Vanishing structure set of Haken 3-manifolds

S. K. Roushon

Abstract

In this article we prove that the surgery groups of the fundamental group of a certain class of Haken 3-manifolds can be computed in terms of a generalized homology theory even if the manifold does not support nonpositively curved metric.

1 Introduction

Let M be a compact connected orientable Haken 3-manifold. Assume that ∂M (if nonempty) is incompressible, in other words $\pi_1(\partial M) \to \pi_1(M)$ is an injective homomorphism. When M is a link complement in the 3-sphere, then we computed the Wall-Novikov surgery groups of $\pi_1(M)$ in [1]. Also an explicit computation of these groups is given in [12] for any subcomplex complement in the 3-sphere. In both the cases we needed to show that stably the structure set vanishes for the spaces involved, i.e., $|\mathcal{S}(M \times \mathbb{D}^n, \partial(M \times \mathbb{D}^n))| = 1$ for large n. Or equivalently the surgery groups can be computed in terms of a generalized homology theory. Waldhausen proved that ([15], [16]) the Whitehead group of the fundamental group of any Haken 3-manifold vanishes. If $\partial M \neq \emptyset$ and incompressible or if M has a hyperbolic piece in its Jaco-Shalen-Johannson and Thurston decomposition then by [9] M supports a complete nonpositively curved Riemannian metric with totally geodesic tori (if any) boundary component. Thus by the Topological Rigidity Theorem (both for the compact [5] and non-compact case [6]) of Farrell and Jones and by [1] and [12] the structure set vanishes stably. So the natural problem at this point to study is:

Question If M is a closed orientable Haken 3-manifold which does not support any nonpositively curved Riemannian metric, is $|\mathcal{S}(M \times \mathbb{D}^n, \partial(M \times \mathbb{D}^n))| = 1$ for large n?

In this article we give a positive answer to the above question under certain condition.

Using Cappell's splitting theorem F. Quinn proved that if $\pi_1(X)$ is square root closed accessible of some order [see [13] for definition] then modulo elements of order a power of 2 the structure set vanishes stably. Using the recent geometric results we show that for Haken 3-manifolds we do not need the assumption on accessibility. Indeed, it is enough to cut the manifold along an embedded torus only once and show that it gives a square root closed decomposition of the fundamental group. More precisely we have:

Theorem 1.1 *Let M be a compact connected orientable Haken 3-manifold with boundary (may be empty). Then $|\mathcal{S}(M \times \mathbb{D}^n, \partial(M \times \mathbb{D}^n))| = 1$ for $n \geq 3$ if one of the following condition is satisfied:*

1: $\partial M \neq \emptyset$

2: $\partial M = \emptyset$ and either M supports a nonpositively curved Riemannian metric or

"(src) there is an incompressible torus T embedded in M such that $\pi_1(T)$ is square root closed in $\pi_1(M)$".

Definition of square root closedness is given in the next section.

We will give a class of examples of Haken manifolds which satisfy (src) and does not support any nonpositively curved Riemannian metric. Also some examples will be given when (src) is not satisfied but still the Question 1 remains open.

Now let $\mathcal{G}$ be the dual graph of the Jaco-Shalen-Johannson and Thurston (JSJT) decomposition of the closed orientable connected Haken 3-manifold M. The vertices of $\mathcal{G}$ correspond to the pieces in the JSJT decomposition and an edge between two vertices corresponds to a common boundary torus between the geometric pieces corresponding to the vertices. We confuse edges and vertices with the corresponding spaces. We prove the following theorem:

Theorem 1.2 *Let M be a closed connected orientable Haken 3-manifold which has more than 2 geometric pieces in its' JSJT decomposition. Assume that the dual graph $\mathcal{G}$ has a vertex which disconnects the graph. Also assume that the fundamental group of any edge emanating from this vertex is square root closed in the fundamental group of the terget vertex. Then $|\mathcal{S}(M \times \mathbb{D}^n, \partial(M \times \mathbb{D}^n))| = 1$ for $n \geq 2$.*

Remark Note that the hypothesis of the Theorem 1.2 contains the important class of manifolds whose dual graph is a tree.

Now if we assume Thurston's Geometrization conjecture, i.e., if M is a closed orientable aspherical 3-manifold then, either M is a Haken manifold, a Seifert fibered manifold or M supports a complete hyperbolic metric, then the following theorem follows from the above theorems:

Theorem 1.3 *Let M be a closed connected orientable aspherical 3-manifold, also assume if M is a Seifert fibered manifold then the base surface is of genus ≥ 1 and M satisfies the hypothesis of Theorems 1.1 and 1.2 in the Haken case. Then $|\mathcal{S}(M \times \mathbb{D}^n, \partial(M \times \mathbb{D}^n))| = 1$ for $n \geq 3$.*

Note that the class of closed Haken 3-manifolds which are not included in Theorem 1.1 and Theorem 1.2 are those which either have exactly two geometric piece (which are Seifert fibered) in its' JSJT decomposition or no vertex disconnects the dual graph and does not satisfy the hypothesis of the Theorem 1.1. So the natural problem at this point is that:

Problem Show that $|\mathcal{S}(M \times \mathbb{D}^n, \partial(M \times \mathbb{D}^n))| = 1$, for large n, where M is as in the above discussion.

Proof of Theorem 1.3 If M is Haken then it follows from the previous theorem. If it is Seifert fibered as in the hypothesis then it follows from [14]. In the hyperbolic case it is a consequence of [5] and [6]. $\square$

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2 Vanishing of the structure set

We recall the definition of the structure set $\mathcal{S}(M, \partial M)$ for a manifold M with boundary ∂M. Consider all objects $(N, \partial N, f)$, where N is a manifold with boundary ∂N and $f : N \to M$ is a homotopy equivalence such that $f|_{\partial N} : \partial N \to \partial M$ is a homeomorphism. Two such objects $(N_1, \partial N_1, f_1)$ and $(N_2, \partial N_2, f_2)$ are equivalent if there is a homeomorphism $g : N_1 \to N_2$ such that the obvious diagram commutes upto homotopy relative to the boundary. The equivalence classes of such objects is the homotopy- topological structure set $\mathcal{S}(M, \partial M)$. One of the main problem in Topology is to find the cardinality of this set.

A subgroup H of a group G is called *square root closed* if $x^2 \in H$ implies $x \in H$ for any $x \in G$. Suppose G can be written in the generalised free product form $G = \pi_1 *_H \pi_2$ or $G = \pi *_H < t >$. Then we recall that (from [3]) H is square root closed in G if and only if H is square root closed in π_1 as well as in π_2 in the first case and the images of the two injections (which gives the HNN extension structure) of H in π are square root closed in the HNN- extension case.

Importance to show the square root closedness of a finitely presented group lies in the work of S. Cappell ([4]), namely the existence of Meyer-Vietoris type exact sequence of surgery groups for such groups. Together with square root closedness if the Whitehead group of this group vanishes then there is a genuine Meyer-Vietoris sequence for surgery groups for these groups.

The main idea to prove the theorem is to use Cappell's splitting theorem and the geometric result of [9].

Proof of the Theorem 1.1 At first assume that $\partial M = \emptyset$. Then by Jaco- Shalen-Johannson and Thurston decomposition there exist an embedded incompressible torus T in M. We can choose this T to satisfy the (src) condition. We cut M along T. Two cases occure according as $\overline{M-T}$ has one component or two components. If $\overline{M-T}$ has two components, say M_1 and M_2 then $\pi_1(M) = \pi_1(M_1) *_{\pi_1(T)} \pi_1(M_2)$ and in the case when $\overline{M-T}$ has only one component then $\pi_1(M)$ is the HNN extension given by the two injections $j_1, j_2 : \pi_1(T) \to \pi_1(\overline{M-T})$, $\pi_1(M) = \pi_1(\overline{M-T})*_{\pi_1(T)} < t >$. Also we have $Wh(\pi_1 M) = 1$. Hence by [3] there is a Meyer-Vietoris exact sequence connecting the surgery groups of $\pi_1 M$, $\pi_1 M_1$, $\pi_1 M_2$ and $\pi_1 T$ in the first case and of $\pi_1 M$, $\pi_1(\overline{M-T})$ and $\pi_1 T$ in the second case. Since $Wh(\pi_1(M)) = 1$, the surgery groups $L_n^h(-)$ for homotopy equivalences and $L_n^s(-)$ for simple homotopy equivalences coincide. We denote it simply by $L_n(-)$.

We consider the first case when $\overline{M-T}$ has two components. The case when $\overline{M-T}$ is connected is similar. We have an exact sequence of the form:

$$\cdots \to L_n(\pi_1 T) \to L_n(\pi_1 M_1) \oplus L_n(\pi_1 M_2) \to L_n(\pi_1 M) \to \cdots$$

Also, since all spaces considered above are aspherical there is a Meyer-Vietoris exact sequence for the homology theory $H_*(-, \mathbb{L}_0)$:

$$\cdots \to H_n(T, \mathbb{L}_0) \to H_n(M_1, \mathbb{L}_0) \oplus H_n(M_2, \mathbb{L}_0) \to H_n(M, \mathbb{L}_0) \to \cdots$$

Now since the manifolds M_1 and M_2 are Haken with nonempty incompressible tori boundary, by [9] both M_1 and M_2 supports complete nonpositively curved Riemannian metric. Also T supports a flat metric. Hence from Topological rigidity theorem and the same sequence of argument as in [1] or [12] show that the assembly map $H_n(-, \mathbb{L}_0) \to L_n(\pi_1(-))$ is an isomorphism for large n for the spaces M_1, M_2, T. Now the naturality of the assembly map and five lemma prove that the assembly map is an isomorphism for M for large n. Now an application of Ranicki surgery exact sequence and noting the fact that $\mathcal{S}_{n+4}(M)$ is in bijection with $\mathcal{S}(M \times \mathbb{D}^n, \partial(M \times \mathbb{D}^n))$ for $n \geq 2$ prove the theorem in the closed case. Here $\mathcal{S}_*(-)$ denotes the total surgery obstruction group of Ranicki ([11]).

Now assume that $\partial M \neq \emptyset$. If ∂M is incompressible then by [9] interior of M supports a complete nonpositively curved Riemannian metric. If ∂M consists of tori only then the metric will have totally geodesic boundary and hence we can take the double with the extended nonpositively curved metric and use the Topological Rigidity Theorem in [5] to conclude that the structure set of M vanishes. Now if one of the component of ∂M is a surface of genus > 1 then the metric will have constant -1 sectional curvature near this boundary component. And the tori boundary components will be totally geodesic. Now if we take the double along the tori boundary components then the extended metric will be complete nonpositively curved and outside of a compact set the metric is of constant -1 sectional curvature. Then we can use the Topological Rigidity Theorem in [6] to prove that the structure set of M vanishes.

Next we assume that ∂M is compressible, i.e., ker $(\pi_1(\partial M) \to \pi_1(M)) \neq 1$. Then by Loop theorem of Papakyriakopoulos for some compressible loop on the boundary of M there is an embedded 2-disc in the interior of M bounding the loop. We cut M along this disc. We iterate this process till we get a Haken 3-manifold whose connected components are either the 3-disc or have incompressible boundary. Here note that we need to cut the manifold only along finitely many 2-disc. Also note that at each stage of the decomposition we get square root closed decomposition (in fact the decomposition is a free product of two torsion free groups or free product of a torsion free group and the infinite cyclic group) of the fundamental group and the decomposed manifolds are aspherical. Hence we have Meyer-Vietoris exact sequence of the surgery groups and similar exact sequence for the homology theory with coefficient in $\mathbb{L}_0$. Hence by induction on the number of discs along which we needed to cut the manifold and using the fact that the assembly map is natural we prove the theorem in this case. $\square$

Proof of Theorem 1.2 If there is a hyperbolic piece in the JSJT decomposition of M then the proof follows from [9] and [5]. So assume all the geometric pieces are Seifert fibered manifolds. Let N be a Seifert fibered piece in the JSJT decomposition of M which disconnects the dual graph. Hence $M - N$ is disconnected. Let N_1 be a connected component of $\overline{M - N}$. Let $P_1 = N_1 \cup N$ under the original identification and $P_2 = \overline{M - N_1}$. Then $M = P_1 \cup P_2$ and $P_1 \cap P_2 = N$. Hence $\pi_1(M) = \pi_1(P_1) *_{\pi_1(N)} \pi_1(P_2)$, because in the JSJT decomposition we cut the manifold only along incompressible torus and hence the homomorphisms (induced by the inclusion map) $\pi_1(N) \to \pi_1(P_1)$ and $\pi_1(N) \to \pi_1(P_2)$ are injective. Choose a base point $x \in N$. Then the centralizer of any element of $\pi_1(N, x)$ in $\pi_1(P_i, x)$ is either contained in $\pi_1(N, x)$ for $i = 1, 2$ or conjugate to a subgroup of $\pi_1(N', x')$, where N and N' have a common boundary and $x' \in N'$. This follows from Theorem VI. 1.6. in [7]. On the other hand if there is an element $\alpha \in \pi_1(P_i, x)$ such that $\alpha^2 \in \pi_1(N, x)$ then obviously α lies in the centralizer of α^2 and hence in $\pi_1(N, x)$, which follows from the hypothesis of the theorem. Hence the decomposition of $\pi_1(M)$ we got above is square root closed. Also by [9] all the manifolds P_1, P_2 and N supports complete nonpositively curved metric with flat torus boundary and near the boundary the metric is flat. Now the proof goes in the same line of the proof of Theorem 1.1 for the closed Haken manifold case. The refined restriction on n comes from the fact that N is a 3-dimensional manifold instead of the 2-dimensional torus as in Theorem 1.1. $\square$

Remark The same sequence of arguments as in the proof of Theroem 1.1 and Theorem 1.2 can be used to prove the isomorphism of the assembly map for connected sum M of orientable Haken 3-manifolds which satisfy the hypothesis of the Theorems, but we have to replace M by $K(\pi_1(M), 1)$ in the assembly map.

3 Geometry and square root closedness

In this section we give examples of a class of orientable Haken 3-manifolds which does (not) satisfy condition (src).

In the following we give a general class of examples of closed Haken 3-manifolds which satisfies (src).

Proposition 3.1 *Let M be a closed oriented Haken 3- manifold. Let $T \subset M$ be an embedded* *incompressible torus and $M_1, M_2 \subset M$ be two geometric pieces in the Jaco-Shalen-Johannson and Thurston decomposition of M such that $T \subset M_1 \cap M_2$ and $\pi_1(T) \to \pi_1(M_1)$ and $\pi_1(T) \to \pi_1(M_2)$ both have square root closed images. Then M satisfies (src).*

Proof For an element $x \in G$ in a group G, let $\mathcal{C}(x, G) = \{y \in G \mid xy = yx\}$ be the centralizer.

We cut M along T to get a Haken manifold N with two boundary components. For simplicity we assume N has two components N_1 and N_2. The case when N is connected is similar. Both N_1 and N_2 have single torus boundary component say T_1 and T_2. Assume the geometric piece in N_i which has T_i as boundary component is M_i for $i = 1, 2$. By hypothesis the image of the injection $\pi_1(T_i) \to \pi_1(M_i)$ is square root closed. Now from the remark above to show that M satisfies (src) it is enough to show that the image $\pi_1(T_i) \to \pi_1(N_i)$ is square root closed. We fix a base point x_i in T_i and consider the fundamental groups with respect to this base point. Let $\alpha \in \pi_1(N_i, x_i)$ such that $\alpha^2 \in \pi_1(T_i, x_i)$. Consider the centralizer $\mathcal{C}(\alpha^2, \pi_1(N_i, x_i))$. Clearly $\alpha \in \mathcal{C}(\alpha^2, \pi_1(N_i, x_i))$. On the other hand by Theorem VI. 1.6. in [7] $\mathcal{C}(\alpha^2, \pi_1(N_i, x_i)) \subset \pi_1(M_i, x_i)$. Hence by the hypothesis $\alpha \in \pi_1(T_i, x_i)$. So M satisfies (src). □

Now we give some concrete examples of Haken 3-manifolds which satisfy the hypothesis of the above Proposition.

Proposition 3.2 *Let $M^3 \to \mathbb{S}^1$ be a smooth fiber bundle with fiber a compact hyperbolic surface F with nonempty boundary and suppose that the monodromy of the bundle is of finite odd order and it keeps the boundary component of F invariant. Let $\partial_1 M$ be a component of ∂M. Choose $x \in \partial_1 M$. Let $\alpha \in \pi_1(M, x)$ such that $\alpha^2 \in \pi_1(\partial_1 M, x)$. Then $\alpha \in \pi_1(\partial_1 M, x)$.*

Proof There is a finite sheeted covering $f : F \times \mathbb{S}^1 \to M$ of odd degree, say r. Note that $X = f^{-1}(\partial_1 M)$ is connected. Hence r is also the degree of the covering $f|_X$.

Clearly $\alpha^{2r} \in (f|_X)_*(\pi_1(X, \tilde{x}))$, where $\tilde{x}$ is a point over x. Also $\alpha^r \in f_*(\pi_1(F \times \mathbb{S}^1))$. Now X is a component of $\partial F \times \mathbb{S}^1$ and hence by the following Lemma (after restricting to the surface case) $\pi_1(X, \tilde{x})$ is square root closed in $\pi_1(F \times \mathbb{S}^1, \tilde{x})$. Therefore $\alpha^r \in (f|_X)_*(\pi_1(X, \tilde{x})) \subset \pi_1(\partial_1 M, x)$. Now since 2 and r are coprime $\alpha \in \pi_1(\partial_1 M, x)$. □

Lemma 3.1 *Let M be an orientable 2-manifold with a boundary component T and assume that the Euler Characteristic of M is negative. Then the image of the injection $\pi_1(T, x) \to \pi_1(M, x)$ is square root closed, i.e., $\alpha^2 \in \pi_1(T, x)$ implies $\alpha \in \pi_1(T, x)$ for any $\alpha \in \pi_1(M, x)$.*

Proof The proof of the Lemma follows from the fact that the centralizer of an element of $\pi_1(M, x)$ is infinite cyclic and the subgroup of $\pi_1(M, x)$ generated by a boundary circle is maximal infinite cyclic. $\square$

There are many examples of closed Haken 3-manifolds which does not support any nonpositively curved Riemannian metric. In the following we give examples of many such manifolds which satisfy the condition (src).

Example 1 Let Σ_i for $i = 1, 2$ be two compact orientable surfaces of genus ≥ 1 with single boundary component. Consider $\Sigma_i \times \mathbb{S}^1$. Then indentifying $\partial(\Sigma_1 \times \mathbb{S}^1)$ and $\partial(\Sigma_2 \times \mathbb{S}^1)$ by a diffeomorphism we get a Haken 3- manifold M which satisfies condition (src) (by the Lemma above). But M. Kapovich and B. Leeb proved in [8] that except for finitely many isotopy classes of diffeomorphisms of the torus boundary, M does not support any nonpositively curved Riemannian metric. In fact in these cases $\pi_1(M)$ is not even a Hadamard group, in the sense of [8], i.e., $\pi_1(M)$ cannot act discretely as a group of semisimple isometry on a complete geodesic metric space which is nonpositively curved in the sense of distance comparison. Also we can construct a more general class of Haken manifold which satisfies (src) but the fundamental group is not Hadamard. Let M be a closed oriented Haken 3-manifold (with no hyperbolic piece in its JSJT decomposition) and $T \subset M$ an incompressible torus such that there are geometric pieces $M_1, M_2 \subset M$ with $T \subset M_1 \cap M_2$ and M_1 and M_2 both fiber over $\mathbb{S}^1$ with monodromy satisfying the hypothesis as in the Proposition 3.2. Then by Proposition 3.1 and 3.2 it follows that M satisfies (src). Now if we cut M along T and glue it differently then it follows ([9]) that for many identification this resulting manifold does not support any nonpositively curved metric and hence by [8] the fundamental group is not Hadamard. But by Proposition 3.1 and 3.2 these manifolds still satisfy (src).

Now we give some examples where the boundary of the manifold is not square root closed. So if we identify any two such manifold along their boundary then the resulting manifold does not satisfy (src). But the Question 1 still remains open in these cases.

Example 2 Let $M(p, q)$ be the complement of a tubular neighbourhood of a (p, q) torus knot in the 3-sphere. Then for even p, $\pi_1(\partial M(p, q))$ is not square root closed in $\pi_1(M(p, q))$. This follows from the following fact: $\pi_1(M(p, q)) =< \alpha, \beta \mid \alpha^p = \beta^q >$ and with respect to suitable base point $< l, m \mid m = \alpha^r \beta^s, l = \alpha^p m^{-pq} >$ is a basis for $\pi_1(\partial M(p, q))$, where $ps + qr = 1$.

Now $\alpha^p = lm^{pq} \in \pi_1(\partial M(p,q))$. But if $p = 2n$ then α^n does not belong to $\pi_1(\partial M(p,q))$. So if we identify any two such torus knot complements along the boundary then we get a Haken manifold where the fundamental group of the obvious incompressible torus is not square root closed.

Example 3 Let $T \to K$ be the principal $\mathbb{Z}_2$ bundle over the Klein bottle. T is a torus. Let M be the associated $[-1,1]$ bundle over K where $\mathbb{Z}_2$ acts on $[-1,1]$ by a flip along 0. Then ∂M is a torus and M supports a natural flat metric with flat torus boundary. On the other hand we have $(\pi_1(M))^2 \subset \pi_1(\partial M)$. Let M' be any Seifert fibered manifold with one boundary component. Now identifying M and M' along their boundary we get a closed Haken 3-manifold which does not satisfy (src). Also not all flat metric on the boundary of a Seifert fibered manifold can be extended to a nonpositively curved metric on the manifold. Thus the resulting manifold may not support nonpositively curved metric either. Hence the proofs of Theorem 1.1 and 1.2 are not applicable in this case.

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Max-Planck-Institut für Mathematik,

Gottfried-Claren-Strasse 26,

53225 Bonn,

Germany.

E-mail:- roushon@mpim-bonn.mpg.de