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EXISTENCE AND COMPACTNESS ESTIMATES FOR THE $\overline{\partial}$-NEUMANN OPERATOR ON q-CONVEX DOMAINS

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Abstract. The aim of this paper is to give a sufficient condition of existence and compactness estimates for the $\overline{\partial}$-Neumann operator N_q on $L^2_{(0,q)}(\Omega)$ in the case Ω is an arbitrary q-convex domain in $\mathbb{C}^n$.

1. Introduction

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. As well known that the $\overline{\partial}$-Neumann operator N_q is a continuous operator from $L^2_{(0,q)}(\Omega)$ to itself. On pseudoconvex domains there are the two important topics concerning to this operator. This is to study conditions under which this operator is compact and to establish its regularity. Pioneer works in the field belong, for instance, to S. Fu, E. J. Straube, D.W. Catlin, J. D. McNeal and some others. Remark that their beautiful results up to now mainly hold on bounded pseudoconvex domains with smooth boundaries in $\mathbb{C}^n$. The reason of this fact is we need to use Rellich's lemma. Recently, K.Gansberger and F. Haslinger studied compactness estimates for the $\overline{\partial}$-Neumann operator in weighted L^2-spaces and the weighted $\overline{\partial}$-Neumann problem on unbounded domains in $\mathbb{C}^n$ (see [4] and [5]). Note that in [4] instead using Rellich's lemma the author gave an strong assumption about the weight function φ with rapidly increasing of gradient $\nabla\varphi$ and Laplace $\triangle\varphi$ at the infinite point and at the boundary of a domain Ω (Proposition 4.5 in [4]). From this it follows that the embedding of $H^1_0(\Omega, \varphi, \nabla\varphi)$ into $L^2(\Omega, \varphi)$ is compact. In this paper, we are interested in the above problems on q-convex domains, an extension of the notion of pseudoconvex domains and, moreover, they may be not bounded. We give the notion of the property (P'_q), a slight more strong condition than the property (P_q) earlier introduced and investigated by D. Catlin in [2] and E. J. Straube in [11] but this is a inside condition for a domain Ω. Moreover, in Corollary 3.9 below we show that every bounded domain $\Omega \subset \mathbb{C}^n$ with smooth boundary having property (P'_q) then $\partial\Omega$ satisfies property (P_q). The main result of the paper is Theorem 4.1. Here we prove that if $\Omega \subset \mathbb{C}^n$ is

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a q-convex domain having property (P'_q) then there exists a bounded $\overline{\partial}$-Neumann operator N_q on $L^2_{(0,q)}(\Omega)$ and N_q is compact.

The paper is organized as follows. In Section 2 we recall some results about q-subharmonic functions and q-convex domains. We show that the Kohn - Murray- Hörmander formula is still true for q-convex domains. Section 3 is devoted to present the property $(P_q^{'})$ and some results concerning to this property. We prove, in Proposition 3.8, that if Ω is a star-shaped bounded domain having the property $(P_q^{'})$ then $\partial\Omega$ has the property (P_q). The existence and compactness estimates of the $\overline{\partial}$-Neumann operator N_q on q-convex domains are presented in Section 4.

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2. Preliminaries

A complex-valued differential form u of type $(0,q)$ on an open subset $\Omega \subset \mathbb{C}^n$ can be expressed as $u = \sum\limits_{|J|=q}{}' u_J d\overline{z}_J$, where J are strictly increasing multi-indices with lengths q and $\{u_J\}$ are defined functions on Ω. Let $\mathcal{C}^\infty_{(0,q)}(\Omega)$ be the space of complex-valued differential forms of class $\mathcal{C}^\infty$ and of type $(0,q)$ on Ω. By $\mathcal{C}^\infty_0(\Omega)$ we denote the space of $\mathcal{C}^\infty$ functions with compact support in Ω. We use $L^2_{(0,q)}(\Omega)$ to denote the space of $(0,q)$-forms on Ω with square-integrable coefficients. If φ is a function in Ω, we denote $L^2_{(0,q)}(\Omega,\varphi)$ the Hilbert space of complex-valued differential forms of type $(0,q)$ on Ω with square integrable coefficients with respect to the density $e^{-\varphi}$. If $u,v \in L^2_{(0,q)}(\Omega,\varphi)$, the weighted L^2-inner product and norms are defined by

$$(u,v)_{\Omega,\varphi} = \int\limits_\Omega \sum_{|J|=q}{}' u_J \overline{v}_J e^{-\varphi} dV \text{ and } \|u\|^2_{\Omega,\varphi} = (u,u)_{\Omega,\varphi},$$

where dV is the volume element of $\mathbb{C}^n$.

The $\overline{\partial}$-operator on $(0,q)$-forms is given by

$$\overline{\partial}\left(\sum_{|J|=q}{}' u_J d\overline{z}_J\right) = \sum_{|J|=q}{}' \sum_{j=1}^n \frac{\partial u_J}{\partial \overline{z}_j} d\overline{z}_j \wedge d\overline{z}_J,$$

where $\sum'$ means that the sum is only taken over strictly increasing multi-indices J. The derivatives are taken in the sense of distributions, and the domain of $\overline{\partial}$ consists of those $(0,q)$-forms for which the right hand side belongs to $L^2_{(0,q+1)}(\Omega,\varphi)$. So $\overline{\partial}$ is a densely defined closed operator, and therefore has an adjoint operator from $L^2_{(0,q+1)}(\Omega,\varphi)$ into $L^2_{(0,q)}(\Omega,\varphi)$ denoted by $\overline{\partial}^*_\varphi$. For $u = \sum\limits_{|J|=q+1}{}' u_J d\overline{z}_J \in dom(\overline{\partial}^*_\varphi)$ one has

$$\overline{\partial}^*_\varphi u = -\sum_{|K|=q}{}' \sum_{j=1}^{n} \left(\frac{\partial u_{jK}}{\partial z_j} - \frac{\partial \varphi}{\partial z_j} u_{jK} \right) d\overline{z}_K.$$

The complex Laplacian on $(0,q)$-forms is defined as

$$\Box_{q,\varphi} := \overline{\partial}\,\overline{\partial}^*_\varphi + \overline{\partial}^*_\varphi \overline{\partial}$$

where the symbol $\Box_{q,\varphi}$ is to be understood as the maximal closure of the operator initially defined on $(0,q)$-forms with coefficients in $\mathcal{C}^\infty_0(\Omega)$. $\Box_{q,\varphi}$ is a selfadjoint and positive operator. The associated Dirichlet form is denoted by

$$Q_\varphi(f,g) = (\overline{\partial} f, \overline{\partial} g)_{\Omega,\varphi} + (\overline{\partial}^*_\varphi f, \overline{\partial}^*_\varphi g)_{\Omega,\varphi},$$

for $f, g \in dom(\overline{\partial}) \cap dom(\overline{\partial}^*_\varphi)$. The weighted $\overline{\partial}$-Neumann operator $N_{q,\varphi}$ is - if it exists - the bounded inverse of $\Box_{q,\varphi}$. Note that when $\varphi \equiv 0$, we denote $N_{q,0}$ by N_q.

As in [5] we notice that equivalent weight functions have the same properties in this regard (see Lemma 2.3 in [5]).

Lemma 2.1. *Let Ω be an open set in $\mathbb{C}^n$ and let φ_1, φ_2 be two equivalent weights in Ω, i.e., $C^{-1}\|.\|_{\Omega,\varphi_1} \leqslant \|.\|_{\Omega,\varphi_2} \leqslant C\|.\|_{\Omega,\varphi_1}$ for some $C > 0$. Suppose that N_{q,φ_2} exists. Then N_{q,φ_1} also exists and N_{q,φ_1} is compact if and only if N_{q,φ_2} is compact.*

Now let $\varphi \in \mathcal{C}^2(\Omega)$. For $j = 1, \ldots, n$, we write $z_j = x_j + iy_j$ and, as in [5], let

$$X_j = \frac{\partial}{\partial x_j} - \frac{\partial \varphi}{\partial x_j} \text{ and } Y_j = \frac{\partial}{\partial y_j} - \frac{\partial \varphi}{\partial y_j}.$$

We define

$$H^1(\Omega,\varphi,\nabla\varphi) = \{f \in L^2(\Omega,\varphi) : X_j f, Y_j f \in L^2(\Omega,\varphi), j = 1,2,\ldots,n\}$$

with the norm

$$\|f\|^2_{H^1(\Omega,\varphi,\nabla\varphi)} := \|f\|^2_\varphi + \sum_{j=1}^{n} (\|X_j f\|^2_\varphi + \|Y_j f\|^2_\varphi), f \in H^1(\Omega,\varphi,\nabla\varphi).$$

Similarly, define $H^1_0(\Omega,\varphi,\nabla\varphi)$ to be the closure of $\mathcal{C}^\infty_0(\Omega)$ under the norm above.

By $H^1_{0,(0,q)}(\Omega,\varphi,\nabla\varphi)$ we denote the space of $(0,q)$-forms on Ω with the coefficients belonging to $H^1_0(\Omega,\varphi,\nabla\varphi)$. Thus each $f\in H^1_{0,(0,q)}(\Omega,\varphi,\nabla\varphi)$ can be expressed as follows

$$f=\sum_{|J|=q}{}' f_J d\bar z_J,$$

where $f_J\in H^1_0(\Omega,\varphi,\nabla\varphi)$. Then we define the inner product on $H^1_{0,(0,q)}(\Omega,\varphi,\nabla\varphi)$ as follows. Let $f,g\in H^1_{0,(0,q)}(\Omega,\varphi,\nabla\varphi)$,

$$f=\sum_{|J|=q}{}' f_J d\bar z_J,\ \ g=\sum_{|J|=q}{}' g_J d\bar z_J.$$

Put

$$(f,g)_{H^1_{0,(0,q)}(\Omega,\varphi,\nabla\varphi)}=\sum_{|J|=q}{}'(f_J,g_J)_{H^1_0(\Omega,\varphi,\nabla\varphi)}.$$

Then $H^1_{0,(0,q)}(\Omega,\varphi,\nabla\varphi)$ is a Hilbert space with the norm

$$\|f\|^2_{H^1_{0,(0,q)}(\Omega,\varphi,\nabla\varphi)}=\sum_{|J|=q}{}'\|f_J\|^2_{H^1_0(\Omega,\varphi,\nabla\varphi)}.$$

Similar as in Definition 4.2 and the Remark after Lemma 4.3 of [4] we can define the dual space of $H^1_{0,(0,q)}(\Omega,\varphi,\nabla\varphi)$. Put $H^{-1}_{0,(0,q)}(\Omega,\varphi,\nabla\varphi)=(H^1_{0,(0,q)}(\Omega,\varphi,\nabla\varphi))'$. Now we have the following.

Proposition 2.2. *Let Ω be a domain in $\mathbb{C}^n$. Assume that φ is a $\mathcal{C}^2$ smooth function in Ω such that for every $M>0$ there exists $\Omega_M\Subset\Omega$ such that $\triangle\varphi>M$ on $\Omega\backslash\Omega_M$. Then the embedding of $H^1_{0,(0,q)}(\Omega,\varphi,\nabla\varphi)$ into $L^2_{(0,q)}(\Omega,\varphi)$ is compact. Therefore, so is the embedding of $L^2_{(0,q)}(\Omega,\varphi)$ into $H^{-1}_{0,(0,q)}(\Omega,\varphi,\nabla\varphi)$*

Proof. Let $\{f^j\}_{j=1}^\infty$ be a sequence bounded in $H^1_{0,(0,q)}(\Omega,\varphi,\nabla\varphi)$. We have to prove that there exists a subsequence of $\{f^j\}_{j=1}^\infty$ which is convergent in $L^2_{(0,q)}(\Omega,\varphi)$.

First by the definition of $H^1_{0,(0,q)}(\Omega,\varphi,\nabla\varphi)$ we notice that the sequences $\{f^j_J\}_{j=1}^\infty$ are bounded in $H^1_0(\Omega,\varphi,\nabla\varphi)$ for every $|J|=q$. Moreover, by the hypothesis we have

$$\lim_{z\in\Omega,|z|\to\infty}\triangle\varphi(z)=+\infty,\quad \lim_{z\in\Omega,z\to\partial\Omega}\triangle\varphi(z)=+\infty$$

so by Proposition 4.5 in [4] we have the embedding of $H^1_0(\Omega,\varphi,\nabla\varphi)$ into $L^2(\Omega,\varphi)$ is compact.

Now, we use the following notion. If J, L are two strictly increasing multi-indices with lengths q then we say that $J<L$ if there exists $k_0\in\{1,\dots,q\}$ such that $j_1=l_1,\dots,j_{k_0-1}=l_{k_0-1},j_{k_0}<l_{k_0}$. Assume that $J_1<J_2<\dots<J_m$ are strictly increasing multi-indices with lengths q. Since $\{f^j_{J_1}\}_{j=1}^\infty$ is

bounded in $H_0^1(\Omega, \varphi, \nabla\varphi)$ so there exists a subsequence $\{f_{J_1}^{1_j}\}_{j=1}^{\infty}$ of $\{f_{J_1}^{j}\}_{j=1}^{\infty}$ such that $\{f_{J_1}^{1_j}\}_{j=1}^{\infty}$ converges in $L^2(\Omega, \varphi)$. Next because $\{f_{J_2}^{1_j}\}_{j=1}^{\infty}$ is bounded in $H_0^1(\Omega, \varphi, \nabla\varphi)$ so there exists a subsequence $\{f_{J_2}^{2_j}\}_{j=1}^{\infty}$ of $\{f_{J_2}^{1_j}\}_{j=1}^{\infty}$ such that $\{f_{J_2}^{2_j}\}_{j=1}^{\infty}$ is convergent in $L^2(\Omega, \varphi)$. Therefore by induction arguments we can find a subsequence $\{f^{m_j}\}_{j=1}^{\infty}$ of $\{f^j\}_{j=1}^{\infty}$ such that $\{f_{J_k}^{m_j}\}_{j=1}^{\infty}$ converges in $L^2(\Omega, \varphi)$ for every $k \in \{1, \dots, m\}$. Hence, the desired subsequence $\{f^{m_j}\}_{j=1}^{\infty}$ is convergent in $L^2_{(0,q)}(\Omega, \varphi)$. The proof is complete. □

Next, we recall the definition of q-subharmonic functions which is an extension of plurisubharmonic functions (see [1], [6], [7]).

Definition 2.3. Let Ω be a domain in $\mathbb{C}^n$. An upper semicontinuous function $u : \Omega \longrightarrow [-\infty, \infty), u \not\equiv -\infty$ is called q-subharmonic if for every q-dimensional complex plane L in $\mathbb{C}^n$, $u|_L$ is a subharmonic function on $L \cap \Omega$.

The set of all q-subharmonic functions on Ω is denoted by $SH_q(\Omega)$.

The function u is called to be strictly q-subharmonic if for every $U \Subset \Omega$ there exists constant $C_U > 0$ such that $u - C_U|z|^2 \in SH_q(U)$.

Remark 2.4. (a) The q-subharmonicity and the strict q-subharmonicity are the local property.

(b) 1-subharmonic functions are exactly plurisubharmonic and n-subharmonic functions are subharmonic.

The following result gives some basic properties of q- subharmonic functions that will be used later on (see [6]).

Proposition 2.5. *Let Ω be an open set in $\mathbb{C}^n$ and let q is an integer with $1 \leqslant q \leqslant n$. Then we have.*

(a) If $u \in SH_q(\Omega)$ then $u \in SH_r(\Omega)$, for every $q \leqslant r \leqslant n$.

(b) If $u, v \in SH_q(\Omega)$ and $\alpha, \beta > 0$ then $\alpha u + \beta v \in SH_q(\Omega)$.

(c) If $\{u_j\}_{j=1}^{\infty}$ is a family of q-subharmonic functions, $u = \sup_j u_j < +\infty$ and u is upper semicontinuous then u is a q-subharmonic function.

(d) If $\{u_j\}_{j=1}^{\infty}$ is a family of nonnegative q-subharmonic functions such that $u = \sum_{j=1}^{\infty} u_j < +\infty$ and u is upper semicontinuous then u is q-subharmonic.

(e) If $\{u_j\}_{j=1}^{\infty}$ is a decreasing sequence of q-subharmonic functions then so is $u = \lim_{j \to +\infty} u_j$.

(f) If $u \in SH_q(\Omega)$ then $u_\varepsilon = u * \rho_\varepsilon$ is smooth q-subharmonic on Ω_ε, where $\Omega_\varepsilon = \{z \in \Omega : d(z, \partial\Omega) > \varepsilon\}$, and $\{u_\varepsilon\}$ decreases to u on Ω as $\varepsilon \downarrow 0$, where $\rho_\varepsilon(z) = \rho(z/\varepsilon)/|\varepsilon|^{2n}$, ρ is a nonnegative smooth radial function in $\mathbb{C}^n$ vanishing outside the unit ball and satisfying $\int_{\mathbb{C}^n} \rho dV = 1$.

(g) If $u \in SH_q(\Omega)$ then for any convex increasing function χ on the range of u we have $\chi \circ u \in SH_q(\Omega)$. Moreover, $\chi \circ u$ is strictly q-subharmonic in Ω if χ' is strictly increasing and u is strictly q-subharmonic in Ω.

(h) If $u \in SH_q(\Omega)$ then for any linear unitary change of coordinates $\varphi : \mathbb{C}^n \to \mathbb{C}^n$, the function $u \circ \varphi \in SH_q(\Omega)$.

We give a following characterization of the q-subharmonicity which is similar to pseudoconvexity (see [1], [7]).

Proposition 2.6. *Let Ω be a domain in $\mathbb{C}^n$ and let q be an integer with $1 \leqslant q \leqslant n$. Let u be a real valued $\mathcal{C}^2$-function defined on Ω. Then the q-subharmonicity of u is equivalent to*

$$\sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} f_{jK} \overline{f}_{kK} \geqslant 0,$$

for every $(0,q)$-form $f = \sum\limits_{|J|=q}{}' f_J d\overline{z}_J$.

We also have the following curious result for q-subharmonic functions.

Proposition 2.7. *Let Ω be an open set in $\mathbb{C}^n$ and let $u \in SH_q(\Omega)$ such that $u - \delta|id_{\mathbb{C}^n}|^2 \in SH_q(\Omega)$ for some $\delta > 0$. Then for every $\varepsilon > 0$ we have $u_\varepsilon - \delta|id_{\mathbb{C}^n}|^2 \in SH_q(\Omega_\varepsilon)$, where $\Omega_\varepsilon := \{z \in \Omega : d(z, \partial\Omega) > \varepsilon\}$.*

Proof. By Proposition 2.5 we have $(u - \delta|id_{\mathbb{C}^n}|^2)_\varepsilon \in SH_q(\Omega_\varepsilon)$. Since

$$\begin{aligned}(u - \delta|id_{\mathbb{C}^n}|^2)_\varepsilon(z) &= u_\varepsilon(z) - \delta \int\limits_{\mathbb{B}(0,\varepsilon)} |z-w|^2 \rho_\varepsilon(w) dV(w) \\ &= u_\varepsilon(z) - \delta|z|^2 - \delta \int\limits_{\mathbb{B}(0,\varepsilon)} (2\Re(z,-w) + |w|^2)\rho_\varepsilon(w) dV(w) \\ &= u_\varepsilon(z) - \delta|z|^2 - v_{(\varepsilon)}(z),\end{aligned}$$

where $v_{(\varepsilon)}(z) := \delta \int\limits_{\mathbb{B}(0,\varepsilon)} (2\Re(z,-w) + |w|^2)\rho_\varepsilon(w) dV(w)$ is a pluriharmonic function in $\mathbb{C}^n$. Hence, $u_\varepsilon - \delta|id_{\mathbb{C}^n}|^2 = (u - \delta|id_{\mathbb{C}^n}|^2)_\varepsilon + v_{(\varepsilon)} \in SH_q(\Omega_\varepsilon)$. This completes the proof. □

The following definition is an extension of pseudoconvexity.

Definition 2.8. A domain $\Omega \subset \mathbb{C}^n$ is said to be q-convex if there exists a q-subharmonic exhaustion function on Ω.

In particular, if Ω is bounded with smooth boundary such that it has a determining function $\varrho \in C^2(\bar{\Omega})$ which is strictly smooth q-subharmonic on a neighborhood of $\partial\Omega$ then Ω is said to be a strictly q-convex domain.

By [6] and Sard's theorem the following holds.

Proposition 2.9. *Let Ω be a q-convex domain in $\mathbb{C}^n$. Then Ω can be written, $\Omega = \bigcup\limits_{j=1}^{\infty} \Omega_j$ such that $\Omega_j \Subset \Omega_{j+1}$ and each Ω_j is a strictly q-convex domain.*

We recall the Kohn-Morrey-Hörmander formula which is true for every domain $\Omega \subset \mathbb{C}^n$ with C^2 boundary (see Proposition 3.3 in [4]).

Proposition 2.10. *Let $\Omega \subset \mathbb{C}^n$ be a domain with $\mathcal{C}^2$ boundary $\partial\Omega$ and ρ be a $\mathcal{C}^2$ defining function for Ω. Let $\varphi \in \mathcal{C}^2(\overline{\Omega})$. Then for every $f = \sum\limits_{|J|=q}{}' f_J d\overline{z}_J \in \mathcal{C}^1(\overline{\Omega}) \cap dom(\overline{\partial}^*_\varphi)$ we have*

$$\begin{aligned}\|\overline{\partial} f\|^2_{\Omega,\varphi} + \|\overline{\partial}^*_\varphi f\|^2_{\Omega,\varphi} &= \sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \int_\Omega \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} f_{jK} \overline{f_{kK}} e^{-\varphi} \\ &+ \sum_{|J|=q}{}' \sum_{j=1}^{n} \int_\Omega \left|\frac{\partial f_J}{\partial \overline{z}_j}\right|^2 e^{-\varphi} \\ &+ \sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \int_{\partial\Omega} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k} f_{jK} \overline{f_{kK}} \frac{e^{-\varphi}}{|\partial\rho|} dS.\end{aligned}$$

From the above proposition the following is valid for all strictly q-convex domains in $\mathbb{C}^n$.

Proposition 2.11. *Let Ω be a strictly q-convex domain in $\mathbb{C}^n$ and let $\varphi \in C^2(\overline{\Omega})$. Then for every $f = \sum\limits_{|J|=q}{}' f_J d\overline{z}_J \in \mathcal{C}^1(\overline{\Omega}) \cap dom(\overline{\partial}^*_\varphi)$ we have*

$$\sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \int_\Omega \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} f_{jK} \overline{f_{kK}} e^{-\varphi} \leqslant \|\overline{\partial} f\|^2_{\Omega,\varphi} + \|\overline{\partial}^*_\varphi f\|^2_{\Omega,\varphi}. \tag{2.1}$$

In particular, if $\varphi - \varepsilon|z|^2 \in SH_q(\Omega)$ then for every $f \in L^2_{(0,q)}(\Omega, \varphi) \cap dom(\overline{\partial}) \cap dom(\overline{\partial}^*_\varphi)$ we have

$$\|f\|^2_{\Omega,\varphi} \leqslant \frac{1}{q\varepsilon}(\|\overline{\partial} f\|^2_{\Omega,\varphi} + \|\overline{\partial}^*_\varphi f\|^2_{\Omega,\varphi}). \tag{2.2}$$

Proof. Let ρ be a C^2 defining function for Ω. Since Ω is a strictly q-convex domain so $\sum\limits_{|K|=q-1}{}' \sum\limits_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k} f_{jK} \overline{f_{kK}} \geqslant 0$ on $\partial\Omega$. Hence (2.1) follows from Proposition 2.10. Now we prove (2.2). If $f \in \mathcal{C}^1(\overline{\Omega}) \cap dom(\overline{\partial}^*_\varphi)$ then (2.2) follows from (2.1). Hence, by Lemma 4.3.2 in [3] we have (2.2) is also valid for every $f \in L^2_{(0,q)}(\Omega, \varphi) \cap dom(\overline{\partial}) \cap dom(\overline{\partial}^*_\varphi)$. The proof is complete. □

3. The property (P'_q)

First we recall an important property introduced and investigated by D. Catlin in [2] and E. J. Straube in [11]. Let X be a compact set in $\mathbb{C}^n$. We say that X satisfies the property (P_q) if the following holds: for every positive number M, there exists a neighborhood U_M of X and a $\mathcal{C}^2$ smooth function λ_M on U_M, such that $0 \leqslant \lambda_M(z) \leqslant 1$, $z \in U_M$, and such that for any $z \in U_M$, the sum of any q (equivalently: the smallest q) eigenvalues of the Hermitian form $(\frac{\partial^2 \lambda_M}{\partial z_j \partial \bar{z}_k}(z))_{j,k}$ is at least M (or, equivalently, $\lambda_M - \frac{M}{q}|z|^2 \in SH_q(U_M)$). Remark that by results in [2] and [10] it follows that if Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$ with the boundary $b\Omega$ having the property (P_q) then $\overline{\partial}$-Neumann operator N_q is compact on Ω.
Now we give the following.

Definition 3.1. Let Ω be an open set in $\mathbb{C}^n$. We say that Ω has the property (P'_q) if there exists a $\mathcal{C}^2$-smooth function $\varphi : \Omega \longrightarrow [0,1]$ such that for every positive number M, we have $\varphi(z) - M|z|^2 \in SH_q(\Omega \setminus \overline{\Omega}_M)$ with some subset $\Omega_M \Subset \Omega$.

Remark 3.2. (a) From Definition 3.1 note that φ only is required to define inside Ω, but not on $\partial\Omega$. Hence, this property is different to property (P_q) in which the function λ_M should be defined on neighborhood of $\partial\Omega$.

(b) The complex plane $\mathbb{C}$ does not have the property P'_1. Assume otherwise, then we can find a smooth subharmonic function φ on $\mathbb{C}$ such that $0 \leq \varphi \leq 1$ and $u(z) := \varphi(z) - |z|^2$ is subharmonic on a neighbourhood of $|z| \geq r$ for some $r > 0$. Define

$$v(z) = u(1/z), 0 < |z| \leq 1/r.$$

Since $\lim\limits_{z \to 0} v(z) = -\infty$, the function v extends through 0 to a subharmonic function on a neighbourhood of $|z| \leq 1/r$. Now for $t \in [0, 1/r]$ we set

$$M(t) = \max\{v(z) : |z| = t\}.$$

It follows that

$$-\frac{1}{t^2} \leq M(t) \leq 1 - \frac{1}{t^2},\ \forall t \leq r. \tag{3.1}$$

On the other hand, we note that M is a convex function of $\log t$, i.e the function $f(\xi) = M(e^\xi)$ is convex in ξ for $\xi \leq -\log r$. In particular, we have

$$2f(\frac{\xi - \log r}{2}) \leq f(-\log r) + f(\xi), \forall \xi < -\log r. \tag{3.2}$$

Combining (3.1) and (3.2) we get

$$\frac{1}{e^{2\xi}} - \frac{2r}{e^\xi} + r^2 < 2,\ \forall \xi < -\log r.$$

This is a contradiction when ξ large enough.

(c) The Property (P'_q) is not preserved under countable unions. Indeed, we can write $\mathbb{C} = \bigcup\limits_{j=1}^{\infty} \mathbb{B}(0,j)$. By Proposition 3.5 below we have each $\mathbb{B}(0,j)$ satisfies property (P'_1) but $\mathbb{C}$ does not satisfy property (P'_1).

Now we show that there exist unbounded q-convex domains having property (P'_q).

Example 3.3. In $\mathbb{C}^n$, $n > 1$ let $\psi \in \mathcal{C}^\infty(\mathbb{C}^n)$ be defined by

$$\psi(z) := \sum_{j=1}^{n} (x_j^2 + 1) y_j^2,$$

where $z = (z_1, \ldots, z_n)$, $z_j = x_j + iy_j$, $j = 1, \ldots, n$. Since

$$i\partial\overline{\partial}\psi(z) = \frac{1}{2}\sum_{j=1}^{n} (|z_j|^2 + 1) i dz_j \wedge d\overline{z}_j$$

so ψ is plurisubharmonic in $\mathbb{C}^n$. Put

$$\Omega := \{z \in \mathbb{C}^n : \psi(z) < 1\}.$$

Then Ω is a unbounded domain in $\mathbb{C}^n$. We prove that Ω satisfies property (P'_1). Indeed, since $0 < e^{2^j(\psi-1)} \leqslant 1$ in Ω, for all $j \in \mathbb{N}^*$ so we can define the function $\varphi : \Omega \longrightarrow \mathbb{R}$ by

$$\varphi(z) := \psi(z) + \sum_{j=1}^{\infty} \frac{e^{2^j(\psi(z)-1)}}{2^j}, \ z \in \Omega.$$

It is easy to see that $\varphi \in \mathcal{C}^\infty(\Omega)$. Since ψ is bounded plurisubharmonic in Ω so we have φ is smooth bounded plurisubharmonic in Ω.

Now we prove that φ satisfies Definition 3.1. Let $M > 2$. First we claim that for each $\xi \in \partial\Omega$ there exists $r_{\xi,M} > 0$ such that $\varphi - M|z|^2 \in PSH(\Omega \cap \mathbb{B}(\xi, r_{\xi,M}))$. Indeed, choose $m \in \mathbb{N}$ such that $m > 2(M+1)$. Put

$$\varphi_m(z) := \sum_{j=1}^{m} \frac{e^{2^j(\psi(z)-1)}}{2^j} \in \mathcal{C}^\infty(\mathbb{C}^n).$$

Since $\psi(\xi) = 1$ so

$$i\partial\overline{\partial}\left(\frac{e^{2^j(\psi-1)}}{2^j}\right)(\xi) \geqslant i\partial\overline{\partial}\psi(\xi), \ \forall j \in \mathbb{N}^*.$$

Hence,

$$i\partial\overline{\partial}\varphi_m(\xi) \geqslant mi\partial\overline{\partial}\psi(\xi) \geqslant \frac{m}{2}\sum_{j=1}^{n} idz_j \wedge d\overline{z}_j \geqslant (M+1)\sum_{j=1}^{n} idz_j \wedge d\overline{z}_j.$$

Thus, $i\partial\overline{\partial}(\varphi_m - (M+1)|z|^2)(\xi) > 0$ so $\varphi_m - M|z|^2$ is strictly plurisubharmonic in a neighbourhood of ξ. Hence, there is a $r_{\xi,M} > 0$ such that $\varphi_m - M|z|^2$ is plurisubharmonic in $\mathbb{B}(\xi, r_{\xi,M})$. Moreover, since

$$\begin{aligned}(\varphi - M|z|^2)|_{\Omega\cap\mathbb{B}(\xi,r_{\xi,M})} &= \psi|_{\Omega\cap\mathbb{B}(\xi,r_{\xi,M})} \\ &+ (\varphi_m - M|z|^2)\Big|_{\Omega\cap\mathbb{B}(\xi,r_{\xi,M})} + \sum_{j=m+1}^{\infty} \frac{e^{2^j(\psi-1)}}{2^j}\Big|_{\Omega\cap\mathbb{B}(\xi,r_{\xi,M})}\end{aligned}$$

so $\varphi - M|z|^2$ is plurisubharmonic in $\Omega \cap \mathbb{B}(\xi, r_{\xi,M})$. This proves the claim.

Next since $\partial\Omega\cap\mathbb{B}(0,M) \Subset \mathbb{C}^n$ so there exists $\xi_1, \xi_2, \ldots, \xi_k \in \partial\Omega$ such that $\partial\Omega \cap \mathbb{B}(0,M) \Subset \bigcup_{j=1}^{k} \mathbb{B}(\xi_j, r_{\xi_j,M})$. Put $\Omega_M := (\Omega \cap \mathbb{B}(0,M)) \setminus \bigcup_{j=1}^{k} \mathbb{B}(\xi_j, r_{\xi_j,M})$. It is clear that $\Omega_M \Subset \Omega$. Since $\psi - M|z|^2$ is a plurisubharmonic function on $\Omega \cap (\mathbb{C}^n \setminus \mathbb{B}(0,M))$ so $\varphi - M|z|^2$ so is. Moreover, since $\Omega\setminus\overline{\Omega}_M \subset \Omega \cap (\mathbb{C}^n\setminus\mathbb{B}(0,M)) \cup \bigcup_{j=1}^{k} \Omega \cap \mathbb{B}(\xi_j, r_{\xi_j,M})$ and $\varphi - M|z|^2$ is plurisubharmonic in $\Omega \cap \mathbb{B}(\xi, r_{\xi_j,M})$ so $\varphi - M|z|^2$ is plurisubharmonic in $\Omega\setminus\overline{\Omega}_M$. Thus Ω satisfies property (P_1'). Therefore, Ω satisfies property (P_q') for every $1 \leqslant q \leqslant n$ (see Proposition 3.4 below).

Next we have the following.

Proposition 3.4. *Let Ω_1, Ω_2 be open subsets in $\mathbb{C}^n$. Then the following holds.*

(a) If Ω_1 satisfies the property (P_q') then so does it the property (P_r') for all $q \leqslant r \leqslant n$.

(b) If Ω_1, Ω_2 have the property (P_q') then so is $\Omega_1 \cap \Omega_2$.

Proof. It is easy to see that (a) follows from the property of q-subharmonic functions (see a) of Proposition 2.5). Now we prove (b). Let φ_1, φ_2 be as in Definition 3.1 of property (P_q'). It is clear that the function $\frac{1}{2}(\varphi_1 + \varphi_2)$ satisfies the definition of (P_q'). □

Proposition 3.5. *Let Ω be a bounded domain in $\mathbb{C}^n$. Assume that there a continuous q-subharmonic function ψ on $\overline{\Omega}$ satisfying the following conditions.*

(a) $\Omega = \{z \in \overline{\Omega} : \psi(z) < 0\}$, $\partial\Omega = \{z \in \overline{\Omega} : \psi(z) = 0\}$.

(b) There is a neighborhood U of $\partial\Omega$ such that ψ is strictly q-subharmonic on U.

Then Ω satisfies property (P_q').

In particular, if Ω is a strictly q-convex domain then Ω satisfies property (P_q').

Proof. It is enough to prove that there exists a bounded smooth q-subharmonic function φ on Ω such that for every $\xi \in \partial\Omega$ and for every $M > 0$, there exits a positive number real $r_{\xi,M}$ such that $\varphi - M|z|^2$ is q-subharmonic in $\Omega \cap \mathbb{B}(\xi, r_{\xi,M})$. Indeed, we will check that φ satisfies Definition 3.1, and hence, Ω satisfies property (P'_q). Given $M > 0$. Since $\partial\Omega \Subset \mathbb{C}^n$ so there exists $\xi_1, \xi_2, \dots, \xi_k \in \partial\Omega$ such that $\partial\Omega \Subset \bigcup_{j=1}^k \mathbb{B}(\xi_j, r_{\xi_j,M})$. Put $\Omega_M := \Omega \setminus \bigcup\limits_{j=1}^{k} \mathbb{B}(\xi_j, r_{\xi_j,M})$. It is clear that $\Omega_M \Subset \Omega$. Since $\Omega\setminus\overline{\Omega}_M \subset \bigcup\limits_{j=1}^{k}(\Omega\cap\mathbb{B}(\xi_j, r_{\xi_j,M}))$ and $\varphi - M|z|^2$ is q-subharmonic in $\Omega \cap \mathbb{B}(\xi, r_{\xi_j,M})$ so $\varphi - M|z|^2$ is q-subharmonic in $\Omega\setminus\overline{\Omega}_M$.

Now we show that there exists a bounded smooth q-subharmonic function φ on Ω such that for every $\xi \in \partial\Omega$ and for every $M > 0$, there exits a positive number real $r_{\xi,M}$ such that $\varphi - M|z|^2$ is q-subharmonic in $\Omega \cap \mathbb{B}(\xi, r_{\xi,M})$. Indeed, we can assume that $\{z \in \Omega : \psi(z) < -1\} \neq \varnothing$. Put $U_j := \{z \in \Omega : \psi(z) < -1/2^j\}$, we have

(i) $U_j \Subset U_{j+1}$ for every $j \in \mathbb{N}^*$.

(ii) $\lim\limits_{\varepsilon\to 0} \sup\limits_{z\in\Omega}\{\psi_\varepsilon(z) - \psi(z)\} = 0$.

(iii) For every $\xi \in \partial\Omega$ there exit positive number reals α_ξ, β_ξ such that $\psi - \alpha_\xi|z|^2$ is q-subharmonic in $\mathbb{B}(\xi, \beta_\xi)$.

From the hypothesis and condition (b) it follows that there exists $\delta > 0$ such that $\psi \in SH_q(\Omega + \mathbb{B}(0, \delta))$. By (i), (ii) we can chose a sequence $\{\delta_j\}_{j=1}^\infty$ such that $\delta_j \downarrow 0$, $0 < \delta_j < \min\{\delta, d(U_{j-1}, \mathbb{C}^n\setminus U_j), d(U_j, \mathbb{C}^n\setminus U_{j+1}), d(U_{j+1}, \mathbb{C}^n\setminus U_{j+2})\}$ and $0 \leqslant \psi_{\delta_j} - \psi \leqslant 2^{-j}$ in Ω. Put $\varphi_j := (\max\{2^j\psi, -1\} + 1)_{\delta_j} \in SH_q(\Omega)$. First we claim that

$$\varphi_j|_{\Omega\setminus U_{j+1}} = 2^j\psi_{\delta_j} + 1. \tag{3.3}$$

Indeed, let $z \in \Omega\setminus U_{j+1}$ and $w \in \mathbb{B}(0, \delta_j)$. If $z - w \in U_j$ then $z = (z - w) + w \in U_j + \mathbb{B}(0, \delta_j) \subset U_{j+1}$. This is impossible. Hence $z - w \notin U_j$ for every $w \in \mathbb{B}(0, \delta_j)$ so $\psi(z - w) \geqslant -\frac{1}{2^j}$ for every $w \in \mathbb{B}(0, \delta_j)$. Thus, we have

$$\begin{aligned}\varphi_j(z) &= \int_{\mathbb{B}(0,\delta_j)} (\max\{2^j\psi(z - w), -1\} + 1)\rho_{\delta_j}(w)dV(w)\\ &= \int_{\mathbb{B}(0,\delta_j)} (2^j\psi(z - w) + 1)\rho_{\delta_j}(w)dV(w)\\ &= 2^j\psi_{\delta_j}(z) + 1,\end{aligned}$$

and the desired conclusion follows.

Next we prove that

$$\varphi_j|_{U_{j-1}} = 0, \tag{3.4}$$

for every $j > 1$. Indeed, assume that $z \in U_{j-1}$ and $w \in \mathbb{B}(0, \delta_j)$. Since $U_{j-1} + \mathbb{B}(0, \delta_j) \subset U_j$ so $z - w \in U_j$. Hence $\psi(z - w) < -\frac{1}{2^j}$. It follows that

$\max\{2^j\psi(z-w),-1\}+1=0$, and hence,

$$\varphi_j(z)=\int_{\mathbb{B}(0,\delta_j)}(\max\{2^j\psi(z-w),-1\}+1)\rho_{\delta_j}(w)dV(w)=0,$$

and (3.4) is proved.

We have $0\leqslant\varphi_j\leqslant 2$. Indeed, it is clear that $\varphi_j\geqslant 0$. Since $U_{j+1}+\mathbb{B}(0,\delta_j)\subset U_{j+2}$ so $\varphi_j\leqslant 1$ on U_{j+1}. Moreover, for every $z\in\Omega\backslash U_{j+1}$ we have $\varphi_j(z)=2^j\psi_{\delta_j}(z)+1\leqslant 2^j\psi(z)+2\leqslant 2$. Thus, $0\leqslant\varphi_j\leqslant 2$ on Ω. Hence, $0\leqslant\sum_{j=1}^\infty\frac{\varphi_j}{2^j}\leqslant 2\sum_{j=1}^\infty\frac{1}{2^j}=2$. Put

$$\varphi:=\sum_{j=1}^\infty\frac{1}{2^j}\varphi_j<+\infty.$$

It is clear that φ is bounded. We claim that $\varphi\in\mathcal{C}^\infty(\Omega)$. Indeed, given $\Omega'\Subset\Omega$. Since $\Omega=\bigcup_{j=1}^\infty U_j$, $U_j\subset U_{j+1}$ so we can choose $j_1>1$ such that $\Omega'\Subset U_{j_1}$. By (3.4) we have $\varphi_j|_{\Omega'}=(\varphi_j|_{U_{j-1}})|_{\Omega'}=0$, for every$j>j_1$. Hence,

$$\varphi|_{\Omega'}=\sum_{j=1}^\infty\frac{1}{2^j}\varphi_j|_{\Omega'}=\sum_{j=1}^{j_1}\frac{1}{2^j}\varphi_j|_{\Omega'}\in\mathcal{C}^\infty(\Omega').$$

Therefore, $\varphi\in\mathcal{C}^\infty(\Omega)$. Now because $\varphi_j\in SH_q(\Omega)$ for all j then Proposition 2.5 implies that $\varphi\in SH_q(\Omega)\cap\mathcal{C}^\infty(\Omega)\cap L^\infty(\Omega)$.

Now we prove that for every $\xi\in\partial\Omega$ and for every $M>0$, there exits a positive number real $r_{\xi,M}$ such that $\varphi-M|z|^2$ is q-subharmonic in $\Omega\cap\mathbb{B}(\xi,r_{\xi,M})$. By (iii) there exit positive number reals α_ξ,β_ξ such that $\psi-\alpha_\xi|z|^2$ is q-subharmonic in $\mathbb{B}(\xi,\beta_\xi)$. Since $\delta_j\downarrow 0$ so there is j_ξ such that $0<\delta_j<\beta_\xi/2$ for every $j\geqslant j_\varepsilon$. Hence, by Proposition 2.7 we have $\psi_{\delta_j}-\alpha_\xi|z|^2$ is q-subharmonic in $\mathbb{B}(\xi,\beta_\xi/2)$ for every $j\geqslant j_\xi$. Choose $m\in\mathbb{N}$ such that $m>M/\alpha_\xi$, and $r_{\xi,M}=\min(\beta_\xi/2,\delta_{j_\xi+m+1})$. Since $\Omega\cap\mathbb{B}(\xi,r_{\xi,M})\subset\Omega\backslash U_j$ for every $0\leqslant j\leqslant j_\xi+m+1$ so by (3.3) we have $\varphi_j-2^j\alpha_\xi|z|^2$ is q-subharmonic in $\mathbb{B}(\xi,\beta_\xi/2)$ for every $j_\xi\leqslant j\leqslant j_\xi+m$. Moreover,

$$(\varphi-m\alpha_\xi|z|^2)|_{\Omega\cap\mathbb{B}(\xi,r_{\xi,M})}=\left(\sum_{j=1}^\infty\frac{\varphi_j}{2^j}-m\alpha_\xi|z|^2\right)\Bigg|_{\Omega\cap\mathbb{B}(\xi,r_{\xi,M})}$$

$$=\left(\sum_{j=j_\xi}^{j_\xi+m}\frac{\varphi_j}{2^j}-m\alpha_\xi|z|^2\right)\Bigg|_{\Omega\cap\mathbb{B}(\xi,r_{\xi,M})}+\sum_{j=1}^{j_\xi-1}\frac{\varphi_j|_{\Omega\cap\mathbb{B}(\xi,r_{\xi,M})}}{2^j}+\sum_{j=j_\xi+m+1}^\infty\frac{\varphi_j|_{\Omega\cap\mathbb{B}(\xi,r_{\xi,M})}}{2^j}$$

$$=\sum_{j=j_\xi}^{j_\xi+m}\frac{1}{2^j}(\varphi_j-2^j\alpha_\xi|z|^2)|_{\Omega\cap\mathbb{B}(\xi,r_{\xi,M})}+\sum_{j=1}^{j_\xi-1}\frac{\varphi_j|_{\Omega\cap\mathbb{B}(\xi,r_{\xi,M})}}{2^j}+\sum_{j=j_\xi+m+1}^\infty\frac{\varphi_j|_{\Omega\cap\mathbb{B}(\xi,r_{\xi,M})}}{2^j}.$$

Therefore $\varphi-m\alpha_\xi|z|^2\in SH_q(\Omega\cap\mathbb{B}(\xi,r_{\xi,M}))$. Hence,

$$\varphi-M|z|^2\in SH_q(\Omega\cap\mathbb{B}(\xi,r_{\xi,M}))$$

because $M < m\alpha_\xi$. Thus, Ω satisfy property (P'_q).

Finally assume that Ω is a strictly q-convex domain. We prove that Ω satisfies all hypotheses of Proposition 3.5. Indeed, let ρ be a C^2 determining function for Ω such that ρ is strictly q-subharmonic in a neighborhood V of $\partial\Omega$. Since $U := \Omega \backslash V \Subset \Omega$ so $c = \sup\limits_U \rho < 0$. Put

$$\widetilde{\rho}(z) = \begin{cases} \rho(z) & \text{if } z \in \{z \in V : \rho(z) > c/2\} \\ c/2 & \text{if } z \in \{z \in \Omega : \rho(z) \leqslant c/2\}. \end{cases}$$

It is easy to see that $\widetilde{\rho} \in SH_q(V)$. Moreover, since $U \Subset \{z \in \Omega : \rho(z) \leqslant c/2\}$ so $\widetilde{\rho} \in SH_q(\Omega \cup V)$ and because $\widetilde{\rho} = \rho$ in $\{z \in V : \rho(z) > c/2\}$ so $\widetilde{\rho}$ strictly q-subharmonic in a neighborhood of $\partial\Omega$. Thus Ω satisfies all hypotheses of Proposition 3.5 and the desired conclusion follows. □

The following proposition is useful for the proof of the main result.

Proposition 3.6. *Let Ω be an open set in $\mathbb{C}^n$ and assume that Ω satisfies the property (P'_q). Then the function φ in Definition 3.1 can be chosen such that $\varphi(z) - \varepsilon|z|^2 \in SH_q(\Omega)$ with some $\varepsilon > 0$.*

We need the lemma following.

Lemma 3.7. *Let $M > 0$. Then for every $r_1 > 0$ we can find a smooth function $\widetilde{\psi} : \mathbb{C} \longrightarrow \mathbb{R}$ such that $\widetilde{\psi} \equiv 0$ in $|w| \geqslant 8(M+2)r_1$, $\widetilde{\psi} + |w|^2 \in SH(\mathbb{C})$ and $\widetilde{\psi} - M|w|^2 \in SH(D(0, r_1))$, where $D(0, r_1)$ is a disc in $\mathbb{C}$ with radii r_1.*

Proof. It suffices to prove that that there is a function ψ such that $\psi \equiv 0$ in $|w| \geqslant 8(M+1)r_1$, $\psi + |w|^2 \in SH(\mathbb{C})$ and $\psi - M|w|^2 \in SH(D(0, r_1))$. Next put $\widetilde{\psi} := \psi * \rho_\varepsilon$ and choose ε sufficient small, then $\widetilde{\psi}$ has all the desired properties. Let $r_2 = 8(M+1)r_1$. Consider $\chi \in \mathcal{C}^1(\mathbb{R})$ defined by

$$\chi(t) = \begin{cases} \frac{2t - r_1 - r_2}{2} & \text{if } t < r_1 \\ \frac{(t - r_2)^2}{2(r_1 - r_2)} & \text{if } r_1 \leqslant t \leqslant r_2 \\ 0 & \text{if } t > r_2. \end{cases}$$

It is easy to see that $0 \leqslant \chi' \leqslant 1$ and χ' is a decreasing function. Hence, we have $\chi \in \mathcal{C}^1(\mathbb{R})$ is a concave increasing function and $\chi(t) \leqslant t - \frac{r_2}{2}$, $\forall t \leqslant r_1$. Now, let $\psi(w) := -\chi(|w|).|w|$, $w \in \mathbb{C}$. By computation we have

$$\begin{aligned} i\partial\overline{\partial}\psi &= -\chi'(|w|)|w| i\partial\overline{\partial}|w| - \chi''(|w|)|w| i\partial|w| \wedge \overline{\partial}|w| \\ &\quad - 2\chi'(|w|) i\partial|w| \wedge \overline{\partial}|w| - \chi(|w|) i\partial\overline{\partial}|w| \\ &\geqslant -\chi'(|w|)|w| i\partial\overline{\partial}|w| - 2\chi'(|w|) i\partial|w| \wedge \overline{\partial}|w| - \chi(|w|) i\partial\overline{\partial}|w| \\ &= \left(-\frac{3\chi'(|w|)}{4} - \frac{\chi(|w|)}{4|w|} \right) i dw \wedge d\overline{w}. \end{aligned}$$

Thus, we have $i\partial\overline{\partial}\psi \geqslant -idw \wedge d\overline{w}$ in $\mathbb{C}$. Moreover, in particular, for every $|w| \leqslant r_1$ we get

$$\begin{aligned} i\partial\overline{\partial}\psi &\geqslant \left(-\frac{3}{4} - \frac{|w| - \frac{r_2}{2}}{4|w|}\right) idw \wedge d\overline{w} = \left(-1 + \frac{r_2}{8|w|}\right) idw \wedge d\overline{w} \\ &\geqslant \left(-1 + \frac{r_2}{8r_1}\right) idw \wedge d\overline{w} = Midw \wedge d\overline{w}. \end{aligned}$$

The proof is complete. □

Proof of Proposition 3.6. Let $\widetilde{\varphi}$ be as in Definition 3.1 and let $U_0 \Subset U \Subset V \Subset \Omega$ such that $\widetilde{\varphi} - 2|z|^2 \in SH_q(\Omega \setminus \overline{U}_0)$. Choose $\chi \in \mathcal{C}_0^\infty(U)$ such that $0 \leqslant \chi \leqslant 1$, $\chi \equiv 1$ on U_0. Let $m_0 > 0$ such that

$$(1-\chi)\widetilde{\varphi} + m_0|z|^2 \in PSH(V).$$

Choose $r_1 > 0$ such that $V \Subset D(0, r_1) \times \ldots \times D(0, r_1)$, where $D(0, r_1)$ is a disc in $\mathbb{C}$. By Lemma 3.7 there exists $\psi \in \mathcal{C}_0^\infty(\mathbb{C})$ such that $\psi + |w|^2 \in SH(\mathbb{C})$, $\psi - (m_0+1)|w|^2 \in SH(D(0, r_1))$. Put

$$\varphi_1(z) := (1 - \chi(z))\widetilde{\varphi}(z) + \sum_{j=1}^{n} \psi(z_j).$$

For each j, we consider the canonical projection

$$\begin{aligned} \pi_j : \mathbb{C}^n &\longrightarrow \mathbb{C} \\ z &\longmapsto z_j \end{aligned}$$

Since $\psi + |w|^2 \in PSH(\mathbb{C})$ so $\psi_j(z) := \psi \circ \pi_j(z) + |z_j|^2 \in PSH(\mathbb{C}^n)$. Hence,

$$\begin{aligned} (\varphi_1 - |z|^2)|_{\Omega\setminus\overline{U}} &= \widetilde{\varphi} + \sum_{j=1}^{n} \psi \circ \pi_j - |z|^2 \\ &= \widetilde{\varphi} - 2|z|^2 + \sum_{j=1}^{n} \psi_j. \end{aligned}$$

Therefore,

$$\varphi_1 - |z|^2 \in SH_q(\Omega\setminus\overline{U}), \tag{3.5}$$

because $\widetilde{\varphi} - 2|z|^2 \in SH_q(\Omega\setminus\overline{U})$ and $\psi_j \in PSH(\mathbb{C}^n)$.

On the other hand, from $\psi - (m_0+1)|w|^2 \in PSH(D(0, r_1))$ we have $\psi \circ \pi_j - (m_0+1)|z_j|^2 \in PSH(\mathbb{C}^{j-1} \times D(0, r_1) \times \mathbb{C}^{n-j})$. Thus, $\psi \circ \pi_j - (m_0 + 1)|z_j|^2 \in PSH(V)$, and therefore, we get

$$\begin{aligned} (\varphi_1 - |z|^2)|_V &= (1-\chi)\widetilde{\varphi} + \sum_{j=1}^{n} \psi \circ \pi_j - |z|^2 \\ &= ((1-\chi)\widetilde{\varphi} + m_0|z|^2) + \sum_{j=1}^{n} (\psi \circ \pi_j - (m_0+1)|z_j|^2). \end{aligned}$$

Moreover, since $((1-\chi)\widetilde{\varphi}+m_0|z|^2)\in SH_q(V)$ so

$$\varphi_1-|z|^2\in SH_q(V). \tag{3.6}$$

From (3.5) and (3.6) we get

$$\varphi_1-|z|^2\in SH_q(\Omega). \tag{3.7}$$

If we choose $C>0$ such that $-C<\varphi_1<C$ on Ω and put

$$\varphi:=\frac{\varphi_1+C}{2C}.$$

Then $0\leqslant\varphi\leqslant 1$ and by (3.7) we have $\varphi-\frac{1}{2C}|z|^2\in SH_q(\Omega)$. Now, we prove that for every $M>0$ there exists $\Omega_M\Subset\Omega$ such that $\varphi-M|z|^2\in SH_q(\Omega\backslash\overline{\Omega}_M)$. Choose $\Omega_M\Subset\Omega$ such that $V\Subset\Omega_M$ and $\widetilde{\varphi}-(2CM+1)|z|^2\in SH_q(\Omega\backslash\overline{\Omega}_M)$. We have

$$\begin{aligned}
(\varphi-M|z|^2)|_{\Omega\backslash\overline{\Omega}_M}&=\frac{1}{2C}(\varphi_1-2CM|z|^2+C)|_{\Omega\backslash\overline{\Omega}_M}\\
&=\frac{1}{2C}(\widetilde{\varphi}+\sum_{j=1}^{n}\psi\circ\pi_j-2CM|z|^2+C)\\
&=\frac{1}{2C}(\widetilde{\varphi}-(2CM+1)|z|^2+\sum_{j=1}^{n}\psi_j+C).
\end{aligned}$$

Hence, $\varphi-M|z|^2\in SH_q(\Omega\backslash\overline{\Omega}_M)$ because $\psi_j\in PSH(\mathbb{C}^n)$, $j=1,\dots,n$ and $\widetilde{\varphi}-(2CM+1)|z|^2\in SH_q(\Omega\backslash\overline{\Omega}_M)$. Thus, φ satisfies Definition 3.1 and $\varphi-\frac{1}{2C}|z|^2\in SH_q(\Omega)$. The proof is complete. □

Next we give the relation between the property (P'_q) and the property (P_q).

Proposition 3.8. *Let Ω be a bounded domain in $\mathbb{C}^n$. Moreover, assume that Ω is star-shaped and Ω satisfies the property (P'_q). Then $\partial\Omega$ satisfies property (P_q).*

Proof. Without loss of generality we can assume that the center at $0\in\Omega$. For every $M>0$ we choose $\Omega_M\Subset\Omega$ such that $\varphi-4M|z|^2\in SH_q(\Omega\backslash\overline{\Omega}_M)$. Put $V_M^\varepsilon:=\{(1+\varepsilon)z:z\in\Omega\backslash\overline{\Omega}_M\}$, where $\varepsilon\in(0,1)$ can be chosen such that $\partial\Omega\Subset V_M^\varepsilon$. Let $\Omega^\varepsilon:=\{(1+\varepsilon)z:z\in\Omega\}$ and let $\varphi_M^\varepsilon\in\mathcal{C}^2(\Omega^\varepsilon)$ defined by $\varphi_M^\varepsilon(z):=\varphi(\frac{z}{1+\varepsilon})$. By computation we have

$$\frac{\partial^2\varphi_M^\varepsilon}{\partial z_j\partial\overline{z_k}}(z)=\frac{1}{(1+\varepsilon)^2}\frac{\partial^2\varphi}{\partial z_j\partial\overline{z_k}}(\frac{z}{1+\varepsilon}).$$

Thus, the sum of q smallest eigenvalues of complex Hessian

$$\left(\frac{\partial^2\varphi_M^\varepsilon}{\partial z_j\partial\overline{z_k}}\right)_{j,k}$$

at z equal to the sum of q smallest eigenvalues of complex Hessian

$$\left(\frac{1}{(1+\varepsilon)^2}\frac{\partial^2\varphi}{\partial z_j\partial\overline{z_k}}\right)_{j,k}$$

at $\frac{z}{1+\varepsilon}$. Moreover, since $\varphi - 4M|z|^2 \in SH_q(\Omega\backslash\overline{\Omega_M})$ so the sum of q smallest eigenvalues of complex Hessian

$$\left(\frac{1}{(1+\varepsilon)^2}\frac{\partial^2\varphi}{\partial z_j\partial\overline{z_k}}\right)_{j,k}$$

more than or equal to $\frac{4qM}{(1+\varepsilon)^2}$ on $\Omega\backslash\overline{\Omega_M}$. Hence, the sum of q smallest eigenvalues of complex Hessian

$$\left(\frac{\partial^2\varphi_M^\varepsilon}{\partial z_j\partial\overline{z_k}}\right)_{j,k}$$

more than or equal to $\frac{4qM}{(1+\varepsilon)^2}$ on V_M^ε. This means that $\varphi_M^\varepsilon - \frac{4M}{(1+\varepsilon)^2}|z|^2 \in SH_q(V_M^\varepsilon)$. Moreover, since $\frac{4M}{(1+\varepsilon)^2} > M$ so $\varphi_M^\varepsilon - M|z|^2 \in SH_q(V_M^\varepsilon)$ and it follows that $\partial\Omega$ satisfies property (P_q). The proof is complete. □

Corollary 3.9. *Let Ω be a bounded domain in $\mathbb{C}^n$ with smooth boundary. Assume that Ω satisfies property (P'_q). Then $\partial\Omega$ satisfies property (P_q).*

Proof. Since Ω has a smooth boundary so by using a partition unity of $\partial\Omega$ it follows that there exists balls B_j, $j = 1, 2, \ldots, m$ such that $\partial\Omega \Subset \bigcup_{j=1}^m B_j$ and $\Omega\cap B_j$ is star-shaped for all j. Moreover, for every j, since B_j is strictly pseudoconvex so it is strictly q-convex domain for all $q \geqslant 1$. By Proposition 3.5 it follows that B_j satisfies property (P'_q). Moreover, since Ω satisfies property (P'_q) so Proposition 3.4 implies that $\Omega\cap B_j$ satisfies property (P'_q). Hence, Proposition 3.8 implies that $\partial(\Omega\cap B_j)$ has property (P_q). Because $\partial\Omega\cap\overline{B_j} \subset \partial(\Omega\cap B_j)$ so $\partial\Omega\cap\overline{B_j}$ has property (P_q). Therefore, Corollary 4.13 in [11] implies that $\partial\Omega$ also has property (P_q). The proof is complete. □

4. EXISTENCE AND COMPACTNESS ESTIMATES OF THE $\overline{\partial}$-NEUMANN OPERATOR ON q-CONVEX DOMAINS

Now we are position to state and to prove the main result of the paper.

Theorem 4.1. *Assume that Ω is a q-convex domain having property (P'_q). Then there exists a bounded $\overline{\partial}$-Neumann N_q on $L^2_{(0,q)}(\Omega)$. Moreover, N_q is compact.*

From Proposition 2.2 and by using notions and notations as in [5] and by repeating the proof of Proposition 4.1 in [5] (also see Proposition 5.1 in [4] and Proposition 4.2 in [11]) we immediately have the following lemma.

Lemma 4.2. *Let Ω be a domain in $\mathbb{C}^n$ and let φ be a $\mathcal{C}^2$-smooth function in Ω such that for every $M>0$ there exists $\Omega_M \Subset \Omega$ such that $\triangle\varphi > M$ on $\Omega\backslash\Omega_M$. Moreover, assume that there exists $N_{q,\varphi}$ on $L^2_{(0,q)}(\Omega,\varphi)$. Then the following are equivalent:*

(a) The $\overline{\partial}$-Neumann operator $N_{q,\varphi}$ is a compact operator from $L^2_{(0,q)}(\Omega,\varphi)$ into itself.

(b) The embedding of the space $dom(\overline{\partial})\cap dom(\overline{\partial}^*_\varphi)$, provided with the graph norm

$$f \longmapsto (\|f\|^2_{\Omega,\varphi}+\|\overline{\partial}f\|^2_{\Omega,\varphi}+\|\overline{\partial}^*_\varphi f\|^2_{\Omega,\varphi})^{1/2}$$

into $L^2_{(0,q)}(\Omega,\varphi)$ is compact.

(c) For each $\varepsilon>0$ there exists a constant $C_\varepsilon>0$ such that

$$\|f\|^2_{\Omega,\varphi} \leqslant \varepsilon(\|\overline{\partial}f\|^2_{\Omega,\varphi}+\|\overline{\partial}^*_\varphi f\|^2_{\Omega,\varphi})+C_\varepsilon\|f\|^2_{H^{-1}_{0,(0,q)}(\Omega,\varphi,\nabla\varphi)},$$

for every $f\in dom(\overline{\partial})\cap dom(\overline{\partial}^*_\varphi)\subset L^2_{(0,q)}(\Omega,\varphi)$.

(d) The canonical solution operators $\overline{\partial}^*_\varphi N_{q,\varphi} : L^2_{(0,q)}(\Omega,\varphi)\cap ker(\overline{\partial}) \longmapsto L^2_{(0,q-1)}(\Omega,\varphi)$ and $\overline{\partial}^*_\varphi N_{q+1,\varphi} : L^2_{(0,q+1)}(\Omega,\varphi)\cap ker(\overline{\partial}) \longmapsto L^2_{(0,q)}(\Omega,\varphi)$ are compact.

We need the following lemma.

Lemma 4.3. *Let Ω be a domain in $\mathbb{C}^n$ and let $\varphi\in\mathcal{C}^2(\Omega)$. Then for any $(0,q)$-form f with compact support in $\Omega'\Subset\Omega$ such that $f\in L^2_{(0,q)}(\Omega,\varphi)\cap dom(\overline{\partial})\cap dom(\overline{\partial}^*_\varphi)$ we have $f\in H^1_{0,(0,q)}(\Omega,\varphi,\nabla\varphi)$ and the following holds*

$$\|f\|^2_{H^1_{0,(0,q)}(\Omega,\varphi,\nabla\varphi)} \leqslant C_{\varphi,\Omega'}\left(\|\overline{\partial}^*_\varphi f\|^2_{\Omega,\varphi}+\|\overline{\partial}f\|^2_{\Omega,\varphi}+\|f\|^2_{\Omega,\varphi}\right),$$

where $C_{\varphi,\Omega'}$ is a constant depending only on φ, Ω' but not on f.

Proof. First we assume that $f\in\mathcal{C}^\infty_{(0,q)}(\Omega)$ with compact support in Ω'. It is easy to see that $f\in H^1_{0,(0,q)}(\Omega,\varphi,\nabla\varphi)$ and

$$\begin{aligned}
\|f\|^2_{H^1_{0,(0,q)}(\Omega,\varphi,\nabla\varphi)} &= {\sum_{|J|=q}}' \left(\|f_J\|^2_{\Omega,\varphi}+\sum_{j=1}^n\left(\|X_jf_J\|^2_{\Omega,\varphi}+\|Y_jf_J\|^2_{\Omega,\varphi}\right)\right)\\
&= {\sum_{|J|=q}}' \left(\|f_J\|^2_{\Omega,\varphi}+\sum_{j=1}^n\left(\|\frac{\partial(e^{-\varphi}f_J)}{\partial x_j}\|^2_{\Omega,-\varphi}+\|\frac{\partial(e^{-\varphi}f_J)}{\partial y_j}\|^2_{\Omega,-\varphi}\right)\right)\\
&\leqslant C_{1,\varphi,\Omega'}\|e^{-\varphi}f\|^2_{H^1_{(0,q)}(\Omega)},
\end{aligned}$$

where $C_{1,\varphi,\Omega'}$ is a constant depending only on φ and Ω'. On the other hand, by (2.19) in [11] (see also Proposition 5.1.1 in [3]) we have

$$\begin{aligned}\|e^{-\varphi}f\|^2_{H^1_{(0,q)}(\Omega)} &\leqslant C\left(\|\overline{\partial}^*(e^{-\varphi}f)\|^2_{\Omega,0}+\|\overline{\partial}(e^{-\varphi}f)\|^2_{\Omega,0}+\|e^{-\varphi}f\|^2_{\Omega,0}\right)\\ &\leqslant C_{2,\varphi,\Omega'}\left(\|\overline{\partial}^*_{\varphi}f\|^2_{\Omega,\varphi}+\|\overline{\partial}f\|^2_{\Omega,\varphi}+\|f\|^2_{\Omega,\varphi}\right),\end{aligned}$$

where $\|.\|_{H^1_{(0,q)}(\Omega)}$ denotes the L^2-Sobolev 1-norm of $(0,q)$-forms, and $C_{2,\varphi,\Omega'}$ is a constant depending only on φ and Ω'. Hence, we get

$$\|f\|^2_{H^1_{0,(0,q)}(\Omega,\varphi,\nabla\varphi)} \leqslant C_{\varphi,\Omega'}\left(\|\overline{\partial}^*_{\varphi}f\|^2_{\Omega,\varphi}+\|\overline{\partial}f\|^2_{\Omega,\varphi}+\|f\|^2_{\Omega,\varphi}\right).$$

Now we assume that f is an arbitrary $(0,q)$-form with compact support in Ω'. Let $\varepsilon_0>0$ such that $0<\varepsilon_0<d(suppf,\partial\Omega')$ and choose a sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ such that $\varepsilon_k\downarrow 0$ as $k\uparrow\infty$, $0<\varepsilon_k<\varepsilon_0$. Put $f_k:=f*\rho_{\varepsilon_k}$. Since $f_k\in\mathcal{C}^{\infty}_{(0,q)}(\Omega)$ with compact support in Ω' so by applying the above result it follows that

$$\|f_k\|^2_{H^1_{0,(0,q)}(\Omega,\varphi,\nabla\varphi)} \leqslant C_{\varphi,\Omega'}\left(\|\overline{\partial}^*_{\varphi}f_k\|^2_{\Omega,\varphi}+\|\overline{\partial}f_k\|^2_{\Omega,\varphi}+\|f_k\|^2_{\Omega,\varphi}\right). \tag{4.1}$$

Therefore, $\{f_k\}_{k=1}^{\infty}$ is a Cauchy sequence in $H^1_{0,(0,q)}(\Omega,\varphi,\nabla\varphi)$ and for each $j\in\{1,\dots,n\}$ it follows that $\{X_jf_k:=\sum_{|J|=q}X_j(f_k)_Jd\overline{z}_J\}_{k=1}^{\infty}$ is also a Cauchy sequence in $L^2_{(0,q)}(\Omega,\varphi)$. Thus $\{X_jf_k\}_{k=1}^{\infty}$ is convergent to $g_j\in L^2_{(0,q)}(\Omega,\varphi)$. Moreover, because $\{X_jf_k\}_{k=1}^{\infty}$ converges to X_jf in the sense of distribution so $X_jf=g_j\in L^2_{(0,q)}(\Omega,\varphi)$. Similarly, we also have $Y_jf\in L^2_{(0,q)}(\Omega,\varphi)$. Hence $f\in H^1_{0,(0,q)}(\Omega,\varphi,\nabla\varphi)$ and $\{f_k\}_{k=1}^{\infty}$ is convergent to f in $H^1_{0,(0,q)}(\Omega,\varphi,\nabla\varphi)$. Finally, from (4.1) by letting $k\to\infty$ we get

$$\|f\|^2_{H^1_{0,(0,q)}(\Omega,\varphi,\nabla\varphi)} \leqslant C_{\varphi,\Omega'}\left(\|\overline{\partial}^*_{\varphi}f\|^2_{\Omega,\varphi}+\|\overline{\partial}f\|^2_{\Omega,\varphi}+\|f\|^2_{\Omega,\varphi}\right).$$

The proof is complete. □

Proof of Theorem 4.1. By Proposition 3.6 we can be choose $\varphi\in\mathcal{C}^2(\Omega)$ satisfying the definition of (P'_q) such that $\varphi-\varepsilon|z|^2\in SH_q(\Omega)$ for some $\varepsilon>0$. Since $0\leqslant\varphi\leqslant 1$ so φ and 0 are two equivalent weights. Hence, by Lemma 2.1 it suffices to prove the existence and compactness estimates of $\overline{\partial}$-Neumann operator $N_{q,\varphi}$ on $L^2_{(0,q)}(\Omega,\varphi)$.

(a) First we prove the existence of $N_{q,\varphi}$. It is easy to see that it is enough to prove

$$\frac{\varepsilon}{2}\|f\|^2_{\Omega,\varphi}\leqslant\|\overline{\partial}f\|^2_{\Omega,\varphi}+\|\overline{\partial}^*_{\varphi}f\|^2_{\Omega,\varphi} \tag{4.2}$$

for every $f\in L^2_{(0,q)}(\Omega,\varphi)\cap dom(\overline{\partial})\cap dom(\overline{\partial}^*_{\varphi})$ and some $\varepsilon>0$. Given $M>0$ and choose $U\Subset\Omega$ such that $\varphi-M|z|^2\in SH_q(\Omega\backslash\overline{U})$. Take $\chi\in\mathcal{C}^{\infty}_0(\Omega)$ such that $0\leqslant\chi\leqslant 1$, $\chi\equiv 1$ on $\overline{U}$. We will prove

$$\frac{M}{2}\|(1-\chi)f\|^2_{\Omega,\varphi}\leqslant\|\overline{\partial}((1-\chi)f)\|^2_{\Omega,\varphi}+\|\overline{\partial}^*_{\varphi}((1-\chi)f)\|^2_{\Omega,\varphi} \tag{4.3}$$

for every $f \in L^2_{(0,q)}(\Omega, \varphi) \cap dom(\overline{\partial}) \cap dom(\overline{\partial}^*_\varphi)$. Assume that (4.3) already has been proved. Then by choosing $M = \varepsilon$, $U = \varnothing$, $\chi \equiv 0$ we obtain (4.2). Hence, it remains to prove (4.3).

Let $V \Subset \Omega$ such that $U \Subset V \Subset \{\chi = 1\}$. By Proposition 2.9 we can choose a sequence $\{\Omega_l\}_{l=1}^{\infty}$ of strictly q-convex domains such that $V \Subset supp\chi \Subset \Omega_l \Subset \Omega_{l+1} \Subset \Omega$, and $\Omega = \bigcup_{l=1}^{\infty} \Omega_l$. By Proposition 3.6 it follows that $\varphi - \varepsilon|z|^2 \in SH_q(\Omega)$ for some $\varepsilon > 0$. Hence $\sum_{|K|=q-1}{}' \sum_{j=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} g_{jK} \overline{g_{kK}} \geqslant 0$ for every $(0,q)$-form g. Now by Proposition 2.11 for every $g \in L^2_{(0,q)}(\Omega_l, \varphi) \cap dom(\overline{\partial}) \cap dom(\overline{\partial}^*_\varphi)$, if we choose $\mu_g \in \mathcal{C}_0^\infty(V)$ such that $0 \leqslant \mu_g \leqslant 1$, $\mu_g \equiv 1$ on U we have

$$\begin{aligned}
M\|g\|^2_{(\Omega_l \setminus \overline{V}),\varphi} &\leqslant M\|(1-\mu_g)g\|^2_{(\Omega_l \setminus \overline{U}),\varphi} \\
&\leqslant \sum_{|K|=q-1}{}' \sum_{j=1}^{n} \int_{\Omega_l \setminus \overline{U}} (1-\mu_g)^2 \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} g_{jK} \overline{g_{kK}} e^{-\varphi} \\
&\leqslant \sum_{|K|=q-1}{}' \sum_{j=1}^{n} \int_{\Omega_l} (1-\mu_g)^2 \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} g_{jK} \overline{g_{kK}} e^{-\varphi} \qquad (4.4) \\
&\leqslant \sum_{|K|=q-1}{}' \sum_{j=1}^{n} \int_{\Omega_l} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} g_{jK} \overline{g_{kK}} e^{-\varphi} \\
&\leqslant \|\overline{\partial} g\|^2_{\Omega_l,\varphi} + \|\overline{\partial}^*_\varphi g\|^2_{\Omega_l,\varphi}.
\end{aligned}$$

Next let $f \in L^2_{(0,q)}(\Omega, \varphi) \cap dom(\overline{\partial}) \cap dom(\overline{\partial}^*_\varphi)$. By [6] and [7] we know that the $\overline{\partial}$- equation has solutions in strictly q-convex domains then it follows that $ker(\overline{\partial}) = Im(\overline{\partial})$ and $ker(\overline{\partial}^*_\varphi) = Im(\overline{\partial}^*_\varphi)$. Next by using arguments as in [11] we have the orthogonal decomposition of $L^2_{(0,q)}(\Omega_l, \varphi)$ as follows

$$\begin{aligned}
L^2_{(0,q)}(\Omega_l, \varphi) &= ker(\overline{\partial}) \oplus ker(\overline{\partial}^*_\varphi) \\
&= Im(\overline{\partial}) \oplus Im(\overline{\partial}^*_\varphi).
\end{aligned}$$

Put $f^l = (1-\chi) f|_{\Omega_l}$. Then we can write

$$f^l = \overline{\partial} v^l + \overline{\partial}^*_\varphi w^l \text{ in } L^2_{(0,q)}(\Omega_l, \varphi),\ v^l \in ker(\overline{\partial})^\perp, w^l \in ker(\overline{\partial}^*_\varphi)^\perp. \qquad (4.5)$$

To estimate the norm of v^l, it suffices to pair with forms in $Im(\overline{\partial}^*_\varphi)$ (since these are dense in $ker(\overline{\partial})^\perp$). Let $\alpha \in L^2_{(0,q)}(\Omega_l, \varphi) \cap dom(\overline{\partial}^*_\varphi) \cap ker(\overline{\partial}^*_\varphi)^\perp \subset$

$ker(\overline{\partial})$. By (4.4) we get

$$\begin{aligned}\left|(v^l,\overline{\partial}^*_\varphi\alpha)_{\Omega_l,\varphi}\right|^2 &= \left|(\overline{\partial}v^l,\alpha)_{\Omega_l,\varphi}\right|^2 = \left|(f^l-\overline{\partial}^*_\varphi w^l,\alpha)_{\Omega_l,\varphi}\right|^2\\ &= \left|(f^l,\alpha)_{\Omega_l,\varphi}\right|^2 = \left|(f^l,\alpha)_{\Omega_l\setminus\overline{V},\varphi}\right|^2\\ &\leqslant \|f^l\|^2_{\Omega_l\setminus\overline{V},\varphi}.\|\alpha\|^2_{\Omega_l\setminus\overline{V},\varphi} \leqslant \frac{1}{M}\|f^l\|^2_{\Omega_l,\varphi}.\|\overline{\partial}^*_\varphi\alpha\|^2_{\Omega_l,\varphi},\end{aligned}$$

because $(\overline{\partial}^*_\varphi w^l,\alpha)_{\Omega_l,\varphi} = (w^l,\overline{\partial}\alpha)_{\Omega_l,\varphi} = 0$ and $\overline{\partial}\alpha = 0$ in Ω_l. Hence,

$$\|v^l\|^2_{\Omega_l,\varphi} \leqslant \frac{1}{M}\|f^l\|^2_{\Omega_l,\varphi}.$$

Extending the v^l by zero outside of Ω_l we obtain a bounded sequences in $L^2_{(0,q-1)}(\Omega_l,\varphi)$. Passing to an appropriate subsequence, if necessary, we obtain the a weak limit v with

$$\|v\|^2_{\Omega,\varphi} \leqslant \frac{1}{M}\|(1-\chi)f\|^2_{\Omega,\varphi}.$$

Using a similar argument for $\|w^l\|^2_{\Omega_l,\varphi}$ we infer that

$$\|w\|^2_{\Omega,\varphi} \leqslant \frac{1}{M}\|(1-\chi)f\|^2_{\Omega,\varphi},$$

where w is a weak limit of the sequence w^l. Because the decomposition in (4.5) is orthogonal, $\overline{\partial}v^l|_{\Omega_l}$ is bounded in $L^2_{(0,q)}(\Omega,\varphi)$ independently to l. This together with the fact that the weak and the distributional limits agree shows that $v \in dom(\overline{\partial})$ and a subsequence of $\{\overline{\partial}v^l\}_{l=1}^\infty$ (extended by zero) converges to $\overline{\partial}v$ weakly. It remains to show that $w \in dom(\overline{\partial}^*_\varphi)$. Indeed, for every $\alpha \in dom(\overline{\partial})$ we have

$$\begin{aligned}|(w,\overline{\partial}\alpha)_{\Omega,\varphi}| &\leqslant \limsup_{l\to\infty}|(w_l,\overline{\partial}\alpha)_{\Omega_l,\varphi}| = \limsup_{l\to\infty}|(\overline{\partial}^*_\varphi w_l,\alpha)_{\Omega_l,\varphi}|\\ &\leqslant (\limsup_{l\to\infty}\|\overline{\partial}^*_\varphi w_l\|_{\Omega_l,\varphi}).\|\alpha\|_{\Omega,\varphi} \leqslant \|(1-\chi)f\|_{\Omega,\varphi}.\|\alpha\|_{\Omega,\varphi}.\end{aligned}$$

At the same time, we note that a subsequence of $\{\overline{\partial}^*_\varphi w^l\}$ is weakly convergent to $\overline{\partial}^*_\varphi w$. Therefore, $(1-\chi)f = \overline{\partial}v + \overline{\partial}^*_\varphi w$ in Ω, $\overline{\partial}v$ and $\overline{\partial}^*_\varphi w$ is orthogonal in $L^2_{(0,q)}(\Omega,\varphi)$, and $\|v\|^2_{\Omega,\varphi} + \|w\|^2_{\Omega,\varphi} \leqslant \frac{2}{M}\|(1-\chi)f\|^2_{\Omega,\varphi}$. Hence we have

$$\begin{aligned}\|(1-\chi)f\|^2_{\Omega,\varphi} &= \|\overline{\partial}v\|^2_{\Omega,\varphi} + \|\overline{\partial}^*_\varphi w\|^2_{\Omega,\varphi} = (\overline{\partial}^*_\varphi\overline{\partial}v,v)_{\Omega,\varphi} + (\overline{\partial}\overline{\partial}^*_\varphi w,w)_{\Omega,\varphi}\\ &\leqslant \|\overline{\partial}^*_\varphi\overline{\partial}v\|_{\Omega,\varphi}.\|v\|_{\Omega,\varphi} + \|\overline{\partial}\overline{\partial}^*_\varphi w\|_{\Omega,\varphi}.\|w\|_{\Omega,\varphi}\\ &\leqslant (\|\overline{\partial}^*_\varphi\overline{\partial}v\|^2_{\Omega,\varphi} + \|\overline{\partial}\overline{\partial}^*_\varphi w\|^2_{\Omega,\varphi})^{1/2}.(\|v\|^2_{\Omega,\varphi} + \|w\|^2_{\Omega,\varphi})^{1/2}\\ &\leqslant \sqrt{\frac{2}{M}}(\|\overline{\partial}^*_\varphi((1-\chi)f)\|^2_{\Omega,\varphi} + \|\overline{\partial}((1-\chi)f)\|^2_{\Omega,\varphi})^{1/2}.\|(1-\chi)f\|_{\Omega,\varphi}.\end{aligned}$$

This shows that

$$\|(1-\chi)f\|^2_{\Omega,\varphi} \leqslant \frac{2}{M}(\|\overline{\partial}^*_\varphi((1-\chi)f)\|^2_{\Omega,\varphi} + \|\overline{\partial}((1-\chi)f)\|^2_{\Omega,\varphi}).$$

Thus, (4.3) is proved.

(b) Next we show that $N_{q,\varphi}$ is compact. By assumption and using Lemma 4.2 it suffices to show that we have a compactness estimate. Given $\varepsilon > 0$. We choose $M > 0$ with $\frac{1}{M} \leqslant \frac{\varepsilon}{10}$ and a smooth bounded domain $\Omega_M \Subset \Omega$ such that $\varphi - 2M|z|^2 \in SH_q(\Omega\backslash\overline{\Omega}_M)$. Let $\chi \in \mathcal{C}_0^\infty(\Omega)$ such that $0 \leqslant \chi \leqslant 1$, $\chi \equiv 1$ on Ω_M. By (4.3) we have the following estimate

$$\begin{aligned}\frac{M}{2}\|f\|^2_{\Omega,\varphi} &\leqslant M\|(1-\chi)f\|^2_{\Omega,\varphi} + M\|\chi f\|^2_{\Omega,\varphi}\\ &\leqslant 2\left(\|\overline{\partial}^*_\varphi((1-\chi)f)\|^2_{\Omega,\varphi} + \|\overline{\partial}((1-\chi)f)\|^2_{\Omega,\varphi}\right) + M\|\chi f\|^2_{\Omega,\varphi}.\end{aligned}$$

Since

$$\begin{aligned}\overline{\partial}^*_\varphi((1-\chi)f) &= -\sum_{|K|=q-1}{}' \sum_{j=1}^{n} e^{\varphi}\frac{\partial(e^{-\varphi}(1-\chi)f_{jK})}{\partial z_j}d\overline{z}_K\\ &= -(1-\chi)\sum_{|K|=q-1}{}' \sum_{j=1}^{n} e^{\varphi}\frac{\partial(e^{-\varphi}f_{jK})}{\partial z_j}d\overline{z}_K + \sum_{|K|=q-1}{}' \sum_{j=1}^{n}\frac{\partial\chi}{\partial z_j}f_{jK}d\overline{z}_K\\ &= -(1-\chi)\overline{\partial}^*_\varphi f + \sum_{|K|=q-1}{}' \sum_{j=1}^{n}\frac{\partial\chi}{\partial z_j}f_{jK}d\overline{z}_K\end{aligned}$$

so

$$\begin{aligned}\|\overline{\partial}^*_\varphi((1-\chi)f)\|^2_{\Omega,\varphi} &\leqslant 2\|(1-\chi)\overline{\partial}^*_\varphi f\|^2_{\Omega,\varphi} + 2\sum_{|K|=q-1}{}' \|\sum_{j=1}^{n}\frac{\partial\chi}{\partial z_j}f_{jK}\|^2_{\Omega,\varphi}\\ &\leqslant 2\|\overline{\partial}^*_\varphi f\|^2_{\Omega,\varphi} + 2\sum_{|K|=q-1}{}' \|(|\partial\chi|.|f|)\|^2_{\Omega,\varphi}.\end{aligned}$$

and

$$\begin{aligned}\|\overline{\partial}((1-\chi)f)\|^2_{\Omega,\varphi} &= \|(1-\chi)\overline{\partial}f - \overline{\partial}\chi\wedge f\|^2_{\Omega,\varphi}\\ &\leqslant 2\|(1-\chi)\overline{\partial}f\|^2_{\Omega,\varphi} + 2\|\overline{\partial}\chi\wedge f\|^2_{\Omega,\varphi}\\ &\leqslant 2\|\overline{\partial}f\|^2_{\Omega,\varphi} + 2\|(|\overline{\partial}\chi|.|f|)\|^2_{\Omega,\varphi}.\end{aligned}$$

Choose $\mu_\chi \in \mathcal{C}_0^\infty(\Omega)$ such that $\mu_\chi = 1$ on the support of χ. Then we get

$$\begin{aligned}\frac{M}{2}\|f\|^2_{\Omega,\varphi} &\leqslant 4\left(\|\overline{\partial}^*_\varphi f\|^2_{\Omega,\varphi} + \|\overline{\partial}f\|^2_{\Omega,\varphi}\right) + M\|\chi f\|^2_{\Omega,\varphi}\\ &\quad + 4\left(\sum_{|K|=q-1}{}' \|(|\partial\chi|.|f|)\|^2_{\Omega,\varphi} + \|(|\overline{\partial}\chi|.|f|)\|^2_{\Omega,\varphi}\right)\\ &\leqslant 4\left(\|\overline{\partial}^*_\varphi f\|^2_{\Omega,\varphi} + \|\overline{\partial}f\|^2_{\Omega,\varphi}\right) + M\|\mu_\chi f\|^2_{\Omega,\varphi}\\ &\quad + 4\left(\left(\sum_{|K|=q-1}{}' \sup|\partial\chi|^2\right).\|\mu_\chi f\|^2_{\Omega,\varphi} + \left(\sup|\overline{\partial}\chi|^2\right).\|\mu_\chi f\|^2_{\Omega,\varphi}\right)\end{aligned}$$

$$\leqslant 4\left(\|\overline{\partial}^*_\varphi f\|^2_{\Omega,\varphi}+\|\overline{\partial}f\|^2_{\Omega,\varphi}\right)+\|\mu_{\chi,M}f\|^2_{\Omega,\varphi},$$

where $\mu_{\chi,M}:=\mu_\chi\sqrt{M+4\sum_{|K|=q-1}{}'\sup|\partial\chi|^2+4\sup|\overline{\partial}\chi|^2}\in\mathcal{C}_0^\infty(\Omega)$ is a positive function depending on χ, M. Moreover, by Lemma 4.3 we have $\mu^2_{\chi,M}f\in H^1_{0,(0,q)}(\Omega,\varphi,\nabla\varphi)$ so we get

$$\begin{aligned}\frac{M}{2}\|f\|^2_{\Omega,\varphi}&\leqslant 4\left(\|\overline{\partial}^*_\varphi f\|^2_{\Omega,\varphi}+\|\overline{\partial}f\|^2_{\Omega,\varphi}\right)\\&+\|\mu^2_{\chi,M}f\|_{H^1_{0,(0,q)}(\Omega,\varphi,\nabla\varphi)}.\|f\|_{H^{-1}_{0,(0,q)}(\Omega,\varphi,\nabla\varphi)}\\&\leqslant 4\left(\|\overline{\partial}^*_\varphi f\|^2_{\Omega,\varphi}+\|\overline{\partial}f\|^2_{\Omega,\varphi}\right)+a\|\mu^2_{\chi,M}f\|^2_{H^1_{0,(0,q)}(\Omega,\varphi,\nabla\varphi)}\\&+\frac{1}{a}\|f\|^2_{H^{-1}_{0,(0,q)}(\Omega,\varphi,\nabla\varphi)},\end{aligned}\tag{4.6}$$

where a is chosen late.

On the other hand, applying Lemma 4.3 and using (4.2) we have

$$\begin{aligned}&\|\mu^2_{\chi,M}f\|^2_{H^1_{0,(0,q)}(\Omega,\varphi,\nabla\varphi)}\\&\leqslant C_{\Omega',\varphi}\left(\|\overline{\partial}^*_\varphi(\mu^2_{\chi,M}f)\|^2_{\Omega,\varphi}+\|\overline{\partial}(\mu^2_{\chi,M}f)\|^2_{\Omega,\varphi}+\|(\mu^2_{\chi,M}f)\|^2_{\Omega,\varphi}\right)\\&\leqslant C_{\Omega',\varphi,\varepsilon,\mu_\chi}\left(\|\overline{\partial}f\|^2_{\Omega,\varphi}+\|\overline{\partial}^*_\varphi f\|^2_{\Omega,\varphi}\right),\end{aligned}$$

where Ω' is a smooth bounded domain such that $\{\mu_{\chi,M}\neq 0\}\Subset\Omega'\Subset\Omega$. Combining this with (4.6) we get

$$\frac{M}{2}\|f\|^2_{\Omega,\varphi}\leqslant\left(4+aC_{\Omega',\varphi,\varepsilon,\mu_\chi}\right)\left(\|\overline{\partial}^*_\varphi f\|^2_{\Omega,\varphi}+\|\overline{\partial}f\|^2_{\Omega,\varphi}\right)+\frac{1}{a}\|f\|^2_{H^{-1}_{0,(0,q)}(\Omega,\varphi,\nabla\varphi)}.$$

Now choose a such that $aC_{\Omega',\varphi,\varepsilon,\mu_\chi}\leqslant 1$ then

$$\|f\|^2_{\Omega,\varphi}\leqslant\varepsilon\left(\|\overline{\partial}^*_\varphi f\|^2_{\Omega,\varphi}+\|\overline{\partial}f\|^2_{\Omega,\varphi}\right)+\frac{2}{aM}\|f\|^2_{H^{-1}_{0,(0,q)}(\Omega,\varphi,\nabla\varphi)}.$$

These estimates and Lemma 4.2 follow the compactness of $N_{q,\varphi}$. The proof is complete. □

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