The Combinatorics of Iterated Loop Spaces.

M.A. Batanin*
Macquarie University, North Ryde, NSW 2109, Australia
e-mail: mbatanin@math.mq.edu.au

Abstract

It is well known since Stasheff's work that 1-fold loop spaces can be described in terms of the existence of higher homotopies for associativity (coherence conditions) or equivalently as algebras of contractible non-symmetric operads. The combinatorics of these higher homotopies is well understood and is extremely useful.

For $n \geq 2$ the theory of symmetric operads encapsulated the corresponding higher homotopies, yet hid the combinatorics and it has remain a mystery for almost 40 years. However, the recent developments in many fields ranging from algebraic topology and algebraic geometry to mathematical physics and category theory show that this combinatorics in higher dimensions will be even more important than the one dimensional case.

In this paper we are going to show that there exists a conceptual way to make these combinatorics explicit using the so called higher nonsymmetric n-operads.

1 Introduction.

1.1 Preoperadic history of the subject.

For the decade beginning around 1955 the question of characterising of loop spaces through algebraic structures was a hot subject in topology. A very nice solution for 1-fold loop spaces was provided by J.Stasheff [26, 27]. Let us sketch it briefly.

Let X be a pointed space with the based point e. Suppose also we have a multiplication

$$\mu : X \times X \to X,$$

$$\mu(a, b) = ab$$

For simplicity we assume also that e is a two sided unit of this multiplication. Then the first condition will be the existence of a homotopy

$$\mu(\mu \times 1) \sim \mu(1 \times \mu).$$

*The author holds the Scott Russell Johnson Fellowship in the Centre of Australian Category Theory at Macquarie University

In other words we have a map

$$\mu_3 : K_3 \times X^3 \to X,$$

where K_3 is the interval $[0,1]$, which provides a path

$$a(bc) \;\text{———}\; (ab)c$$

for any $a, b, c \in X$.

We should not stop here. If we consider all possible bracketings of 4 symbols we thus have a pentagon of homotopies

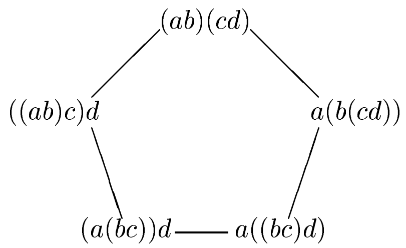


Figure I:

Our next condition is: we should be able to extend these homotopies to the map

$$\mu_4 : K_4 \times X^4 \to X,$$

where K_4 is the pentagon above. It is clear how to proceed now. In general there exists a sequence of convex polytopes K_n for all n ($K_0 = K_1 = K_2 = \star$) called associahedra. The vertices of K_n correspond to all binary bracketings of a string of n letters.

Definition 1.1 (Stasheff [26, 27]) *A connected pointed space X with a multiplication μ is called an A_∞-space if there exists a sequence of continuous maps*

$$\mu_n : K_n \times X^n \to X$$

where μ_n is an extension of a map from the boundary of $K_n \times X^n$ which can be constructed from low dimensional μ_d.

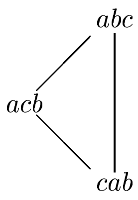
Theorem 1.1 (Stasheff [26, 27]) *A connected topological space X is a 1-fold loop space if and only if it admits a structure of an A_∞-space.*

Stasheff's approach to recognition of loop spaces turned out to be exceptionally fruitful and was and still is a source of inspiration for many breakthrough discoveries.

What about double loop spaces? It is clear that the multiplication in a double loop space should be homotopy commutative. For simplicity, let us suppose that it is strictly associative. So we have a homotopy:

$$ab \;\text{———}\; ba$$

Next, we should have a homotopy filling in the triangle



and also

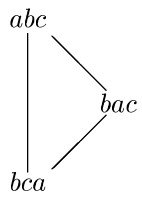
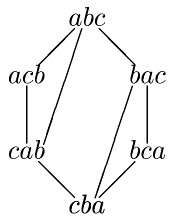


Figure II:

These homotopies and the homotopy for commutativity allow us to fill in the hexagon in two essentially different ways (compare with two different methods of proving the Yang-Baxter equation in a braided monoidal category).



and

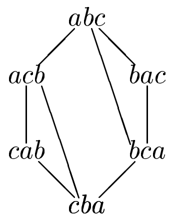


Figure III:

So we should be able to find a homotopy which fills in the three dimensional ball with boundary subdivided according to the above pictures (S.Crans has considered such an axiom for his theory of teisi in [13]).

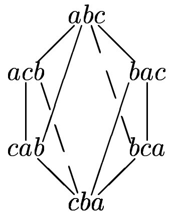


FIGURE IV:

If we try to proceed further in this direction the situation quickly becomes unmanageable. Stasheff himself noticed that there should be a way to write down these conditions explicitly but 'it is tediously difficult' [28].

1.2 Operads and homotopy coherence.

To handle this situation the language of operads was invented by M.Boardman, R.Vogt and P.May [12, 23].

Definition 1.2 *A nonsymmetric (topological) operad is a sequence of topological spaces*

$$A_0, A_1, A_2, \ldots$$

(called the underlying collection) together with unit $e \in A_1$ and multiplication

$$\mu : A_k \times A_{i_1} \times \ldots \times A_{i_k} \to A_{i_1 + \ldots + i_k}$$

which satisfies associativity and unitarity conditions.

The nonsymmetric operads form a category. The morphisms are levelwise morphisms of underlying collections which preserve multiplication and units.

There are two important examples. The first is the endomorphism operad $E(X)$ of a topological space X:

$$E(X)_n = Top(X^n, X).$$

The unit is given by the identity morphism and multiplication by iterated composition of n-ary maps. The second example is the sequence of Stasheff's associahedra. Here multiplication is given by an appropriate inclusion of $K_k \times K_{i_1} \times \ldots \times K_{i_k}$ to the boundary of $K_{i_1+\ldots+i_k}$ defined by Stasheff [26, 27].

Definition 1.3 *An algebra of an operad A is a topological space X equipped with a morphism of operads*

$$A \to End(X).$$

The Stasheff's recognition principle for a topological space X can be formulated now as the existence of a K-algebra structure on X. Moreover, one can use any other contractible nonsymmetric operad for the recognition principle. The operad K is then an initial (up to higher homotopies) object in the category of contractible nonsymmetric operads (if we ignore units). So the language of nonsymmetric operads is esentially equivalent to the direct combinatorial approach of Stasheff.

However, for the study of homotopy commutativity the nonsymmetric operads are not enough.

Definition 1.4 *A symmetric (topological) operad is a sequence of topological spaces*

$$A_0, A_1, A_2, \ldots$$

(called the underlying collection) together with unit $e \in A_1$ and multiplication

$$\mu : A_k \times A_{i_1} \times \ldots \times A_{i_k} \to A_{i_1+\ldots+i_k}$$

and actions of the symmetric groups S_n on A_n which satisfy associativity, unitarity and equivariancy conditions.

The endomorphism operad has a natural action of the symmetric groups and can be completed to a symmetric operad structure. It is now easy to define algebras of a symmetric operads.

Theorem 1.2 (May [23]) *A connected topological space X is an infinite loop space if and only if it has an action of a contractible symmetric operad.*

For finite values of n there are symmetric operads which detect n-fold loop spaces: the so called little n-cube operads $\mathcal{C}^n$ and anything equivalent to them (*E_n-operads*). Yet, they are very complicated from a homotopy point of view: $\mathcal{C}^n_k$ is homotopy equivalent to the space of configurations of k-distinct points in n-dimensional real space. *There is no known characterisation of E_n-operads similar to the characterisation of E_1 and E_∞ operads except for $n = 2$* [14].

Now, if we try to write down explicitly the coherence laws for an n-fold loop space using the action of an E_n-operad we will have the same trouble as in the previous subsection. So the approach based on symmetric operads, being very powerful in many respects, still tells almost nothing about the combinatorics of higher homotopies for $n > 1$. We quote J.Baez and P.May: '...work of Boardman and Vogt, May, and Segal gave conceptual encapsulations that hid the implicit higher homotopies, whose combinatorial structure is still somewhat obscure' [2].

1.3 The approach of this paper.

The main goal of the paper is to give a conceptual approach to the combinatorics of n-fold loop spaces. It has two essential ingredients.

First we suggest to replace symmetric operads by higher nonsymmetric operads which appeared in higher category theory [6]. The intuition behind this approach is that E_n-space must be equivalent to *n-tuple weak ω-groupoid* i.e. a weak ω-groupoid with one object, one arrow, ..., one $(n-1)$-arrow, as have been conjectured by J.Baez and J.Dolan [1]. The decisive step to make this intuition precise was made by us in [7]. Here we already proved that in this theory we have a simple characterisation of so called $(n-1)$-terminal n-operads (see Definition 2.3) which detect n-fold loop spaces: they are just the contractible n-operads.

The second ingredient is the idea to consider some sort of 'universal categorical model' for a given theory of operads. To explain this better let us recall a well known fact that the poset of faces of the associahedron can be described in terms of planar trees. Moreover, the planar trees can be organised into a category, and even a categorical nonsymmetric operad $\mathbf{k}$ such that the classifying space of $\mathbf{k}$ is exactly K [25]. The classical tree formalism is based on the observation that every operad produces a functor on $\mathbf{k}$. In [7] we observed that the existence of this formalism is due to the fact that the categorical operad $\mathbf{k}$ is actually a categorical nonsymmetric operad (without nullary operations) freely generated by an internal nonunital nonsymmetric operad. Actually, in [7] we work with the full operadic structure with nullary opertions and units and we construct a categorical symmetric operad $\mathbf{h}^1$ freely generated by an internal nonsymmetric operad. It is easy to make appropriate modifications if we want to produce $\mathbf{k}$.

In some sense the combinatorics of E_1-spaces is governed by the categorical operad $\mathbf{k}$ or $\mathbf{h}^1$ if we want to work with the full structure including homotopy units.

Using these ideas we show that there exist appropriate n-dimensional analogues of $\mathbf{h}^1$ and the combinatorics of E_n-spaces is governed by this categorical n-operad. Even though this combinatorics is much more complicated than in Stasheff's case we show that there is no obstacle to writing it down explicitly.

We also consider the relationships between n-operads and symmetric operads more closely. In [7] we established an adjunction

$$Des_n : SOper(W) \leftrightarrows Oper_n(\Sigma^n W) : EH_n$$

between these two categories of operads, which is the usual adjunction between symmetric and nonsymmetric operads in a symmetric monoidal category W when $n=1$. In this paper we define model category structures on the categories of topological $(n-1)$-terminal n-operads and topological symmetric operads and show that the adjunction above gives rise to a Quillen adjunction [16].

Using this techniques we demonstrate that the classifying space $B(\mathbf{h}^n)$ of the categorical operad $\mathbf{h}^n$ constructed in [7] is a cofibrant replacement for $B(\mathcal{M}^n)$, where $\mathcal{M}^n$ is the operad of iterated monoidal categories introduced in [3].

This allows us to look closer at the combinatorial structure of E_n-operads and we show that such an operad is actually homotopically freely generated by its internal n-operad.

In the last section we illustrate on some examples how this theory can be applied to produce some coherence laws. We are going to develop this theme in a further paper.

2 Higher operads.

2.1 Trees, their morphisms and n-operads.

In this section we give a brief overview of the theory developed in [6, 7] to make our paper relatively self contained. We do not give full definitions of monoidal globular category and n-operad here because we need only some properties of them established in [6, 7].

For a natural number n we will denote by $[n]$ the ordinal

$$1 \ < \ 2 \ < \ \dots \ < n.$$

In particular $[0]$ will denote the empty ordinal.

Definition 2.1 *A tree of height n (or simply n-tree) is a chain of order preserving maps of ordinals*

$$T = [k_n] \xrightarrow{\rho_{n-1}} [k_{n-1}] \xrightarrow{\rho_{n-2}} \dots \xrightarrow{\rho_0} [1]$$

If $i \in [k_m]$ and there is no $j \in [k_{m+1}]$ such that $\rho_m(j) = i$ then we call i *a leaf of T of height i*. We will call the leaves of T of height n *the tips* of T. If for an n-tree T all its leaves are tips we call such a tree *pruned.*

We illustrate the definition in a picture

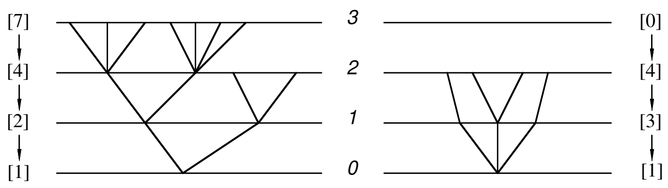


FIGURE V:

The tree on the left side of the picture is not pruned since it has two leaves which are not tips. The tree on the right side has the empty ordinal at the highest level; we will call such trees *degenerate.* There is actually an operation on trees which we denote by $z(-)$ which assigns to the n-tree $[k_n] \to [k_{n-1}] \to \dots \to [1]$ the $(n+1)$-tree

$$[0] \longrightarrow [k_n] \longrightarrow [k_{n-1}] \longrightarrow \dots \longrightarrow [1].$$

Another operation $\partial(-)$ on trees is *truncation*

$$\partial([k_n] \to [k_{n-1}] \to \dots \to [1]) = ([k_{n-1}] \to \dots \to)[1].$$

We now define both the source and target of a tree T to be equal to $\partial(T)$. So we have a globular structure on the set of all trees. We actually have more. The trees form an ω-category Tr with the set of n-cells being equal to the set of the trees of height n. If two n-trees S and T have the same k-sources and k-targets (i.e. $\partial^{n-k}T = \partial^{n-k}S$) then they can be composed, and the composite will be denoted by $S \otimes_k T$. Then $z(T)$ is the operation of taking the identity of the n-cell T. Here is an example of the 2-categorical operations on trees

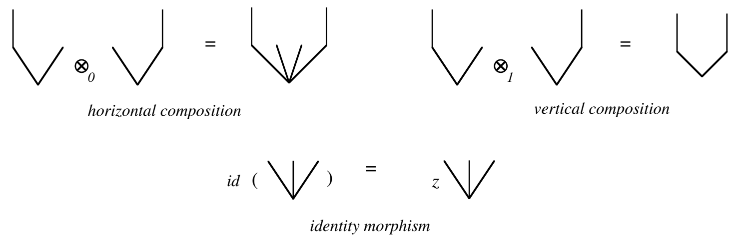


FIGURE VI:

The ω-category Tr is actually the free ω-category generated by the terminal globular set. Every n-tree can be considered as a special sort of n-pasting diagram called *globular.* This construction was called the $\star$-construction in [6]. Here are a couple of examples.

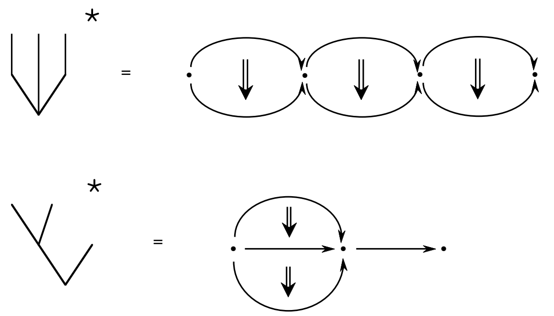


FIGURE VII:

For a globular set X one can then form the set $D(X)$ of all globular pasting diagrams labelled in X. This is the free ω-category generated by X. In this way we have a monad (D, μ, ϵ) on the category of globular sets, which plays a central role in [6].

In particular, $D(1) = Tr$. We also can consider $D(Tr) = D^2(1)$. It was observed in [6] that the n-cells of $D(Tr)$ can be identified with the morphisms of another category introduced by A.Joyal in [18]. This category was called Ω_n and some properties of it were studied later in [9]. More precisely, it was found that the collection of categories Ω_n forms an ω-category in Cat and, moreover, it is freely generated by an internal ω-category. So it is a higher dimensional analogue of the algebraic simplicial category Δ (which is of course the free monoidal category generated by a monoid [24]). The Ω_n will be of primary importance for us.

Definition 2.2 *The category Ω_n has as objects the trees of height n. The morphisms of Ω_n are commutative diagrams*

$$\begin{tikzcd}
{[k_n]} \arrow[r, "\rho_{n-1}"] \arrow[d, "\sigma_n"'] & {[k_{n-1}]} \arrow[r, "\rho_{n-2}"] \arrow[d, "\sigma_{n-1}"'] & \cdots \arrow[r, "\rho_0"] & {[1]} \arrow[d, "\sigma_0"'] \\
{[s_n]} \arrow[r, "\xi_{n-1}"] & {[s_{n-1}]} \arrow[r, "\xi_{n-2}"] & \cdots \arrow[r, "\xi_0"] & {[1]}
\end{tikzcd}$$

where vertical maps are not necessary order preserving but for all i and all $j \in [k_{i-1}]$ the restriction of σ_i on $\rho_{i-1}^{-1}(j)$ preserves the natural order on it.

The category Ω_n has terminal object

$$U_n = [1] \longrightarrow \ldots \longrightarrow [1].$$

Let T be an n-tree and let i be a leaf of height m of T. Then i determines a unique morphism $\xi_i : z^{n-m}U_m \to T$ in Ω_n such that $\xi_m(1) = i$. We will often identify the leaf with this morphism.

Let $\sigma : T \to S$ be a morphism in Ω_n and let i be a leaf of T . Then *the fiber of σ over i* is the following pullback in Ω_n

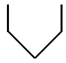
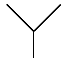
$$\begin{tikzcd}
\sigma^{-1}(i) \arrow[r] \arrow[d] & z^{n-m}U_m \arrow[d, "\xi_i"] \\
T \arrow[r, "\sigma"] & S
\end{tikzcd}$$

which can be calculated as a levelwise pullback in Set.

Now, for such a $\sigma : T \to S$ one can construct a labelling of the pasting scheme $S^\star$ in the ω-category Tr by associating to a vertex i from S the fiber of σ over i. The result of the pasting operation will be exactly T. The inverse process also works i.e. every pasting diagram of trees determines a unique morphism in

Ω_n. Because of this we can specify a morphism of trees by the list of its fibers. For example, the two diagrams

FIGURE VIII:

determine two morphisms in Ω_2 from  to . At level 2 the first morphism is an identity while the second is the switch map. Analogously we can present a chain of morphisms $T_1 \stackrel{\sigma_1}{\to} T_2 \stackrel{\sigma_2}{\to} \dots \stackrel{\sigma_{m-1}}{\to} T_m$ by specifying fibers of σ_{m-1}, fibers of σ_{m-2} over fibers of σ_{m-1} and so on.

Now an n-operad in an augmented monoidal n-globular category V can be defined as a collection of objects $A_T,\ T \in Tr_k\ ,\ 0 \le k \le n$, which respects source and target functors (*n-collection*), together with an appropriate unit and multiplication which make A a monoid in the monoidal category of n-collections [6].

In any augmented monoidal n-globular category V there exists an operad $\mathrm{T}(V)$ which algebras are *n-globular monoids in V* [9]. In a particular case $V = Span(C)$, where C is cartesian closed category, this operads is just the terminal operad (and the n-globular monoids are internal n-categories in C).

A funny but important example of an n-operad in the augmented monoidal globular category $Span(CAT)$ (*categorical n-operad for simplicity*) is the following. Let V be a strict augmented n-globular monoidal category [6]. Then we put

$$op(V)_T = V_n,\ T \in Tr_n.$$

The multiplication in $op(V)$ is given by iterated tensor products in V and the unit object is given by $z^k(I)$ where I is the unit of V_0.

2.2 Tree formalism.

In [7] we showed that the tree formalism for operads is just another way to represent the corresponding category of operads.

There exists a categorical n-operad $\mathcal{H}^n$ which represents the category of n-operads in the following sense.

There are two inverse natural isomorphisms of categories

$$Oper_n(V) \leftrightarrows CATOper_n(\mathcal{H}^n, op(V)).$$

Here $Oper_n(V)$ is the category of n-operads in V and $CATOper_n(A,B)$ is the category of operadic functors from a categorical n-operad A to a categorical n-operad B and their operadic natural transformations.

We can describe the operad $\mathcal{H}^n$ explicitly. An object of arity $T \in Tr_n$ of $\mathcal{H}^n$ is a chain of morphisms

$$\delta = (T \to T_1 \to \ldots \to T_m \to U_n)$$

in Ω_n. There is a morphism $\delta \to \delta'$ if δ' can be obtained from δ by a chain of composites of several morphisms in the chain δ and of insertion identity morphisms. The category $\mathcal{H}^n_T$ is contractible since $(T \to U_n)$ is the terminal object of it.

Every augmented monoidal globular category V has its $(n-1)$-trancation $tr_{n-1}V$ defined by $(tr_{n-1}V)_k = V_k,\ 0 \geq k \leq n-1$. Analogously we can trancate an n-operad by considering its restriction on the trees of height less or equal to $n-1$.

Definition 2.3 *An n-operad is called $(n-1)$-terminal if its $(n-1)$-trancation is* $\mathrm{T}(tr_{n-1}V)$ *[7].*

Remark 2.1 If $V = \Sigma(W)$ for a braided monoidal category W the categories of 1-operads in V and of 0-terminal 1-operads in V coincide and are isomorphic to the category of nonsymmetric operads in W [7]. If $V = Span(C)$ and $n = 1$ the category of 0-terminal 1-operads in V is the same as the category of nonsymmetric operads in C. However, in general the category of 1-operads is larger than the category of 0-terminal 1-operads.

In [7] we introduced the categorical n-operad $\mathbf{H}^n$ which represents $(n-1)$-terminal n-operads in the same sense as $\mathcal{H}^n$ represent all n-operads. The operad $\mathbf{H}^n$ is also a contractible n-operad. Let V have globular colimits [6] and let $Oper_n^{(n-1)}(V)$ denote the category of $(n-1)$-terminal n-operads in V. Then we have a pair of adjoint functors

$$\tau_n : Oper_n^{(n-1)}(V) \leftrightarrows Oper_n(V) : \lambda_n,$$

where the right adjoint τ_n is the obvious inclusion.

Also in [7] we constructed a categorical symmetric operad $\mathbf{h}^n$ which represents the category of n-operads in a symmetric categorical operad. The operad $\mathbf{h}^n$ has the homotopy type of the little n-cube operad. We also constructed a pair of adjoints $Des_n \vdash EH_n$:

$$Des_n : SOper(W) \leftrightarrows Oper_n(\Sigma^n W) : EH_n$$

which generalizes the classical adjunction between symmetric and nonsymmetric operads

$$SOper(W) \leftrightarrows Oper_1(\Sigma W).$$

Here $SOper(W)$ is the category of symmetric operads in a symmetric monoidal category W,

In the particular case $V = Span(C)$ [6], where C is a cartesian closed category, we can identify $Oper_n(\Sigma^n C)$ with $Oper_n^{(n-1)}(Span(C))$ so we have a chain of adjunctions

$$SOper(C) \leftrightarrows Oper_n^{(n-1)}(Span(C)) \leftrightarrows Oper_n(Span(C)).$$

Moreover, we have a theorem

Theorem 2.1 ([7]) *The functors τ_n and Des_n preserve endomorphism operads, so the category of $(n-1)$ terminal algebras of an n-operad A (i.e. algebras with terminal $(n-1)$-skeleton) is isomorphic to the category of algebras of the symmetric operad $EH_n(\lambda_n(A))$.*

Finally, it was proved in [7][Example 10.2] that

$$EH_n(\mathbf{H}^n) \simeq \mathbf{h}^n, \tag{1}$$

and argument uses only the universal properties of $\mathbf{H}^n$ and $\mathbf{h}^n$. Analogously, one can easily show that

$$\lambda(\boldsymbol{\mathcal{H}}^n) \simeq \mathbf{H}^n. \tag{2}$$

Remark 2.2 According to (1) the operad $\mathbf{h}^1$ is just a symmetrisation of the nonsymmetric operad $\mathbf{H}^1$. Yet, the operad $\boldsymbol{\mathcal{H}}^1$ is different from $\mathbf{H}^1$. Roughly speaking, $\mathbf{H}^1$ is the classical tree operad (if we ignore units) with morphisms being contractions of internal edges [25]. The objects of the operad $\boldsymbol{\mathcal{H}}^1$ are 'regular trees' in the sense that they are constructed from corollas

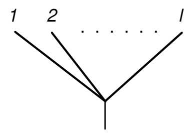


FIGURE IX:

by **simultaneous** grafting at every vertex. The morphisms are simultaneous contraction of all internal edges on a given level. Every 'regular tree' has its length and so we have a map l from the set of regular trees to the set of natural numbers. The monoid of natural numbers is actually a part of the structure of $\boldsymbol{\mathcal{H}}^1$; it is the category $\boldsymbol{\mathcal{H}}^1_{U_0}$. The map l is the map of source and target in $\boldsymbol{\mathcal{H}}^1$. The substitution in $\boldsymbol{\mathcal{H}}^1$ is possible only if source is equal to target and, therefore, does not destroys regularity of trees.

3 Model structure on the category of $(n-1)$-terminal n-operads

In this section we adopt the theory of [11] to the case of $(n-1)$-terminal n-operads. Let V be a cartesian closed model category that is cofibrantly generated and has the terminal object cofibrant, and let V have a symmetric monoidal fibrant replacement functor. Then the category of $(n-1)$-terminal n-collections has an obvious fiberwise model structure. So we define a fibration (weak equivalence) of $(n-1)$-terminal n-operads to be an operadic morphism such that its underlying morphism is a fibration (weak equivalence) of n-collections. The argument from [11] works without changes, and we have:

Proposition 3.1 *The category of $(n-1)$-terminal n-operads in V is a cofibrantly generated model category.*

Of special interest for us is $V = Top$, the category of compactly generated Hausdorf topological. The weak equivalences are weak homotopy equivalences and the fibrations are Serre fibrations. In this case we can define the bar-construction $B(F,F,X)$ for an n-operad X, where F is the free $(n-1)$-terminal n-operad functor.

Theorem 3.1 *Let X be an $(n-1)$-terminal n-operad with cofibrant underlying n-collection. Then the canonical operad morphism*

$$B(F,F,X) \longrightarrow X$$

is a cofibrant replacement for X.

Proof. We have to prove that $B(F,F,X)$ is a cofibrant n-operad. Let $f : E \to B$ be a trivial fibration of n-operads.

We have to prove that any operadic map $B(F,F,X) \to B$ can be lifted to E

$$\begin{tikzcd} & E \arrow[d, "f"] \\ B(F,F,X) \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

By construction this amounts to the following lifting problem in the category of cosimplicial spaces

$$\begin{tikzcd} & Oper_n^S(F^\star X, E) \arrow[d, "f^\star"] \\ \Delta \arrow[r] \arrow[ur, dashed] & Oper_n^S(F^\star X, B) \end{tikzcd}$$

Here Δ is the cosimplicial simplicial set consisting of standard simplices and $Oper_n^S$ means the simplicial enriched Hom-functor on the category of $(n-1)$-terminal n-operads. Since Δ is cofibrant in the Reedy model structure [16] it remains to show that $f^\star$ is a trivial fibration.

We follow a method developed in [5]. We have to prove that in the diagram

$$\begin{tikzcd}
Oper_n^S(F^{i+1}X,E) \arrow[dr, "\omega_i"] \arrow[drr] \arrow[ddr] & & \\
& W_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & M_i(Oper_n^S(F^{\star}X,E)) \arrow[d, "M_i f^{\star}"] \\
& Oper_n^S(F^{i+1}X,B) \arrow[r] & M_i(Oper_n^S(F^{\star}X,B))
\end{tikzcd}$$

the canonical map to the pullback is a trivial fibration. In this diagram $M_i(-)$ is the i-th matching object of the corresponding cosimplicial object [16].

According to Lemma 2.3 from [5] the diagram above is isomorphic to the diagram

$$\begin{tikzcd}
Call_n^S(F^{i}X,E) \arrow[dr, "\omega_i"] \arrow[drr, "\phi_i"] \arrow[ddr, "\psi_i"'] & & \\
& W_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Call_n^S(L_{i-1}F^{\star-1}X,E) \arrow[d, "M_i f^{\star}"] \\
& Call_n^S(F^{i}X,B) \arrow[r] & Call_n^S(L_{i-1}F^{\star-1}X,B)
\end{tikzcd}$$

Here, $Call_n^S$ means the simplicial enriched Hom-functor on the category of n-collections, $L_i(-)$ is the latching object for augmented (!) cosimplicial objects [16] and ϕ_1, ψ_i are generated by the canonical morphism

$$\lambda_{i-1} : L_{i-1}F^{\star-1}X \to F^{i}X.$$

If this morphism were a cofibration, then ω_i would be a trivial fibration by the axiom for simplicial model category.

We actually will prove that λ_{i-1} is an isomorphism onto a summand. For an n-collection X, let $\tilde{X}$ be an operadic functor from $\mathbf{H}_d^n$ to $op(Top)$ where $\mathbf{H}_d^n$ is a discretisation of the categorical operad $\mathbf{H}^n$. Recall [7] that $\mathbf{H}_d^n$ is a free operad generated by the terminal collection, so every topological n-collection considered as a morphism $1 \to op(Top)$ of n-collections indeed generates an operadic functor from $\mathbf{H}_d^n$.

Then it is not hard to check that the free n-operad functor on an n-collection can be defined by the following formula:

$$F(X)_T = \coprod_{T \leftarrow W} \tilde{X}(W) \tag{3}$$

where the coproduct is taken over all morphisms in $\mathbf{H}_T^n$ (recall that T is the terminal object of $\mathbf{H}_T^n$). It follows that the augmented cosimplicial space

$(F^{\star -1}X)_T$ in dimension i is just the coproduct

$$\coprod_{T\leftarrow W_0\leftarrow \ldots \leftarrow W_i} \tilde{X}(W_i).$$

The coface operators are canonical inclusions on the summands corresponding to the operators of insertion of the identities to the chain

$$T \leftarrow W_0 \leftarrow \ldots \leftarrow W_i.$$

The rest of the proof follows in complete analogy with Lemma 4.1 of [4].

$\square$

Recall [7] that on the level of collections the functor Des_n is defined as

$$Des_n(X)_T = X_{|T|},$$

where $|T|$ is the number of tips of T. Therefore, Des_n preserves fibrations and weak equivalences and so $Des_n \vdash EH_n$ is a Quillen adjunction. So the functor EH_n preserves cofibrations and, in particular, it maps cofibrant n-operads to cofibrant symmetric operads.

Corollary 3.1.1 *The operad $B(\mathbf{H}^n)$ is a cofibrant $(n-1)$-terminal n-operad. The operad $N(\mathbf{h}^n)$ is a cofibrant replacement for the operad $B(\mathcal{M}^n)$ of [3].*

Proof. It is not hard to see from the formula (3) that the operad $B(\mathbf{H}^n)$ is isomorphic to $B(F,F,1)$. But $B(\mathbf{h}^n) \simeq EH_n(B(\mathbf{H}^n))$ by (1) and so it is cofibrant. The Theorem 9.2 from [7] states that the the nerve of the canonical morphism $\mathbf{h}^n \to \mathcal{M}^n$ is a trivial fibration.

$\square$

Corollary 3.1.2 *The restriction of the total left derived functor of EH_n to the subcategory of contractible n-operads induces an equivalence between the homotopy category of E_n-operads and homotopy category of contractible $(n-1)$-terminal n-operads.*

Theorem 3.2 *The homotopy category of E_n-spaces is equivalent to the following three categories*

- *the homotopy category of $B(\mathbf{h}^n)$-algebras;*
- *the homotopy category of $B(\mathbf{H}^n)$-algebras;*
- *the homotopy category of $(n-1)$-terminal $B(\mathcal{H}^n)$-algebras.*

4 Internal n-operads.

In this section we show that the categorical theory of internal operads developed in [7] has its topological analogue. Here "n-operad" always means "$(n-1)$-terminal n-operad".

Definition 4.1 *An internal n-operad in a topological symmetric operad A is a coherent n-operadic functor*

$$1 \to Des_n(A);$$

or equivalently, it is an operadic map

$$B(F, F, 1) \to Des_n(A).$$

For any symmetric topological operad C consider the space $\mathrm{OPER}_n(C)$ of all internal operads in C i.e. the simplicial set

$$\mathrm{OPER}_n(C) = Oper_n^S(B(F, F, 1), Des_n(C)).$$

Observe that the operad $B(\mathbf{h}^n)$ contains a canonical internal n-operad given by the unit

$$B(F, F, 1) \to Des_n(EH_n(B(F, F, 1)).$$

Theorem 4.1 *Let C be an E_n-operad. Then C contains an internal n-operad.*

Proof. Since $B(\mathbf{h}^n)$ is a cofibrant E_n-operad we have an operadic equivalence

$$B(\mathbf{h}^n) \to C.$$

Since $B(\mathbf{h}^n)$ contains an n-operad so does C.

$\square$

Theorem 4.2 *The functor $\mathrm{OPER}_n(C)$ is representable on the category of topological symmetric operads. The representing object is $B(\mathbf{h}^n)$.*

Proof. Obviously the adjunction isomorphism

$$(Oper_n^S)_0(B(F, F, 1), Des_n(C)) \simeq (SOper^S)_0(EH_n(B(F, F, 1)), C),$$

where $SOper^S$ is the simplicially enriched Hom-functor on the category of symmetric operads, can be extended to the simplicial adjunction

$$Oper_n^S(B(F, F, 1), Des_n(C)) \simeq SOper^S(EH_n(B(F, F, 1)), C) \simeq$$

$$\simeq SOper^S(B(\mathbf{h}^n), C)$$

$\square$

Remark 4.1 It is instructive to try to write down an internal n-operad in the little n-cube operad $\mathcal{C}^n$.

First let us choose an orientation of the space R^n. For a tree T we can produce a subdivision of the unit n-cube I^n in the following way. If $T = T_1 \otimes_0 \ldots \otimes_0 T_k$ is the canonical decomposition of T [7] then we subdivide I^n into k parallelepipeds by $k-1$ hyperplanes

$$x_1 = i/k, \; 1 \leq i \leq k-1.$$

Then we consider the canonical decomposition of $T_i = T_{i1} \otimes_1 \ldots \otimes_1 \ldots T_{il}$ and the corresponding subdivision of the i-th parallelepiped by hyperplanes

$$x_2 = i/l,\ 1 \leq i \leq l-1.$$

We proceed by induction and get the required subdivision of I^n. The interiors of these parallelepipeds labelled by natural order on tips of T gives a configuration of little n-cubes a_T which we take as the vertex of the internal operad corresponding to the tree T. Figure (X) illustates the procedure.

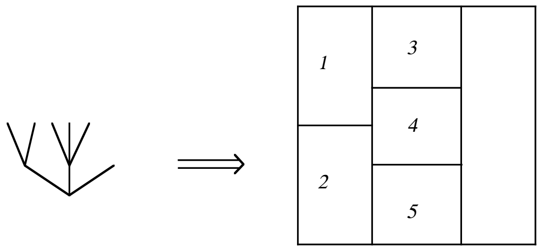


FIGURE X:

Now for every morphism of trees $\sigma : T \to S$ we have to specify a homotopy $h_\sigma(t)$

$$h_\sigma(0) = \mu(a_S; a_{T_1}, \ldots, a_{T_k})\ ;\ h_\sigma(1) = \pi(\sigma)a_T,$$

where μ is the multiplication in $\mathcal{C}^n$, $T_1, \ldots, T_k$ are fibers of σ, and $\pi(\sigma)$ is the permutation which corresponds to σ [7].

This is more complicated combinatorially and we refer the reader to [7][Theorem 9.1] for the details in the case of $N(\mathcal{M}^n)$. The corresponding homotopies for $\mathcal{C}^n$ can be constructed analogously.

For example, for the morphisms in Ω_2 presented in Figure XI

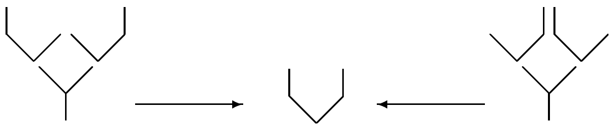


FIGURE XI:

the corresponding homotopies are

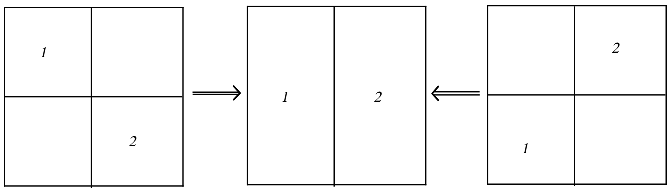


FIGURE XII:

The next piece of structure will be a homotopy

$$h_{\sigma_1,\sigma_2} : \Delta(2) \to \mathcal{C}^n_k$$

defined for every chain

$$T \xrightarrow{\sigma_1} S \xrightarrow{\sigma_2} R$$

which agrees with the appropriate h_σ on the faces of the 2-simplex. This process continues.

Theorem 4.2 shows that the combinatorial structure of an internal n-operad is actually the quintessence of the algebraic structure which E_n-operads were invented for. We can reformulate Theorem 4.2 in the following way

Theorem 4.3 *Every cofibrant E_n-operad A is homotopically freely generated by its internal n-operad, in the sense that there is a homotopy equivalence of Kan simplicial sets*

$$\mathrm{O}{\scriptstyle \mathrm{PER}}_n(C) \simeq SOper^S(A,C).$$

An arbitrary E_n-operad is the representable object for the total left derived functor of $\mathrm{O}{\scriptstyle \mathrm{PER}}_n(-)$ on the homotopy category of symmetric operads.

Remark 4.2 According to this theorem every endomorphism of E_n-operads can be identified (up to higher homotopies) with an internal operad in the little n-cube operad. This shows that the homotopy type of $\mathrm{O}{\scriptstyle \mathrm{PER}}_2(\mathcal{C}^2)$ is closely related to the Grothendieck-Teichmüller group. It would be very interesting to study these connections more closely.

5 Coherence laws for n-fold loop spaces.

It is established in the previous section that the categorical n-operad $\mathcal{H}^n$ provides us with all possible coherence laws. But this operad is very big and in practice we only need some suboperad or even quotient of a suboperad of $\mathcal{H}^1$. We will demonstrate this point with several examples.

Example 5.1 Let us begin from $n = 1$. One can consider a suboperad $\mathcal{SH}^1$ of $\mathcal{H}^1$ which is generated by the collection which contains one operation in every strictly positive arity. So we do not consider nullary operations. This suboperad is easy to describe: the objects are chains of **surjective** morphisms in Ω_1 (see the description of $\mathcal{H}^n$ in section 2.2).

Moreover, we want to have only one unary operation, so we take quotion of this suboperad by the corresponding relation. This means that we ask for the morphisms

$$(T \to U_n \to U_n) \longrightarrow (T \to U_n) \tag{4}$$

and

$$(T \overset{id}{\to} T \to U_n) \longrightarrow (T \to U_n), \tag{5}$$

as well as their tensor products, to be identity morphisms in our factor operad.

Then we have a poset operad which in arity l is the poset of faces of the associahedron K_l (see the classical description of this poset in terms of planar trees [25]).

For example, for $l = 3$ the picture of the corresponding poset is

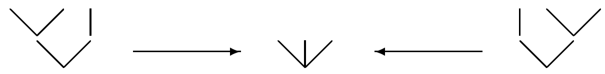


FIGURE XIII:

The result for $l = 4$ is presented in Figure XV.

Remark 5.1 The result of our identification is the contraction of the top face of the hexagon on Figure XV. In general if we do not require the tensor product of the morphisms (4),(5) to be the identity, we get the collection of permutohedra $P_\star$ and a quotient map

$$q : P_\star \to K_\star,$$

which is of some importance as well [29, 22]. The collection of permutohedra does not have a structure of nonsymmetric operad so this quotient map is not a map of operads. Nevertheless, we can define yet another notion of *noncommutative nonsymmetric operad* by dropping the two extreme conditions for associativity in the definition of *Markl's pseudooperad* [25][Definition 1.16]. Every nonsymmetric operad is, of course, a noncommutative operad. The collection $P_\star$ is an example of a noncommutative operad and q is an operadic map of noncommutative operads. Recently W.Joyce used this notion of noncommutative operad implicitly in his work [19].

Example 5.2 Let us go to higher dimensions.

As with $n = 1$ we do not consider nullary operations, so we consider a suboperad $\mathcal{SH}^n$ of $\mathcal{H}^n$ based on morphisms of trees which are surjective in dimension n. We also require conditions (4) and (5). Yet, we want a little bit more.

The operad $\mathcal{H}^n$ has as generating objects all trees including non-pruned trees. This corresponds to possibility of having l-ary operations on n-fold loop space which depend on 'low dimensional units'. Generally speaking, this can be the case in the nerve of an n-tuply monoidal ω-category. In an algebra of the little 2-cube operad, for example, we have different binary multiplications which correspond to the following trees.

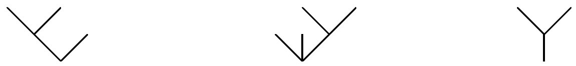


FIGURE XIV:

However, if we are interested in the coherence laws for a unique binary multiplication on an n-fold loop space, we want these low dimensional units to be strict.

Formally we can do this as follows. Observe, that the trees above has a common maximal pruned subtree. In general, for a tree T let $T^{(p)}$ be its maximal pruned subtree. We have a morphism $i^{(p)} : T^{(p)} \to T$ in Ω_n. It is not hard to see that $(-)^{(p)}$ can be extended to a functor (even an involutive comonad) on Ω_n. Therefore, it gives an endofunctor on $\mathcal{H}^n$. Here we consider $\mathcal{H}^n$ as a category

$$\coprod_T \mathcal{H}^n_T.$$

Now we will factorise our suboperad $\mathcal{SH}^n$ by the equivalence relation generated by (4),(5) and identification to the identity of those morphisms of trees σ for which $\sigma^{(p)} = id$.

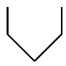
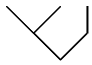
When $n = 2$ the resulting category in the arity  has already appeared in Figure XI.

Figure XVI shows the corresponding category at the arity  . The reader can notice that this is actually the poset of faces of one of the hexagons from the definition of the braided monoidal category [17]. If we want the associativity of the multiplication to be strict we have to factorise further and we will obtain the first of the triangles from Figure II.

Using different degrees of factorisation we can get many interesting polytopes which appear in the literature like permutoassociahedra [20], resultohedra [15, 21] and their genearalisations. We are going to study their properties in a future paper [8].

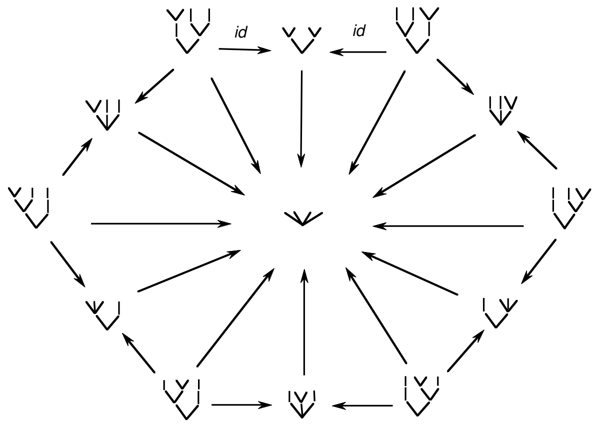


Figure XV:

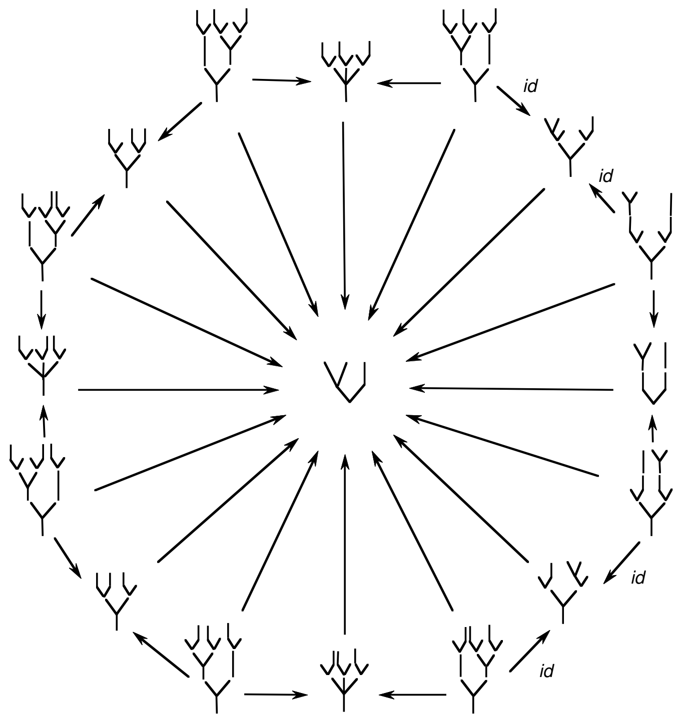


Figure XVI:

Acknowledgements. The first version of this paper was written during my visit to the Max-Plank Institut in Bonn in the automn of 2002 and I gratefully acknowledge their support. I also would like to thank C.Berger, A.Joyal, A.Davydov, A.Makovecky, P.May, J.McClure, J.Smith, and R. Street for many stimulating discussions during my work on this paper. Finally, I acknowledge the financial support of the Scott Russell Johnson Memorial Foundation and the Macquarie University Research Commitee.

References

[1] Baez J., Dolan J., Higher-dimensional algebra and topological quantum field theory, *Journal Math. Phys.* **36** (1995), 6073-6105.

[2] Baez J., May P., Higher categorical structures and their applications,*proposed research*, available at http://www.math.uchicago.edu/ may/

[3] Balteanu C., Fiedorowicz Z., Schwänzl R., Vogt R., Iterated Monoidal Categories, preprint, (1998), 1-55, to appear.

[4] Batanin M.A., Coherent categories with respect to monads and coherent prohomotopy theory,*Cahiers Topologie Geom. Diff. Cat.,* XXXIV-4 (1993), 270-304.

[5] Batanin M.A. Homotopy coherent category theory and A_∞-structures in monoidal categories, *Journal of Pure and Appl. Algebras,* **123** (1998), 67-103.

[6] Batanin M.A., Monoidal globular categories as a natural environment for the theory of weak n-categories, *Adv. Math.* **136** (1998), pp. 39-103.

[7] Batanin M.A., The Eckmann-Hilton argument, higher operads and E_n-spaces, preprint, math.CT/0207281 .

[8] Batanin M.A., Shuffle polytopes, cooperative games and 2-dimensional coherence for higher dimensional categories, in preparation.

[9] Batanin M.A., Street R., The universal property of the multitude of trees, *Journal of Pure and Appl. Algebra* **154** (2000), 3-13.

[10] Berger C., Combinatorial models for real configuration spaces and E_n-operads, *Contemp. Math.* **202** (1997), 37-52.

[11] Berger C, Moerdijk I., Axiomatic homotopy theory for operads, preprint, math.AT/0206094

[12] Boardman J.M., Vogt R.M., Homotopy invariant algebraic structures in topological spaces,*Lectures Notes in Math.,* **347**, Springer, Berlin-Heidelberg,-New York, (1973).

[13] Crans S., Higher-dimensional Mac Lane's pentagon and Zamolodchikov equations, *J. Pure Appl. Algebra,* **168** (2002)

[14] Fiedorowicz S., The symmetric bar construction, preprint, 1998.

[15] Gelfand I.M., Kapranov M.M., Zelevinsky A.V., Newton polytopes of classical determinant and resultant, *Adv. Math.,* **84** (1990), 255-271,

[16] Hovey M., Model Categories, *Math. Surveys and Monographs*, **v.63**, AMS, 1998.

[17] Joyal A., Street R., Braided tensor categories, *Adv. Math.,* **102** (1993), 20-78.

[18] Joyal A., Disks, duality and Θ-categories, preprint, 1997.

[19] Joyce W.P., Natural Associativity without the Pentagon condition, preprint, math.CT/0109088 .

[20] Kapranov M., The permutoassociahedron, Mac Lane's coherence theorem and asymptotic zones for the KZ-equation, *Journal of Pure and Appl. Algebra*, **85**, (1993), 119-142.

[21] Kapranov M., Voevodsky V, 2-Categories and Zamolodchikov tetrahedra equations, in *Proc. Symp. Pure Math.,* **56** p.2 (1994), AMS, Providence, pp. 177-260.

[22] Loday J.-L., Realization of the Stasheff polytope, preprint, math.AT/0212126

[23] May P., The Geometry of Iterated Loop Spaces, *Lecture Notes in Math.,* **271**, Springer Verlag, Berlin, (1972).

[24] Mac Lane S., Categories for the Working Mathematician, *Second edition. Graduate Texts in Mathematics,* **5** Springer-Verlag, New York, 1998.

[25] Markl M., Shnider S., Stasheff J., Operads in Algebra, Topology and Physics, *Math. Surveys and Monographs*, **v.96**, AMS, 2002.

[26] Stasheff J.D., Homotopy associativity of H-spaces, I, *Trans. Amer. Math. Soc.*, **108**, (1963), 275-292.

[27] Stasheff J.D., Homotopy associativity of H-spaces, II, *Trans. Amer. Math. Soc.*, **108**, (1963), 293-312.

[28] Stasheff J.D., H-spaces from a homotopy point of view, *Lecture Notes in Math.*, **vol 161**, Springer-Verlag, 1970.

[29] Tonks A., Relating the associahedron and the permutohedron, *Operads: Proceedings of Renaissance Conferences (Hartford, CT/Luminy, 1995), Contemp. Math.*, **202**, AMS, Providence, (1997), 33-36.