n$. We denote by δ the operator $z\frac{d}{dz}$. Let further $l = 0, 1, 2$, $k \in \mathfrak{K}_l$, $|z| > 1, \nu \in \mathbb{N}$, and let $Y_{l,k}(z;\nu)$ be the columnn with $4 + 2l$ elements, i-th of which is equal to $(\nu^{-1}\delta)^{i-1} f^{\vee}_{l,k}(z,\nu)$ for $i = 1, ..., 4 + 2l$.

Theorem 1. *The following equalities hold*

$$A^{\sim}_l(z;\nu) Y_{l,k}(z;\nu) = T_{4+2l,1-\nu^{-1}} Y_{l,k}(z;\nu-1), \tag{17}$$

$$Y_{l,k}(z;\nu) = T_{4+2l,-1} A^{\sim}_l(z;-\nu) T_{4+2l,-1+\nu^{-1}} Y_{l,k}(z;\nu-1), \tag{18}$$

where $l = 0, 1, 2$, $k \in \mathfrak{K}_l$, $|z| > 1, \nu \in \mathbb{N}, \nu \geq 2$,

$$A^{\sim}_l(z;\nu) = S^{\sim}_l + z \sum_{i=0}^{1+l} \nu^{-i} V^{\sim *}_l(i) \tag{19}$$

with

$$S^{\sim}_0 = \begin{pmatrix} 1 & -4 & 8 & -12 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{20}$$

(21) $$S_1^{\sim} = \begin{pmatrix} -1 & 6 & -18 & 38 & -66 & 102 \\ 0 & -1 & 6 & -18 & 38 & -66 \\ 0 & 0 & -1 & 6 & -18 & 38 \\ 0 & 0 & 0 & -1 & 6 & -18 \\ 0 & 0 & 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

(22) $$S_2^{\sim} = \begin{pmatrix} 1 & -8 & 32 & -88 & 192 & -360 & 608 & -952 \\ 0 & 1 & -8 & 32 & -88 & 192 & -360 & 608 \\ 0 & 0 & 1 & -8 & 32 & -88 & 192 & -360 \\ 0 & 0 & 0 & 1 & -8 & 32 & -88 & 192 \\ 0 & 0 & 0 & 0 & 1 & -8 & 32 & -88 \\ 0 & 0 & 0 & 0 & 0 & 1 & -8 & 32 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$V_0^{\sim *}(0) = 4 \begin{pmatrix} 4 & -5 & -2 & 3 \\ -3 & 4 & 1 & -2 \\ 2 & -3 & 0 & 1 \\ -1 & 2 & -1 & 0 \end{pmatrix},$$

$$V_0^{\sim *}(1) = 4 \begin{pmatrix} 3 & -6 & 3 & 0 \\ -2 & 4 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$V_1^{\sim *}(0) = \begin{pmatrix} 146 & -198 & -180 & 268 & 66 & -102 \\ -102 & 146 & 108 & -180 & -38 & 66 \\ 66 & -102 & -52 & 108 & 18 & -38 \\ -38 & 66 & 12 & -52 & -6 & 18 \\ 18 & -38 & 12 & 12 & 2 & -6 \\ -6 & 18 & -20 & 12 & -6 & 2 \end{pmatrix},$$

$$V_1^{\sim *}(1) = \begin{pmatrix} 240 & -516 & 108 & 372 & -204 & 0 \\ -160 & 348 & -84 & -236 & 132 & 0 \\ 96 & -212 & 60 & 132 & -76 & 0 \\ -48 & 108 & -36 & -60 & 36 & 0 \\ 16 & -36 & 12 & 20 & -12 & 0 \\ 0 & -4 & 12 & -12 & 4 & 0 \end{pmatrix},$$

$$V_1^{\sim *}(2) = \begin{pmatrix} 102 & -306 & 306 & -102 & 0 & 0 \\ -66 & 198 & -198 & 66 & 0 & 0 \\ 38 & -114 & 114 & -38 & 0 & 0 \\ -18 & 54 & -54 & 18 & 0 & 0 \\ 6 & -18 & 18 & -6 & 0 & 0 \\ -2 & 6 & -6 & 2 & 0 & 0 \end{pmatrix},$$

$$V_2^{\sim *}(0)=8\begin{pmatrix}176&-249&-364&545&280&-431&-76&119\\-119&176&227&-364&-169&280&45&-76\\76&-119&-128&227&92&-169&-24&45\\-45&76&61&-128&-43&92&11&-24\\24&-45&-20&61&16&-43&-4&11\\-11&24&-1&-20&-5&16&1&-4\\4&-11&8&-1&4&-5&0&1\\-1&4&-7&8&-7&4&-1&0\end{pmatrix},$$

(23)
$$V_2^{\sim *}(1)=8\begin{pmatrix}455&-1020&-113&1552&-603&-628&357&0\\-300&682&44&-996&404&394&-228&0\\185&-428&-3&592&-253&-228&135&0\\-104&246&-16&-316&144&118&-72&0\\51&-124&19&144&-71&-52&33&0\\-20&50&-12&-52&28&18&-12&0\\5&-12&1&16&-9&-4&3&0\\0&-2&8&-12&8&-2&0&0\end{pmatrix},$$

(24)
$$V_2^{\sim *}(2)=8\begin{pmatrix}400&-1243&972&542&-1028&357&0&0\\-259&808&-642&-332&653&-228&0&0\\156&-489&396&186&-384&135&0&0\\-85&268&-222&-92&203&-72&0&0\\40&-127&108&38&-92&33&0&0\\-15&48&-42&-12&33&-12&0&0\\4&-13&12&2&-8&3&0&0\\-1&4&-6&4&-1&0&0&0\end{pmatrix},$$

(25)
$$V_2^{\sim *}(3)=8\begin{pmatrix}119&-476&714&-476&119&0&0&0\\-76&304&-456&304&-76&0&0&0\\45&-180&270&-180&45&0&0&0\\-24&96&-144&96&-24&0&0&0\\11&-44&66&-44&11&0&0&0\\-4&16&-24&16&-4&0&0&0\\1&-4&6&-4&1&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}.$$

The above matices $A_l^{\sim}(z;\nu), S_l^{\sim}$ and $V_l^{\sim *}(i)$ have the following properties:

(26)
$$A_l^{\sim}(z;-\nu)T_{4+2l,-1}A_l^{\sim}(z;\nu)=T_{4+2l,-1},$$

(27)
$$S_l^{\sim}T_{4+2l,-1}=(S_l^{\sim}T_{4+2l,-1})^{-1}$$

(28)
$$S_l^{\sim}T_{4+2l,-1}V_l^{\sim *}(i)=-(-1)^iV_l^{\sim *}(i)T_{4+2l,-1}S_l^{\sim},$$

(29)
$$V_l^{\sim *}(i)T_{4+2l,-1}V_l^{\sim *}(k)=0T_{4+2l,-1},$$

where

$$l = 0,\ 1,\ 2,\ i \in [0, 1+l] \cap \mathbb{Z},\ k \in [0, 1+l] \cap \mathbb{Z}.$$

Proof. Full proof can be found in [5] – [10] . Some arithmetical applications are given in [10] –[12]. To present other arithmetical applications I need some generalisation of the result of the Theorem 1. Here I begin to realize this goal.

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§9.1. More general my auxiliary functions in the case $k = 1,\ 2,\ 3$.

We fix $\alpha \in \mathbb{N}_0$.
Let

$$f^{\vee}_{\alpha,l,1}(z,\nu) = f_{\alpha,l,1}(z,\nu) = -(-1)^{\nu l} \times \tag{30}$$

$$G^{(1,2+l)}_{4+2l,4+2l}\left(-(-1)^l z \left| \begin{array}{l} \overbrace{-\nu, \quad \dots \quad , -\nu,}^{l+2 \text{ times}} \overbrace{\nu+1+\alpha, \quad \dots \quad , \nu+1+\alpha}^{l+2 \text{ times}} \\ 0, \quad \dots \quad , 0, \quad\quad 0, \quad\quad \dots \quad\quad , 0 \end{array}\right.\right) =$$

$$\frac{-(-1)^{l\nu}}{2i\pi} \int\limits_{L_1} g^{(1,2+l)}_{4+2l,4+2l} \times$$

$$\left(-(-1)^l z \left| \begin{array}{l} \overbrace{-\nu, \quad \dots \quad , -\nu,}^{l+2 \text{ times}} \overbrace{\nu+1+\alpha, \quad \dots \quad , \nu+1+\alpha}^{l+2\text{times}} \\ 0, \quad \dots \quad , 0, \quad\quad 0, \quad\quad \dots \quad\quad , 0 \end{array}\right| s \right) ds,$$

where $l = 0, 1, 2$, $\nu \in [-alpha, +\infty) \cap \mathbb{Z}$,

$$g^{(1,2+l)}_{4+2l,4+2l} = g^{(1,2+l)}_{4+2l,4+2l}(s) =$$

$$g^{(1,2+l)}_{4+2l,4+2l}\left(-(-1)^l z \left| \begin{array}{l} \overbrace{-\nu, \quad \dots \quad , -\nu,}^{l+2 \text{ times}} \overbrace{\nu+1+\alpha, \quad \dots \quad , \nu+1+\alpha}^{l+2 \text{ times}} \\ 0, \quad \dots \quad , 0, \quad\quad 0, \quad\quad \dots \quad\quad , 0 \end{array}\right| s \right) =$$

$$(-(-1)^l z)^s \Gamma(-s)(\Gamma(1+s))^{-3-2l}(\Gamma(1+\nu+s))^{2+2l}(\Gamma(1+\nu+\alpha-s))^{-2-2l},$$

and the curve L_1 passes from $+\infty$ to $+\infty$ in the negative direction such that the set $[0, +\infty) \cap \mathbb{Z}$ lies to the right from it, but the set $(-\infty, -1) \cap \mathbb{Z}$ lies to the left from it. The set of all unremovable singular points of $g^{(1,2+l)}_{4+2l,4+2l}(s)$ encircled by the curve L_1, consists of the points $s = 0, \dots , \nu + 2\alpha$, each of these points is a pole of the first order. Therefore

$$Res(g^{(1,2+l)}_{4+2l,4+2l}; k) =$$

$$\lim_{s \to k}((s+k) g^{(1,2+l)}_{4+2l,4+2l}(s)),$$

where $l = 0, 1, 2$ and $k = 0, \dots, \nu$. Let

$$s = k + u,\ H_{l,1}(u, k, \nu) =$$

$$g^{(1,2+l)}_{4+2l,4+2l}(k+u) =$$

$$(-(-1)^l z)^{k+u}\Gamma(-k-u)(\Gamma(1+k+u))^{-3-2l}\times$$

$$(\Gamma(1+\nu+k+u))^{2+2l}(\Gamma(1+\nu+\alpha-k-u))^{-2-2l} =$$

$$\prod_{\kappa=1}^{k}(-k+\kappa-u)^{-1}(-(-1)^l z)^{k+u}\times$$

$$\Gamma(1-u)(\Gamma(1+k+u))^{-3-2l}(\Gamma(1+\nu+k+u))^{2+l}(\Gamma(1+\nu+\alpha-k-u))^{-2-l},$$

where $l = 0, 1, 2$ and $k \in [0, +\infty) \cap \mathbb{Z}$. Hence,

$$\frac{(\nu+\alpha)!^{2+l}}{\nu!^{2+l}}Res(g^{(1,2+l)}_{4+2l,4+2l}; k) = \lim_{u\to 0}(ug^{(1,2+l)}_{4+2l,4+2l}(k+u)) =$$

$$-(-1)^{lk}(\nu+\alpha)!^{2+l}(k!)^{-4-2l}((\nu+k)!)^{2+l}((\nu+\alpha-k)!)^{-2-l} =$$

$$-(-1)^l kz^k\binom{\nu+\alpha}{k}^{2+l}\binom{\nu+k}{k}^{2+2l},$$

where $l = 0,\ 1, 2$ and $k = 0, \dots, \nu$. Consequently,

$$f^{*\vee}_{\alpha,l,1}(z,\nu) := f^{*}_{\alpha,l,1}(z,\nu) := \tag{31}$$

$$\frac{(\nu+\alpha)!^{2+l}}{\nu!^{2+l}}f_{\alpha,l,1}(z,\nu) =$$

$$\sum_{k=0}^{\nu+\alpha}(-1)^{(\nu+k)l}(z)^k\binom{\nu+\alpha}{k}^{2+l}\binom{\nu+k}{\nu}^{2+l}$$

, where $l = 0,\ 1,\ 2,\ \nu \in [0, +\infty) \cap \mathbb{Z}$. Let

$$f_{\alpha,l,2}(z,\nu)^{\vee} = f_{l,2}(\alpha, z, \nu) = -(-1)^{l\nu}\times \tag{32}$$

$$G^{(3+l,2+l)}_{4+2l,4+2l}\left(-z\left|\begin{matrix}\overbrace{-\nu, & \dots & , -\nu,}^{l+2\text{ times}} & \overbrace{\nu+1+\alpha, & \dots & , \nu+1+\alpha}^{l+2\text{ times}}\\ 0, & \dots & , 0, & 0, & \dots & , 0\end{matrix}\right.\right) =$$

$$\frac{-(-1)^{l\nu}}{2i\pi}\times$$

$$\int\limits_{L_2} g^{(3+l,2+l)}_{4+2l,4+2l}\left(-z\left|\begin{matrix}\overbrace{-\nu, & \dots & , -\nu,}^{l+2\text{ times}} & \overbrace{\nu+1+\alpha, & \dots & , \nu+1+\alpha}^{l+2\text{ times}}\\ 0, & \dots & , 0, & 0, & \dots & , 0\end{matrix}\right|s\right)ds,$$

where $l = 0,\ 1,\ 2,\ \nu \in [0, +\infty) \cap \mathbb{Z}$,

$$g^{(3+l,2+l)}_{4+2l,4+2l} = g^{(3+l,2+l)}_{4+2l,4+2l}(s) =$$

$$g^{(3+l,2+l)}_{4+2l,4+2l}\left(-z\left|\begin{matrix}\overbrace{-\nu, & \ldots, & -\nu,}^{l+2\text{ times}} & \overbrace{\nu+1+\alpha, & \ldots & ,\nu+1+\alpha}^{l+2\text{ times}}\\ 0, & \ldots & ,0, & 0, & \ldots & ,0\end{matrix}\right|s\right)=$$

$$(-z)^s(\Gamma(-s))^{3+l}(\Gamma(1+s))^{-1-l}(\Gamma(1+\nu+s))^{2+l}(\Gamma(1+\nu+\alpha-s))^{-2-l},$$

and the curve L_2 passes from $-\infty$ to $-\infty$ in the positive direction such that the set $[0,+\infty)\cap\mathbb{Z}$ lies to the right from it but the set $\mathbb{Z}\cap(-\infty,-1]$ lies to the left from it. The set of all unremovable singular points of $g^{(3+l,2+l)}_{4+2l,4+2l}(s)$ encircled by L_2, consists of all the $s=-1-\nu-k$ with $k\in[0,+\infty)\cap\mathbb{Z}$; each of these points is a pole of the first order. Therefore

$$Res(g^{(3+l,2+l)}_{4+2l,4+2l};-1-d_1\nu-k)=\lim_{s\to-\nu-1-k}((s+\nu+1+k)g^{(3+l,2+l)}_{4+2l,4+2l}(s)),$$

where $l=0,\,1,\,2$ and $k\in[0,+\infty)\cap\mathbb{Z}$. Let $\nu\in[0,+\infty)\cap\mathbb{Z}$, Let

$$s=-\nu-1-k+u, H^*_l(\alpha,u,k,\nu)=(\Gamma(\nu+1+k-u))^{4+2l}\times$$

$$(\Gamma(1+k-u))^{-2-l}(\Gamma(2+2\nu+\alpha+k-u))^{-2-l},$$

and

(33) $$R(\alpha,t,\nu)=\frac{\prod\limits_{j=1}^{\nu}(t-j)}{\prod\limits_{j=0}^{\nu+\alpha}(t+j)}.$$

where $\nu\in[0,+\infty)\cap\mathbb{Z}$. Let further $T=-s=\nu+1+k-u,$ Then

(34)

$$H^*_l(\alpha,u,k,\nu)=\left(\frac{\prod\limits_{j=1}^{\nu}(1+\nu+k-j-u)}{\prod\limits_{j=0}^{\nu+\alpha}(1+\nu+k+j-u)}\right)^{2+l}=(R(\alpha,T,\nu))^{2+l},$$

where $\nu\in[0,+\infty)\cap\mathbb{Z}$. Let

$$H_{l,2}(\alpha,u,k,\nu)=g^{(3+l,2+l)}_{4+2l,4+2l}(-\nu-1-k+u)=$$

$$(-z)^{-\nu-1-k+u}(\Gamma(\nu+1+k-u))^{3+l}\times$$

$$(\Gamma(-\nu-k+u))^{-1-l}(\Gamma(-k+u))^{2+l}(\Gamma(2+2\nu+\alpha+k-u))^{-2-l}=$$

$$(-1)^{k+(l+1)nu}(-z)^{-\nu-1-k+u}\frac{\pi}{\sin(u\pi)}(\Gamma(\nu+1+k-u))^{4+2l}\times$$

$$(\Gamma(1+k-u))^{-2-l}(\Gamma(2+2\nu+\alpha+k-u))^{-2-l}=$$

$$(-1)^{k+(l+1)nu}(-z)^{-\nu-1-k+u}\frac{\pi}{\sin(u\pi)}H^*_l(u,k,\nu),$$

where $l=0,\,1,\,2$ and $k\in[0,+\infty)\cap\mathbb{Z}$. Therefore

$$Res(g^{(3+l,2+l)}_{4+2l,4+2l};-1-\nu-k)=$$

$$\lim_{u\to 0}(uH_{l,2}(\alpha,u,k,\nu)) =$$
$$-(-1)^{l\nu}z^{-(1+\nu+k)}(R(\alpha,1+\nu+k,\nu))^{2+l},$$

where $l=0,\ 1,\ 2$ and $k\in[0,+\infty)\cap\mathbb{Z}$. Consequently, if

(35)
$$f^{\vee *}_{\alpha,l,2}(z,\nu) := f^{*}_{\alpha,l,2}(z,\nu) :=$$
$$\frac{(\nu+\alpha)!^{2+l}}{(\nu!)^{2+l}}f_{\alpha,l,2}(z,\nu),$$

then

(36)
$$f^{*}_{\alpha,l,2}(z,\nu) =$$
$$\sum_{k=0}^{+\infty}z^{-(1+\nu+k)}\frac{(\nu+\alpha)!^{2+l}}{(\nu!)^{2+l}}(R(\alpha,1+\nu+k,\nu))^{2+l},$$

where $l=0,1,2$ and $k\in[0,+\infty)\cap\mathbb{Z}$. Let $t=1+\nu+k$ with $k\in[0,+\infty)\cap\mathbb{Z}$; in view of (33) and (34), it follows that

(37)
$$f^{*}_{\alpha,l,2}(z,\nu) =$$
$$\sum_{t=1+\nu}^{+\infty}z^{-t}\frac{(\nu+\alpha)!^{2+l}}{(\nu!)^{2+l}}(R(\alpha,t,\nu))^{2+l},$$

where $l=0,\ 1,\ 2$ and $\nu\in[0,+\infty)\cap\mathbb{Z}$. Since $(R(\alpha,t,\nu))^{2+l}$ for $\nu\in\mathbb{N}$ has in the points $t=1,\ \dots,\ \nu$, the zeros of the order $2+l$, it follows that

(38)
$$f^{*}_{\alpha,l,2}(z,\nu) =$$
$$\sum_{t=1}^{+\infty}z^{-t}\frac{(\nu+2\alpha)!^{2+l}}{(\nu!)^{2+l}}(R(\alpha,t,\nu))^{2+l} =$$
$$f^{\vee *}_{\alpha,l,2}(z,\nu) = f^{*}_{\alpha,l,2}(z,\nu).$$

where $l=0,\ 1,\ 2$ and $\nu\in[0,+\infty)\cap\mathbb{Z}$.

Let

(39)
$$f^{\vee}_{\alpha,l,3}(z,\nu) = f_{l,3}(z,\nu) = (-1)^{l\nu}\times$$
$$G^{(4+l,2+l)}_{4+2l,4+2l}\left(z\left|\begin{matrix}\overbrace{-\nu, & \dots & ,-\nu,}^{l+2\text{ times}} & \overbrace{\nu+1+\alpha, & \dots & ,\nu+1+\alpha}^{l+2\text{ times}}\\ 0, & \dots & ,0, & 0, & \dots & ,0\end{matrix}\right.\right) =$$
$$\frac{(-1)^{l\nu}}{2i\pi}\int\limits_{L_2}g^{(4+l,2+l)}_{4+2l,4+2l}\left(-z\left|\begin{matrix}\overbrace{-\nu, & \dots & ,-\nu,}^{l+2\text{ times}} & \overbrace{\nu+1+\alpha, & \dots & ,\nu+\alpha}^{l+2\text{ times}}\\ 0, & \dots & ,0, & 0, & \dots & ,0\end{matrix}\right|s\right)ds,$$

where $l=0,\ 1,\ 2,\ \nu[0,+\infty)\cap\mathbb{Z}$,

$$g^{(4+l,2+l)}_{4+2l,4+2l} = g^{(4+l,2+l)}_{4+2l,4+2l}(s) =$$

$$g_{4+2l,4+2l}^{(4+l,2+l)}\left(-z\left|\begin{matrix}\overbrace{-\nu, & \dots & ,-\nu,}^{l+2 \text{ times}} & \overbrace{\nu+1+\alpha, & \dots & ,\nu+1+\alpha}^{l+2 \text{ times}}\\ 0, & \dots & ,0, & 0, & \dots & ,0\end{matrix}\right|s\right)=$$

$$(z)^s(\Gamma(-s))^{4+l}(\Gamma(1+s))^{-l}(\Gamma(1+\nu+s))^{2+l}(\Gamma(1+\nu+alpha-s))^{-2-l}.$$

The set of all unremovable singular points of the function $g_{4+2l,4+2l}^{(4+l,2+l)}(s)$ encircled by the curve L_2, consists of the points $s=-1-\nu-k$ with $k\in[0,+\infty)\cap\mathbb{Z}$; each of these points is a pole of the second order. Therefore

$$Res(g_{4+2l,4+2l}^{(4+l,2+l)};-1-\nu-k)=$$

$$\lim_{s\to-\nu-1-k}\left(\frac{\partial}{\partial s}((s+\nu+1+k)^2g_{4+2l,4+2l}^{(4+l,2+l)}(s))\right),$$

where $l=0,\ 1,\ 2$ and $k\in[0,+\infty)\cap\mathbb{Z}$. Let

$$s=-\nu-1-k+u,\ H_{l,3}(\alpha,u,k,nu)=g_{4+2l,4+2l}^{(4+l,2+l)}(-\nu-1-k+u)=(z)^{-\nu-1-k+u}\times$$

$$(\Gamma(\nu+1+k-u))^{4+l}(\Gamma(-\nu-k+u))^{-l}(\Gamma(-k+u))^{2+l}(\Gamma(2+2\nu+\alpha+k-u))^{-2-l}=$$

$$(-1)^{l\nu}\left(\frac{\pi}{\sin(u\pi)}\right)^2(z)^{-\nu-1-k+u}(\Gamma(\nu+1+k-u))^{4+2l}\times$$

$$(\Gamma(1+k-u))^{-2-l}(\Gamma(2+2\nu+k+\alpha-u))^{-2-l}=$$

$$(-1)^{l\nu}\left(\frac{\pi}{\sin(u\pi)}\right)^2(z)^{-\nu-1-k+u}H_l^*(\alpha,u,k,nu)=$$

$$(-1)^{l\nu}\left(\frac{\pi}{\sin(u\pi)}\right)^2(z)^{-\nu-1-k+u}(R(\alpha,T,\nu))^{2+l},$$

where $T=\nu+1+k-u, l=0,\ 1,\ 2$ and $k\in[0,+\infty)\cap\mathbb{Z}$. Therefore

$$Res(g_{4+2l,4+2l}^{(4+l,2+l)};-1-d_1\nu-k)=$$

$$\lim_{u\to 0}\left(\frac{\partial}{\partial u}(u^2H_{l,3}(\alpha,u,k,nu)\right)=$$

$$(-1)^{l\nu}(\log(z))(z)^{-\nu-1-k}H_l^*(\alpha,0,k,\nu)+$$

$$(-1)^{l\nu}(z)^{-\nu-1-k}\left(\frac{\partial}{\partial u}H_l^*\right)(\alpha,0,k,\nu)=$$

$$(-1)^{l\nu}(\log(z))(z)^{-\nu-1-k}(R(\alpha,1+\nu+k,\nu))^{2+l}-$$

$$(-1)^{l\nu}(z)^{-\nu-1-k}\left(\frac{\partial}{\partial t}(R)^{2+l}\right)(\alpha,1+\nu+k,\nu)$$

because $(\pi u/(\sin(\pi u))^2$ is a even function. Thus, if

$$f_{\alpha,l,3}^{\vee *}(z,\nu):=f_{\alpha,l,3}^*(z,\nu):= \tag{40}$$

$$\frac{(\nu+\alpha)!^{2+l}}{(\nu!)^{2+l}}f_{\alpha,l,3}(z,\nu),$$

then

(41)
$$f^*_{\alpha,l,3}(z,\nu)=$$
$$(\log(z))\frac{(\nu+\alpha)!^{2+l}}{(\nu!)^{2+l}}\sum_{t=1+\nu}^{+\infty}z^{-t}(R(\alpha,t,\nu))^{2+l}-$$
$$\frac{(\nu+\alpha)!^{2+l}}{(\nu!)^{2+l}}\sum_{t=1+\nu}^{+\infty}z^{-t}\left(\frac{\partial}{\partial t}(R^{2+l})\right)(\alpha,t,\nu),$$

where $l=0,\,1,\,2$ and $\nu\in[0,+\infty)\cap\mathbb{Z}$. Let

(42)
$$f_{\alpha,l,4}(z,\nu)=-\sum_{t=1+\nu}^{+\infty}z^{-t}\left(\frac{\partial}{\partial t}(R^{2+l})\right)(\alpha,t,\nu),$$

(43)
$$f^*_{\alpha,l,4}(z,\nu)=\frac{(\nu+\alpha)!^{2+l}}{(\nu!)^{2+l}}f^*_{\alpha,l,4}(z,\nu),$$

where $l=0,\,1,\,2$ and $\nu\in[0,+\infty)\cap\mathbb{Z}$. Then in view of (35), (37), (40) – (43)

(44)
$$f_{\alpha,l,3}=(\log(z))f_{\alpha,l,2}(z,\nu)+f_{\alpha,l,4}(z,\nu).$$

(45)
$$f^*_{\alpha,l,3}(z,\nu)=(\log(z))f^*_{\alpha,l,2}(z,\nu)+f^*_{\alpha,l,4}(z,\nu).$$

Since $(R(\alpha,t,\nu))^{2+l}$ for $\nu\in\mathbb{N}$ have in the points $t=1,\,\dots\,,\,\nu$, the zeros of the order $2+l$, it follows that

(46)
$$f^*_{\alpha,l,3}(z,\nu)==(\log(z))\sum_{t=1}^{+\infty}z^{-t}\frac{(\nu+\alpha)!^{2+l}}{(\nu!)^{2+l}}(R(\alpha,t,\nu))^{2+l}-$$
$$\sum_{t=1}^{+\infty}z^{-t}\frac{(\nu+\alpha)!^{2+l}}{(\nu!)^{2+l}}\left(\frac{\partial}{\partial t}(R^{2+l})\right)(\alpha,t,\nu),$$

(47)
$$f_{\alpha,l,4}(z,\nu)=-\sum_{t=1}^{+\infty}z^{-t}\left(\frac{\partial}{\partial t}(R^{2+l})\right)(\alpha,t,\nu),$$

where $l=0,\,1,\,2$ and $\nu\in[0,+\infty)\cap\mathbb{Z}$. Let

(48)
$$S_i(\alpha,t,\nu)=\left(\frac{\partial}{\partial t}\right)^i\left(\left(\sum_{k=1}^{\nu}\frac{1}{t-k}\right)-\sum_{k=0}^{\nu+\alpha}\frac{1}{t+k}\right)=$$
$$(-1)^{i-1}(i-1)!\left(\left(\sum_{k=1}^{\nu}\frac{1}{(t-k)^i}\right)-\sum_{k=0}^{\nu+\alpha}\frac{1}{(t+k)^i})\right),$$

(49)
$$H^\wedge_i(\alpha,\nu)=S_i(\alpha,\nu,\nu-1)-S_i(\alpha,\nu+1,\nu)$$

$$(-1)^{i-1}(i-1)!\left(\left(\sum_{k=1}^{\nu-1}\frac{1}{k^i}\right)-\sum_{k=\nu}^{2\nu-1+\alpha}\frac{1}{k^i})\right)-$$

$$(-1)^{i-1}(i-1)!\left(\left(\sum_{k=1}^{\nu}\frac{1}{k^i}\right)-\sum_{k=\nu+1}^{2\nu+1+\alpha}\frac{1}{k^i})\right)=$$

$$(-1)^{i-1}(i-1)!\left(-\frac{2}{\nu^i}+\frac{1}{(2\nu+\alpha)^i}+\frac{1}{(2\nu+1+\alpha)^i}\right).$$

Then

(50) $$\left(\frac{\partial}{\partial t}\right)(R(\alpha,t,\nu))^{2+l})=(R(\alpha,t,\nu))^{2+l}(2+l)S_1(\alpha,t,\nu),$$

(51) $$\left(\frac{\partial}{\partial t}\right)^2(R(t,\nu))^{2+l})=$$

$$(R(\alpha,t,\nu))^{2+l}((2+l)^2(S_1^2(\alpha,t,\nu))+(2+l)S_2(\alpha,t,\nu)),$$

(52) $$\left(\frac{\partial}{\partial t}\right)^3(R(\alpha,t,\nu))^{2+l}=$$

$$(R(\alpha,t,\nu))^{2+l}((2+l)^3S_1^3(\alpha,t,\nu))^{2+l}+$$
$$3(2+l)^2S_1(\alpha,t,\nu))S_2(t,\nu)+(2+l)S_3(t,\nu)).$$

§9.2. Some relations for the functions, considered in §9.1 in the case $l=0$.

Let $\nu\in M_\alpha=((-\infty,-1-\alpha]\cup[0,+\infty))\cap\mathbb{Z}$,

(53) $$P_{\alpha,0}(w,\nu)=\nu^3(\nu^2+2\alpha\nu+\alpha^2)-$$
$$2\nu^2(2\nu^2+3\alpha\nu+\alpha^2)w+\nu(8\nu^2+10\nu\alpha+3\alpha^2)w^2-$$
$$2(6\nu^2+7\alpha\nu+2\alpha^2)w^3=$$
$$\nu^3(\nu+\alpha)^2-2\nu^2(\nu+\alpha)(2\nu+\alpha)w+$$
$$\nu(2\nu+\alpha)(4\nu+3\alpha)w^2-2(2\nu+\alpha)(3\nu+2\alpha)w^3,$$

(54) $$Q_{\alpha,0}(w,\nu)=\nu(16\nu^2+18\alpha\nu+5\alpha^2)+$$
$$2(6\nu^2+7\alpha\nu+2\alpha^2)w=$$
$$\nu(8\nu+5\alpha)(2\nu+\alpha)+2(2\nu+\alpha)(3\nu+2\alpha)w=$$
$$(2\nu+\alpha)(\nu(8\nu+5\alpha)+2(3\nu+2\alpha)w).$$

Lemma 9.2.1. *The following eqality holds*

(55) $$T_0^\wedge(\alpha,w,\nu):=$$
$$(w+nu)^2P_{\alpha,0}(w,\nu)+w^4Q_{\alpha,0}(w,\nu)-\nu^5(w-\nu-\alpha)^2=0.$$

Proof. I view of (53), (54) and (left) definition in (55),

$$\deg_\alpha(T_0^\wedge(\alpha, w, \nu)) = 2.$$

Therfore it is sufficient to check the equality (55) for

$$\alpha = -2\nu,\ \alpha = -\nu \text{ and } \alpha = w - \nu.$$

If $\alpha = -2\nu$, then

$$P_{\alpha,0}(w, \nu) = \nu^5,\ Q_{\alpha,0}(w, \nu) = 0,\ (w - \nu - \alpha)^2 = (w + \nu)^2$$

and (55) holds. If $\alpha = -\nu$, then

$$P_{\alpha,0}(w, \nu) = \nu^2 w^2(\nu - 2w),\ Q_{\alpha,0}(w, \nu) = \nu^2(3\nu + 2w),\ (w - \nu - \alpha)^2 = w^2,$$

$$(w + \nu)^2\nu^2 w^2(\nu - 2w) + w^4\nu^2(3\nu + 2w) =$$
$$w^2\nu^2((w + \nu)(\nu^2 - w\nu - 2w^2) + w^2(3\nu + 2w)) =$$
$$w^2\nu^2(\nu(nu^2 - w^2) - 2w^2(w + \nu)) + w^2(3\nu + 2w)) = w^2\nu^5.$$

If $\alpha = w - \nu$, then

$$P_{\alpha,0}(w, \nu) =$$
$$\nu^3 w^2 - 2\nu^2(\nu + w)w^2 + \nu(\nu + w)(\nu + 3w)w^2 - 2(\nu + w)(\nu + 2w)w^3 =$$
$$\nu^3 w^2 + (\nu + w)(-2\nu^2 + \nu^2 + 3w\nu - 2w\nu - 4w^2)w^2 =$$
$$\nu^3 w^2 + (\nu + w)(-\nu^2 + w\nu - 4w^2)w^2 =$$
$$\nu^3 w^2 + \nu(-\nu^2 + w^2)w^2 + (\nu + w)(-4w^2)w^2 = -(3w^2\nu + 4w^2)w^2 = -(3\nu + 4w)w^4,$$
$$Q_{\alpha,0}(w, \nu) = \nu(3\nu + 5w)(\nu + w) + 2(\nu + w)(\nu + 2w)w =$$
$$(\nu + w)(3\nu^2 + 7w\nu + 4w^2) = (\nu + w)(\nu + w)(3\nu + 4w),$$
$$w^4(\nu + w)^2(3\nu + 4w),\ (w - \nu - \alpha)^2 = 0,$$

and (55) holds. ■.

Remark 9.1. In (53) – (54),

$$P_{0,0}(w, \nu) = \nu^7 - 2\nu^4 w + 8\nu^3 w^2 - 12\nu^2 w^3 = \tag{56}$$
$$\nu^2(\nu^5 - 2\nu^2 w + 8\nu w^2 - 12w^3) = \nu^2 P_0(\nu, w)$$

$$Q_{0,0}(w, \nu) = 16\nu^3 + 12\nu^2 w = \nu^2(16\nu + 12w) = \nu^2 Q_0(\nu, w), \tag{57}$$

where $P_0(\nu, w)$ and $Q_0(\nu, w)$ are defined in (13) of [5]. Consequently, the equality (55) with $\alpha = 0$ is equivalent to (47) in [5] with $l = 0$. The appearance of the multiplier ν^2 determines a seeming distinction between considered below formulas with $\alpha = 0$ and formulas obtained in [5]. This distinction expires in the final result.

Let (see (3.1.52) in [2] with $c_1(\nu) = c_2(\nu) = \nu$, $\beta^\wedge = \alpha$, $m = n = 2$)

$$D_{\alpha,0}(z, \nu, w) = z(w - \nu - \alpha)^2(w + \nu + 1)^2 - w^4. \tag{58}$$

and w is independent variable. Clearly,

$$D_{\alpha,0}(z,\nu,w)=D_{\alpha,0}(z,-\nu-1-\alpha,w) \tag{59}$$

for $\nu\in\mathbb{Z}$, Let further $\delta:=z\frac{\partial}{\partial z}$. Then (see (3.1.64) in [2])

$$D_{\alpha,0}(z,\nu,\delta)f^{\vee}_{\alpha,0,k}(z,\nu)=0, \tag{60}$$

where $\nu\in[0,+\infty)\cap\mathbb{Z}$, $k\in\mathfrak{K}_0=\{1,\,2,\,3\}$. It follows from the general properties of the Mejer functions that

$$(\delta+\nu+1)^2f^{\vee}_{\alpha,0,k}(z,\nu)=(\delta-\nu-1-\alpha)^2f^{\vee}_{\alpha,0,k}(z,\nu+1), \tag{61}$$

where $\nu\in[0,+\infty)\cap\mathbb{Z}$, $k\in\mathfrak{K}_0$. Therefore

$$(\delta+\nu)^2f^{\vee}_{\alpha,0,k}(z,\nu-1)=(\delta-\nu-\alpha)^2f^{\vee}_{\alpha,0,k}(z,\nu), \tag{62}$$

where $\nu\in\cap\mathbb{N}$, $k\in\mathfrak{K}_0$. Let

$$f^{\vee}_{\alpha,0,k}(z,-\nu-1-\alpha)=f^{\vee}_{\alpha,0,k}(z,\nu), \tag{63}$$

where $\nu\in[0,+\infty)\cap\mathbb{Z}$, $k\in\mathfrak{K}_0$. Then $f^{\vee}_{\alpha,0,k}(z,\nu)$ is defined for $\nu\in M_\alpha$. Moreover, if $\nu\in(-\infty,-1-\alpha]\cap\mathbb{Z}$, then $\nu_1=-\nu-1-\alpha\in[0,+\infty))\cap\mathbb{Z}$, and, in view of (63),

$$f^{\vee}_{\alpha,0,k}(z,\nu)=f^{\vee}_{\alpha,0,k}(z,-\nu_1-1-\alpha)=f^{\vee}_{\alpha,0,k}(z,\nu_1)f^{\vee}_{\alpha,0,k}(z,-\nu-1-\alpha).$$

Therefore (63) holds for all the $\nu\in M_\alpha$ (as before, $\alpha\in\mathbb{N}_0$). Furthermore,

$$\delta^sf^{\vee}_{\alpha,0,k}(z,-\nu-1-\alpha)=\delta^sf^{\vee}_{\alpha,0,k}(z,\nu), \tag{64}$$

where $\{s,\,\alpha\}\subset\mathbb{N}_0$, $\nu\in M_\alpha$, $k\in\mathfrak{K}_0$. In view of (59), (64), the equality (60) holds for $\nu\in M_\alpha$. Moreover, if $\nu\in(-\infty,-1-\alpha]\cap\mathbb{Z}$, then

$$\nu_1=-\nu-1-\alpha\in[0,+\infty))\cap\mathbb{Z},-\nu-\alpha\in\mathbb{N},$$

and in view of (61), (64)

$$\begin{gathered}(\delta-\nu-\alpha)^2f^{\vee}_{\alpha,0,k}(z,\nu)=\\ (\delta-\nu-\alpha)^2f^{\vee}_{\alpha,0,k}(z,-\nu-1-\alpha)=\\ (\delta+\nu_1+1)^2f^{\vee}_{\alpha,0,k}(z,\nu_1)=(\delta-\nu_1-1-\alpha)^2f^{\vee}_{\alpha,0,k}(z,\nu_1+1)=\\ (\delta+\nu)^2f^{\vee}_{\alpha,0,k}(z,-\nu-\alpha)=(\delta+\nu)^2f^{\vee}_{\alpha,0,k}(z,\nu-1);\end{gathered} \tag{65}$$

if $\nu\in(-\infty,-2-\alpha]\cap\mathbb{Z}$, then $\nu+1\in(-\infty,-1-\alpha]\cap\mathbb{Z}$, and, in view (65)

$$\begin{gathered}(\delta-\nu-1-\alpha)^2f^{\vee}_{\alpha,0,k}(z,\nu+1)=\\ (\delta+\nu+1)^2f^{\vee}_{\alpha,0,k}(z,\nu).\end{gathered} \tag{66}$$

So, the equality (62) holds for $\nu\in((-\infty,-1-\alpha]\cup[1,+\infty))\cap\mathbb{Z}$, and (61) holds for $\nu\in((-\infty,-2-\alpha]\cup[0,+\infty))\cap\mathbb{Z}$.

In view of (55), (62), (65), (58), (60),

$$\nu^5(\delta+\nu)^2 f^{\vee}_{\alpha,0,k}(z,\nu-1)= \tag{67}$$

$$((\delta+nu)^2P_{\alpha,0}(\delta,\nu)+\delta^4Q_{\alpha,0}(\delta,\nu))f^{\vee}_{\alpha,0,k}(z,\nu)=$$

$$((\delta+nu)^2P_{\alpha,0}(\delta,\nu)+Q_{\alpha,0}(\delta,\nu)\delta^4)f^{\vee}_{\alpha,0,k}(z,\nu)=$$

$$((\delta+nu)^2P_{\alpha,0}(\delta,\nu)+Q_{\alpha,0}(\delta,\nu)z(\delta+\nu+1)^2(\delta-\nu-\alpha)^2)f^{\vee}_{\alpha,0,k}(z,\nu).$$

Clearly,

$$z(\delta+\nu+1)^2=(\delta+\nu)^2,\ Q_{\alpha,0}(\delta,\nu)(\delta+\nu)^2= \tag{68}$$

$$(\delta+\nu)^2Q_{\alpha,0}(\delta,\nu),\ Q_{\alpha,0}(\delta,\nu))(\delta+\nu)^2,\ Q_{\alpha,0}(\delta,\nu)z=$$

$$zQ_{\alpha,0}(\delta+1,\nu).$$

Therefore, in view of (67), (68)

$$(\delta+\nu)^2\nu^5 f^{\vee}_{\alpha,0,k}(z,\nu-1)= \tag{69}$$

$$(\delta+\nu)^2(P_{\alpha,0}(\delta,\nu)+zQ_{\alpha,0}(\delta+1,\nu)(\delta-\nu-\alpha)^2))f^{\vee}_{\alpha,0,k}(z,\nu),$$

where $\nu\in((-\infty,-1-\alpha]\cup[1,+\infty))\cap\mathbb{Z},\ k\in\mathfrak{K}_0$.

If $\nu\in((-\infty,-2-\alpha]\cup[0,+\infty))\cap\mathbb{Z},\ k\in\mathfrak{K}_0$, then

$$-\nu-2-\alpha\in((-\infty,-2-\alpha]\cup[0,+\infty))\cap\mathbb{Z}\subset M_\alpha,$$

$$\nu_1=-\nu-1-\alpha\in((-\infty,-1-\alpha]\cup[1,+\infty))\cap\mathbb{Z},$$

and therefore, in view of (69), (64)

$$(\delta-\nu-1-\alpha)^2(-\nu-1-\alpha)^5 f^{\vee}_{\alpha,0,k}(z,\nu+1)= \tag{70}$$

$$(\delta-\nu-1-\alpha)^2(-\nu-1-\alpha)^5 f^{\vee}_{\alpha,0,k}(z,-\nu-2-\alpha)=$$

$$(\delta+\nu_1)^2\nu_1^5 f^{\vee}_{\alpha,0,k}(z,\nu_1-1)=$$

$$(\delta+\nu_1)^2(P_{\alpha,0}(\delta,\nu_1)+zQ_{\alpha,0}(\delta+1,\nu_1)(\delta-\nu_1-\alpha)^2))f^{\vee}_{\alpha,0,k}(z,\nu_1)=$$

$$(\delta-\nu-1-\alpha)^2\times$$

$$(P_{\alpha,0}(\delta,-\nu-1-\alpha)+zQ_{\alpha,0}(\delta+1,-\nu-1-\alpha)(\delta+\nu+1)^2))f^{\vee}_{\alpha,0,k}(z,-\nu-1-\alpha)=$$

$$(\delta-\nu-1-\alpha)^2\times$$

$$(P_{\alpha,0}(\delta,-\nu-1-\alpha)+zQ_{\alpha,0}(\delta+1,-\nu-1-\alpha)(\delta+\nu+1)^2))f^{\vee}_{\alpha,0,k}(z,\nu).$$

Let

$$\mathfrak{W}^{\vee}_{\alpha,0,k}(z,\nu)=\nu^5 f^{\vee}_{\alpha,0,k}(z,\nu-1)- \tag{71}$$

$$(P_{\alpha,0}(\delta,\nu)+zQ_{\alpha,0}(\delta+1,\nu)(\delta-\nu-\alpha)^2))f^{\vee}_{\alpha,0,k}(z,\nu),$$

where $\nu\in M^*_\alpha=((-\infty,-1-\alpha]\cup[1,+\infty))\cap\mathbb{Z},\ k\in\mathfrak{K}_0,\ |z|>1$. Let further

$$\mathfrak{W}^{\wedge}_{\alpha,0,k}(z,\nu)=(-\nu-1-\alpha)^5 f^{\vee}_{\alpha,0,k}(z,\nu+1)- \tag{72}$$

$$(P_{\alpha,0}(\delta,-\nu-1-\alpha)+zQ_{\alpha,0}(\delta+1,-\nu-1-\alpha)(\delta+\nu+1)^2))f^{\vee}_{\alpha,0,k}(z,\nu),$$

where $\nu \in ((-\infty, -2-\alpha] \cup [0, +\infty)) \cap \mathbb{Z}$, $k \in \mathfrak{K}_0$.

In view of (64), (71), (72),

(73)
$$\mathfrak{W}^{\vee}_{\alpha,0,k}(z, -\nu-1-\alpha) = (-\nu-1-\alpha)^5 f^{\vee}_{\alpha,0,k}(z, -\nu-2-\alpha) -$$
$$(P_{\alpha,0}(\delta, -\nu-1-\alpha) + zQ_{\alpha,0}(\delta+1, -\nu-1-\alpha)(\delta+1)^2))f^{\vee}_{\alpha,0,k}(z, -\nu-1-\alpha) =$$
$$\mathfrak{W}^{\vee}_{\alpha,0,k}(z, -\nu-1-\alpha) = (-\nu-1-\alpha)^5 f^{\vee}_{\alpha,0,k}(z, \nu+1) -$$
$$(P_{\alpha,0}(\delta, -\nu-1-\alpha) + zQ_{\alpha,0}(\delta+1, -\nu-1-\alpha)(\delta+1)^2))f^{\vee}_{\alpha,0,k}(z, \nu) =$$
$$\mathfrak{W}^{\wedge}_{\alpha,0,k}(z, \nu),$$

where $\nu \in ((-\infty, -2-\alpha] \cup [0, +\infty)) \cap \mathbb{Z}$,

(74)
$$\mathfrak{W}^{\wedge}_{\alpha,0,k}(z, -\nu-1-\alpha) = \nu^5 f^{\vee}_{\alpha,0,k}(z, -\nu-\alpha) -$$
$$(P_{\alpha,0}(\delta, \nu) + zQ_{\alpha,0}(\delta+1, \nu)(\delta-\nu-\alpha)^2))f^{\vee}_{\alpha,0,k}(z, -\nu-1-\alpha) =$$
$$\nu^5 f^{\vee}_{\alpha,0,k}(z, \nu-1) -$$
$$(P_{\alpha,0}(\delta, \nu) + zQ_{\alpha,0}(\delta+1, \nu)(\delta-\nu-\alpha)^2))f^{\vee}_{\alpha,0,k}(z, \nu) =$$
$$\mathfrak{W}^{\vee}_{\alpha,0,k}(z, \nu),$$

where $\nu \in ((-\infty, -1+\alpha] \cup [1, +\infty)) \cap \mathbb{Z}$. Then (69) can be rewritten in the form

(75)
$$(\delta+\nu)^2 \mathfrak{W}^{\vee}_{\alpha,0,k}(z, \nu) = 0,$$

where $\nu \in ((-\infty, -1-\alpha] \cup [1, +\infty)) \cap \mathbb{Z}$, $k \in \mathfrak{K}_0$, and (70) can be rewritten in the form

(76)
$$(\delta-\nu-1-\alpha)^2 \mathfrak{W}^{\wedge}_{\alpha,0,k}(z, \nu) = 0,$$

where $\nu \in ((-\infty, -2-\alpha] \cup [0, +\infty)) \cap \mathbb{Z}$, $k \in \mathfrak{K}_0$,

We want to prove the equality

(77)
$$\mathfrak{W}^{\vee}_{\alpha,0,k}(z, \nu) = 0,$$

where $\nu \in M^*_\alpha$, $k \in \mathfrak{K}_0$, $|z| > 1$, and to prove the equality

(78)
$$\mathfrak{W}^{\wedge}_{\alpha,0,k}(z, \nu) = 0,$$

where $\nu \in ((-\infty, -2-\alpha] \cup [0, +\infty)) \cap \mathbb{Z}$, $k \in \mathfrak{K}_0$, $|z| > 1$. In view of (31), (71), the function $\mathfrak{W}^{\vee}_{\alpha,0,1}(z, \nu)$ belongs to $\mathbb{C}[z]$ for $\nu \in ((-\infty, -1-\alpha] \cup [1, +\infty)) \cap \mathbb{Z}$, if $\nu \in \mathbb{N}$, then null-space of the operator $(\delta+\nu)^2$ (as linear operator on $\mathbb{C}[z]$) is equual to zero element in $\mathbb{C}[z]$. Hence the equality (77) holds for $\nu \in \mathbb{N}$, $k = 1$.

If $\nu \in (-\infty, -2-\alpha] \cap \mathbb{Z}$, then $\nu_1 = -\nu-1-\alpha \in \mathbb{N}$; consequently, in view of (77) and (73), $\mathfrak{W}^{\wedge}_{\alpha,0,k}(z, \nu) = 0$. Therefore, if $\nu \in (-\infty, -2-\alpha] \cap \mathbb{Z}$, then 78) holds for $k = 1$. In view of (54),

(79)
$$Q_{\alpha,0}(w, -\nu-1-\alpha) =$$
$$-(2\nu+2+\alpha)((\nu+1+\alpha)((8\nu+8+3\alpha) - (6\nu+6+2\alpha)w),$$

(80)
$$Q_{\alpha,0}(\delta+1,-\nu-1-\alpha)z^{\nu+\alpha}=$$
$$-(\nu+1+\alpha)(2\nu+2+\alpha)^2z^{\nu+\alpha}.$$

If $\nu\in\mathbb{N}_0$, then n view of (31), (71),
$$\deg_z\mathfrak{W}^{\wedge}_{\alpha,0,k}(z,\nu)=\nu+1+\alpha;$$
therefore, in view of (80), to establish (78) in this case, we must check the equality
$$\frac{(\nu+1)!^2}{(\nu+1+\alpha)!^2}\binom{2\nu+2+\alpha}{\nu+1+\alpha}^2\times$$
$$(-\nu-1-\alpha)^5=-\frac{(\nu)!^2}{(\nu+\alpha)!^2}\times$$
$$\binom{2\nu+\alpha}{\nu+\alpha}^2(2\nu+1+\alpha)(\alpha+1+\nu)(2\nu+2+\alpha)^2,$$
which is equivalent to the equality
$$\frac{(\nu+1)^2}{(\nu+1+\alpha)^2}(2\nu+2+\alpha)^2(2\nu+1+\alpha)\times$$
$$(\nu+1)^{-2}(\nu+1+\alpha)^{-2}(\nu+1\alpha)^5=$$
$$(\alpha+1+\nu)(2\nu+2+\alpha)^3,$$
and the last equality, clearly, holds. So, (78) holds for $k=1$.

If $\nu\in(-\infty,-1-\alpha]\cap\mathbb{Z}$, then $\nu_1=-\nu-1-\alpha\in\mathbb{N}_0$, and since the equality (78) holds for $k=1$, it follows from (74) that $\mathfrak{W}^{\vee}_{\alpha,0,1}(z,\nu)=0$. So, the equality (77) holds for $k=1$. In view of (37), (72),
$$\mathfrak{W}^{\wedge}_{\alpha,0,2}(z,\nu)\in\mathbb{C}[[z^{-1}]],$$
where $\nu\in\mathbb{N}_0$, $|z|>1$ and $\mathbb{C}[[x]]$ denotes the linear space (and also ring) of all the formal power series over $\mathbb{C}$ with variable x, moreover
$$z^{\nu+1}\mathfrak{W}^{\wedge}_{\alpha,0,2}(z,\nu)=\in\mathbb{C}[[z^{-1}]],$$
where $\nu\in\mathbb{N}_0$, $|z|>1$.

If $\{\nu,\alpha\}\subset\mathbb{N}_0$, then null-space of the operator $(\delta-\nu-1-\alpha)^{2+l}$ (as operaotr on linar over $\mathbb{C}$ space $\mathbb{C}[[z^{-1}]]$) coincides with 0. Therefore, in view of (76), the equality (78) holds for $k=2$, $\nu\in\mathbb{N}_0$ and $|z|>1$.

If $\nu\in(-\infty,-1-\alpha]\cap\mathbb{Z}\,|z|>1$, then $\nu_1=-\nu-1-\alpha\in\mathbb{N}_0$; since (78) holds for $k=2$, it follows from (74) that $\mathfrak{W}^{\vee}_{\alpha,0,2}(z,\nu)=0$.

So, the equality (77) holds for $k=2$, $\nu\in(-\infty,-1-\alpha]\cap\mathbb{Z}\,|z|>1$.

In view of (54),

(81)
$$Q_{\alpha,0}(\delta+1,\nu)z^{-\nu-1}=$$
$$(2\nu+\alpha)(\nu(8\nu+5\alpha)+2(3\nu+2\alpha)(-nu))=$$
$$\nu(2\nu+\alpha)^2.$$

In view of (33),

$$(R(\alpha,\nu+1,\nu))^2=\frac{(\nu!)^4}{((2\nu+1+\alpha)!)^2}, \tag{82}$$

$$(R(\alpha,\nu,\nu-1))^2=\frac{((\nu-1)!)^4}{((2\nu-1+\alpha)!)^2} \tag{83}$$

In view of (37), (71),

$$\mathfrak{W}^{\vee}_{\alpha,0,2}(z,\nu)\in\mathbb{C}[[z^{-1}]],$$

where $\nu\in\mathbb{N}$, $|z|>1$, moreover

$$z^{\nu}\mathfrak{W}^{\wedge}_{\alpha,0,2}(z,\nu)=\in\mathbb{C}[[z^{-1}]],$$

where $\nu\in\mathbb{N}$, $|z|>1$; therefore, in view of (71), (37), (81), (82), (83), to establish (77) in this case, we must check the equality

$$z^{-\nu}\nu^5(R(\alpha,\nu,\nu-1))^2=$$

$$z^{-\nu}(R(\alpha,\nu+1,\nu))^2(2\nu+1+\alpha)^2)\times$$

$$\nu(2\nu+\alpha)^2,$$

i.e. the equality

$$\nu^5\frac{((\nu-1)!)^4}{((2\nu-1+\alpha)!)^2}=$$

$$\frac{((\nu!)^4}{((2\nu+1+\alpha)!)^2}\times$$

$$(2\nu+1+\alpha))^2\nu(2\nu+\alpha)^2,$$

which, evidently, holds. So, the equality (77) holds for $k=2$.

Therefore, if $\nu\in(-\infty,-2-\alpha]\cap\mathbb{Z}$, then then $\nu_1=-\nu-1-\alpha\in\mathbb{N}$; since (77) holds for $k=2$, it follows from (73) that $\mathfrak{W}^{\wedge}_{\alpha,0,2}(z,\nu)=0$. So, theequality (78) holds for $k=2$.

In view of (41),

$$\mathfrak{W}^{\wedge}_{\alpha,0,3}(z,\nu)\in\mathbb{C}[[z^{-1}]]+(\log(z))\mathbb{C}[[z^{-1}]],$$

where $\nu\in\mathbb{N}_0$, $|z|>1$, moreover

$$z^{\nu+1}\mathfrak{W}^{\wedge}_{\alpha,0,2}(z,\nu)\in\mathbb{C}[[z^{-1}]]+(\log(z))\mathbb{C}[[z^{-1}]],$$

where $\nu\in\mathbb{N}_0$, $|z|>1$. We can interpret $\mathbb{C}[[z^{-1}]]+(\log(z))\mathbb{C}[[z^{-1}]]$, as linear over $\mathbb{C}$ space

$$\mathbb{C}[[z^{-1}]]\oplus\mathbb{C}[[z^{-1}]]$$

with linear operator δ, which acts according to the formula

$$\delta\begin{pmatrix}T_1\\T_2\end{pmatrix}=\begin{pmatrix}\delta T_1+T_2\\\delta T_2\end{pmatrix}, \tag{84}$$

where $T_k \in \mathbb{C}[[z^{-1}]]$ for $k = 1,\, 2$. For those, who find previous argument not sufficient we add, that if

$$h(z) = T_1(z) + (\log(z))T_2(z), \tag{85}$$

where $T_1(z)$ and $T_2(z)$ are regular for $|z| > 1$, then the equality (85) can by means of analytic continuation prolonged to the equality

$$H((r, \varphi)) = T_1(r\exp(i\varphi)) + i\varphi T_2(r\exp(i\varphi)), \tag{86}$$

where $Z = (r, \phi)$ lie on Rimannian surface of $Log(z)$, $r > 1$, $\phi \in \mathbb{R}$ and H is uniquely defined by h.

Then the equality $H((r, \varphi + 2\pi)) - H((r, \varphi)) = 2i\pi T_2(r\exp(i\varphi))$ show that T_1 and T_2 are uniquely defined by h. If $\{\nu, \alpha\} \subset \mathbb{N}_0$, then null-space of the operator $(\delta - \nu - 1 - \alpha)^{2+l}$ (as operator, which acts on $\mathbb{C}[[z^{-1}]] \oplus \mathbb{C}[[z^{-1}]]$ according to (85)) coincides with 0. Therefore, in view of of the equality (76), the equality (78) holds for $k = 3$, $\nu \in \mathbb{N}_0$ and $|z| > 1$.

If $\nu \in (-\infty, -1 - \alpha] \cap \mathbb{Z}\, |z| > 1$, then $\nu_1 = -\nu - 1 - \alpha \in \mathbb{N}_0$; since (78) holds for $k = 3$, it follows from (74) that $\mathfrak{W}^{\vee}_{\alpha,0,3}(z, \nu) = 0$.

So, (77) holds for $k = 3$, $\nu \in (-\infty, -1 - \alpha] \cap \mathbb{Z}\, |z| > 1$. Let

$$H(w) \in \mathbb{C}[w]$$

Then, according to the Leibnitz formula

$$H(\delta)((\log(z)f(z)) = (\log(z))H(\delta)f(z) + \left(\frac{d}{dw}H\right)\big|_{w=\delta} f(z).$$

If $\nu \in \mathbb{N}$, $|z| > 1$, then, since (77) holds for $k = 2$, it follows from (71), (45) that

$$\begin{gathered}
\mathfrak{W}^{\vee}_{\alpha,0,3}(z, \nu) = (\log(z))\mathfrak{W}^{\vee}_{\alpha,0,2}(z, \nu) + \\
\nu^5 f^{\vee}_{\alpha,0,4}(z, \nu - 1) - \\
(P_{\alpha,0}(\delta, \nu) + zQ_{\alpha,0}(\delta + 1, \nu)(\delta - \nu - \alpha)^2))f^{\vee}_{\alpha,0,4}(z, \nu) - \\
\left(\frac{d}{dw}P_{\alpha,0}(w, \nu)\big|_{w=\delta} + z\frac{d}{dw}(Q_{\alpha,0}(w + 1, \nu)(w - \nu - \alpha)^2))\big|_{w=\delta}\right) f^{\vee}_{\alpha,0,2}(z, \nu) = \\
\nu^5 f^{\vee}_{\alpha,0,4}(z, \nu - 1) - \\
(P_{\alpha,0}(\delta, \nu) + zQ_{\alpha,0}(\delta + 1, \nu)(\delta - \nu - \alpha)^2))f^{\vee}_{\alpha,0,4}(z, \nu) - \\
\left(\frac{d}{dw}P_{\alpha,0}(w, \nu)\big|_{w=\delta} + z\frac{d}{dw}(Q_{\alpha,0}(w + 1, \nu)(w - \nu - \alpha)^2))\big|_{w=\delta}\right) f^{\vee}_{\alpha,0,2}(z, \nu).
\end{gathered} \tag{87}$$

In view of (75) to establish (77) in the case $\nu \in \mathbb{N}$, $|z| > 1$, we must check the equality

$$\begin{gathered}
-\nu^5\frac{\partial}{\partial t}(R(\alpha, t, \nu - 1))^2\big|_{t=\nu} z^{-nu} - \\
\left(-\frac{\partial}{\partial t}(R(\alpha, t, \nu))^2\big|_{t=\nu+1}\right) \times
\end{gathered} \tag{88}$$

$$zQ_{\alpha,0}(\delta+1,\nu)(\delta-\nu-\alpha)^2))z^{-nu-1}-$$
$$(R(\alpha,\nu+1,\nu))^2 z\frac{d}{dw}(Q_{\alpha,0}(w+1,\nu)(w-\nu-\alpha)^2))\big|_{w=\delta}z^{-\nu+1}=0.$$

In view of (42), (49), (50), (82), (83), (54),

$$\text{(89)}\qquad -\nu^5\frac{\partial}{\partial t}(R(\alpha,t,\nu-1))^2\big|_{t=\nu}z^{-nu}-$$
$$\left(-\frac{\partial}{\partial t}(R(\alpha,t,\nu))^2\big|_{t=\nu+1}\right)\times$$
$$zQ_{\alpha,0}(\delta+1,\nu)(\delta-\nu-\alpha)^2))z^{-nu-1}=$$
$$\nu\frac{2(\nu!)^4}{(2\nu-1+\alpha)^2}\times$$
$$\left(\frac{2}{\nu}-\frac{1}{2\nu+\alpha}-\frac{1}{2\nu+\alpha+1}\right)=$$
$$\frac{2(\nu!)^4}{(2\nu-1+\alpha)^2}\times$$
$$\frac{4\nu^2+6\nu\alpha+2\alpha^2+3\nu+2\alpha}{(2\nu+\alpha)(2\nu+\alpha+1)}.$$

In view of (54),

$$\frac{d}{dw}(Q_{\alpha,0}(w+1,\nu)(w-\nu-\alpha)^2))=$$
$$(2\nu+\alpha)(\nu(8\nu+5\alpha)+2(3\nu+2\alpha)(w+1))2(w-\nu-\alpha)+$$
$$(2\nu+\alpha)(2(3\nu+2\alpha))(w-\nu-\alpha)^2,$$

$$\text{(90)}\qquad \frac{d}{dw}(Q_{\alpha,0}(w+1,\nu)(w-\nu-\alpha)^2))\big|_{w=\delta}z^{-\nu-1}=$$
$$-2(2\nu+\alpha)(2\nu+\alpha+1)(\nu(2\nu+\alpha)-(3\nu+2\alpha)(2\nu+\alpha+1))z^{-\nu-1}=$$
$$2(2\nu+\alpha)(2\nu+\alpha+1)(4\nu^2+6\nu\alpha+\alpha^2+3\nu+2\alpha)z^{-\nu-1},$$

and

$$\text{(91)}\qquad (R(\alpha,\nu+1,\nu))^2 z\times$$
$$\frac{d}{dw}(Q_{\alpha,0}(w+1,\nu)(w-\nu-\alpha)^2))\big|_{w=\delta}z^{-\nu+1}=$$
$$\frac{(\nu!)^4}{((2\nu+1+\alpha)!)^2}\times$$
$$2(2\nu+\alpha)(2\nu+\alpha+1)(4\nu^2+6\nu\alpha+\alpha^2+3\nu+2\alpha)z^{-\nu-1},$$

According to (89) and (91), the equality (88) holds, and, consequently, the equality (77) holds for $k=3$, $|z|>1$.

If $\nu\in(-\infty,-2-\alpha]\cap\mathbb{Z}$, then then $\nu_1=-\nu-1-\alpha\in\mathbb{N}$; since (77) holds for $k=3$, it follows from (73) that $\mathfrak{W}^{\wedge}_{\alpha,0,3}(z,\nu)=0$.

So, (78) holds for $k=3$, $|z|>1$.

§9.3. Passing to the system of difference equations for the functions, considered in §9.2.

Let

$$\mu = \mu_\alpha(\nu) = (\nu+\alpha)(\nu+1),\ \tau = \tau_a lpa(\nu) = \nu + \frac{1+\alpha}{2} \tag{92}$$

where $\nu \in \mathbb{Z}$. Then

$$\tau^2 = \mu_\alpha + (1-\alpha)^2/4, \tag{93}$$

and, in view of (58),

$$D_{\alpha,0}(z,\nu,w) = z(w^2 + w(1-\alpha) - \mu_\alpha)^2 - w^4, \tag{94}$$

$$D_0(z,\nu,w) = z(\mu_\alpha)^2 - 2z(1-\alpha)w(\mu_\alpha) +$$
$$zw^2((1-\alpha)^2 - 2\mu_\alpha) + 2(1-\alpha)zw^3 + (z-1)w^4.$$

Let

$$b_{\alpha,0,1}(z;\nu) = -(z-1)^{-1}z\mu_\alpha^2 = -(z-1)^{-1}z\times$$
$$(\alpha^2 + 2(\alpha+1)\alpha\nu + (\alpha 2 + 3\alpha + 1)\nu^2 + 2(\alpha+1)\nu^3 + \nu^4),$$
$$b_{\alpha,0,2}(z;\nu) = (z-1)^{-1}2z(1-\alpha)\mu_\alpha,$$
$$b_{\alpha,0,3}(z;\nu) = -(z-1)^{-1}z((1-\alpha)^2 - 2\mu_\alpha) = -(z-1)^{-1}z\times$$
$$(1 - 4\alpha + \alpha^2 + 2(1+\alpha)\nu - 2\nu^2),$$
$$b_{\alpha,0,4}(z;\nu) = -(z-1)^{-1}2(1-\alpha)z,$$
$$B_{\alpha,0}(z;\nu) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \\ b_{\alpha,0,1}(z;\nu) & b_{\alpha,0,1}(z;\nu) & b_{\alpha,0,1}(z;\nu) & b_{\alpha,0,1}(z;\nu) \end{pmatrix}$$
$$X_{\alpha,0,k}(z;\nu) = \begin{pmatrix} f^*_{\alpha,0,k}(z,\nu) \\ \delta f^*_{\alpha,0,k}(z,\nu) \\ \delta^2 f^*_{\alpha,0,k}(z,\nu) \\ \delta^3 f^*_{\alpha,0,k}(z,\nu) \end{pmatrix}$$

where $k \in \mathfrak{K}_0$, $|z| > 1$. Then

$$X_{\alpha,0,k}(z;-\nu-1-\alpha) = X_{\alpha,0,k}(z;\nu), \tag{95}$$

$$\delta X_{\alpha,0,k}(z;\nu) = B_{\alpha,0}(z;\nu)X_{\alpha,0,k}(z;\nu), \tag{96}$$

where $k \in \mathfrak{K}_0$, $|z| > 1$, $\nu \in M_\alpha$.

Since, in view of (54),

$$Q^*_0(\nu, w+1) = (2\nu+\alpha)(\nu(8\nu+5\alpha) + 2(3\nu+2\alpha)(w+1)) =$$

$$(2\nu+\alpha)(\nu(8\nu+5\alpha) + 2(3\nu+2\alpha)(w+1)) = (2\nu+\alpha)\times$$
$$(8\nu^2 + (6+5\alpha))\nu + 4\alpha + (6\nu+4\alpha)w) =$$

$$16\nu^3 + (12 + 18\alpha)\nu^2 + (14\alpha + 5\alpha^2)\nu + 4\alpha^2+$$
$$(12\nu^2 + 14\nu\alpha + 4\alpha^2)w,$$

it follows that

$$Q^*_{\alpha,0}(\nu, w + 1)(w - \nu - \alpha)^2 =$$
$$\sum_{k=0}^{3} q^*_{\alpha,0,k}(\nu)w^k,$$

where

$$q^*_{\alpha,0,0}(\nu) = 4\alpha^4 + (22 + 5\alpha)\alpha^3\nu + (44 + 28\alpha)\alpha^2\nu^2+$$
$$(38 + 57\alpha)\alpha\nu^3 + (12 + 50\alpha)\nu^4 + 16\nu^5 =$$
$$\frac{1}{2}(12 + 26\alpha + 131\alpha^2 + 115\alpha^3 + 121\alpha^4 + 40\alpha^5 - 25\alpha^6)+$$
$$\frac{1}{2}(34 - 5\alpha + 33\alpha^2 - 59\alpha^3 - 125\alpha^4)\mu_\alpha+$$
$$(12 - 16\alpha - 50\alpha^2)\mu_\alpha^2+$$
$$(-12 + 14\alpha + 53\alpha^2 + 64\alpha^3 + 46\alpha^4 + 25\alpha^5)\tau_\alpha+$$
$$(14 + 57\alpha + 81\alpha^2 + 75\alpha^3)\mu_\alpha\tau_\alpha+$$
$$(16 + 50\alpha)\mu_\alpha^2\tau_\alpha,$$

$$q^*_{\alpha,0,1}(\nu) = -(8 - 4\alpha)\alpha^3 - (36 - 12\alpha)\alpha^2\nu - (52 + 2\alpha)\alpha\nu^2-$$
$$(24 + 30\alpha)\nu^3 - 20\nu^4 =$$
$$-(12 + 31\alpha - 10\alpha^2 + 27\alpha^3 + 8\alpha^4)-$$
$$(39 + 82\alpha + 27\alpha^2)\mu_\alpha - 20\mu_\alpha^2+$$
$$(-24 + 18\alpha - 26\alpha^2 - 14\alpha^3 + 4\alpha^4)\tau_\alpha-$$
$$(34 + 20\alpha)\mu_\alpha\tau_\alpha,$$

$$q^*_{\alpha,0,2}(\nu) = 4\alpha^2 - 8\alpha^3 + 14\alpha\nu - 31\alpha^2\nu+$$
$$12\nu^2 - 34\alpha\nu^2 =$$
$$-\frac{1}{2}(8 + 48\alpha - 17\alpha^2 + 19\alpha^3 - frac12(16 + 68\alpha)\mu_\alpha-$$
$$(8 - 40\alpha + \alpha^2 + 8\alpha^3)\tau_\alpha - 8\mu_\alpha\tau_\alpha,$$

$$q^*_{\alpha,0,3}(\nu) = 4\alpha^2 + 14\alpha\nu + 12\nu^2 =$$
$$frac12(12 - 14\alpha + 6\alpha^2 + 24\mu_\alpha)-$$
$$(12 - 2\alpha - 4\alpha^2)\tau_\alpha.$$

Let

$$q^\vee_{\alpha,0,0}(\nu) = \frac{1}{2}(12 + 26\alpha + 131\alpha^2 + 115\alpha^3 + 121\alpha^4 + 40\alpha^5 - 25\alpha^6)+$$

$$\frac{1}{2}(34-5\alpha+33\alpha^2-59\alpha^3-125\alpha^4)\mu_\alpha+$$
$$(12-16\alpha-50\alpha^2)\mu_\alpha^2,$$

$$q^\vee_{\alpha,0,1}(\nu)=-(12+31\alpha-10\alpha^2+27\alpha^3+8\alpha^4)-$$
$$(39+82\alpha+27\alpha^2)\mu_\alpha-20\mu_\alpha^2,$$

$$q^\vee_{\alpha,0,2}(\nu)=-\frac{1}{2}(8+48\alpha-17\alpha^2+19\alpha^3-frac12(16+68\alpha)\mu_\alpha,$$

$$q^\vee_{\alpha,0,3}(\nu)=frac12(12-14\alpha+6\alpha^2+24\mu_\alpha),$$
$$(12-2\alpha-4\alpha^2)\tau_\alpha,$$

$$q^\wedge_{\alpha,0,0}(\nu)=-12+14\alpha+53\alpha^2+64\alpha^3+46\alpha^4+25\alpha^5+$$
$$(14+57\alpha+81\alpha^2+75\alpha^3)\mu_\alpha+$$
$$(16+50\alpha)\mu_\alpha^2,$$

$$q^\wedge_{\alpha,0,1}(\nu)=$$
$$-24+18\alpha-26\alpha^2-14\alpha^3+4\alpha^4-$$
$$(34+20\alpha)\mu_\alpha,$$

$$q^\wedge_{\alpha,0,2}(\nu)=-8+40\alpha-\alpha^2-8\alpha^3-8\mu_\alpha,$$

$$q^\wedge_{\alpha,0,3}(\nu)=-12+2\alpha+4\alpha^2.$$

Then

$$q^*_{\alpha,0,k}(\nu)=q^\vee_{\alpha,0,k}(\nu)+q^\wedge_{\alpha,0,k}(\nu)\tau_\alpha$$

for $k=1,\ 2,\ 3$. In view of (53),

(97) $$P_{\alpha,0}(w,\nu)=\sum_{k=0}^{3}p^*_{\alpha,k}(\nu)w^k,$$

where

$$p^*_{\alpha,0,0}(\nu)=\frac{1}{4}(\alpha^2-\alpha^3-\alpha^4+\alpha^5)-$$
$$\frac{1}{4}(1+\alpha-\alpha^2-5\alpha^3)\mu_\alpha-4\alpha\mu_\alpha^2)+$$
$$\frac{1}{2}(-\alpha^2+\alpha^4+(1+2\alpha+3\alpha^2)\mu_\alpha+2\mu_\alpha^2)\tau_\alpha,$$

$$p^*_{\alpha,0,1}(\nu)=-\alpha-\alpha^3-2\alpha^4-$$
$$(3+6\alpha+7\alpha^2)\mu_\alpha-4\mu_\alpha^2)-$$

$$2(\alpha+\alpha^2+\alpha^3)\tau_\alpha-(2+4\alpha)\mu_\alpha\tau_\alpha,$$

$$p^*_{\alpha,0,2}(\nu)=\frac{1}{2}(8+10\alpha+5\alpha^2+7\alpha^3)+$$
$$(8+10\alpha)\mu_\alpha+$$
$$(8-2\alpha+\alpha^2+8\mu_\alpha)\tau_\alpha,$$

$$p^*_{\alpha,0,3}(\nu)=-6+7\alpha-3\alpha^2+7\alpha^3-12\mu_\alpha+$$
$$(12-2\alpha-4\alpha^2)\tau_\alpha.$$

Let

$$p^\vee_{\alpha,0,0}(\nu)=\frac{1}{4}(\alpha^2-\alpha^3-\alpha^4+\alpha^5)-$$
$$\frac{1}{4}(1+\alpha-\alpha^2-5\alpha^3)\mu_\alpha)+\alpha\mu_\alpha^2,$$

$$p^\vee_{\alpha,0,1}(\nu)=-\alpha-\alpha^3-2\alpha^4-$$
$$(3+6\alpha+7\alpha^2)\mu_\alpha-4\mu_\alpha^2,$$

$$p^\vee_{\alpha,0,2}(\nu)=\frac{1}{2}(8+10\alpha+5\alpha^2+7\alpha^3)+$$
$$(8+10\alpha)\mu_\alpha,$$

$$p^\vee_{\alpha,0,3}(\nu)=-6+7\alpha-3\alpha^2+7\alpha^3-12\mu_\alpha,$$

$$p^\wedge_{\alpha,0,0}(\nu)=\frac{1}{2}(-\alpha^2+\alpha^4+(1+2\alpha+3\alpha^2)\mu_\alpha+2\mu_\alpha^2),$$

$$p^\wedge_{\alpha,0,1}(\nu)=-2(\alpha+\alpha^2+\alpha^3)-(2+4\alpha)\mu_\alpha,$$

$$p^\wedge_{\alpha,0,2}(\nu)=8-2\alpha+\alpha^2+8\mu_\alpha,$$

$$p^\wedge_{\alpha,0,3}(\nu)=12-2\alpha-4\alpha^2.$$

Then

$$p^*_{\alpha,0,k}(\nu)=p^\vee_{\alpha,0,k}(\nu)+p^\wedge_{\alpha,0,k}(\nu)\tau_\alpha$$

for $k=0,\ 1,\ 2,\ 3$. We denote by

$$\bar{a}^*_{\alpha,0,1}(z;\nu),\ \bar{a}^\vee_{\alpha,0,1}(z;\nu),\ \bar{a}^\wedge_{\alpha,0,1}(z;\nu)$$

the row with 4 elements, $(k+1)$-th of which is equal respectively

$$p^*_{\alpha,0,k}(\nu)+zq^*_{l,k}(\nu),$$
$$p^\vee_{\alpha,0,k}(\nu)+zq^\vee_{l,k}(\nu),$$
$$p^\wedge_{\alpha,0,k}(\nu)+zq^\wedge_{l,k}(\nu),$$

where $k = 0, 1, 2, 3$. Let

(98) $$\bar{a}^*_{\alpha,0,i+1}(z;\nu) = \delta\bar{a}^*_{\alpha,0,i}(z;\nu) + \bar{a}^*_{\alpha,0,i}(z;\nu)B_{\alpha,0}(z;\nu),$$

(99) $$\bar{a}^\vee_{\alpha,0,i+1}(z;\nu) = \delta\bar{a}^*_{\alpha,0,i}(z;\nu) + \bar{a}^\vee_{\alpha,0,i}(z;\nu)B_l(z;\nu),$$

(100) $$\bar{a}^\wedge_{\alpha,0,i+1}(z;\nu) = \delta\bar{a}^\wedge_{\alpha,0,i}(z;\nu) + \bar{a}^\wedge_{\alpha,0,i}(z;\nu)B_l(z;\nu),$$

where $i = 0, ..., 3$. Clearly,

$$\bar{a}^*_{\alpha,0,k}(z;\nu) = \bar{a}^\vee_{\alpha,0,k}(z;\nu) + \bar{a}^\wedge_{\alpha,0,k}(z;\nu)\tau$$

for $k = 0, ..., 3$. We denote by

$$a^*_{\alpha,0,i,k}(z;\nu),\ a^\vee_{\alpha,0,i,k}(z;\nu),\ a^\wedge_{\alpha,0,i,k}(z;\nu)$$

the k-th elements of the rows respectively

$$\bar{a}^*_{\alpha,0,i}(z;\nu),\ \bar{a}^\vee_{\alpha,0,i}(z;\nu),\ \bar{a}^\wedge_{\alpha,0,i}(\nu),$$

where $i = 1, ..., 4$, $k = 1, ..., 4$. Then,

$$a^\vee_{\alpha,0,1,1}(z,\nu) = \frac{1}{2}(-1 + 2\alpha - \alpha^2 - 5\mu + 3\alpha\mu - 5\mu^2 - \alpha\mu^2) + \frac{z}{2}(-4 + 12\alpha - 13\alpha^2 + 6\alpha^3 - \alpha^4) + \frac{z}{2}\mu(-32 + 54\alpha - 29\alpha^2 + 5\alpha^3 - 56\mu + 20\alpha\mu),$$

$$a^\vee_{\alpha,0,1,2}(z;\nu) = -2 + 3\alpha - \alpha^2 - 8\mu + \alpha\mu - \alpha^2\mu - 4\mu^2 + z(2 - 11\alpha + 17\alpha^2 - 10\alpha^3 + 2\alpha^4 - 4\mu - 11\alpha\mu + 3\alpha^2\mu - 20\mu^2),$$

$$a^\vee_{\alpha,0,1,3}(z;\nu) = -4 + 5\alpha - \frac{3}{2}\alpha^2 - \frac{1}{2}\alpha^3 - 12\mu - 2\alpha\mu + z\left(10 - 24\alpha + \frac{37}{2}\alpha^2 - \frac{11}{2}\alpha^3 + 24\mu - 22\alpha\mu\right),$$

$$a^\vee_{\alpha,0,1,4}(z;\nu) = (z - 1)(6 - 7\alpha + 3\alpha^2 + 12\mu),$$

$$a^\wedge_{\alpha,0,1,1}(z;\nu) = 1 - \alpha + 3\mu + \mu^2 + z(4 - 8\alpha + 5\alpha^2 - \alpha^3 + 24\mu - 22\alpha\mu + 5\alpha^2\mu + 16\mu^2),$$

$$a^\wedge_{\alpha,0,1,2}(z;\nu) = 4 - 2\alpha + 8\mu + 2\alpha\mu + z(-4 + 18\alpha - 16\alpha^2 + 4\alpha^3 + 16\mu + 10\alpha\mu),$$

$$a^{\wedge}_{\alpha,0,1,3}(z;\nu) = 8 - 2\alpha + \alpha^2 + 8\mu + z(-20 + 28\alpha - 5\alpha^2 - 8\mu),$$

$$a^{\wedge}_{\alpha,0,1,4}(z;\nu) = -(z-1)(12 - 2\alpha),$$

and, in view of (98) – (100),

$$a^{\vee}_{\alpha,0,2,1}(z;\nu) = \frac{z}{2}(-4 + 12\alpha - 13\alpha^2 + 6\alpha^3 - \alpha^4) +$$
$$\frac{z}{2}\mu(-32 + 54\alpha - 29\alpha^2 + 5\alpha^3 - 68\mu + 34\alpha\mu - 6\alpha^2\mu - 24\mu^2),$$

$$a^{\vee}_{\alpha,0,2,2}(z;\nu) = \frac{1}{2}(-1 + 2\alpha - \alpha^2 - 5\mu + 3\alpha\mu - 5\mu^2 - \alpha\mu^2) +$$
$$\frac{z}{2}(-10\alpha + 21\alpha^2 - 14\alpha^3 + 3\alpha^4) +$$
$$\frac{z}{2}\mu(-16 - 20\alpha + 17\alpha^2 - 7\alpha^3 - 48\mu - 28\alpha\mu),$$

$$a^{\vee}_{\alpha,0,2,3}(z;\nu) = -2 + 3\alpha - \alpha^2 - 8\mu + \alpha\mu - \alpha^2\mu - 4\mu^2 +$$
$$\frac{z}{2}(12 - 32\alpha + 25\alpha^2 - 5\alpha^3 - 2\alpha^4) +$$
$$z\mu(20 - 23\alpha - 3\alpha^2 + 4\mu),$$

$$a^{\vee}_{\alpha,0,2,4}(z;\nu) = \frac{1}{2}(-8 + 10\alpha - 3\alpha^2 - \alpha^3 - 24\mu - 4\alpha\mu) +$$
$$\frac{z}{2}(8 - 10\alpha + 3\alpha^2 + \alpha^3 + 24\mu + 4\alpha\mu),$$

$$a^{\vee}_{\alpha,0,3,1}(z;\nu) = \frac{z}{2}(-4 + 12\alpha - 13\alpha^2 + 6\alpha^3 - \alpha^4) +$$
$$\frac{z}{2}\mu(-32 + 54\alpha - 29\alpha^2 + 5\alpha^3 - 76\mu + 44\alpha\mu - 9\alpha^2\mu - \alpha^3\mu - 48\mu^2 - 4\alpha\mu^2),$$

$$a^{\vee}_{\alpha,0,3,2}(z;\nu) = z(-2 + \alpha + 4\alpha^2 - 4\alpha^3 + \alpha^4) +$$
$$z\mu(-16 - \alpha + 7\alpha^2 - 3\alpha^3 - \alpha^4 - 34\mu - 17\alpha\mu - 7\alpha^2\mu - 12\mu^2),$$

$$a^{\vee}_{\alpha,0,3,3}(z;\nu) = \frac{1}{2}(-1 + 2\alpha - \alpha^2 - 5\mu + 3\alpha\mu - 5\mu^2 - \alpha\mu^2) +$$
$$\frac{z}{2}(4 - 16\alpha + 15\alpha^2 - 4\alpha^3 - \alpha^5) +$$
$$\frac{z}{2}\mu(16 - 42\alpha + \alpha^2 - 9\alpha^3 + 8\mu - 20\alpha\mu),$$

$$a^{\vee}_{\alpha,0,3,4}(z;\nu) = -2 + 3\alpha - \alpha^2 - 8\mu + \alpha\mu - \alpha^2\mu - 4\mu^2 +$$
$$z(2 - 3\alpha + \alpha^2 + 8\mu - \alpha\mu + \alpha^2\mu + 4\mu^2),$$

$$a^{\vee}_{\alpha,0,4,1}(z;\nu) = \frac{z}{2}(-4 + 12\alpha - 13\alpha^2 + 6\alpha^3 - \alpha^4) +$$

$$\frac{z}{2}\mu(-32+54\alpha-29\alpha^2+5\alpha^3-80\mu+50\alpha\mu-11\alpha^2\mu-\alpha^3\mu)+$$
$$z\mu^3(-32-\alpha-\alpha^2-4\mu),$$

$$a^{\vee}_{\alpha,0,4,2}(z;\nu)=\frac{z}{2}(-8+14\alpha-5\alpha^2-2\alpha^3+\alpha^4)+$$
$$\frac{z}{2}\mu(-56+32\alpha+\alpha^2-5\alpha^3-2\alpha^4-112\mu-26\alpha\mu-15\alpha^2\mu-5\alpha^3\mu)+$$
$$z\mu^3(-56-20\alpha),$$

$$a^{\vee}_{\alpha,0,4,3}(z;\nu)=\frac{z}{2}(-4+5\alpha^2-2\alpha^3-\alpha^5)+$$
$$\frac{z}{2}\mu(-24-22\alpha-3\alpha^2-9\alpha^3-4\alpha^4-36\mu-42\alpha\mu-18\alpha^2\mu-8\mu^2),$$

$$a^{\vee}_{\alpha,0,4,4}(z;\nu)=\frac{1}{2}(-1+2\alpha-\alpha^2-5\mu+3\alpha\mu-5\mu^2-\alpha\mu^2)+$$
$$\frac{z}{2}(-2\alpha+\alpha^2-\alpha^5)+$$
$$\frac{z}{2}\mu(-8\alpha-5\alpha^2-5\alpha^3-4\alpha\mu),$$

$$a^{\wedge}_{\alpha,0,2,1}(z;\nu)=z(4-8\alpha+5\alpha^2-\alpha^3)+$$
$$z\mu(24-22\alpha+5\alpha^2+28\mu-2\alpha\mu),$$

$$a^{\wedge}_{\alpha,0,2,2}(z;\nu)=1-\alpha+3\mu+\mu^2+$$
$$z(10\alpha-11\alpha^2+3\alpha^3)+$$
$$z\mu(52-38\alpha+25\alpha^2-6\alpha^3-40\mu-24\alpha\mu),$$

$$a^{\wedge}_{\alpha,0,2,3}(z;\nu)=4-2\alpha+8\mu+2\alpha\mu+$$
$$z(-12+20\alpha-5\alpha^2+\alpha^3)+z\mu(-16+14\alpha),$$

$$a^{\wedge}_{\alpha,0,2,4}(z;\nu)=(8-2\alpha+\alpha^2+8\mu)(1-z),$$

$$a^{\wedge}_{\alpha,0,3,1}(z;\nu)=z(4-8\alpha+5\alpha^2-\alpha^3)+$$
$$z\mu(24-22\alpha+5\alpha^2+36\mu-4\alpha\mu+\alpha^2\mu+8\mu^2),$$

$$a^{\wedge}_{\alpha,0,3,2}(z;\nu)=z(4+2\alpha-6\alpha^2+\alpha^3)+$$
$$z\mu(60-40\alpha+24\alpha^2-4\alpha^3+52\mu-10\alpha\mu),$$

$$a^{\wedge}_{\alpha,0,3,3}(z;\nu)=1-\alpha+3\mu+\mu^2+$$
$$z(-4+12\alpha-3\alpha^2+\alpha^3+\alpha^4)+$$
$$z\mu(28-36\alpha+31\alpha^2-6\alpha^3+24\mu-24\alpha\mu),$$

$$a^{\wedge}_{\alpha,0,3,4}(z;\nu) = (4-2\alpha+8\mu+2\alpha\mu)(1-z),$$

$$a^{\wedge}_{\alpha,0,4,1}(z;\nu) = z(4-8\alpha+5\alpha^2-\alpha^3)+$$
$$z\mu(24-22\alpha+5\alpha^2+40\mu-6\alpha\mu+\alpha^2\mu)+z\mu^3(16+2\alpha),$$

$$a^{\wedge}_{\alpha,0,4,2}(z;\nu) = z(8-6\alpha-\alpha^2+\alpha^3)+$$
$$z\mu(76-50\alpha+25\alpha^2-4\alpha^3+72\mu-2\alpha\mu+5\alpha^2\mu)+8z\mu^3,$$

$$a^{\wedge}_{\alpha,0,4,3}(z;\nu) = z(4+4\alpha-\alpha^2-\alpha^4)+$$
$$z\mu(88-86\alpha+59\alpha^2-8\alpha^3+60\mu-38\alpha\mu,$$

$$a^{\wedge}_{\alpha,0,4,4}(z;\nu) = 1-\alpha+3\mu+\mu^2+$$
$$z(2\alpha+\alpha^2+\alpha^3+\alpha^4)+$$
$$z\mu(36-50\alpha+27\alpha^2-6\alpha^3+24\mu-24\alpha\mu),$$

We denote by

$$A^*_{\alpha,0}(z;\nu),\ A^*_{\alpha,0}(z;\nu)^{\vee}(z;\nu),\ A^*_{\alpha,0}(z;\nu)^{\wedge}(z;\nu)$$

the 4×4-matrix, such that its element in i-th row and k-th column is equal respectively to the k-th elements of the rows respecively

$$\bar{a}^*_{\alpha,0,i}(z;\nu),\ \bar{a}^{\vee}_{\alpha,0,i}(z;\nu),\ a^{\wedge}_{\alpha,0,i}(z;\nu)$$

where $i=1,\ ...,\ 4,\ k=1,\ ...,\ 4$

Clearly,

$$A^*_{\alpha,0}(z;\nu) = A^{\vee}_{\alpha,0}(z;\nu)+\tau A^{\wedge}_{\alpha,0}(z;\nu). \tag{101}$$

Let

$$S^{\vee}_0(0,0) = \begin{pmatrix} -1 & -4 & 16 & 24 \\ 0 & -1 & -4 & -8 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & -1 \end{pmatrix},\ V^{\vee}_0(0,0) = \begin{pmatrix} -4 & 4 & -40 & -24 \\ -4 & 0 & 12 & 8 \\ -4 & -4 & 4 & 4 \\ -4 & -8 & -4 & 0 \end{pmatrix},$$

$$S^{\vee}_0(1,0) = \begin{pmatrix} 2 & 6 & -4 & -4 \\ 0 & 2 & 6 & 10 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 2 \end{pmatrix},\ V^{\vee}_0(1,0) = \begin{pmatrix} 12 & -22 & 56 & 4 \\ 12 & -10 & -32 & -10 \\ 12 & 2 & -16 & -2 \\ 12 & 14 & 0 & -2 \end{pmatrix},$$

$$S^{\vee}_0(2,0) = \begin{pmatrix} -1 & -2 & 2 & 0 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & -1 \end{pmatrix},\ V^{\vee}_0(2,0) = \begin{pmatrix} -13 & 34 & -10 & 0 \\ -13 & 21 & 25 & 3 \\ -13 & 8 & 15 & 2 \\ -13 & -5 & 5 & 1 \end{pmatrix},$$

$$S_0^{\vee}(3,0)=\begin{pmatrix}0&0&0&0\\0&0&0&-1\\0&0&0&0\\0&0&0&0\end{pmatrix},\ V_0^{\vee}(3,0)=\begin{pmatrix}6&-20&0&0\\6&-14&-5&1\\6&-8&-4&0\\6&-2&-2&0\end{pmatrix},$$

$$V_0^{\vee}(4,0)=\begin{pmatrix}-1&3&0&0\\-1&3&-2&0\\-1&2&0&0\\-1&1&0&0\end{pmatrix},\ V_0^{\vee}(5,0)=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&-1&0\\0&0&-1&-1\end{pmatrix},$$

$$S_0^{\vee}(0,1)=\begin{pmatrix}-5&-16&16&0\\0&-5&-16&-24\\0&0&-5&-16\\0&0&0&-5\end{pmatrix},\ V_0^{\vee}(0,1)=\begin{pmatrix}-32&-8&-16&0\\-32&-16&40&24\\-32&-32&16&16\\-32&-56&-24&0\end{pmatrix},$$

$$S_0^{\vee}(1,1)=\begin{pmatrix}3&2&0&0\\0&3&2&-4\\0&0&3&-2\\0&0&0&3\end{pmatrix},\ V_0^{\vee}(1,1)=\begin{pmatrix}54&-22&0&0\\54&-20&-46&-4\\54&-2&-42&2\\54&32&-22&-8\end{pmatrix},$$

$$S_0^{\vee}(2,1)=\begin{pmatrix}0&-2&0&0\\0&0&-2&0\\0&0&-1&-2\\0&0&0&0\end{pmatrix},\ V_0^{\vee}(2,1)=\begin{pmatrix}-29&6&0&0\\-29&17&-6&0\\-29&14&1&2\\-29&1&-3&-5\end{pmatrix},$$

$$V_0^{\vee}(3,1)=\begin{pmatrix}5&0&0&0\\5&-7&0&0\\5&-6&-9&0\\5&-5&-9&-5\end{pmatrix},\ V_0^{\vee}(4,1)=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&-2&0&0\\0&-2&-4&0\end{pmatrix},$$

$$S_0^{\vee}(0,2)=\begin{pmatrix}-5&0&0&0\\0&-5&-8&0\\0&0&-5&-8\\0&0&0&-5\end{pmatrix},\ V_0^{\vee}(0,2)=\begin{pmatrix}-56&0&0&0\\-68&-48&8&0\\-76&-68&8&8\\-80&-112&-36&0\end{pmatrix},$$

$$S_0^{\vee}(1,2)=\begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix},\ V_0^{\vee}(1,2)=\begin{pmatrix}20&0&0&0\\34&-28&0&0\\44&-34&-20&0\\50&-26&-42&-4\end{pmatrix},$$

$$V_0^{\vee}(2,2)=\begin{pmatrix}0&0&0&0\\-6&0&0&0\\-9&-14&0&0\\-11&-15&-18&0\end{pmatrix},\ V_0^{\vee}(3,2)=\begin{pmatrix}0&0&0&0\\0&0&0&0\\-1&0&0&0\\-1&-5&0&0\end{pmatrix},$$

$$S_0^{\vee}(0,3)=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\ V_0^{\vee}(0,3)=\begin{pmatrix}0&0&0&0\\-24&0&0&0\\-48&-24&0&0\\-64&-112&-8&0\end{pmatrix},$$

$$V_0^{\vee}(1,3)=\begin{pmatrix}0&0&0&0\\0&0&0&0\\-4&0&0&0\\-2&-40&0&0\end{pmatrix},\ V_0^{\vee}(2,3)=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\-2&0&0&0\end{pmatrix},$$

$$V_0^{\vee}(0,4)=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\-8&0&0&0\end{pmatrix},$$

$$S_0^{\wedge}(0,0)=\begin{pmatrix}1&4&8&12\\0&1&4&8\\0&0&1&4\\0&0&0&1\end{pmatrix},\ V_0^{\wedge}(0,0)=\begin{pmatrix}4&-4&-20&-12\\4&0&-12&-8\\4&4&-4&-4\\4&8&4&0\end{pmatrix},$$

$$S_0^{\wedge}(1,0)=\begin{pmatrix}-1&-1&-4&-2\\0&-1&-2&-2\\0&0&-1&-2\\0&0&0&-1\end{pmatrix},\ V_0^{\wedge}(1,0)=\begin{pmatrix}-8&18&28&2\\-8&10&20&-10\\-8&2&12&2\\-8&-6&4&2\end{pmatrix},$$

$$S_0^{\wedge}(2,0)=\begin{pmatrix}0&0&1&0\\0&0&-2&1\\0&0&0&0\\0&0&0&0\end{pmatrix},\ V_0^{\wedge}(2,0)=\begin{pmatrix}5&-16&-5&0\\5&-11&-5&-1\\5&-6&15&1\\5&-1&-1&1\end{pmatrix},$$

$$V_0^{\wedge}(3,0)=\begin{pmatrix}-1&3&0&0\\-1&3&1&0\\-1&1&-4&1\\-1&1&0&0\end{pmatrix},\ V_0^{\wedge}(4,0)=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&-1&0&1\end{pmatrix},$$

$$S_0^{\wedge}(0,1)=\begin{pmatrix}3&8&8&0\\0&3&8&8\\0&0&3&8\\0&0&0&3\end{pmatrix},\ V_0^{\wedge}(0,1)=\begin{pmatrix}24&16&-8&0\\24&52&-16&-8\\24&76&88&-8\\24&-56&-24&36\end{pmatrix},$$

$$S_0^{\wedge}(1,1)=\begin{pmatrix}0&2&0&0\\0&0&2&-4\\0&0&0&2\\0&0&0&0\end{pmatrix},\ V_0^{\wedge}(1,1)=\begin{pmatrix}-22&10&0&0\\-22&-38&14&0\\-22&-40&-36&-2\\-22&-50&-86&-50\end{pmatrix},$$

$$V_0^{\wedge}(2,1)=\begin{pmatrix}5&0&0&0\\5&25&0&0\\5&24&31&0\\5&25&59&27\end{pmatrix},\ V_0^{\wedge}(3,1)=\begin{pmatrix}0&0&0&0\\0&-6&0&0\\0&-6&-6&0\\0&-4&-8&-6\end{pmatrix},$$

$$S_0^{\wedge}(0,2)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\ V_0^{\wedge}(0,2)=\begin{pmatrix}16&0&0&0\\28&-40&0&0\\36&72&60&0\\40&-112&-36&24\end{pmatrix},$$

$$V_0^{\wedge}(1,2)=\begin{pmatrix}0&0&0&0\\-2&-28&0&0\\-4&-10&-20&0\\-6&-2&-38&-24\end{pmatrix},\ V_0^{\wedge}(2,2)=\begin{pmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\1&5&0&0\end{pmatrix},$$

$$S_{\alpha,0}^{\vee\vee}(\mu)=\left(\sum_{i=0}^{3}\alpha^i S_0^{\vee}(i,0)\right)+$$

$$\left(\sum_{i=0}^{2}\alpha^i\mu S_0^{\vee}(i,1)\right)+\left(\sum_{i=0}^{1}\alpha^i\mu^2 S_0^{\vee}(i,2)\right),$$

$$V_{\alpha,0}^{\vee\vee}(\mu)=\left(\sum_{i=0}^{5}\alpha^i V_0^{\vee}(i,0)\right)+$$

$$\left(\sum_{i=0}^{4}\alpha^i\mu V_0^{\vee}(i,1)\right)+\left(\sum_{i=0}^{3}\alpha^i\mu^2 V_0^{\vee}(i,2)\right)+$$

$$\left(\sum_{i=0}^{2}\alpha^i\mu^3 V_0^{\vee}(i,3)\right)+\mu^4 V_0^{\vee}(0,4),$$

$$S_{\alpha,0}^{\wedge\wedge}(\mu)=\left(\sum_{i=0}^{2}\alpha^i S_0^{\wedge}(i,0)\right)+$$

$$\left(\sum_{i=0}^{1}\alpha^i\mu S_0^{\wedge}(i,1)\right)+\mu^2 S_0^{\wedge}(0,2),$$

$$U_{\alpha,0}^{\vee}(z,\mu)=S_{\alpha,0}^{\vee\vee}(\mu)+zV_{\alpha,0}^{\vee\vee}(\mu), \tag{102}$$

$$V_{\alpha,0}^{\wedge\wedge}(\mu)=\left(\sum_{i=0}^{4}\alpha^i V_0^{\wedge}(i,0)\right)+$$

$$\left(\sum_{i=0}^{3}\alpha^i\mu V_0^{\wedge}(i,1)\right)+$$

$$\left(\sum_{i=0}^{2}\alpha^i\mu^2 V_0^{\wedge}(i,2)\right)+$$

$$\left(\sum_{i=0}^{1}\alpha^i\mu^3 V_0^{\wedge}(i,3)\right),$$

$$U_{\alpha,0}^{\wedge}(z,\mu)=S_{\alpha,0}^{\wedge\wedge}(\mu)+zV_{\alpha,0}^{\wedge\wedge}(\mu), \tag{103}$$

Comparing the above

$$a_{\alpha,0,i,k}^{*}(z;\nu),\ a_{\alpha,0,i,k}^{\vee}(z;\nu),\ a_{\alpha,0,i,k}^{\wedge}(z;\nu)$$

where $i=1,\ ...,\ 4,\ k=1,\ ...,\ 4$, with the elements of the matrices respectively $U_l^{\vee}(z,\mu)$, $U_l^{\wedge}(z,\mu)$, we see that

$$A_{\alpha,0}^{\vee}(z;\nu)=(1/2)U_{\alpha,0}^{\vee}(z;\nu),\ A_{\alpha,0}^{\wedge}(z;\nu)=U_{\alpha,0}^{\wedge}(z;\nu), \tag{104}$$

and, in view of (101) ,

$$A_{\alpha,0}^{*}(z;\nu)=(1/2)U_{\alpha,0}^{\vee}(z;\nu)+U_{\alpha,0}^{\wedge}(z;\nu)\tau.$$

In view of (71), (77), (96),

$$\nu^5 X_{\alpha,0,k}(z;\nu-1) = A^*_{\alpha,0}(z;\nu)X_{\alpha,0,k}(z;\nu), \tag{105}$$

where $\nu \in M^*_\alpha = (-\infty, -1-\alpha] \cup [1, +\infty)) \cap \mathbb{Z}$, $k \in \mathfrak{K}_0$, $|z| > 1$.

Replacing in (105) ν by $-\nu-\alpha$ and taking in the account (95), we see that

$$(-\nu-\alpha)^3 X_{\alpha,0,k}(z;-\nu-\alpha-1) = A^*_{\alpha,0}(z;-\nu-\alpha)X_{\alpha,0,k}(z;-\nu-alpha),$$

and taking in the account (95),

$$(-\nu-\alpha)^5 X_{\alpha,0,k}(z;\nu) = A^*_{\alpha,0}(z;-\nu-\alpha)X_{\alpha,0,k}(z;\nu-1), \tag{106}$$

where $\nu \in M^*_\alpha$, $k \in \mathfrak{K}_0$, $|z| > 1$. Therefore the following equality must be fulfilled:

$$-\nu^5(\nu+\alpha)^5 E_4 = A^*_{\alpha,0}(z;-\nu-\alpha)A^*_{\alpha,0}(z;\nu), \tag{107}$$

where E_4 is the 4×4 unit matrix, $z \in \mathbb{C}$, $\nu \in \mathbb{C}$. This equality opens the possibility for us to check independently the previous calculations. Let

$$S^{**}_{\alpha,0}(\nu) = \frac{1}{2}S^{\vee\vee}_{\alpha,0}(\mu) + \tau V^{\vee\vee}_{\alpha,0}(\mu), \tag{108}$$

$$V^{**}_{\alpha,0}(\nu) = \frac{1}{2}S^{\wedge\wedge}_{\alpha,0}(\mu) + \tau V^{\wedge\wedge}_{\alpha,0}(\mu), \tag{109}$$

where, as before

$$\mu = (\nu+1)(\nu+\alpha),\ \tau = \nu + (1+\alpha)/2.$$

§9.4. Passing from $(\tau,\ \mu)$, to $(\alpha,\ \nu)$ in the case $l = 0$.

In view of (101), (102), (103), (104), (108), (109),

$$A^*_{\alpha,0}(z;\nu) = S^{**}_{\alpha,0}(\nu) + zV^{**}_{\alpha,0}(\nu) \tag{110}$$

As above, $a^*_{\alpha,0,i,k}(z;\nu)$ stands in the matrix $A^*_{\alpha,0}(z;\nu)$ on the intersection of i-th row and k-th column of the matrix $A^*_{\alpha,0}(z;\nu)$, where $\{i,\ k\} \subset \{1,\ 2,\ 3,\ 4\}$. We denote further by $s^{**}_{\alpha,0,i,k}(\nu)$, $v^{**}_{\alpha,0,i,k}(\nu)$ the element, which stands in the matrix respectively $S^{**}_{\alpha,0}(\nu)$, $V^{**}_{\alpha,0}(\nu)$, on the intersection of i-th row and k-th column, where $\{i,\ k\} \subset \{1,\ 2,\ 3\,,4\}$. Since

$$\mu^n = \left(\sum_{i=0}^{n}\binom{n}{i}\alpha^{n-i}\nu^i\right)\left(\sum_{\kappa=0}^{n}\binom{n}{\kappa}\alpha^\kappa\nu^{n-\kappa}\right) =$$

$$\left(\sum_{k=0}^{2n}\left(\sum_{i=0}^{k}\binom{n}{i}\binom{n}{k-i}\alpha^{n-i}\right)\nu^k\right) =$$

$$\left(\sum_{k=0}^{2n}\left(\sum_{\kappa=\max(n-k,0)}^{\min(2n-k,n)}\binom{n}{\kappa}\binom{n}{k+\kappa-n}\alpha^{\kappa}\right)\nu^k\right),$$

$$\mu=\alpha+\nu+\alpha\nu+\nu^2=\alpha+(1+\alpha)\nu+\nu^2,$$

$$\mu^2=\alpha^2+2\alpha\nu+2\alpha^2\nu+\nu^2+4\alpha\nu^2+\alpha^2\nu^2+$$
$$2\nu^3+2\alpha\nu^3+\nu^4=$$
$$\alpha^2+2(1+\alpha)\alpha\nu+(1+4\alpha+\alpha^2)\nu^2+2(1+\alpha)\nu^3+\nu^4,$$

$$\mu^3=\alpha^3+3\alpha^2\nu+3\alpha^3\nu+3\alpha\nu^2+9\alpha^2\nu^2+$$
$$3\alpha^3\nu^2+\nu^3+9\alpha\nu^3+9\alpha^2\nu^3+\alpha^3\nu^3+3\nu^4+$$
$$9\alpha\nu^4+3\alpha^2\nu^4+3\nu^5+3\alpha\nu^5+\nu^6=$$
$$\alpha^3+3(1+\alpha)\alpha\nu+3(1+3\alpha+\alpha^2)\nu^2+$$
$$(1+9\alpha+9\alpha^2+\alpha^3)\nu^3+3(1+3\alpha+\alpha^2)\nu^4+$$
$$3(1+\alpha)\nu^5+\nu^6,$$

$$\mu^4=\alpha^4+4\alpha^3\nu+4\alpha^4\nu+6\alpha^2\nu^2+16\alpha^3\nu^2+$$
$$6\alpha^4\nu^2+4\alpha\nu^3+24\alpha^2\nu^3+24\alpha^3\nu^3+4\alpha^4\nu^3+$$
$$\nu^4+16\alpha\nu^4+36\alpha^2\nu^4+16\alpha^3\nu^4+\alpha^4\nu^4+$$
$$4\nu^5+24\alpha\nu^5+24\alpha^2\nu^5+4\alpha^3\nu^5+$$
$$6\nu^6+16\alpha\nu^6+6\alpha^2\nu^6+4\nu^7+4\alpha\nu^7+\nu^8=$$
$$\alpha^4+4(\alpha+1)\alpha^3\nu+2(3+8\alpha+3\alpha^2)\alpha^2\nu^2+$$
$$4(1+6\alpha+6\alpha^2+\alpha^3)\alpha\nu^3+$$
$$(1+16\alpha+36\alpha^2+16\alpha^3+\alpha^4)\nu^4+$$
$$4(1+6\alpha+6\alpha^2+\alpha^3)\nu^5+2(3+8\alpha+3\alpha^2)\nu^6+$$
$$4(1+\alpha)\nu^7+\nu^8,$$

$$\tau=\frac{1}{2}(1+\alpha)+\nu,$$

$$\mu\tau=\frac{1}{2}\alpha(1+\alpha)+\left(\alpha+\frac{1}{2}(1+\alpha)^2\right)\nu+$$
$$\left(\frac{3}{2}(1+\alpha)\right)\nu^2+\nu^3=$$
$$\frac{1}{2}(\alpha(1+\alpha)+(1+4\alpha+\alpha^2)\nu+3(1+\alpha)\nu^2)+\nu^3,$$

$$\mu^2\tau=\frac{1}{2}(1+\alpha)\alpha^2+\alpha(1+3\alpha+\alpha^2)\nu+$$
$$\frac{1}{2}(1+9\alpha+9\alpha^2+\alpha^3)\nu^2+2(1+3\alpha+\alpha^2)\nu^3+$$
$$\frac{5}{2}(1+\alpha)\nu^4+\nu^5$$

$$\mu^3\tau = \frac{1}{2}(1+\alpha)\alpha^3 + \frac{1}{2}(3+8\alpha+3\alpha^2)\alpha^2\nu +$$
$$\frac{3}{2}(1+6\alpha+6\alpha^2+\alpha^3)\alpha)\nu^2 +$$
$$\frac{1}{2}(1+16\alpha+36\alpha^2+16\alpha^3+\alpha^4)\nu^3 +$$
$$\frac{5}{2}(1+6\alpha+6\alpha^2+\alpha^3)\nu^4 +$$
$$\frac{3}{2}(3+8\alpha+3\alpha^2)\nu^5 + \frac{7}{2}(1+\alpha)\nu^6 + \nu^7,$$

it follows that

(111) $$s^{**}_{\alpha,0,1,1}(\nu) = s^{**}_{\alpha,0,2,2}(\nu) =$$
$$s^{**}_{\alpha,0,3,3}(\nu) = s^{**}_{\alpha,0,4,4}(\nu) = \nu^3(\nu+\alpha)^2,$$

(112) $$s^{**}_{\alpha,0,1,2}(\nu) = s^{**}_{\alpha,0,2,3}(\nu) =$$
$$s^{**}_{\alpha,0,3,4}(\nu) = -2\nu^2(2\nu+\alpha)(\nu+\alpha),$$

(113) $$s^{**}_{\alpha,0,1,3}(\nu) = s^{**}_{\alpha,0,2,4}(\nu) = \nu(2\nu+\alpha)(4\nu+3\alpha),$$

(114) $$s^{**}_{\alpha,0,1,4}(\nu) = -2(2\nu+\alpha)(3\nu+2\alpha),$$

(115) $$s^{**}_{\alpha,0,2,1}(\nu) = s^{**}_{\alpha,0,3,1}(z;\nu) = s^{**}_{\alpha,0,3,2}(\nu) =$$
$$s^{**}_{\alpha,0,4,1}(\nu) = s^{**}_{\alpha,0,4,2}(\nu) = s^{**}_{\alpha,0,4,3}(\nu) = 0,$$

(116) $$v^{**}_{\alpha,0,1,1}(\nu) = (\nu+\alpha)^2(2\nu+\alpha)(8\nu^2+(6+5\alpha)\nu+4\alpha),$$

(117) $$v^{**}_{\alpha,0,1,2}(\nu) =$$
$$2(\nu+\alpha)(2\nu+\alpha)(-5\nu^2-6\nu-4\alpha+2\alpha^2),$$

(118) $$v^{**}_{\alpha,0,1,3}(\nu) =$$
$$-(2\nu+\alpha)(4\nu^2-6\nu+15\nu\alpha-4\alpha+8\alpha^2),$$

(119) $$v^{**}_{\alpha,0,1,4}(\nu) = 2(2\nu+\alpha)(3\nu+2\alpha),$$

(120) $$v^{**}_{\alpha,0,2,1}(\nu) =$$
$$-\nu(\nu+\alpha)^2(2\nu+\alpha)(6\nu^2+4(1+\alpha)\nu+3\alpha),$$

(121)
$$v^{**}_{\alpha,0,2,2}(\nu) =$$
$$-\nu(\nu+\alpha)(2\nu+\alpha)(-8\nu^2-8\nu-\alpha\nu-6\alpha+3\alpha^2),$$

(122)
$$v^{**}_{\alpha,0,2,3}(\nu) = \nu(2\nu+\alpha)(2\nu^2-4\nu+10\alpha\nu-3\alpha+6\alpha^2),$$

(123)
$$v^{**}_{\alpha,0,2,4}(\nu) = -\nu(2\nu+\alpha)(4\nu+3\alpha),$$

(124)
$$v^{**}_{\alpha,0,3,1}(\nu) =$$
$$\nu^2(\nu+\alpha)^2(2\nu+\alpha)(4\nu^2+(2+3\alpha)\nu+2\alpha),$$

(125)
$$v^{**}_{\alpha,0,3,2}(\nu) =$$
$$2\nu^2(\nu+\alpha)(2\nu+\alpha)(-3\nu^2-2\nu-\alpha\nu-2\alpha+\alpha^2),$$

(126)
$$v^{**}_{\alpha,0,3,3}(\nu) = -\nu^2(2\nu+\alpha)(-2\nu+5\alpha\nu-2\alpha+4\alpha^2),$$

(127)
$$v^{**}_{\alpha,0,3,4}(\nu) = 2\nu^2(2\nu+\alpha)(\nu+\alpha),$$

(128)
$$v^{**}_{\alpha,0,4,1}(\nu) = -\nu^3(\nu+\alpha)^2(2\nu+\alpha)(2\nu^2+2\alpha\nu+\alpha),$$

(129)
$$v^{**}_{\alpha,0,4,2}(\nu) =$$
$$-\nu^3(\nu+\alpha)(2\nu+\alpha)(-4\nu^2-3\alpha\nu-2\alpha+\alpha^2),$$

(130)
$$v^{**}_{\alpha,0,4,3}(\nu) = \nu^3(2\nu+\alpha)(-2\nu^2-\alpha+2\alpha^2),$$

(131)
$$v^{**}_{\alpha,0,4,4}(\nu) = -\nu^3\alpha(2\nu+\alpha),$$

In view of (110), the equality (107) is equivalent to the following system of equalities

(132)
$$\begin{cases} S^{**}_{\alpha,0}(\nu)S^{**}_{\alpha,0}(-\nu-\alpha) = -\nu^3(\nu+\alpha)^3E_4, \\ S^{**}_{\alpha,0}(\nu)V^{**}_{\alpha,0}(-\nu-\alpha) + V^{**}_{\alpha,0}(\nu)S^{**}_{\alpha,0}(-\nu-\alpha) = 0E_4 \\ V^{**}_{\alpha,0}(z;\nu)V^{**}_{\alpha,0}(z;-\nu-\alpha) = 0E_4. \end{cases}$$

We shall check the equalities (132) in the next part.

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