BOUNDARY BEHAVIOR OF SPECIAL COHOMOLOGY CLASSES ARISING FROM THE WEIL REPRESENTATION

JENS FUNKE* AND JOHN MILLSON**

Abstract. In our previous paper [9], we established a correspondence between vector-valued holomorphic Siegel modular forms and cohomology with local coefficients for local symmetric spaces X attached to real orthogonal groups. This correspondence is realized using integral operators whose kernels are theta functions associated to explicitly constructed "special" Schwartz forms. Furthermore, the theta functions give rise to generating series of Poincaré dual classes of certain "special cycles" in X with coefficients arising from embeddings of smaller orthogonal groups.

In this paper, we compute the restriction of these theta functions to the Borel-Serre boundary of X. The restriction to each face is again a theta series as in [9], now for a smaller orthogonal group and a larger coefficient system.

As application we establish the cohomological nonvanishing of the special (co)cycles when passing to an appropriate finite cover of X. In particular, the (co)homology groups in question do not vanish.

1. Introduction

The cohomology of arithmetic quotients $X = \Gamma\backslash D$ of a symmetric space D associated to a reductive Lie group G is of fundamental interest in number theory and for the field of automorphic forms. One very attractive topic in this area is the construction of non-trivial cohomology classes via the embedding $H \hookrightarrow G$ of suitable subgroups H. Classes of these geometrically defined cycles are often called (generalized) modular symbols or also "special" cycles.

Since the work of Hirzebruch and Zagier [14] on certain algebraic cycles, the "Hirzebruch-Zagier curves", in Hilbert modular surfaces and Shintani [23] for the "classical" modular symbols inside modular curves, the occurrence of intersection numbers of geometrically defined cycles as Fourier coefficients of automorphic forms has been widely studied. Kudla and Millson (see e.g. [18]) carried out an extensive program to explain and generalize the work of Hirzebruch-Zagier from the point of view of Riemannian geometry and the theory of reductive dual pairs and the theta correspondence. They obtain analogues of the results of [14] for orthogonal, unitary, and symplectic groups of arbitrary dimension and signature. In particular, their work gives rise to a lift from the cohomology with compact supports for the associated locally symmetric spaces to spaces of holomorphic Siegel and Hermitian modular forms.

Date: November 24, 2008.

* Partially supported by NSF grants DMS-0305448 and DMS-0710228.

** Partially supported by NSF grant DMS-0405606, NSF FRG grant DMS-0554254, and the Simons Foundation.

Note however, that the restriction to cohomology with compact supports implies that their results actually do not include the one obtained by Hirzebruch-Zagier (which deals with a smooth compactification of the Hilbert modular surface).

We let V be a rational quadratic space of signature (p,q). Then X is a (suitable) arithmetic quotient of the symmetric space associated to $G = \underline{G}(\mathbb{R})_0 \simeq \mathrm{SO}_0(p,q)$, the connected component of the identity of the real points of the orthogonal group $\underline{G} = \mathrm{SO}(V)$. Special cycles Z_U arise from the embedding $G_U \hookrightarrow G$ of the stabilizer of a positive definite rational subspace $U \subset V$ of dimension n. Hence G_U is an orthogonal group of signature $(p-n,q)$. The special cycles Z_U for varying U give rise to a family of composite cycles Z_T parametrized by symmetric positive definite integral $n \times n$ matrices T. We obtain classes $[Z_T]$ in $H^{nq}(X,\mathbb{Z})$. Then the Kudla-Millson result in this situation roughly states that the generating series $\sum_{T\geq 0}[Z_T]q^T$ is a holomorphic Siegel modular form of degree n with values in $H^{nq}(X,\mathbb{Z})$.

In our previous paper [9], we extended the theta lift of Kudla and Millson and obtained a correspondence between vector-valued holomorphic Siegel modular forms of degree n (associated to a dominant weight λ of $\mathrm{GL}_n(\mathbb{C})$) and cohomology classes of X with local coefficients in a finite dimensional irreducible representation of $\mathrm{O}(V)$ with a corresponding highest weight $[\lambda]$. This correspondence (like the one for trivial coefficients) is realized using integral operators whose kernels are theta functions associated to explicitly constructed "special" Schwartz forms. The Fourier coefficients of these theta functions represent Poincaré dual classes for the special cycles $Z_{T,[\lambda]}$ (to which we have assigned local coefficients). Then the main result of [9] is that $\sum_{T\geq 0}[Z_{T,[\lambda]}]q^T$ is a vector-valued Siegel modular form of degree n with values in the cohomology of degree nq with non-trivial coefficients.

In this paper, we compute the restriction of these theta functions to the Borel-Serre boundary of X. The result for a maximal face can roughly be stated as follows.

Theorem 1.1. *The restriction to a maximal face corresponding to a parabolic subgroup of $\underline{G}$ stabilizing a rational totally isotropic subspace of V of dimension ℓ is again a theta series as in* [9]*, now for the smaller orthogonal group $\mathrm{O}(p-\ell,q-\ell)$ and a larger coefficient system with highest weight $[\ell\varpi_n + \lambda]$. Here ϖ_n denotes the n-th fundamental weight of $\mathrm{GL}_n(\mathbb{C})$.*

As application we establish the cohomological non-vanishing of the special (co)cycles in a certain range when passing to an appropriate finite cover of X.

For trivial coefficients, our non-vanishing result is

Theorem 1.2. *Assume that the $\mathbb{Q}$-rank and the $\mathbb{R}$-rank of $\underline{G}$ coincide. Then if $n \leq \left[\frac{p-q}{2}\right]$ for $q \geq 2$ and $n \leq p-1$ for $q = 1$, there exists a finite cover X' of X such that the cycles Z_T define non-vanishing classes in $H^{nq}(X',\mathbb{Z})$ for infinitely many T.*

The basic idea for the proof of this result is as mentioned above that the T-th Fourier coefficient of the theta series represents the Poincaré dual class of the cycle Z_T. We study the restriction of the theta series to a face of the Borel-Serre compactification of X associated to a minimal rational parabolic subgroup. At such a face, the theta series becomes positive definite, and we establish the non-vanishing there.

Bergeron [1] in the compact case established non-vanishing of the classes introduced by Kudla and Millson (with trivial coefficient system) by considering the analogous classes in $\mathrm{U}(p,q)$ and using the non-vanishing result in the unitary case by Kazhdan, see [5]. Li [20] also used the theta correspondence to establish non-vanishing for the cohomology of orthogonal groups, again in the compact (or L^2)-case (without giving a geometric interpretation of the classes).

Speh and Venkataramana [24] gave in general a criterion for the non-vanishing of certain modular symbols on locally symmetric spaces in terms of the compact dual. In contrast to our result, their non-vanishing occurs from classes defined by invariant forms on D.

Furthermore, we indicate how one can use our work to obtain a proof of the results of Hirzebruch-Zagier using the theta correspondence and local geometric arguments. This will be carried out in a different paper [10]. In fact, our treatment of Hirzebruch-Zagier represents the solution of a special case of the main motivation for our work, which can be roughly stated as follows. Extend the lift by Kudla and Millson (and our extension to non-trivial coefficient systems) from the cohomology of compact supports to cohomology groups of the space X which capture its boundary. The results of this paper should be viewed in this context. For this program, we expect the interpretation of our results in terms of weighted cohomology to be very fruitful. For example, for $p > q$, the classes are square integrable if $p > 2n+1$.

Finally, we mention that [8] gives an introductory survey of the results obtained in this paper.

1.1. **Statement of the main result.** Let V be a rational non-degenerate quadratic space of dimension m and signature (p,q) and let $\underline{G} = \mathrm{SO}(V)$. We let $G = \underline{G}(\mathbb{R})_0 = \mathrm{SO}_0(V_{\mathbb{R}})$. Let $D_V = D = G/K$ be the symmetric space of G with $K = K_V$ a maximal compact subgroup. We let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the associated Cartan decomposition of the Lie algebra of G associated to a Cartan involution θ. We let Γ be an appropriate congruence subgroup of $\underline{G}(\mathbb{Z}) \cap G$ and write $X = X_V = \Gamma \backslash D$ for the associated locally symmetric space. We let $G' = \mathrm{Mp}(n,\mathbb{R})$ denote the metaplectic covering group of the symplectic group $\mathrm{Sp}(n,\mathbb{R})$, and we let K' be the 2-fold covering of $\mathrm{U}(n)$ in G'. Note that K' admits a character $\det^{-1/2}$ of K'. We let $U = \bigwedge^n(\mathbb{C}^n)^*$ and define an action of K' on the tensor power $T^j(U)$ by requiring that K' acts on $T^j(U)$ by $\det^{-j-(p-q)/2}$. Furthermore, we let $\mathcal{W}_{n,V}$ be the K'-finite vectors of the Weil representation of $G' \times G$.

Every partition λ of a non-negative integer ℓ' into at most n parts gives rise to a dominant weight λ of $\mathrm{GL}(n)$. We write $i(\lambda)$ for the number of nonzero entries of λ. We explicitly realize the corresponding irreducible representation of highest weight λ as the image $\mathbb{S}_\lambda(\mathbb{C}^n)$ of the Schur functor $\mathbb{S}_\lambda(\cdot)$ for (a standard filling of) the Young diagram $D(\lambda)$ associated to λ applied to the tensor space $T^{\ell'}(\mathbb{C}^n)$. We can apply the same Schur functor to $T^{\ell'}(V_{\mathbb{C}})$ to obtain the space $\mathbb{S}_\lambda(V_{\mathbb{C}})$. We let $\mathcal{H} : T^{\ell'}(V_{\mathbb{C}}) \to V_{\mathbb{C}}^{[\ell']}$ be the orthogonal projection to the harmonic ℓ'-tensors $V_{\mathbb{C}}^{[\ell']}$. Then, if the sum of the lengths of the first two columns of $D(\lambda)$ is at most m, applying $\mathcal{H}$ to $\mathbb{S}_\lambda(V_{\mathbb{C}})$ one obtains the nonzero irreducible representation $\mathbb{S}_{[\lambda]}(V_{\mathbb{C}})$ for $\mathrm{O}(V_{\mathbb{C}})$, see [7] section 19.5. Furthermore, this gives rise to an irreducible representation for G of a certain highest weight $\widetilde{\lambda}$ (unless m is even and $i(\lambda) = \frac{m}{2}$, in that case it splits into two irreducible representations). If $i(\lambda) \leq [\frac{m}{2}]$, then $\tilde{\lambda}$ has the same nonzero entries as λ (when $\tilde{\lambda}$ is expressed in coordinates relative to the standard basis $\{\epsilon_i\}$ of [6], Planche II and IV).

Note that in our previous work [9], we proceeded the other way starting from a dominant weight λ for G to obtain a weight λ' for $\mathrm{GL}(n,\mathbb{C})$ such that λ and λ' have the same nonzero entries (and subsequently did not distinguish between λ and λ' writing $\lambda = \lambda'$). For the purposes of this paper going from $\mathrm{GL}(n,\mathbb{C})$ to G is more convenient. The results of [9] are still valid in this more general setting.

The main point of [9] was the construction of certain $(\mathfrak{g}, K)$-cocycles

$$\varphi^V_{nq,[\lambda]} \in C_V^{q,nq,[\lambda]}, \tag{1.1}$$

where $C_V^{\bullet,[\lambda]}$ is the complex given by

$$\begin{aligned} C_V^{j,r,[\lambda]} &= \left[T^j(U) \otimes \mathbb{S}_\lambda(\mathbb{C}^n)^* \otimes \mathcal{W}_{n,V} \otimes \mathcal{A}^r(D) \otimes \mathbb{S}_{[\lambda]}(V_\mathbb{C})\right]^{K'\times G} \\ &\simeq \left[T^j(U) \otimes \mathbb{S}_\lambda(\mathbb{C}^n)^* \otimes \mathcal{W}_{n,V} \otimes \bigwedge\nolimits^r(\mathfrak{p}_\mathbb{C}^*) \otimes \mathbb{S}_{[\lambda]}(V_\mathbb{C})\right]^{K'\times K_V}. \end{aligned} \tag{1.2}$$

Here $\mathcal{A}^{nq}(D)$ denotes the differential nq-forms on D. The isomorphism is given by evaluation at the base point of D. Furthermore, K' acts on the first three tensor factors, while G (resp. K) acts on the last three. The differential is the usual relative Lie algebra differential for the action of $\mathrm{O}(V)$. These forms vanish if $n > p$.

These classes generalize the work of Kudla and Millson (see e.g. [18]) to the case of nontrivial coefficients systems $\mathbb{S}_{[\lambda]}(V_\mathbb{C})$. In [9], we showed how theta series $\theta(\varphi^V_{nq,[\lambda]})$ associated to $\varphi^V_{nq,[\lambda]}$ give rise to holomorphic vector-valued Siegel modular forms whose Fourier expansions involve periods over certain "special" cycles with coefficients.

The purpose of this paper is to study the boundary behavior of these classes.

We let $P = \underline{P}(\mathbb{R})_0$ be the connected component of the identity of the real points of a rational parabolic subgroup $\underline{P}$ in $\underline{G}$ stabilizing a flag $\mathbf{F}$ of totally isotropic rational subspaces in V. We can choose $\mathbf{F}$ in such a way such that the θ-stable subgroup of $\underline{P}$ forms a Levi subgroup. We let $P = NAM$ be the associated (rational) Langlands decomposition, and we let $\mathfrak{m}$ and $\mathfrak{n}$ be the Lie algebras of M and N respectively. We set $\mathfrak{p}_M = \mathfrak{p} \cap \mathfrak{m}$. Let E be the largest element in the isotropic flag $\mathbf{F}$ with dimension ℓ. We let $W = E^\perp/E$, which is naturally a quadratic space of signature $(p-\ell, q-\ell)$, and we realize W as a subspace of V such that the Cartan involution θ for $\mathrm{O}(V)$ restricts to one for $\mathrm{O}(W)$. Then M splits naturally into the product of $\mathrm{SO}_0(W_\mathbb{R})$ with a product of special linear groups of subquotients of $E_\mathbb{R}$, and consequently we have a projection map from the symmetric space D_M for M to D_W, the symmetric space for $\mathrm{SO}_0(W_\mathbb{R})$. We let $e(\underline{P}) = NM/K_P$ be the associated face of the Borel-Serre enlargement of D with $K_P = M \cap K$, see [3].

We consider an analogous complex $A_P^{\bullet,[\lambda]}$ at the boundary given by

$$\begin{aligned} A_P^{j,r,[\lambda]} &= \left[T^j(U) \otimes \mathbb{S}_\lambda(\mathbb{C}^n)^* \otimes \mathcal{W}_{n,W} \otimes \mathcal{A}^r(e(\underline{P})) \otimes \mathbb{S}_{[\lambda]}(V_\mathbb{C})\right]^{K'\times NM} \\ &\simeq \left[T^j(U) \otimes \mathbb{S}_\lambda(\mathbb{C}^n)^* \otimes \mathcal{W}_{n,W} \otimes \bigwedge\nolimits^r(\mathfrak{n} \oplus \mathfrak{p}_M)^*_\mathbb{C} \otimes \mathbb{S}_{[\lambda]}(V_\mathbb{C})\right]^{K'\times K_P} \end{aligned} \tag{1.3}$$

with coefficients in the Weil representation for $G' \times \mathrm{O}(W_\mathbb{R})$. We introduce a local restriction map of complexes

$$r_P : C_V^\bullet \to A_P^\bullet, \tag{1.4}$$

induced by a natural (restriction) map $\bigwedge^r(\mathfrak{p}^*) \to \bigwedge^r(\mathfrak{n} \oplus \mathfrak{p}_M)^*$ and by a $G' \times NM$-intertwiner from $\mathcal{W}_{n,V}$ to $\mathcal{W}_{n,W}$ using a so-called mixed model of $\mathcal{W}_{n,V}$.

We also construct an inclusion map ι_P of complexes from the relative Lie algebra complex $C_W^\bullet$ for W into $A_P^\bullet$ with the property

$$\iota_P : C_W^{j-\ell,r,[\ell\varpi_n+\lambda]} \hookrightarrow A_P^{j,r+n\ell,[\lambda]}. \tag{1.5}$$

Here $\varpi_n = (1,\ldots,1)$ is the n-th fundamental weight for $\mathrm{GL}(n)$, so that the Young diagram associated to $\ell\varpi_n$ is an n by ℓ rectangle. The map ι_P is a $G' \times \mathrm{O}(W_\mathbb{R})$-intertwiner. The map ι_P is induced by the pullback of D_W to D_M and the explicit construction of an embedding of $\mathbb{S}_{[\ell\varpi_n+\lambda]}(W_\mathbb{C})$ into the cocyles in the nilpotent Lie algebra complex $\bigwedge^{n\ell}(\mathfrak{n}_\mathbb{C}^*) \otimes S_{[\lambda]}(V_\mathbb{C})$. In fact, we have an embedding of M-modules

$$\mathbb{S}_{[\ell\varpi_n+\lambda]}(W_\mathbb{C}) \hookrightarrow H^{n\ell}(\mathfrak{n}, \mathbb{S}_{[\lambda]}(V_\mathbb{C})) \tag{1.6}$$

into the nilpotent cohomology.

We explicitly describe the local restriction $r_P(\varphi_{nq,[\lambda]}^V)$ in the complex $A_P^\bullet$. As a consequence we obtain

Theorem 1.3. *Let $i(\lambda) \leq n \leq p$. Then the cohomology class $[r_P(\varphi_{nq,[\lambda]}^V)]$ satisfies*

$$[r_P(\varphi_{nq,[\lambda]}^V)] = [\iota_P(\varphi_{n(q-\ell),[\ell\varpi_n+\lambda]}^W)].$$

In particular, $[r_P(\varphi_{nq,[\lambda]}^V)] = 0$ for $n > \min\left(p, \left[\frac{m}{2}\right]\right) - \ell$ (if $\ell \geq 2$) and $n > p-1$ or $n > m-2-i(\lambda)$ (if $\ell = 1$).

On the other hand, let P be a (real) parabolic subgroup as above such that the associated space W is positive definite. Assume

$$i(\lambda) \leq n \leq \begin{cases} \left[\frac{p-q}{2}\right] & \text{if } q \geq 2 \\ p-1-i(\lambda) & \text{if } q = 1. \end{cases} \tag{1.7}$$

Then

$$[r_P(\varphi_{nq,[\lambda]}^V)] \neq 0.$$

The connection between the local restriction map and the global restriction to a face of the Borel-Serre compactification $\overline{X}$ is the following. We let $e'(P)$ be the corner corresponding to $e(P)$ in the Borel-Serre compactification $\overline{X}$ of X, and let $\tilde{r}_P$ be the restriction map from X to the corner $e'(P)$. There is a global version of (1.5) which on the level of cohomology induces a map

$$\begin{aligned} \widetilde{\iota}_P : H^{n(q-\ell)}(X_W, \mathbb{S}_{[\ell\varpi_n+\lambda]}(W_\mathbb{C})) &\hookrightarrow H^{n(q-\ell)}(X_M, H^{n\ell}(\mathfrak{n}, \mathbb{S}_{[\lambda]}(V_\mathbb{C}))) \\ &\hookrightarrow H^{nq}(e'(P), \mathbb{S}_{[\lambda]}(V_\mathbb{C})). \end{aligned} \tag{1.8}$$

Here X_M and X_W are suitable arithmetic quotients of D_M and D_W respectively.

We let $\theta_{\mathcal{L}_V}(g', \varphi_{nq,[\lambda]}^V)$ with $g' \in G'$ be the theta series associated to the Schwartz form $\varphi_{nq,[\lambda]}$ and a theta distribution associated to a suitable (coset of a) lattice $\mathcal{L}_V$ in V stabilized by Γ viewed as a closed differential nq-form on X. The theta series is termwise moderately increasing. However, switching to the mixed model of the Weil representation we show

Theorem 1.4. *The form $\theta_{\mathcal{L}_V}(\varphi^V_{nq,[\lambda]})$ extends to a differential form on the Borel-Serre compactification $\overline{X}$. Moreover, for a given face $e'(P)$, there exists a theta distribution $\widehat{\mathcal{L}}_W$ for W such that*

$$[\tilde{r}_P(\theta_{\mathcal{L}_V}(\varphi^V_{nq,[\lambda]}))] = [\theta_{\widehat{\mathcal{L}}_W}(r_P(\varphi^V_{nq,[\lambda]}))]$$

and

$$[\tilde{r}_P(\theta_{\mathcal{L}_V}(\varphi^V_{nq,[\lambda]}))] = [\widetilde{\iota}_P(\theta_{\widehat{\mathcal{L}}_W}(\varphi^W_{n(q-\ell),[\ell\varphi_n+\lambda]}))].$$

Loosely speaking Theorem 1.3 and Theorem 1.4 can be summarized that the restriction of our theta series for $\mathrm{O}(V)$ to a face of the Borel-Serre compactification is the theta series for $\mathrm{O}(W)$ of the same type corresponding to an enlarged coefficient system corresponding to placing an n by ℓ rectangle on the left of the Young diagram corresponding to λ to obtain a bigger Young diagram corresponding to $\ell\varpi_n + \lambda$.

1.2. **Applications.**

1.2.1. *Nonvanishing of the special cycles.* Restricting to a face $e'(P)$ such that the associated subspace W of V is positive definite, we obtain

Theorem 1.5. *Assume that the $\mathbb{Q}$-rank and the $\mathbb{R}$-rank of $\underline{G}$ coincide. Then in the nonvanishing range* (1.7) *of Theorem 1.3, there exists a finite cover X' of X such that*

$$[\theta(\varphi^V_{nq,[\lambda]})] \neq 0.$$

In particular,

$$H^{nq}(X', \mathbb{S}_{[\lambda]}(V_{\mathbb{C}})) \neq 0.$$

Furthermore, $H^{nq}(X', \mathbb{S}_{[\lambda]}(V_{\mathbb{C}}))$ is not spanned by classes arising from invariant forms on D.

When viewed as a function of $\tau \in \mathbb{H}_n$, the Siegel upper half space, it is shown in [9] that the theta series $[\theta_{\mathcal{L}_V}(\tau, \varphi^V_{nq,[\lambda]})]$ is a holomorphic vector-valued Siegel modular form for the representation $S_\lambda(\mathbb{C}^n) \otimes \det^{m/2}$ with values in $H^{nq}(X, \mathbb{S}_{[\lambda]}(V_{\mathbb{C}}))$. The Fourier expansion of such a form is parametrized by (integral) positive semi-definite matrices T. Furthermore, for T positive definite, the T-th Fourier coefficient of $\theta_{\mathcal{L}_V}(\tau, \varphi^V_{nq,[\lambda]})$ represents a Poincaré dual form for certain special cycles

$$Z_{T,[\lambda]} \in \mathrm{Hom}\left(S_\lambda(\mathbb{C}^n), H_{p(n-q)}(X, \partial X, S_{[\lambda]}(V_{\mathbb{C}}))\right). \tag{1.9}$$

These cycles arise from embedded orthogonal groups associated to positive definite rational subspaces $U \subset V$ of signature $(p-n, q)$. Namely, the embedding of the stabilizer G_U of such a subspace in to G defines a subsymmertic space D_U in D and a cycles $Z_U = (\Gamma \cap G_U)\backslash D_U$ in X. For $\mathbf{x} = (x_1, \dots, x_n) \in \mathcal{L}$, we let $(\mathbf{x}, \mathbf{x}) = (x_i, x_j)_{i,j}$ be its inner product matrix. Then (for trivial coefficients), the cycles Z_T are given by

$$Z_T = \sum_{\substack{\mathbf{x} \in \Gamma\backslash\mathcal{L} \\ \frac{1}{2}(\mathbf{x},\mathbf{x}) = T}} Z_{\mathrm{span}(\mathbf{x})}. \tag{1.10}$$

We then obtain

Corollary 1.6. *Under the hypotheses of Theorem 1.5, there exists a finite cover X' of X such that*

$$Z_{T,[\lambda]} \neq 0$$

for infinitely many T in cohomology.

Note that in the case of nontrivial coefficients for compact hyperbolic manifolds (when $q = 1$), Millson [21] proved the cohomological nonvanishing of the special cycles in codimension n in the range $i(\lambda) \leq n \leq p - i(\lambda)$. The shift in the noncompact case for the upper bound for nonvanishing is of course expected by Borel-Serre [4].

1.2.2. *Hirzebruch-Zagier.* We first describe the general main motivation for our work.

The theta series $\theta_{\mathcal{L}}(\varphi_{nq,0})$ (for simplicity, we only consider here the trivial coefficient system) gives rise to a lift

$$\Lambda_{nq}(\eta, \tau) = \int_X \eta \wedge \theta_{\mathcal{L}}(\varphi_{nq,0})(\tau) \tag{1.11}$$

for η a closed $(p-n)q$-form on X with compact support. This map factors through cohomology, and we obtain a map

$$\Lambda_{nq} : H_c^{(p-n)q}(X, \mathbb{C}) \to M^{(n)}_{\frac{m}{2}}(\Gamma') \tag{1.12}$$

from the cohomology with compact supports to the space of holomorphic Siegel modular forms of degree n of weight $m/2$ for a certain congruence subgroup $\Gamma' \subseteq \mathrm{Sp}(n, \mathbb{Z})$.

We are interested in the problem of extending the lift by Kudla and Millson (and our extension to non-trivial coefficient systems) to other cohomology groups of the space X.

The point is that the original work of Hirzebruch-Zagier provides a lift from the full cohomology of a Hilbert modular surface to classical holomorphic modular forms. For the following, see [14, 25]. For signature $(2, 2)$ and $\mathbb{Q}$-rank 1, X is a Hilbert modular surface, and the cycles in question for $n = 1$ are the famous Hirzebruch-Zagier curves T_N parameterized by positive integers N. These cycles define classes in $H^{1,1}(X, \mathbb{Q})$. Hirzebruch-Zagier consider the smooth (toroidal) compactification $\widetilde{X}$ of X. Then the cohomology $H^2(\widetilde{X}, \mathbb{Q})$ naturally splits into the orthogonal direct sum of the classes obtained by the compactifying divisors and the image of $H_c^2(X, \mathbb{Q})$ inside $H^2(\widetilde{X}, \mathbb{Q})$ induced by the projection $\widetilde{X} \to Y$, the Baily-Borel compactification of X. Hirzebruch-Zagier define the cycle T_N^c by the projection onto the part coming from the interior of X. They then show by explicit calculation of intersection numbers that the generating series of the truncated cycles T_N^c is a holomorphic modular form of weight 2. In particular, for fixed $M \geq 0$,

$$P_M(\tau) := \sum_{N=0}^{\infty} \langle T_M, T_N^c \rangle e^{2\pi i N \tau} \in M_2(\Gamma'). \tag{1.13}$$

Since the curves T_M generate $H^{1,1}(\widetilde{X}, \mathbb{Q})$, one obtains a lift

$$P : H^{1,1}(\widetilde{X}, \mathbb{Q}) \to M_2(\Gamma'). \tag{1.14}$$

Moreover, the *holomorphic* form P_M can be written as the sum of two non-holomorphic modular forms

$$P_M(\tau) = H_M(\tau) + I_M(\tau), \tag{1.15}$$

where H_M incorporates the intersection numbers of T_N and T_M in the interior of X (suitably defined) and I_M the boundary contributions.

In a different paper [10], we will obtain the work of Hirzebruch-Zagier from a topological point of view using our current work. We give a brief sketch.

The theta series $\theta_{\mathcal{L}_V}(\varphi^V_{2,0})$ associated to the given data now has weight 2. In this case, the restriction of the classes $\theta_{\mathcal{L}_V}(\varphi^V_{2,0})$ to any boundary face $e'(P)$ is a theta series for the associated space W of signature $(1,1)$. We have

Theorem 1.7. *The restriction of $\theta_{\mathcal{L}_V}(\varphi^V_{2,0})$ to ∂X gives an exact differential form on ∂X. For each boundary face $e'(P)$ there exists a theta series $\theta_{\widehat{\mathcal{L}}_W}(\phi^W)$ for signature $(1,1)$ of weight 2 for a certain $\phi \in L^2(W_{\mathbb{R}})$ with values in the differential 1-forms on $e(P)$ such that $\theta_{\widehat{\mathcal{L}}_W}(\phi^W)$ is a primitive for $\tilde{r}_P\theta_{\mathcal{L}_V}(\varphi^V_{2,0})$ on the boundary face $e'(P)$:*

$$d(\theta_{\widehat{\mathcal{L}}_W}(\phi^W)) = \tilde{r}_P\theta_{\mathcal{L}_V}(\varphi^V_{2,0}).$$

Note this also shows that the hypotheses of Theorem 1.5 are necessary in general.

The boundary faces are all isolated, and for simplicity we now assume that there is only one cusp, that is, $\partial X = e'(P)$ for one parabolic P. Via the usual mapping cone construction we then view the pair $[\theta_{\mathcal{L}_V}(\varphi^V_{2,0}), \theta_{\widehat{\mathcal{L}}_W}(\phi^W)]$ as an element of $H^2_c(X,\mathbb{C})$. Explictly, let η be a closed 2-form on the Borel-Serre compactification $\overline{X}$ representing a class $[\eta]$ in the full cohomology $H^2(X,\mathbb{C})$. The pairing between $[\theta_{\mathcal{L}_V}(\varphi^V_{2,0}), \theta_{\widehat{\mathcal{L}}_W}(\phi^W)]$ and $[\eta]$ is given by

$$\langle [\eta], [\theta_{\mathcal{L}_V}(\varphi^V_{2,0}), \theta_{\widehat{\mathcal{L}}_W}(\phi^W)]\rangle = \int_X \eta \wedge \theta_{\mathcal{L}_V}(\varphi^V_{2,0}) - \int_{\partial X} \tilde{r}_P(\eta) \wedge \theta_{\widehat{\mathcal{L}}_W}(\phi^W).$$

This provides the envisioned extension of the geometric theta lift. In particular, we recover Hirzebruch-Zagier:

Theorem 1.8. *The pairing $\langle [\eta], [\theta_{\mathcal{L}_V}(\varphi^V_{2,0}), \theta_{\widehat{\mathcal{L}}_W}(\phi^W)]\rangle$ takes values in the holomorphic modular forms of weight 2. Hence the pair $[\theta_{\mathcal{L}_V}(\varphi^V_{2,0}), \theta_{\widehat{\mathcal{L}}_W}(\phi^W)]$ induces a map on the full cohomology of X to the space of holomorphic modular forms:*

$$\Lambda : H^2(X,\mathbb{C}) \to M_2(\Gamma').$$

Moreover, when viewed as an element of $H^2(\widetilde{X},\mathbb{C})$, we have

$$[\theta_{\mathcal{L}_V}(\varphi^V_{2,0}), \theta_{\widehat{\mathcal{L}}_W}(\phi^W)] = \sum_{N \geq 0} T^c_N e^{2\pi i N\tau}.$$

In particular,

$$\langle T_M, [\theta_{\mathcal{L}_V}(\varphi^V_{2,0}), \theta_{\widehat{\mathcal{L}}_W}(\phi^W)]\rangle = \int_{T_M} \theta_{\mathcal{L}_V}(\varphi^V_{2,0}) - \int_{\partial T_M} \theta_{\widehat{\mathcal{L}}_W}(\phi^W) = P_M,$$

and the two integrals are the two summands H_M and I_M of P_M, see (1.15).

There are other instances when the restriction to the boundary vanishes in cohomology at each face, and it is natural to expect that a similar analysis can be carried out. Particularly attractive is the case $n = p-1$ for quotients of the Hermitian domain associated to $\mathrm{O}(p,2)$. In this situation the cycles are modular or Shimura curves.

For the general case, we anticipate that the interpretation of the special classes in weighted cohomology should be the proper setting, see Remark 9.6.

The paper is organized as follows. In section 2, we establish the basic notation of the paper. In particular, we introduce the locally symmetric space X and its Borel-Serre compactification and give an explicit description of the parabolic subgroups of $\underline{G}$. In section 3, we briefly review some basics of the representation theory of $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{O}(V_{\mathbb{C}})$ via the Schur functor $\mathbb{S}(\cdot)$. In section 4, we study various models of the Weil representation and introduce the Weil representation restriction map $r_P^{\mathcal{W}}$ from $\mathcal{S}(V_{\mathbb{R}}^n)$ to $\mathcal{S}(W_{\mathbb{R}}^n)$ and study $r_P^{\mathcal{W}}$ for a certain class of Schwartz functions. In section 5, we introduce the complexes $C_V^{\bullet}$ and $A_P^{\bullet}$ and define the map ι_P from $C_W^{\bullet}$ to $A_P^{\bullet}$. In section 6, we study certain aspects of the cohomology of $A_P^{\bullet}$ arising from the fiber N_P. In particular we introduce the map ι_P (1.5). In sections 7 and 8, we study the special Schwartz form $\varphi_{nq,[\lambda]}$ (and variants of it) and establish a first version of the local restriction formula on the level of differential forms and then establish Theorem 1.3. In section 9, we turn to the global situation. We introduce the theta series $\theta(\varphi_{nq,[\lambda]}^V])$ and prove Theorem 1.4. From this, we obtain the nonvanishing result Theorem 1.6.

We would like to thank G. Gotsbacher, L. Saper, and J. Schwermer for fruitful discussions and also E. Freitag and R. Schulze-Pillot for answering a question on positive definite theta series. As always it is a pleasure to thank S. Kudla for his encouragement.

The work on this paper has greatly benefitted from three visits of the first named author at the Max Planck Institute from 2005 to 2008. He gratefully acknowledges the excellent research environment in Bonn.

2. Basic Notations

2.1. **Orthogonal Symmetric Spaces.** Let V be a rational vector space of dimension $m = p+q$ and let $(\ ,\ )$ be a non-degenerate symmetric bilinear form on V with signature (p,q). We fix a standard orthogonal basis $e_1, \dots, e_p, e_{p+1}, \dots, e_m$ of $V_{\mathbb{R}}$ such that $(e_\alpha, e_\alpha) = 1$ for $1 \leq \alpha \leq p$ and $(e_\mu, e_\mu) = -1$ for $p+1 \leq \mu \leq m$. (We will use "early" Greek letters to denote indices between 1 and p, and "late" ones for indices between $p+1$ and m). With respect to this basis the matrix of the bilinear form is given by the matrix $I_{p,q} = \begin{pmatrix} 1_p & \\ & -1_q \end{pmatrix}$.

We let $\underline{G} = \mathrm{SO}(V)$ viewed as an algebraic group over $\mathbb{Q}$. We let $G := G(\mathbb{R})_0$ be the connected component of the identity of $G(\mathbb{R})$ so that $G \simeq \mathrm{SO}_0(p,q)$. We let K be the maximal compact subgroup of G stabilizing $\mathrm{span}\{e_\alpha; 1 \leq \alpha \leq p\}$. Thus $K \simeq \mathrm{SO}(p) \times \mathrm{SO}(q)$. Let $D = G/K$ be the symmetric space of dimension pq associated to G. We realize D as the space of negative q-planes in $V_{\mathbb{R}}$:

$$D \simeq \{z \subset V_{\mathbb{R}} : \dim z = q;\ (\ ,\ )|_z < 0\}. \tag{2.1}$$

Thus $z_0 = \operatorname{span}\{e_\mu; p+1 \leq \mu \leq m\}$ is the base point of D. Furthermore, we can also interpret D as the space of minimal majorants for $(\ ,\ )$. That is, $z \in D$ defines a majorant $(\ ,\ )_z$ by $(x,x)_z = -(x,x)$ if $x \in z$ and $(x,x)_z = (x,x)$ if $x \in z^{\perp}$. We write $(\ ,\ )_0$ for the majorant associated to the base point z_0.

The Cartan involution θ_0 of G corresponding to the basepoint z_0 is obtained by conjugation by the matrix $I_{p,q}$. We will systematically abuse notation below and write $\theta_0(v)$ for the action of the linear transformation of V with matrix $I_{p,q}$ relative to the above basis acting on $v \in V$. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{k}$ be the one of K. We obtain the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \tag{2.2}$$

where

$$\mathfrak{p} = \operatorname{span}\{X_{\alpha\mu} := e_\alpha \wedge e_\mu; 1 \leq \alpha \leq p, p+1 \leq \mu \leq m\}. \tag{2.3}$$

Here $w \wedge w' \in \bigwedge^2 V_{\mathbb{R}}$ is identified with an element of $\mathfrak{g}$ via

$$(w \wedge w')(v) = (w,v)w' - (w',v)w. \tag{2.4}$$

We let $\{\omega_{\alpha\mu}\}$ be the dual basis of $\mathfrak{p}^*$ corresponding to $\{D_{\alpha\mu}\}$. Finally note that we can identify $\mathfrak{p}$ with the tangent space $T_{z_0}(D)$ at the base point z_0 of D.

Let $L \subset V$ be an even $\mathbb{Z}$-lattice of full rank, i.e., $(x,x) \in 2\mathbb{Z}$ for $x \in L$. In particular, $L \subset L^{\#}$, the dual lattice. We denote by $\Gamma(L)$ the stabilizer of the lattice L and fix a neat subgroup Γ of finite index in $\Gamma(L) \cap G$ which acts trivially on $L^{\#}/L$. We let $X = \Gamma\backslash D$ be the locally symmetric space. We assume that X is non-compact. It is well known that this is the case if and only if V has an isotropic vector over $\mathbb{Q}$. We let r be the Witt rank of V, i.e., the dimension of a maximal totally isotropic subspace of V over $\mathbb{Q}$.

Let F be an isotropic subspace of V of dimension ℓ. Then we can describe the ℓ-dimensional isotropic subspace $\theta_0(F)$ as follows. For U a subspace of V, let $U^{\perp}$, resp. $U^{\perp_0}$ be the orthogonal complement of U for the form $(\ ,\ )$, resp. $(\ ,\ )_0$. Then

$$\theta_0(F) = (F^{\perp})^{\perp_0}.$$

We fix a maximal totally isotropic subspace E_r and choose a basis $u_1, u_2, \dots, u_r$ of E_r. Let $E'_r = \theta_0(E_r)$. We pick a basis $u'_r, \cdots, u'_1$ of E'_r such that $(u_i, u'_j) = \delta_{ij}$. More generally, we let

$$E_\ell := \operatorname{span}\{u_1, \dots, u_\ell\}, \tag{2.5}$$

and we call E_ℓ a *standard* totally isotropic subspace. Furthermore, we set $E'_\ell = \theta_0(E_\ell) = \operatorname{span}(u'_\ell, \dots, u'_1)$. Note that E'_ℓ can be naturally identified with the dual space of E_ℓ. We can assume that with respect to the standard basis of $V_{\mathbb{R}}$ we have

$$e_\alpha = \frac{1}{\sqrt{2}}(u_\alpha - u'_\alpha) \qquad \text{and} \qquad e_{m+1-\alpha} = \frac{1}{\sqrt{2}}(u_\alpha + u'_\alpha). \tag{2.6}$$

for $\alpha = 1, \dots, \ell$. We let

$$W_\ell = E_\ell^{\perp}/E_\ell, \tag{2.7}$$

and note that W_ℓ is a non-degenerate space of signature $(p-\ell, q-\ell)$. We can realize W_ℓ as a subspace of V by

$$W_\ell = (E_\ell \oplus E'_\ell)^\perp, \tag{2.8}$$

where the orthogonal complement is either with respect to $(\, , \,)$ or $(\, , \,)_0$. This gives a θ_0-invariant Witt splitting

$$V = E_\ell \oplus W_\ell \oplus E'_\ell. \tag{2.9}$$

Note that with these choices θ_0 restricts to a Cartan involution for $\mathrm{O}(W_\ell(\mathbb{R}))$. We obtain a Witt basis $u_1, \dots, u_\ell, e_{\ell+1}, ..., e_{m-\ell}, u'_\ell, \dots, u'_1$ for $V_\mathbb{R}$. We will denote coordinates with respect to the Witt basis with y_i and coordinates with respect to the standard basis with x_i. Note that with respect to the Witt basis, the bilinear form $(\, , \,)$ has Gram matrix

$$(\, , \,) \sim \begin{pmatrix} & & J \\ & 1_{W_\ell} & \\ J & & \end{pmatrix} \tag{2.10}$$

with $J = \begin{pmatrix} & & 1 \\ & \cdots & \\ 1 & & \end{pmatrix}$.

We often drop the subscript ℓ and just write E, E', and W.

2.2. **Parabolic Subgroups.** We now describe the parabolic subgroups of $\underline{G}$. We follow in part [3]. We let $\mathbf{F}$ be a flag of totally isotropic subspaces $0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_k \subset F_k^\perp \subset \cdots \subset F_1^\perp \subset V$ of V over $\mathbb{Q}$. Then we let $\underline{P} = \underline{P}_\mathbf{F}$ be the parabolic subgroup of $\underline{G}$ stabilizing the flag $\mathbf{F}$:

$$\underline{P}_\mathbf{F} = \{g \in \underline{G}; gF_i = F_i\}. \tag{2.11}$$

We let $\underline{N}_P$ be the unipotent radical of $\underline{P}$. It acts trivially on all quotients of the flag. We let $\underline{L}_P = \underline{N}_P \backslash \underline{P}$ and let $\underline{S}_P$ be the split center of $\underline{L}_P$ over $\mathbb{Q}$. Note that $\underline{S}_P$ acts by scalars on each quotient. Let $\underline{M}_P = \cap_{\chi \in D(\underline{L}_P)} \mathrm{Ker}(\chi^2)$. We let $N = N_P$ and $L = L_{\underline{P}}$ be their respective real points in G, and let $P = P_\mathbf{F} = (\underline{P}_\mathbf{F}(\mathbb{R}))_0$, $M = M_P = (\underline{M}_P(\mathbb{R}))_0$, and $A = A_{\underline{P}} = (\underline{S}_P(\mathbb{R}))_0$ be the connected component of the identity in $\underline{P}(\mathbb{R})$, $\underline{M}_P(\mathbb{R})$, and $\underline{S}_P(\mathbb{R})$ respectively.

By conjugation, we can assume that the flag $\mathbf{F}$ consists of standard totally isotropic subspaces E_i (2.5) and call $\underline{P}_\mathbf{F}$ a *standard* $\mathbb{Q}$-parabolic. In that case, using the Cartan involution θ_0, we realize $\underline{L}_P$ (and also $\underline{S}_P, \underline{M}_P$) as θ_0-stable subgroups of $\underline{P}$:

$$\underline{L}_P = \underline{P} \cap \theta_0(\underline{P}). \tag{2.12}$$

Then $\underline{M}_P$ is the semi-simple part of the centralizer of $\underline{S}_P$ in $\underline{P}$. We will regularly drop the subscripts $\mathbf{F}$, $\underline{P}$, and P.

We obtain the (rational) Langlands decomposition of P:

$$P = NAM \simeq N \times A \times M, \tag{2.13}$$

and we write $\mathfrak{n}$, $\mathfrak{a}$, and $\mathfrak{m}$ for their respective Lie algebras. The map $P \to N \times A \times M$ is equivariant with the P-action defined by

$$n'a'm'(n,a,m) = (n' Ad(a'm')(n), a'a, m'm)\,. \tag{2.14}$$

We let $\mathbf{F}$ be a standard rational totally isotropic flag $0 = E_0 \subset E_{i_1} \subset \cdots \subset E_{i_k} = E_\ell = E$ and assume that the last (biggest) totally isotropic space in the flag $\mathbf{F}$ is equal to E_ℓ for some ℓ.

Let $U_{i_j} = \operatorname{span}(u_{i_{j-1}+1}, \dots, u_{i_j})$ be the orthogonal complement of $E_{i_{j-1}}$ in E_{i_j} with respect to $(\ ,\ )_0$ and U'_{i_j} be the orthogonal complement of E'_{i_j} in $E'_{i_{j+1}}$ and let $W = W_\ell = (E_\ell \oplus E'_\ell)^\perp$. We obtain a refinement of the Witt decomposition of V such that the subspaces U_{i_j}, U'_{i_s}, and W are mutually orthogonal for $(\ ,\ )_0$ and defined over $\mathbb{Q}$:

$$V = \left(\bigoplus_{i_j=1}^{k} U_{i_j} \right) \oplus W \oplus \left(\bigoplus_{i_j=1}^{k} U'_{i_j} \right). \tag{2.15}$$

Then $\underline{L}_P$ is the subgroup of $\underline{P}$ that stabilizes each of the subspaces in the above decomposition of V. In what follows we will describe matrices in block form relative to the above direct sum decomposition of V.

We first note that we naturally have $\mathrm{O}(W) \times \mathrm{GL}(E) \subset \mathrm{O}(V)$ via

$$\left\{ \begin{pmatrix} g & & \\ & h & \\ & & \tilde{g} \end{pmatrix}; h \in \mathrm{O}(W), g \in \mathrm{GL}(E) \right\}, \tag{2.16}$$

where $\tilde{g} = J g^* J$ and $g^* = {}^t g^{-1}$. In particular, we can view the corresponding Lie algebras $\mathfrak{o}(W_\mathbb{R})$ and $\mathfrak{gl}(E_\mathbb{R})$ as subalgebras of $\mathfrak{g}$. Namely,

$$\mathfrak{o}(W_\mathbb{R}) \simeq \operatorname{span}\{e_i \wedge e_j;\ \ell < i < j \leq m - \ell\}, \tag{2.17}$$

$$\mathfrak{gl}(E_\mathbb{R}) \simeq \operatorname{span}\{u'_i \wedge u_j;\ i, j \leq \ell\}. \tag{2.18}$$

via the identification $\mathfrak{g} \simeq \bigwedge^2 V_\mathbb{R}$.

We let $\underline{S}$ be the maximal $\mathbb{Q}$-split torus of $\underline{G}$ given by

$$\underline{S} = \left\{ a(t_1, \dots, t_r) := \begin{pmatrix} \operatorname{diag}(t_1,\dots,t_r) & & \\ & 1 & \\ & & \operatorname{diag}(t_r^{-1},\dots,t_1^{-1}) \end{pmatrix} \right\}. \tag{2.19}$$

We write $\mathbf{t} = (t_1, \dots, t_r)$ and $\widetilde{\mathbf{t}} = \mathbf{t}J = (t_r, \dots, t_1)$. Note

$$\exp(u'_i \wedge u_i) = a(0, \dots, 0, 1, 0, \dots, 0). \tag{2.20}$$

The set of simple rational roots for $\underline{G}$ with respect to $\underline{P}$ and $\underline{S}$ is given by $\Delta = \Delta(\underline{S}, \underline{G}) = \{\alpha_1, \dots, \alpha_r\}$, where

$$\alpha_i(a) = t_i t_{i+1}^{-1}, \qquad (1 \leq i \leq r-1) \tag{2.21}$$

$$\alpha_r(a) = \begin{cases} t_r & \text{if } W_r \neq 0 \\ t_{r-1} t_r & \text{if } W_r = 0. \end{cases} \tag{2.22}$$

We write $\Phi(P, A_{\underline{P}})$ for the roots of P with respect to $A_{\underline{P}}$ and $\Delta(P, A_{\underline{P}})$ for the simple roots of P with respect to $A_{\underline{P}}$, which are those $\alpha \in \Delta$ which act nontrivially on $\underline{S}_{\underline{P}}$. We have

$$\Delta(P, A_{\underline{P}}) = \{\alpha_{i_1}, \dots, \alpha_{i_k}\}. \tag{2.23}$$

Note that in the $\mathbb{Q}$-split case for $\mathrm{O}(p,p)$, the case $E = E_{p-1}$ can never occur. We rather have the maximal parabolics stabilizing $E = E_p = \operatorname{span}(u_1, \dots, u_p)$ (with $\Delta(P, A_{\underline{P}}) = \{\alpha_p\}$) and $\operatorname{span}(u_1, \dots, u_{p-1}, u'_p)$ (with $\Delta(P, A_{\underline{P}}) = \{\alpha_{p-1}\}$).

We let $\underline{Q}$ be the standard maximal parabolic stabilizing the single totally isotropic rational subspace E_ℓ of dimension $\ell \leq r$. In that case, we have $\underline{S}_\ell = \underline{S} = \{a(t)\}$ with $a(t) = a(t, \dots, t)$.

We now consider the isotropic flag $\mathbf{F}$ in V as a flag $\mathbf{F}(E) : (0) \subset E_{i_1} \subset \cdots \subset E_{i_k} = E$ inside E. We let $\underline{P}'$ be the parabolic subgroup of $\mathrm{GL}(E)$ stabilizing $\mathbf{F}(E)$. Then for the real points $P' = (\underline{P}'(\mathbb{R}))_0$, we have

$$P' = N_{P'} A M_{P'}, \tag{2.24}$$

with unipotent radical $N_{P'}$ and Levi factor

$$M_{P'} = \prod_{j=1}^{k} \mathrm{SL}(U_{i_j}(\mathbb{R})). \tag{2.25}$$

Here A is as above, viewed as a subgroup of $\mathrm{GL}_+(E_\mathbb{R})$. Note $M_{Q'} \simeq \mathrm{SL}(E_\mathbb{R})$. Furthermore, we can view $\underline{P}'$ and its subgroups naturally as subgroups of of $\underline{P}$ via the embbeding of $\mathrm{GL}(E)$ into $\mathrm{O}(V)$ given by (2.16).

Returning to $\underline{P}$, we have

$$\underline{L} \simeq \left\{ \begin{pmatrix} g & & \\ & h & \\ & & \tilde{g} \end{pmatrix} ; h \in \mathrm{SO}(W), g = \mathrm{diag}(g_1, \dots, g_k) \in \prod_{j=1}^{k} \mathrm{GL}(U_{i_j}), \right\}. \tag{2.26}$$

Thus

$$M \simeq \mathrm{SO}_0(W_\mathbb{R}) \times M_{P'}. \tag{2.27}$$

We also define

$$\mathfrak{p}_M = \mathfrak{p} \cap \mathfrak{m}, \tag{2.28}$$

and we write

$$\mathfrak{p}_M = \mathfrak{p}_W \oplus \mathfrak{p}_E, \tag{2.29}$$

where $\mathfrak{p}_E = \mathfrak{sl}(E) \cap \mathfrak{p}$ and

$$\mathfrak{p}_W = \mathfrak{o}_W \cap \mathfrak{p} = \mathrm{span}\{D_{\alpha\mu} = e_\alpha \wedge e_\mu;\ \ell+1 \leq \alpha \leq p,\ p+1 \leq \mu \leq m-\ell\}. \tag{2.30}$$

We naturally view $\underline{N_{P'}} \subset \mathrm{SL}(E)$ as a subgroup of the unipotent radical $\underline{N_P}$ via

$$n' \mapsto N(n') := \begin{pmatrix} n' & & \\ & 1 & \\ & & \widetilde{n'} \end{pmatrix} \in \underline{N_P}. \tag{2.31}$$

We then have a semidirect product decomposition

$$\underline{N_P} = \underline{N_{P'}} \ltimes \underline{N_Q}, \tag{2.32}$$

where $\underline{Q}$ is as above the maximal parabolic that stabilizes E. Furthermore, we let $\underline{Z_Q}$ be the center of $\underline{N_Q} \subseteq \underline{N_P}$. It is given by

$$\underline{Z_Q} = \left\{ z(b) := \begin{pmatrix} 1 & & b \\ & 1 & \\ & & 1 \end{pmatrix} ;\ J^t b J = -b \right\}. \tag{2.33}$$

Then for the coset space $\underline{N_P}/(\underline{N_{P'}} \ltimes \underline{Z_Q})$, we have

$$\underline{N_P}/(\underline{N_{P'}} \ltimes \underline{Z_Q}) \simeq \underline{N_Q}/\underline{Z_Q} \simeq W \otimes E \tag{2.34}$$

as vector spaces. Explicitly, the basis of E gives rise to an isomorphism $W \otimes E \simeq W^\ell$. Then for $(w_1, \dots, w_\ell) \in W^\ell$, the corresponding coset is represented by

$$n(w_1, \dots, w_\ell) := \begin{pmatrix} I_\ell & (\cdot, w_1) & & & -w_1^2 \\ & \vdots & & \cdot & \\ & (\cdot, w_\ell) & -w_\ell^2 & & \\ & I_W & -w_\ell & \dots & -w_1 \\ & & & I_\ell & \end{pmatrix}. \tag{2.35}$$

Here we write $w_i^2 = \frac{1}{2}(w_i, w_i)$ for short.

On the Lie algebra level, we let $\mathfrak{z}_Q$ be the center of $\mathfrak{n}_Q \subseteq \mathfrak{n}_P$, whence corresponding to (2.33)

$$\mathfrak{z}_Q \simeq \bigwedge^2 E_{\mathbb{R}}. \tag{2.36}$$

We let $\mathfrak{n}_{P'}$ be the Lie algebra of $N_{P'}$; thus $\mathfrak{n}_{P'} \subset E'_{\mathbb{R}} \wedge E_{\mathbb{R}} = \mathfrak{gl}(E_{\mathbb{R}})$. Corresponding to (2.35), we can realize $W_{\mathbb{R}} \otimes E_{\mathbb{R}}$ as a subspace of $\mathfrak{n}$. Namely, we obtain an embedding

$$W_{\mathbb{R}} \otimes E_{\mathbb{R}} \hookrightarrow \mathfrak{n}, \tag{2.37}$$

$$w \otimes u \to w \wedge u =: \mathfrak{n}_u(w), \tag{2.38}$$

and we denote this subspace by $\mathfrak{n}_W$, which we frequently identify with $W_{\mathbb{R}} \otimes E_{\mathbb{R}}$. Furthermore, this embedding is $\mathfrak{o}(W_{\mathbb{R}}) \oplus \mathfrak{gl}(E_{\mathbb{R}})$-equivariant, i.e.,

$$[X, \mathfrak{n}_u(w)] = \mathfrak{n}_u(Xw) \qquad [Y, \mathfrak{n}_u(w)] = \mathfrak{n}_{Yu}(w) \tag{2.39}$$

for $X \in \mathfrak{o}(W_{\mathbb{R}})$ and $Y \in \mathfrak{gl}(E_{\mathbb{R}})$. We easily see

$$\exp(\mathfrak{n}_{u_i}(w)) = n(0, \dots, w, \dots, 0). \tag{2.40}$$

A standard basis of $\mathfrak{n}_W$ is given by

$$X_{\alpha i} := n_{u_i}(e_\alpha) = e_\alpha \wedge u_i, \qquad X_{\mu i} := n_{u_i}(e_\mu) = e_\mu \wedge u_i \tag{2.41}$$

with $1 \leq i \leq \ell$, $\ell + 1 \leq \alpha \leq p$, and $p + 1 \leq \mu \leq m - \ell$. The dual space $\mathfrak{n}_W^*$ we can identify with $W_{\mathbb{R}} \otimes E'_{\mathbb{R}}$, and we denote the elements of the corresponding dual basis by $\nu_{\alpha i} = e_\alpha \wedge u'_i$ and $\nu_{\mu i} = -e_\mu \wedge u'_i$.

Summarizing, we obtain

Lemma 2.1. *We have a direct sum decomposition (of vector spaces)*

$$\mathfrak{n}_P = \mathfrak{n}_{P'} \oplus \mathfrak{n}_W \oplus \mathfrak{z}_Q.$$

Furthermore, the adjoint action of $\mathfrak{o}(W_{\mathbb{R}}) \oplus \mathfrak{gl}(E_{\mathbb{R}})$ on $\mathfrak{n}_P$ induces an action on the space $\mathfrak{n}_P/(\mathfrak{n}_{P'} \oplus \mathfrak{z}_Q) \simeq \mathfrak{n}_W$ such that

$$\mathfrak{n}_W \simeq W_{\mathbb{R}} \otimes E_{\mathbb{R}}$$

as $\mathfrak{o}(W_{\mathbb{R}}) \oplus \mathfrak{gl}(E_{\mathbb{R}})$-representations. Finally, the action of A on $\mathfrak{n}_W^*$ has weights $\alpha_{i_1}^{-1}, \dots, \alpha_{i_k}^{-1}$ with respective weight spaces $W \otimes U'_{i_1}, \dots, W \otimes U'_{i_k}$.

2.3. **The Maurer Cartan forms and horospherical coordinates.** The Langlands decomposition of P gives rise to the (rational) horospherical coordinates on D associated to P by

$$\sigma = \sigma_P : N \times A \times D_{\underline{P}} \longrightarrow D \tag{2.42}$$

given by

$$\sigma(n,a,m) = n\,a\,mz_0. \tag{2.43}$$

Here

$$D_{\underline{P}} = M_P/K_P \tag{2.44}$$

is the boundary symmetric space associated to $\underline{P}$. Here $K_P = M \cap K$.

We note that the boundary symmetric space D_P always factors into a product of symmetric spaces for special linear groups and one orthogonal factor, namely, the symmetric space associated to $\mathrm{SO}(W)$. We call the associated symmetric space D_W the *orthogonal factor* in the boundary symmetric space D_P. We have

$$D_P = D_W \times \prod_{j=1}^{k} D_{U_{i_j}}, \tag{2.45}$$

where $D_{U_{i_j}}$ denotes the symmetric space associated to $\mathrm{SL}(U_{i_j})$.

We now describe how the basic cotangent vectors $\omega_{\alpha\mu} = (e_\alpha \wedge e_\mu)^* \in \mathfrak{p}^* \simeq T^*_{z_0}(D)$ look like in NAM coordinates. We extend σ to $N \times A \times M \times K \longrightarrow G$ by $\sigma(n,a,m,k) = namk$, and this induces an isomorphism between the left-invariant forms on NAM (which we identify with $\mathfrak{n}^* \oplus \mathfrak{a}^* \oplus \mathfrak{p}^*_M$) and the horizontal left-invariant forms on G (which we identify with $\mathfrak{p}^*$). Thus we have an isomorphism

$$\sigma^* : \mathfrak{p}^* \longrightarrow \mathfrak{n}^* \oplus \mathfrak{a}^* \oplus \mathfrak{p}^*_M. \tag{2.46}$$

Explicitly, we have

Lemma 2.2. *Let $1 \leq i \leq \ell$. For the preimage under σ^* of the elements in $\mathfrak{n}^*_W$ coming from $W_+ \otimes E$, we have*

$$\sigma^*\, \omega_{\alpha m+1-i} = -\tfrac{1}{\sqrt{2}}\nu_{\alpha i}, \tag{2.47}$$

where $\ell + 1 \leq \alpha \leq p$. Furthermore, for the ones coming from $W_- \otimes E$, we have

$$\sigma^*\omega_{i\mu} = \tfrac{1}{\sqrt{2}}\nu_{\mu i}, \tag{2.48}$$

where $p+1 \leq \mu \leq m+1-\ell$. On $\mathfrak{p}^*_M$, the map σ^* is the identity. In particular, for $\ell+1 \leq \alpha \leq p$ and any $\mu \geq p+1$, we have

$$\sigma^*\omega_{\alpha\mu} \in \mathfrak{p}^*_W \oplus \mathfrak{n}^*_W. \tag{2.49}$$

The remaining elements of $\mathfrak{p}^*$ are of the form $\omega_{i\mu}$ with $p+1 \leq \mu \leq m+1-\ell$. These elements are mapped under σ^* to $\mathfrak{n}^*_{P'} \oplus \mathfrak{a}^* \oplus \mathfrak{p}^*_E \subset \mathfrak{gl}(E_{\mathbb{R}})^*$.

2.4. **Borel-Serre Compactification.** We now describe the Borel-Serre compactification of D and of $X = \Gamma\backslash D$. We follow [3], III.9. We first partially compactify the symmetric space D. For any rational parabolic $\underline{P}$, we define the boundary component

$$e(\underline{P}) = N_P \times D_{\underline{P}} \simeq P/A_{\underline{P}}K_{\underline{P}}. \tag{2.50}$$

Then as a set the (rational) Borel-Serre enlargement $\overline{D}^{BS} = \overline{D}$ is given by

$$\overline{D} = D \cup \coprod_{\underline{P}} e(\underline{P}), \tag{2.51}$$

where $\underline{P}$ runs over all rational parabolic subgroups of $\underline{G}$. As for the topology of $\overline{D}$, we first note that D and $e(\underline{P})$ have the natural topology. Furthermore, a sequence of $y_j = \sigma_P(n_j, a_j, z_j) \in D$ in horospherical coordinates of D converges to a point $(n, z) \in e(\underline{P})$ if and only if $n_j \to n$, $z_j \to z$ and $\alpha(a_j) \to \infty$ for all roots $\alpha \in \Phi(\underline{P}, A_{\underline{P}})$. For convergence within boundary components, see [3], III.9.

With this, $\overline{D}$ has a canonical structure of a real analytic manifold with corners. Moreover, the action of $\underline{G}(\mathbb{Q})$ extends smoothly to $\overline{D}$. The action of $g = kp = kman \in KMAN = G$ on $e(\underline{P})$ is given by

$$g \cdot (n', z') = k \cdot (Ad(am)(nn'), mz') \in e(Ad(k)\underline{P}) = e(Ad(g)\underline{P}) \tag{2.52}$$

with $k \cdot (n', z') = (Ad(k)n, Ad(k)mK_{Ad(k)\underline{P}}) \in e(Ad(k)\underline{P})$. Finally,

$$\overline{X} := \Gamma\backslash\overline{D} \tag{2.53}$$

is the Borel-Serre compactification of $X = \Gamma\backslash D$ to a manifold with corners. If $\underline{P}_1, \dots, \underline{P}_k$ is a set of representatives of Γ-conjugacy classes of rational parabolic subgroups of $\underline{G}$, then

$$\Gamma\backslash\overline{D} = \Gamma\backslash D \cup \coprod_{i=1}^{k} \Gamma_{P_i}\backslash e(\underline{P})_i, \tag{2.54}$$

with $\Gamma_{P_i} = \Gamma \cap P_i$. We will write $e'(\underline{P}) = \Gamma_{\underline{P}}\backslash e(\underline{P})$. We write Γ_M for the image of Γ_P under the quotient map $P \to P/N$. Furthermore, we Γ_P acts on the quotient$E_{\mathbb{R}}^{\perp}/E_{\mathbb{R}}$, and we denote this transformation group by Γ_W. Note that Γ_M and Γ_W when viewed as subgroups of P contain $\Gamma \cap M$ and $\Gamma \cap \mathrm{SO}_0(W_{\mathbb{R}})$ respectively as subgroups of finite index.

We now describe Siegel sets. For $t \in \mathbb{R}_+$, let

$$A_{P,t} = \{a \in A_P;\ \alpha(a) > t \text{ for all } \alpha \in \Delta(P, A_{\underline{P}})\}, \tag{2.55}$$

and for bounded sets $U \subset N_P$ and $V \subset D_P$, we define the Siegel set

$$\mathfrak{S}_{P,U,t,V} = U \times A_{P,t} \times V \subset N_P \times A_P \times D_P. \tag{2.56}$$

Note that for t sufficiently large, two Siegel sets for different parabolic subgroups are disjoint. Furthermore, if $P_1, \dots, P_k$ are representatives of the $\underline{G}(\mathbb{Q})$-conjugacy classes of rational parabolic subgroups of G, then there are Siegel sets $\mathcal{S}_i$ associated to P_i such that the union $\bigcup \pi(\mathcal{S}_i)$ is a fundamental set for Γ. Here π denotes the projection $\pi : D \to \Gamma\backslash D$.

3. Review of representation theory for general linear and orthogonal groups

In this section, we will briefly review the construction of the irreducible finite dimensional (polynomial) representations of $\mathrm{GL}(\mathbb{C}^n)$ and $\mathrm{O}(V)$. Here, in this section, we assume that V is complex space of dimension m. Basic references are [7], §4.2 and §6.1, [11], §9.3.1-9.3.4 and [2], Ch. V, §5 to which we refer for details.

3.1. **Representations of** $\mathrm{GL}_n(\mathbb{C})$**.** Let $\lambda = (b_1, b_2, \cdots, b_n)$ be a partition of ℓ'. We assume that the b_i's are arranged in decreasing order. We will use $D(\lambda)$ to denote the Young diagram associated to λ. We will identify the partition λ with the dominant weight λ for $GL(n)$ in the usual way. A standard filling λ of the Young diagram $D(\lambda)$ by the elements of the set $[\ell] = \{1, 2, \cdots, \ell'\}$ is an assignment of each of the numbers in $[\ell']$ to a box of $D(\lambda)$ so that the entries in each row strictly increase when read from left to right and the entries in each column strictly increase when read from top to bottom. A Young diagram equipped with a standard filling will be also called a standard tableau.

We let $s(t(\lambda))$ be the idempotent in the group algebra of the symmetric group $S_{\ell'}$ associated to a standard tableau T with ℓ' boxes corresponding to a standard filling $t(\lambda)$ of a Young diagram $D(\lambda)$. Note that $S_{\ell'}$ acts on the space of ℓ'-tensors $T^{\ell'}(\mathbb{C}^n)$ in the natural fashion on the factors of $T^{\ell'}(\mathbb{C}^n)$. Therefore $s(t(\lambda))$ gives rise to a projection operator in $End(T^{\ell'}(\mathbb{C}^n))$, which by slight abuse of notation we also denote by $s(t(\lambda))$. We write

$$\mathbb{S}_{t(\lambda)}(\mathbb{C}^n) = s(t(\lambda))(T^{\ell'}(\mathbb{C}^n)). \tag{3.1}$$

We have a direct sum decomposition

$$T^{\ell'}(\mathbb{C}^n) = \bigoplus_{\lambda} \bigoplus_{t(\lambda)} \mathbb{S}_{t(\lambda)}(\mathbb{C}^n), \tag{3.2}$$

where λ runs over all partitions of ℓ' and $t(\lambda)$ over all standard fillings of $D(\lambda)$. This gives the decomposition of $T^{\ell'}(\mathbb{C}^n)$ into irreducible constituents, i.e, for every standard filling $t(\lambda)$, the $GL(\mathbb{C}^n)$-module $\mathbb{S}_{t(\lambda)}(\mathbb{C}^n)$ is irreducible with highest weight λ. In particular, $\mathbb{S}_{t(\lambda)}(\mathbb{C}^n)$ and $\mathbb{S}_{t'(\lambda)}(\mathbb{C}^n)$ are isomorphic for two different standard fillings $t(\lambda)$ and $t'(\lambda)$. We denote this isomorphism class by $\mathbb{S}_{\lambda}(\mathbb{C}^n)$ (or if we do not want to specify the standard filling).

Explicitly, we let A be the standard filling of a Young diagram $D(A)$ corresponding to the partition λ with less than or equal to n rows and ℓ' boxes by $1, 2, \cdots, \ell'$ obtained by filling the rows in order beginning at the top with $1, 2, \cdots, \ell'$. We let $R(A)$ be the subgroup of $S_{\ell'}$ which preserves the rows of A and $C(A)$ be the subgroup that preserves the columns of A. We define elements $r(A)$ and $c(A)$ by

$$r(A) = \sum_{s \in R(A)} s \quad \text{and} \quad c(A) = \sum_{s \in C(A)} \mathrm{sgn}(s) s. \tag{3.3}$$

Let $h(A)$ be the product of the hook lengths of the boxes in $D(A)$, see [7], page 50. Then the idempotent $s(A)$ is given

$$s(A) = \frac{1}{h(A)} c(A)r(A). \tag{3.4}$$

We will also need the "dual" idempotent $s(A)^*$ given by

$$s(A)^* = \frac{1}{h(A)} r(A)c(A). \tag{3.5}$$

We let $\varepsilon_1, \dots, \varepsilon_n$ denote the standard basis of $\mathbb{C}^n$ and $\theta_1, \dots, \theta_n \in (\mathbb{C}^n)^*$ be its dual basis. We set

$$\varepsilon_A = \varepsilon_1^{b_1} \otimes \cdots \otimes \varepsilon_n^{b_n} \tag{3.6}$$

and let θ_A be the corresponding element in $T^{\ell'}(\mathbb{C}^n)^*$. Then $s(A)(\varepsilon_A)$ is a highest weight vector in $\mathbb{S}_A(\mathbb{C}^n)$, see [11], §9.3.1. We have

Lemma 3.1.

$$s(A)^*\theta_A(s(A)\varepsilon_A) = \frac{|R(A)|}{h(A)}.$$

Here $|R(A)|$ is the order of $R(A)$.

Proof. We compute

$$s(A)^*\theta_A(s(A)\varepsilon_A) = \theta_A(s(A)^2\varepsilon_A) = \theta_A(s(A)\varepsilon_A) = \frac{|R(A)|}{h(A)}\theta_A(c(A)\varepsilon_A) = \frac{|R(A)|}{h(A)}\theta_A(\varepsilon_A).$$

The last equation holds because $\theta_A(q\varepsilon_A) = 0$ for any nontrivial q in the column group of A as the reader will easily verify. We have used $r(A)\varepsilon_A = |R(A)|\varepsilon_A$ (since all row permutations fix ε_A) and $s(A) = \frac{1}{h(A)}c(A)r(A)$. □

3.2. **Enlarging the Young diagram.** The following will be important later.

We let $B = B_{n,\ell}$ be the standard tableau with underlying shape $D(B)$ an n by ℓ rectangle with the standard filling obtained by putting 1 through ℓ in the first row, $\ell+1$ through 2ℓ in the second row etc. Then $D(B)$ is the Young diagram corresponding to the dominant weight $\ell\varpi_n$. Here $\varpi_n = (1, 1, \cdots, 1)$ is the n-th fundamental weight for $GL(n)$. We note that we have $\varepsilon_B = \epsilon_1^\ell \otimes \cdots \otimes \epsilon_n^\ell$ and $\theta_B = \theta_1^\ell \otimes \cdots \otimes \theta_n^\ell$.

Lemma 3.2. (1) *$s(B)T^{n\ell}(\mathbb{C}^n)$ is one-dimensional and*

$$s(B)T^{n\ell}(\mathbb{C}^n) = \mathbb{C}s(B)\varepsilon_B$$

as $\mathrm{GL}(n, \mathbb{C})$*-modules.*

(2) *$s(B)^*T^{n\ell}(\mathbb{C}^n)^*$ is one-dimensional and*

$$s(B)^*T^{n\ell}(\mathbb{C}^n)^* = \mathbb{C}s(B)^*\theta_B$$

as $\mathrm{GL}(n, \mathbb{C})$*-modules. In particular, we have*

$$s(B)^*T^{n\ell}(\mathbb{C}^n)^* \cong \left(\bigwedge^n (\mathbb{C}^n)^*\right)^{\otimes \ell}.$$

We let A be the standard filling of the Young diagram $D(\lambda)$ as above. Then $B|A$ denotes the standard tableau with underlying shape $D(B|A)$ given by making the shape of A abut B (on the right), using the above filling for B and filling A in the standard way (as above) with $n\ell+1$ through $n\ell+\ell'$. For example, if

$$B = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline \end{array} \quad \text{and} \quad A = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} \end{array}, \quad \text{then} \quad B|A = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 10 & 11 & 12 \\ \hline 4 & 5 & 6 & 13 & 14 \\ \cline{1-5} 7 & 8 & 9 \\ \cline{1-3} \end{array}$$

We have an idempotent $s(B|A)$ in the group ring of $S_{n\ell+\ell'}$ and the $nl+\ell'$ tensor $\varepsilon_{B|A} \in T^{n\ell+\ell'}(\mathbb{C}^n)$, which gives rise to a highest weight vector $s(B|A)\varepsilon_{B|A}$ in $s(B|A)(T^{n\ell+\ell'}(\mathbb{C}^n))$. Note

$$\varepsilon_B \otimes \varepsilon_A = \varepsilon_{B|A}. \tag{3.7}$$

Lemma 3.3. *There is a positive number $c(A,B)$ such that*

$$s(B)\varepsilon_B \otimes s(A)\varepsilon_A = c(A,B)s(B|A)\varepsilon_{B|A}.$$

Proof. Since the Young diagrams $D(B)$ and $D(A)$ are abutted along their vertical borders, we see

$$c(B|A) = (c(B)\otimes 1_{\ell'})\circ(1_{n\ell}\otimes c(A)) = (1_{n\ell}\otimes c(A))\circ(c(B)\otimes 1_{\ell'}). \tag{3.8}$$

Also (for any standard tableau C)

$$r(C)\varepsilon_C = |R(C)|\varepsilon_C. \tag{3.9}$$

Then we compute (using the three equations (3.7), (3.8),(3.9))

$$\begin{aligned} s(B)\varepsilon_B \otimes s(A)\varepsilon_A &= \frac{1}{h(B)}c(B)r(B)\varepsilon_B \otimes \frac{1}{h(A)}c(A)r(A)\varepsilon_A \\ &= \frac{1}{h(B)h(A)}(c(B)\otimes 1_{\ell'})\circ(1_{n\ell}\otimes c(A))((r(B)\varepsilon_B \otimes r(A)\varepsilon_A) \\ &= \frac{|R(B)|R(A)|}{h(B)h(A)}c(B|A)(\varepsilon_B\otimes\varepsilon_A) = \frac{|R(B)|R(A)|}{h(B)h(A)}c(B|A)\varepsilon_{B|A} \\ &= \frac{|R(B)|R(A)|}{h(B)h(A)}\frac{1}{|R(B|A)|}c(B|A)r(B|A)\varepsilon_{B|A} = \frac{h(B|A)}{h(B)h(A)}\frac{|R(B)||R(A)|}{|R(B|A)|}s(B|A)\varepsilon_{B|A}. \end{aligned}$$

□

Corollary 3.4. *Under the identification of $T^{n\ell}(\mathbb{C}^n)\otimes T^{\ell'}(\mathbb{C}^n) \to T^{n\ell+\ell'}(\mathbb{C}^n)$ given by tensor multiplication, we have the equality of maps*

$$s(B)\otimes s(A) = s(B|A).$$

That is,

$$\mathbb{S}_B(\mathbb{C}^n)\otimes\mathbb{S}_A(\mathbb{C}^n) = \mathbb{S}_{B|A}(\mathbb{C}^n)$$

as (physical) subspaces of $T^{n\ell+\ell'}(\mathbb{C}^n)$. The same statements hold for the dual space $\mathbb{S}^*_{B|A}(\mathbb{C}^{n\ell+\ell'})^*$ etc.

Proof. Since $\mathbb{S}_B(\mathbb{C}^n)$ is one-dimensional, the tensor product $\mathbb{S}_B(\mathbb{C}^n) \otimes \mathbb{S}_A(\mathbb{C}^n)$ defines an irreducible representation for $\mathrm{GL}_n(\mathbb{C}^n)$ (under the tensor multiplication map $T^{n\ell}(\mathbb{C}^n) \otimes T^{\ell'}(\mathbb{C}^n)$ inside $T^{n\ell+\ell'}(\mathbb{C}^n)$). But by Lemma 3.3 it has nonzero intersection with the irreducible $\mathrm{GL}_n(\mathbb{C})$-representation $\mathbb{S}_{B|A}(\mathbb{C}^n)$ inside $T^{n\ell+\ell'}(\mathbb{C}^n)$. Hence the two subspaces coincide. □

3.3. **Representations of** $\mathrm{O}(V)$**.** We extend the bilinear form $(\ ,\ )$ on V to $T^{\ell'}(V)$ as the ℓ'-fold tensor product and note that the action of $S_{\ell'}$ on $T^{\ell'}(V)$ is by isometries. We let $V^{[\ell']}$ be the space of harmonic ℓ'-tensors (which are those ℓ'-tensors which are annihilated by all contractions with the form $(\ ,\ )$). We let $\mathcal{H}$ be the orthogonal projection $\mathcal{H} : T^{\ell'}(V) \to V^{[\ell']}$ onto the harmonic ℓ'-tensors of V. Note that the space of harmonic ℓ'-tensors is invariant under the action of S'_ℓ. We then define for λ as above the harmonic Schur functor $\mathbb{S}_{[t(\lambda)]}(V)$ by

$$\mathbb{S}_{[t(\lambda)]}(V) = \mathcal{H}\mathbb{S}_{t(\lambda)}(V). \tag{3.10}$$

If the sum of the lengths of the first two columns of $D(\lambda)$ is at most m, then $\mathbb{S}_{[t(\lambda)]}(V_{\mathbb{C}})$ is a nonzero irreducible representation for $\mathrm{O}(V_{\mathbb{C}})$, see [7] section 19.5. Otherwise, it vanishes. Of course, for different fillings $t(\lambda)$ of $D(\lambda)$, these representations are all isomorphic and we write $\mathbb{S}_{[\lambda]}(V)$ for the isomorphism class. Furthermore, it is also irreducible when restricted to G unless m is even and $i(\lambda) = \frac{m}{2}$, in which case it splits into two irreducible representations. If $i(\lambda) \leq [\frac{m}{2}]$, then the corresponding highest weight $\tilde{\lambda}$ for the representation $\mathbb{S}_{[\lambda]}(V)$ of G ihas the same nonzero entries as λ.

4. The Weil representation

We review different models of the Weil representation. In this section, V denotes a real quadratic space of signature (p, q) and dimension m.

We let V' be a real symplectic space of dimension $2n$. We denote by $G' = \mathrm{Mp}(n, \mathbb{R})$ the metaplectic cover of the symplectic group $\mathrm{Sp}(V') = \mathrm{Sp}(n, \mathbb{R})$ and let $\mathfrak{g}'$ be its Lie algebra. We let K' be the inverse image of the standard maximal compact $\mathrm{U}(n) \subset \mathrm{Sp}(n, \mathbb{R})$ under the covering map $\mathrm{Mp}(n, \mathbb{R}) \to \mathrm{Sp}(n, \mathbb{R})$. Note that K' admits a character $\det^{1/2}$, i.e., its square descends to the determinant character of $\mathrm{U}(n)$. The embedding of $\mathrm{U}(n)$ into $\mathrm{Sp}(n, \mathbb{R})$ is given by $A + iB \mapsto \left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right)$.

We write $\mathcal{W}_{n,V}$ for (an abstract model of) the K'-finite vectors of the restriction of the Weil representation of $\mathrm{Mp}(V' \otimes V)$ to $\mathrm{Mp}(n, \mathbb{R}) \times \mathrm{O}(V)$ associated to the additive character $t \mapsto e^{2\pi i t}$.

4.1. **The Schrödinger model.** We let V'_1 be a Langrangian subspace of V'. Then $V \otimes V'_1$ is a Langrangian subspace of $V' \otimes V$ (which is naturally a symplectic space of dimension $2nm$). The Schrödinger model of the Weil representation consists of the space of (complex-valued) Schwartz functions on the Langrangian subspace $V'_1 \otimes V \simeq V^n$. We write $\mathcal{S}(V^n)$ for the space of Schwartz functions on V^n and write $\omega = \omega_{n,V}$ for the action.

The Siegel parabolic $P' = M'N'$ has Levi factor

$$M' = \left\{ m'(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix}; a \in \mathrm{GL}(n, \mathbb{R}) \right\} \tag{4.1}$$

and unipotent radical

$$N' = \left\{ n'(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}; b \in \mathrm{Sym}_n(\mathbb{R}) \right\}. \tag{4.2}$$

It is well known that we can embed P' into $\mathrm{Mp}(n, \mathbb{R})$, and the action of P' on $S(V^n)$ is given by

$$\omega\left(m'(a)\right)\varphi(\mathbf{x}) = (\det a)^{m/2}\varphi(\mathbf{x}a) \qquad (\det a > 0), \tag{4.3}$$

$$\omega\left(n'(b)\right)\varphi(\mathbf{x}) = e^{\pi i tr(b(\mathbf{x},\mathbf{x}))}\varphi(\mathbf{x}) \tag{4.4}$$

with $\mathbf{x} = (x_1, \dots, x_n) \in V^n$. The central $\mathbb{C}^1$ acts by

$$\omega((1,t))\varphi = \begin{cases} t\varphi & \text{if } m \text{ is odd} \\ \varphi & \text{if } m \text{ is even} \end{cases} \tag{4.5}$$

for all $t \in \mathbb{C}^1$. The orthogonal group G acts on $\mathcal{S}(V^n)$ via

$$\omega(g)\varphi(\mathbf{x}) = \varphi(g^{-1}\mathbf{x}), \tag{4.6}$$

which commutes with the action G'. The standard Gaussian is given by

$$\varphi_0(\mathbf{x}) = e^{-\pi tr(\mathbf{x},\mathbf{x})_{z_0}} \in \mathcal{S}(V^n)^K. \tag{4.7}$$

Here $(\mathbf{x}, \mathbf{x})$ is the inner product matrix $(x_i, x_j)_{ij}$.

We let $S(V^n)$ be the space of smooth, i.e., K'-finite, vectors inside the space of Schwartz functions on V^n. It consists of those Schwartz functions of the form $p(\mathbf{x})\varphi_0(\mathbf{x})$, where p is a polynomial function on V^n.

4.2. **The mixed model and the definition of local restriction for the Weil representation.** We now describe a different model for the Weil representation, the so-called mixed model. Furthermore, we will define a "local" restriction $r_P^{\mathcal{W}}$ from $\mathcal{S}(V^n)$ to the space of Schwartz functions $\mathcal{S}(W^n)$ for W, a (real) subspace of signature $(p-\ell, q-\ell)$.

4.2.1. *The mixed model.* We let $E = E_\ell$ be one of the standard totally isotropic subspaces of V, see (2.5). As before, we identify the dual space of E_ℓ with E'_ℓ. Accordingly, for the decomposition $V = E \oplus W \oplus E'$, we write $\mathbf{x} = \left(\begin{smallmatrix} u \\ \mathbf{x}_W \\ u' \end{smallmatrix}\right)$ for $\mathbf{x} \in V^n$, where $u \in E^n$, $u' \in (E')^n$, and $\mathbf{x}_W \in W^n$. We then have an isomorphism of two models of the Weil representation given by

$$\begin{aligned} \mathcal{S}(V^n) &\longrightarrow \mathcal{S}((E')^n) \otimes S(W^n) \otimes \mathcal{S}((E')^n) \\ \varphi &\longmapsto \widehat{\varphi} \end{aligned} \tag{4.8}$$

given by the partial Fourier transform operator

$$\widehat{\varphi}\begin{pmatrix} \xi \\ \mathbf{x}_W \\ u' \end{pmatrix} = \int_{E^n} \varphi\begin{pmatrix} u \\ \mathbf{x}_W \\ u' \end{pmatrix} e^{-2\pi i tr(u,\xi)} du \tag{4.9}$$

with $\xi, u' \in (E')^n$ and $\mathbf{x}_W \in W^n$.

We will need some formulae relating the action of ω in the two models. We have

Lemma 4.1. *Let* $\begin{pmatrix} \xi \\ \mathbf{x}_W \\ u' \end{pmatrix} \in (E' \oplus W \oplus E')^n$.

(i) *Let* $n \in N_Q$ *and write* $n(u')_W$ *for the image of* $n(u')$ *under the orthogonal projection onto* W *Then*

$$\widehat{n\varphi}({}^t(\xi, \mathbf{x}_W, u')) = e\left(\operatorname{tr}(n(\mathbf{x}_W + u'), \xi)\right) \widehat{\varphi}({}^t(\xi, \mathbf{x}_W + n(u')_W, u')).$$

(ii) *For* $g \in \mathrm{SL}(E) \subset G$ *(in particular,* $g \in N_{P'}$ *or* $g \in M_{P'}$*) we have*

$$\widehat{g\varphi}({}^t(\xi, \mathbf{x}_W, u')) = \widehat{\varphi}({}^t(\widetilde{g}\xi, \mathbf{x}_W, \widetilde{g}u'))$$

with $\widetilde{g} = Jg^*J$ *and* $g^* = {}^tg^{-1}$.

(iii) *For* $\mathbf{t} = (t_1, \dots, t_\ell)$, *set* $\widetilde{\mathbf{t}} = \mathbf{t}J = (t_\ell, \dots, t_1)$ *and* $|\mathbf{t}| = t_1 \cdot t_2 \cdots t_\ell$. *Then for* $a(\mathbf{t})$, *we have*

$$\widehat{a(\mathbf{t})\varphi}({}^t(\xi, \mathbf{x}_W, u')) = |\mathbf{t}|^n \widehat{\varphi}({}^t(\widetilde{\mathbf{t}}\xi, \mathbf{x}_W, \widetilde{\mathbf{t}}u')).$$

(iv) *For* $h \in \mathrm{SO}_0(W) \subset M$, *we have*

$$\widehat{h\varphi}({}^t(\xi, \mathbf{x}_W, u')) = \widehat{\varphi}({}^t(\xi, h^{-1}\mathbf{x}_W, u')).$$

(v) *For* $m'(a) = \left(\begin{smallmatrix} a & 0 \\ 0 & {}^ta^{-1} \end{smallmatrix}\right) \in M' \subset Sp(n, \mathbb{R})$ *with* $a \in GL_n^+(\mathbb{R})$,

$$(\widehat{m'(a)\varphi})({}^t(\xi, \mathbf{x}_W, u')) = (\det a)^{\frac{m}{2}-\ell} \widehat{\varphi}({}^t(\xi a^*, \mathbf{x}_W a, u'a))$$

(vi) *For* $n'(b) = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \in N' \subset Sp(n, \mathbb{R})$ *with* $b \in \mathrm{Sym}_n(\mathbb{R})$,

$$(\widehat{n'(b)\varphi})({}^t(\xi, \mathbf{x}_W, u')) = e\left(tr(b\tfrac{(\mathbf{x}_W, \mathbf{x}_W)}{2})\right) \widehat{\varphi}({}^t(\xi - u'b, \mathbf{x}_W, u')).$$

Proof. This is straightforward. □

We obtain

Proposition 4.2. *Let* $\varphi \in \mathcal{S}(V^n)$. *Then the restriction of* $\widehat{\varphi}$ *to* W^n,

$$\varphi \mapsto \widehat{\varphi}|_{W^n},$$

defines a $G' \times MN$ *intertwiner from* $\mathcal{S}(V^n)$ *to* $\mathcal{S}(W^n)$. *Here, we identify* W *with* $E^\perp/E$ *to define the action of* MN *on* W. *In particular,* N *and* $M_{P'}$ *(see 2.27) act trivially on* $\mathcal{S}(W^n)$.

4.2.2. *Weil representation restriction.*

Definition 4.3. Let $\varphi \in \mathcal{S}(V^n)$ and let P a standard parabolic of G, and let $E = E_\ell$ be the biggest totally isotropic subspace of V in the flag stabilized by P with Witt decomposition $V = E \oplus W \oplus E'$. We then define the "local" restriction $r_P^{\mathcal{W}}(\varphi) \in \mathcal{S}(W^n)$ with respect to P for the Schrödinger model of the Weil representation $\mathcal{W}$ by

$$r_P^{\mathcal{W}}(\varphi) = \widehat{\varphi}|_{W^n}.$$

We now describe this restriction on a certain class of Schwartz functions on V^n. For $\mathbf{x} = (x_1, \dots, x_n) \in V^n$, we write $\begin{pmatrix} x_{1j} \\ \vdots \\ x_{mj} \end{pmatrix}$ for the standard coordinates of x_j. We define a family of commuting differential operators on $\mathcal{S}(V^n)$ by

$$\mathcal{H}_{rj} = \left(x_{rj} - \frac{1}{2\pi}\frac{\partial}{\partial x_{rj}}\right), \tag{4.10}$$

where $1 \leq r \leq m$ and $1 \leq j \leq n$. Then there exists a polynomial $\widetilde{H}_k$ such that

$$\mathcal{H}_{rj}^k \varphi_0(\mathbf{x}) = \widetilde{H}_k(x_{rj})\varphi_0(\mathbf{x}), \tag{4.11}$$

where $\varphi_0(\mathbf{x})$ is the standard Gaussian, see (4.7). In fact, it is easy to see that $\widetilde{H}_k$ is essentially the k-th Hermite polynomial $H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2}$:

$$\widetilde{H}_k(x) = (2\pi)^{-k/2} H_k\left(\sqrt{2\pi}x\right). \tag{4.12}$$

We let $\Delta \in M_{m\times n}(\mathbb{Z}) = (\delta_{rj})$ be an integral matrix with non-negative entries and split Δ into $\Delta_+ \in M_{p\times n}(\mathbb{Z})$ and $\Delta_- \in M_{q\times n}(\mathbb{Z})$ into its "positive" and "negative" part, where Δ_+ consists of the first p rows of Δ and Δ_- of the last q. (Recall $m = p + q$). We define operators

$$\mathcal{H}_\Delta = \prod_{\substack{1\leq r\leq m\\ 1\leq j\leq n}} \mathcal{H}_{rj}^{\delta_{rj}}, \qquad \mathcal{H}_{\Delta_+} = \prod_{\substack{1\leq \alpha\leq p\\ 1\leq j\leq n}} \mathcal{H}_{\alpha j}^{\delta_{\alpha j}}, \qquad \mathcal{H}_{\Delta_-} = \prod_{\substack{p+1\leq \mu\leq m\\ 1\leq j\leq n}} \mathcal{H}_{\mu j}^{\delta_{\mu j}},$$

so that $\mathcal{H}_\Delta = \mathcal{H}_{\Delta_+}\mathcal{H}_{\Delta_-}$. Here again we make use of our convention to use early Greek letters for the "positive" indices of V and late ones for the "negative" indices.

Definition 4.4. For Δ as above, we define the Schwartz function φ_Δ by

$$\varphi_\Delta(\mathbf{x}) = \mathcal{H}_\Delta \varphi_0(\mathbf{x}) = \prod_{\substack{1\leq \alpha\leq p\\ p+1\leq \mu\leq m\\ 1\leq j\leq n}} \widetilde{H}_{\delta_{\alpha j}}(x_{\alpha j})\widetilde{H}_{\delta_{\mu j}}(x_{\mu j})\varphi_0(\mathbf{x}).$$

We now describe φ_Δ^V in the mixed model. The superscript V emphasizes that the Schwartz function is associated to the space V. We begin with some auxiliary considerations. The following little fact will be crucial for us.

Lemma 4.5. *For a Schwartz function $f \in \mathcal{S}(\mathbb{R})$, let $\widehat{f}(\xi) = \int_{\mathbb{R}} f(y)e^{-2\pi i y\xi}dy$ be its Fourier transform. Let $g_k(y) = \widetilde{H}_k(-\frac{y}{\sqrt{2}})e^{-\pi y^2}$. Then*

$$\widehat{g_k}(\xi) = (-\sqrt{2}i\xi)^k e^{-\pi\xi^2}.$$

Proof. We use induction and differentiate the equation $\widehat{(\widehat{g_k})}(-y) = \widetilde{H}_k(\frac{y}{\sqrt{2}})e^{-\pi y^2}$. The assertion follows from the recursion $\widetilde{H}_{k+1}(y) = 2y\widetilde{H}_k(y) - \frac{1}{2\pi}\widetilde{H}_k'(y)$, which is immediate from the definition of $\widetilde{H}_k$. The claim also follows easily from [19], (4.11.4). □

Remark 4.6. Recall that on the other hand that $\widetilde{H}_k(y)e^{-\pi y^2}$ is an eigenfunction under the Fourier transform with eigenvalue $(-i)^k$, see [19], (4.12.3). This fact is underlying the automorphic properties of the theta series associated to the special Schwartz forms $\varphi_{nq,[\lambda]}$.

The Gaussian is given in standard coordinates by $\varphi_0^V(\mathbf{x}) = \exp(-\pi\sum_{j=1}^n\sum_{i=1}^m x_{ij}^2)$. In Witt coordinates, we have $x_{rj} = \frac{1}{\sqrt{2}}(y_{rj} - y_{(m-r)j})$ and $x_{(m-r)j} = \frac{1}{\sqrt{2}}(y_{rj} + y_{(m-r)j})$; thus $x_{rj}^2 + x_{(m-r),j}^2 = y_{rj}^2 + y_{(m-r)j}^2$ for $r \leq \ell$. Thus

$$\varphi_0^V\begin{pmatrix} u \\ \mathbf{x}_W \\ u' \end{pmatrix} = \exp\left(-\pi\sum_{j=1}^n\sum_{r=1}^{\ell}(y_{rj}^2 + y_{(m-r)j}^2)\right)\varphi_0^W(\mathbf{x}_W). \tag{4.13}$$

We write slightly abusing

$$\varphi_0^E(u,u') := \varphi_0^V \begin{pmatrix} u \\ 0 \\ u' \end{pmatrix} = \exp\left(-\pi \sum_{j=1}^n \sum_{r=1}^{\ell} (y_{rj}^2 + y_{(m-r)j}^2)\right). \tag{4.14}$$

We let Δ' be the truncated matrix of size $(m-2\ell)\times n$ given by eliminating the first and the last ℓ rows from Δ. We let Δ'' be the matrix of these eliminated rows. Note that $\mathcal{H}_{\Delta'}$ now defines an operator on $\mathcal{S}(W^n)$ and $\mathcal{H}_{\Delta''}$ on $\mathcal{S}((E\oplus E')^n)$. We also obtain matrices Δ'_+ of size $(p-\ell)\times n$ and Δ'_- of size $(q-\ell)\times n$ by eliminating the first ℓ and the last ℓ rows from Δ_+ and Δ_- respectively. Similarly we obtain Δ''_+ and Δ''_-. With these notations we obtain

Lemma 4.7. (i)

$$\widehat{\varphi_{\Delta}^V} \begin{pmatrix} \xi \\ \mathbf{x}_W \\ u' \end{pmatrix} = \varphi_{\Delta'}^W(\mathbf{x}_W)\widehat{\varphi_{\Delta''}^E}(\xi,u').$$

(ii)

$$r_P^{\mathcal{W}}\left(\varphi_{\Delta}^V\right)(\mathbf{x}_W) = \varphi_{\Delta'}^W(\mathbf{x}_W)\widehat{\varphi_{\Delta''}^E}(0,0).$$

In our applications all entries of Δ_- will be zero, so $\Delta=\Delta_+$ (by abuse of notation). We first note

Lemma 4.8.

$$\widehat{\varphi_{\Delta''_+}^E}(\xi,0) = \left(\prod_{j=1}^n \prod_{\alpha=1}^{\ell} (-\sqrt{2}i\xi_{\alpha j})^{\delta_{\alpha j}}\right)\varphi_0^E(\xi,0).$$

In particular, if in addition all *entries of* Δ''_+ *vanish, then*

$$\widehat{\varphi_{\Delta_+}^V} \begin{pmatrix} \xi \\ \mathbf{x}_W \\ 0 \end{pmatrix} = \varphi_{\Delta'_+}^W(\mathbf{x}_W)\varphi_0^E(\xi,0).$$

Proof. This follows from applying Lemma 4.5. □

We conclude

Proposition 4.9. (i) *Assume that* one *of the entries of* Δ''_+ *is nonzero, then*

$$r_P^{\mathcal{W}}(\varphi_{\Delta_+}^V) = 0.$$

(ii) *If all of the entries of* Δ''_+ *vanish, then*

$$r_P^{\mathcal{W}}\left(\varphi_{\Delta_+}^V\right) = \varphi_{\Delta'_+}^W.$$

Remark 4.10. Of course the analogous result holds for $r_P^{\mathcal{W}}(\varphi_{\Delta_-}^V)$. However, a general formula for the restriction of $r_P^{\mathcal{W}}(\varphi_{\Delta}^V)$, i.e., for $\widehat{\varphi_{\Delta''}^E}(0,0)$, is more complicated (and is not needed in this paper).

4.3. **The Fock model.** It will be also convenient to consider the Fock model $\mathcal{F} = \mathcal{F}_{n,V}$ of the Weil representation. For more details for what follows, see the appendix of [9].

There is an intertwining map $\iota : S(V^n) \to \mathcal{P}(\mathbb{C}^{n(p+q)})$ from the K'-finite Schwartz functions to the infinitesimal Fock model of the Weil representation acting on the space of complex polynomials $\mathcal{P}(\mathbb{C}^{n(p+q)})$ in $n(p+q)$ variables such that $\iota(\varphi_0) = 1$. We denote the variables in $\mathcal{P}(\mathbb{C}^{n(p+q)})$ by $z_{\alpha i}$ $(1 \le \alpha \le p)$ and $z_{\mu i}$ $(p+1 \le \mu \le p+q)$ with $i = 1, \dots, n$. Moreover, the intertwining map ι satisfies

$$\iota \left(x_{\alpha i} - \frac{1}{2\pi} \frac{\partial}{\partial x_{\alpha i}} \right) \iota^{-1} = \frac{1}{2\pi i} z_{\alpha i}, \qquad \iota \left(x_{\mu j} - \frac{1}{2\pi} \frac{\partial}{\partial x_{\mu j}} \right) \iota^{-1} = -\frac{1}{2\pi i} z_{\mu j}.$$

By slight abuse of notation, we use the same symbol for corresponding objects in the Schrödinger and Fock model.

In the Fock model, φ_Δ^V looks as follows.

Lemma 4.11.

$$\varphi_\Delta^V = \prod_{\substack{1 \le \alpha \le p \\ p+1 \le \mu \le m \\ 1 \le j \le n}} \left(\frac{1}{2\pi i} z_{\alpha j} \right)^{\delta_{\alpha j}} \left(-\frac{1}{2\pi i} z_{\mu j} \right)^{\delta_{\mu j}}.$$

Proposition 4.9 translates to

Proposition 4.12. *If* one *of the entries of* Δ''_+ *is nonzero, then*

$$r_P^{\mathcal{W}} \left(\varphi_{\Delta_+}^V \right) = 0.$$

If all of the entries of Δ''_+ *vanish, then*

$$r_P^{\mathcal{W}} \left(\varphi_{\Delta_+}^V \right) = \prod_{\substack{\ell+1 \le \alpha \le p \\ 1 \le j \le n}} \left(\frac{1}{2\pi i} z_{\alpha j} \right)^{\delta_{\alpha j}}.$$

5. Differential graded algebras associated to the Weil representation

In this section, we construct the differential graded algebras $C_V^\bullet$ and $A_P^\bullet$, which were introduced in the introduction and define the the local restriction map r_P from $C_V^\bullet$ to $A_P^\bullet$. Again V will denote a nondegenerate real quadratic space of dimension m and signature (p,q).

5.1. **Relative Lie algebra complexes.** For convenience of the reader, we very briefly review some basic facts about relative Lie algebra complexes, see e.g., [5]. For this subsection, we deviate from the notation of the paper and let $\mathfrak{g}$ be any real Lie algebra $\mathfrak{g}$ and let $\mathfrak{k}$ be any subalgebra. We let (U,π) be a representation of $\mathfrak{g}$. We set

$$C^q(\mathfrak{g},\mathfrak{k};U) = \left[\operatorname{Hom} \left(\bigwedge^q (\mathfrak{g}/\mathfrak{k}), U \right) \right]^{\mathfrak{k}} \simeq \left[\bigwedge^q (\mathfrak{g}/\mathfrak{k})^* \otimes U \right]^{\mathfrak{k}}, \tag{5.1}$$

where the action of $\mathfrak{k}$ on $\bigwedge^q(\mathfrak{g}/\mathfrak{k})$ is induced by the adjoint representation. Thus $C^q(\mathfrak{g},\mathfrak{k};U)$ consists of those $\varphi \in \operatorname{Hom}\left(\bigwedge^q(\mathfrak{g}/\mathfrak{k}), U\right)$ such that

$$\sum_{i=1}^{q} \varphi(X_1,\dots,[X,X_i],\dots,X_q) = X\cdot\varphi(X_1,\dots,X_q) \qquad (X\in\mathfrak{k}). \tag{5.2}$$

The differential $d: C^q \to C^{q+1}$ is defined by

$$\begin{aligned} d\varphi(X_0,X_1,\dots,X_q) &= \sum_{i=0}^{q}(-1)^i X_i\cdot\varphi(X_0,\dots,\hat{X}_i,\dots,X_q) \\ &\quad + \sum_{i<j}(-1)^{i+j}\varphi([X_i,X_j],X_0,\dots,\hat{X}_i,\dots,\hat{X}_j\dots,X_q) \end{aligned} \tag{5.3}$$

for $X_0,\dots,X_q \in \mathfrak{g}/\mathfrak{k}$. We let $\{X_i\}$ be a basis of $\mathfrak{g}/\mathfrak{k}$ and let $\{\omega_i\}$ be the dual basis. Then in the setting of $\left[\bigwedge^q(\mathfrak{g}/\mathfrak{k})^*\otimes U\right]^{\mathfrak{k}}$, the differential d is given by

$$d = \sum_i A(\omega_i)\otimes\pi(X_i) + \frac{1}{2}\sum_i A(\omega_i)\,\mathrm{ad}^*(X_i)\otimes 1. \tag{5.4}$$

Here $A(\omega_i)$ denotes the left multiplication with ω_i in $\bigwedge^\bullet(\mathfrak{g}/\mathfrak{k})^*$, and $\mathrm{ad}^*(X)$ is the dual of the adjoint action on $\bigwedge^\bullet(\mathfrak{g}/\mathfrak{k})^*$, that is, $(\mathrm{ad}^*(X)(\alpha))(Y_1,\cdots,Y_q) = \sum_{i=1}^q \alpha(Y_1,\dots,[Y_i,X],\dots,Y_q)$. We easily see

Lemma 5.1. *Consider two relative Lie algebra complexes $C^\bullet(\mathfrak{g},\mathfrak{k};U)$ and $C^\bullet(\mathfrak{g}',\mathfrak{k}';U')$. Then the following datum,*

(i) *$\rho:\mathfrak{g}\to\mathfrak{g}'$, a Lie algebra homomorphism such that $\rho(\mathfrak{k})\subseteq\mathfrak{k}'$,*
(ii) *$T: U'\to U$, an intertwining map with respect to ρ (i.e., $T(\rho(X)\cdot u') = X\cdot T(u')$ for $X\in\mathfrak{g}$),*

induces a natural map of complexes

$$C^\bullet(\mathfrak{g}',\mathfrak{k}';U')\to C^\bullet(\mathfrak{g},\mathfrak{k};U)$$

given by

$$\varphi\mapsto T\circ\varphi\circ\rho.$$

When realizing φ as an element $\left[\bigwedge^q(\mathfrak{g}'/\mathfrak{k}')^*\otimes U'\right]^{\mathfrak{k}'}$, then the map is given by

$$\rho^*\otimes T,$$

where $\rho^*:(\mathfrak{g}'/\mathfrak{k}')^*\to(\mathfrak{g}/\mathfrak{k})^*$ is the dual map.

Now let G be any real Lie group with Lie algebra $\mathfrak{g}$ and let K be a closed connected subgroup of G (not necessarily compact) with Lie algebra $\mathfrak{k}$. For U a smooth G-module, we let $\mathcal{A}^q(G/K;U)$ be the U-valued differential q-forms on G/K (with the usual exterior differentiation). The G-action on $\mathcal{A}^q(G/K;U)$ is given by

$$(g\circ w)_x(X) = g(\omega_{g^{-1}\cdot x}(g^{-1}\cdot X)), \tag{5.5}$$

for $\omega \in \mathcal{A}^q(G/K;U)$, $x \in G/K$, and $X \in T_x^q(G/K)$. Then evaluation at the base point of G/K gives rise to an isomorphism of complexes

$$\mathcal{A}^\bullet(G/K;U)^G \simeq C^\bullet(\mathfrak{g},\mathfrak{k};U) \tag{5.6}$$

of the G-invariant forms on G/K with $C^\bullet(\mathfrak{g},\mathfrak{k};U)$.

5.2. **The differential graded algebra $C_V^\bullet$.** We begin this section by defining a differential graded (but not graded-commutative) algebra $C_V^\bullet$. We first define the underlying complex.

The complex $C_V^\bullet$ is essentially the relative Lie algebra complex for $\mathrm{O}(V)$ with values in $\mathcal{W}_{n,V}$ tensored with the tensor algebra of $V_\mathbb{C}$ and twisted by some factors associated to $\mathbb{C}^n$. Precisely, it is the complex given by

$$\begin{aligned} C_V^{j,r,k} &= \Big[T^j(U)[-\tfrac{p-q}{2}] \otimes T^k(\mathbb{C}^n)^* \otimes \mathcal{W}_{n,V} \otimes \bigwedge\nolimits^r \mathfrak{p}_\mathbb{C}^* \otimes T^k(V_\mathbb{C})\Big]^{K'\times K\times S_k} \\ &\simeq \big[T^j(U)[-\tfrac{p-q}{2}] \otimes T^k(\mathbb{C}^n)^* \otimes \mathcal{W}_{n,V} \otimes \mathcal{A}^r(D) \otimes T^k(V_\mathbb{C})\big]^{K'\times G\times S_k}. \end{aligned}$$

Here j,r,k are nonnegative integers and $\mathcal{A}^r(D)$ denotes the space of complex-valued differential r-forms on D. We let $U = \bigwedge^n(\mathbb{C}^n)^*$, and we define the action of K' on $T^\bullet(U)[\frac{p-q}{2}]$ by requiring K' to act by the character $\det^{-j-\frac{p-q}{2}}$ on $T^j(U)$. Thus K' acts by algebra homomorphisms shifted by the character $\det^{-\frac{p-q}{2}}$. We will usually drop the $[\frac{p-q}{2}]$ in what follows. Also note that all tensor products are over $\mathbb{C}$. The differential is the usual relative Lie algebra differential for the action of $\mathrm{O}(V)$. The group K' acts on the first three factors, while the maximal compact subgroup $K_V = K$ of $\mathrm{SO}_0(V)$ fixing the basepoint z_0 acts on the last three factors. Finally, the symmetric group S_k acts on the second and the last factor.

We now give the complex $C_V^\bullet$ an associative multiplication. In order to give the complex the structure of a graded algebra we choose as a model for the Weil representation that has an algebra structure, the Fock model $\mathcal{F}_{n,V}$, the multiplication law is multiplication of polynomials. However, it is important to observe that K' does not act on $\mathcal{F}_{n,V}$ by algebra homomorphisms (but rather by homomorphisms twisted by the character $\det^{\frac{p-q}{2}}$). Now the vector space underlying $C_V^\bullet$ is a subspace (of invariants under a group action) of a tensor product of graded algebras. Thus it remains to prove that the group acts by homomorphisms of the product multiplication.

Lemma 5.2. *The group $K'\times K\times S_k$ acts by algebra homomorphisms on the tensor product of algebras $T^\bullet(U)\otimes T^\bullet(\mathbb{C}^n)^* \otimes \mathcal{W}_{n,V} \otimes \bigwedge^\bullet \mathfrak{p}_\mathbb{C}^* \otimes T^\bullet(V_\mathbb{C})$.*

Proof. The statement is obvious except possibly for the action of the group K'. The group K' acts on the algebra $\mathcal{F}_{n,V}$ by algebra homomorphisms twisted by the character $\det^{\frac{p-q}{2}}$. It acts on the tensor product $T^\bullet(U)$ by algebra homomorphisms twisted by the inverse character $\det^{-\frac{p-q}{2}}$. The two twists cancel on the tensor product and we find that K' acts by algebra homomorphisms. ∎

Sometimes it is more convenient to view an element $\varphi \in C_V^{j,r,k}$ as an element in

$$\Big[\mathrm{Hom}\Big(T^k(\mathbb{C}^n); T^j(U)\otimes \mathcal{W}_{n,V} \otimes \bigwedge\nolimits^r \mathfrak{p}_\mathbb{C}^* \otimes T^k(V_\mathbb{C})\Big)\Big]^{K'\times K\times S_k}. \tag{5.7}$$

For $w \in T^k(\mathbb{C}^n)$, we write $\varphi(w)$ for its value in $T^j(U) \otimes \mathcal{W}_{n,V} \otimes \bigwedge^r \mathfrak{p}^*_{\mathbb{C}} \otimes T^k(V_{\mathbb{C}})$.

By Schur-Weyl theory, see [7], Lecture 6, we have the decomposition

$$T^k(\mathbb{C}^n)^* \simeq \bigoplus_{\lambda} s(t(\lambda))(T^k(\mathbb{C}^n)^*) \otimes V^*_{\lambda}. \tag{5.8}$$

Here the sum is over the Young diagrams λ with k boxes and no more than n rows, $t(\lambda)$ is a chosen standard filling of λ for each λ and V_λ is the irreducible representation of S_k corresponding to λ. We also have the corresponding decomposition

$$T^k(V_{\mathbb{C}}) \simeq \bigoplus_{\mu} s(t'(\mu))(T^k(V_{\mathbb{C}})) \otimes V_{\mu}. \tag{5.9}$$

Combining the two decompositions we obtain

(5.10)
$$C_V^{j,r,k} \simeq \bigoplus_{\lambda,\mu} \Big[T^j(U) \otimes \mathbb{S}_{t(\lambda)}(\mathbb{C}^n)^* \otimes V^*_\lambda \otimes \mathcal{W}_{n,V} \otimes \bigwedge^r \mathfrak{p}^*_{\mathbb{C}} \otimes \mathbb{S}_{t'(\mu)}(V_{\mathbb{C}}) \otimes V_\mu\Big]^{K' \times K \times S_k}.$$

Noting that

$$(V^*_\lambda \otimes V_\mu)^{S_k} \simeq \begin{cases} 0 & \text{if } \lambda \neq \mu \\ \mathbb{C} & \text{if } \lambda = \mu, \end{cases} \tag{5.11}$$

we obtain

Lemma 5.3.
$$C_V^{j,r,k} \simeq \bigoplus_{\lambda} \Big[T^j(U) \otimes \mathbb{S}_{t(\lambda)}(\mathbb{C}^n)^* \otimes \mathcal{W}_{n,V} \otimes \bigwedge^r \mathfrak{p}^*_{\mathbb{C}} \otimes \mathbb{S}_{t(\lambda)}(V_{\mathbb{C}})\Big]^{K' \times K}.$$

We have assumed (as we may do) that the fillings $t(\lambda)$ and $t'(\lambda)$ are the same. For the summands in the lemma we write $C_V^{j,r,t(\lambda)}$ (or just $C_V^{j,r,\lambda}$ if we do not want to specify the filling) and obtain the complex $C_V^{\bullet,\lambda}$ introduced in the introduction. The application of the Schur functor $\mathbb{S}^*_{t(\lambda)}(\cdot)$ on $T^k(\mathbb{C}^n)^*$ or equivalently applying $\mathbb{S}_{t(\lambda)}(\cdot)$ on $T^k(V_{\mathbb{C}})$, gives rise to a projection map

$$\pi_{t(\lambda)} : C_V^{j,r,k} \longrightarrow C_V^{j,r,t(\lambda)}. \tag{5.12}$$

That is,

$$\begin{aligned} \pi_{t(\lambda)} &= 1_U \otimes s(t(\lambda))_{(\mathbb{C}^n)^*} \otimes 1_{\mathcal{W}_{n,V}} \otimes 1_{\mathfrak{p}^*} \otimes 1_V \\ &= 1_U \otimes 1_{\mathbb{C}^n} \otimes 1_{\mathcal{W}_{n,V}} \otimes 1_{\mathfrak{p}^*} \otimes s(t(\lambda))_V. \end{aligned} \tag{5.13}$$

Here we have used subscripts to indicate which spaces the respective identity transformations 1 operate on. We will do this henceforth. We apply the harmonic projection $\mathcal{H}_V$, see (3.10), on the last factor to obtain $\mathbb{S}_{[t(\lambda)]}(V_{\mathbb{C}})$, and we obtain a complex $C_V^{\bullet,[t(\lambda)]}$ (or $C_V^{\bullet,[\lambda]}$) and a projection map

$$\pi_{[t(\lambda)]} : C_V^{j,r,k} \longrightarrow C^{j,r,[t(\lambda)]}. \tag{5.14}$$

That is,

$$\begin{aligned}\pi_{[t(\lambda)]} &= 1_U \otimes 1_{\mathbb{C}^n} \otimes 1_{\mathcal{W}_{n,V}} \otimes 1_{\mathfrak{p}^*} \otimes s([t(\lambda)])_V \\ &= \big(1_U \otimes 1_{\mathbb{C}^n} \otimes 1_{\mathcal{W}_{n,V}} \otimes 1_{\mathfrak{p}^*} \otimes \mathcal{H}_V\big) \circ \pi_{t(\lambda)}.\end{aligned} \tag{5.15}$$

Remark 5.4. We can interpret an element $\varphi \in C_V^{j,r,k}$ as a $K' \times K \times S_k$-invariant homomorphism from $T^k(\mathbb{C}^n)$ to $T^j(U) \otimes \mathcal{W}_{n,V} \otimes \bigwedge^r \mathfrak{p}_{\mathbb{C}}^* \otimes T^k(V_{\mathbb{C}})$, see (5.7). In this setting, we can interpret $\pi_{t(\lambda)}\varphi$ as the restriction of the homomorphism φ to the $\mathbb{S}_{t(\lambda)}(\mathbb{C}^n)$. From this point of view, Lemma 5.3 states that the homomorphism $\pi_{t(\lambda)}\varphi$ on $\mathbb{S}_{t(\lambda)}(\mathbb{C}^n)$ automatically takes values in $\mathcal{W}_{n,V} \otimes \bigwedge^r \mathfrak{p}_{\mathbb{C}}^* \otimes \mathbb{S}_{t(\lambda)}(V_{\mathbb{C}})$.

5.3. **The face differential graded algebra $A_P^\bullet$ and the map r_P.** In this section we assume P is the stabilizer of a standard flag $E_{i_1} \subset E_{i_2} \subset \cdots \subset E_{i_k} = E_\ell$ and N_P is the unipotent radical of P. We will abbreviate E_{i_k} to E and let Q be the stabilizer of E (a maximal parabolic subgroup). We will now construct a differential graded algebra $A_P^\bullet$, which is the relative Lie algebra version of a differential graded subalgebra of the de Rham complex of the face $e(P)$ of the Borel-Serre enlargement of D. We will continue with the notation of section 2.

We define the differential graded algebra $A_P^\bullet$ associated to the face $e(P)$ of the Borel-Serre boundary corresponding to P by

$$\begin{aligned}A_P^{j,r,k} &= \Big[T^j(U) \otimes T^k(\mathbb{C}^n)^* \otimes \mathcal{W}_{n,W} \otimes \bigwedge^r (\mathfrak{n} \oplus \mathfrak{p}_M)_{\mathbb{C}}^* \otimes T^k(V_{\mathbb{C}})\Big]^{K' \times K_P \times S_k} \\ &\simeq \big[T^j(U) \otimes T^k(\mathbb{C}^n)^* \otimes \mathcal{W}_{n,W} \otimes \mathcal{A}^r(e(P)) \otimes T^k(V_{\mathbb{C}})\big]^{K' \times NM \times S_k}.\end{aligned} \tag{5.16}$$

Furthermore, we define $A_P^{\bullet;\lambda}$ and $A_P^{\bullet;[\lambda]}$ as for $C_V^\bullet$.

Definition 5.5. The "local" restriction map of de Rham algebras with coefficients

$$r_P : C_V^\bullet \to A_P^\bullet$$

of de Rham algebras with coefficients is given by

$$1 \otimes 1 \otimes r_P^{\mathcal{W}} \otimes \iota^* \otimes 1,$$

where

$$\iota : \mathfrak{n} \oplus \mathfrak{m} \hookrightarrow \mathfrak{g}$$

is the underlying Lie algebra homomorphism, and the map from the coefficients of $C_V^\bullet$ to the coefficients of $A_P^\bullet$ is given by the tensor product

$$1 \otimes 1 \otimes r_P^{\mathcal{W}} \otimes 1,$$

where $r_P^{\mathcal{W}} : \mathcal{W}_{n,V} \to \mathcal{W}_{n,W}$ is the restriction map of the Weil representation (see Definition 4.3). By Lemma 5.1 we therefore see that r_P is a map of complexes. We note that $r_P^{\mathcal{W}}$ is not a ring homomorphism so r_P is not a map of algebras. Since r_P commutes with the action of the symmetric group S_k, we obtain maps $C_V^{\bullet;\lambda} \to A_P^{\bullet;\lambda}$ and $C_V^{\bullet;[\lambda]} \to A_P^{\bullet;[\lambda]}$ as well, which we also denote by r_P.

Note that the induced map $\iota^* : (\mathfrak{g}/\mathfrak{k})^* \simeq \mathfrak{p}^* \to ((\mathfrak{n} \oplus \mathfrak{m})/\mathfrak{k}_M)^* \simeq (\mathfrak{n} \oplus \mathfrak{p}_M)^*$ is the composition of the isomorphism $\sigma^* : \mathfrak{p}^* \longrightarrow (\mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{p}_M)^*$, see (2.46), with the restriction $(\mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{p}_M)^* \to (\mathfrak{n} \oplus \mathfrak{p}_M)^*$.

Finally observe that on the level of homogeneous spaces, the map r_P arises by realizing $e(P)$ as the orbit of the basepoint z_0 under the group NM. So in this setting, we are no longer thinking of $e(P)$ as being at the boundary of D; we have pushed $e(P)$ far inside D.

6. Aspects of the Cohomology of $\mathfrak{n}_P$ and the map ι_P

6.1. **Cohomology of the nilpotent.** In what follows, we discuss some aspects of the Lie algebra cohomology of the nilpotent Lie algabra $\mathfrak{n}_P$ which we need later.

As before, we let P be a standard parabolic subgroup of G. Recall (see section 2) that we have the decomposition of vector spaces $\mathfrak{n}_P = \mathfrak{n}_{P'} \oplus \mathfrak{n}_Q$, where Q is the maximal parabolic containing P. For the two-step nilpotent algebra $\mathfrak{n}_Q$, we have the central extension $\mathfrak{z}_Q \to \mathfrak{n}_Q \to \mathfrak{n}_W$ with $\mathfrak{z}_Q \simeq \bigwedge^2 E$ and $\mathfrak{n}_W \simeq W \otimes E$. On the other hand, $\mathfrak{n}_{P'}$ is a nilpotent subgroup of $\mathfrak{sl}(E) \subset E' \otimes E$.

We assume for the next subsections that V, W, E, E', $\mathfrak{n}_P$ etc. are defined over $\mathbb{C}$. We let

$$\mathcal{C}^{\bullet,\ell'} = \bigwedge^{\bullet}(\mathfrak{n}_P) \otimes T^{\ell'}(V)$$

be the complex for the nilpotent cohomology with coefficients in $T^{\ell'}(V)$ and define analogously $\mathcal{C}^{r,\lambda}$ and $\mathcal{C}^{r,[\lambda]}$ for $\mathbb{S}_\lambda(V)$ and $\mathbb{S}_{[\lambda]}(V)$ respectively.

We will be interested in certain cohomology classes arising from $\bigwedge^r \mathfrak{n}_W^*$. By Lemma 2.1 $\mathfrak{n}_W^* \simeq W \otimes E'$ as $\mathrm{O}(W) \times \mathrm{GL}(E)$-modules. Furthermore, see for example [7], p. 80,

$$\bigwedge^r(\mathfrak{n}_W^*) \simeq \bigwedge^r (W \otimes E') \simeq \bigoplus_{\mu} \mathbb{S}_\mu(W) \otimes \mathbb{S}_{\mu'}(E'), \tag{6.1}$$

as $\mathrm{O}(W) \times \mathrm{GL}(E)$-modules. Here the sum extends over all partitions μ of r with at most $\dim W = m - 2\ell$ rows and at most $\dim E = \ell$ columns, and μ' denotes the conjugate partition of μ.

We will be mainly interested in the case $r = n\ell$. Then we can take $\mu = \ell\varpi_n = (\ell, \ell, \dots, \ell)$, so that $\mu' = n\varpi_\ell = (n, n, \dots, n)$ and $\mathbb{S}_{\mu'}(E') = \left(\bigwedge^\ell E'\right)^{\otimes n} \simeq \mathbb{C}$ is the trivial (one-dimensional) $\mathrm{SL}(E)$-module. We obtain

$$\mathbb{S}_B(W) \otimes \mathbb{S}_{B'}(E') \simeq \left[\bigwedge^{n\ell}(W \otimes E')\right]^{SL(E)} \simeq \left[\bigwedge^{n\ell}(\mathfrak{n}_W^*)_{\mathbb{C}}\right]^{\mathrm{SL}(E)} \tag{6.2}$$

as $\mathrm{O}(W) \times \mathrm{SL}(E)$-modules. Here $B = B_{n,\ell}$ is the filling of the Young diagram associated to μ described in section 3.2.

To realize this isomorphism, we define a $\mathrm{GL}(W) \times \mathrm{GL}(E)$ intertwining map

$$\tau_{r,\ell'} : T^r(W) \otimes T^{\ell'}(W) \otimes T^r(E') \to \bigwedge^r (W \otimes E') \otimes T^{\ell'}(V) \subset \mathcal{C}^{r,\ell'} \tag{6.3}$$

given by

$$\tau_{r,\ell'}((w_1 \otimes \cdots \otimes w_r) \otimes \bar{\mathbf{w}} \otimes (v'_1 \otimes \cdots \otimes v'_r)) = (w_1 \otimes v'_1) \wedge \cdots \wedge (w_r \otimes v'_r) \otimes \bar{\mathbf{w}},$$

where $\bar{\mathbf{w}} \in T^{\ell'}(W)$. We also write τ_r for $\tau_{r,0}$. We immediately see

Lemma 6.1. *The map $\tau_{r,\ell'}$ is $\mathrm{O}(W) \times \mathrm{SL}(E) \times S_{r+\ell'} \times S_r$-equivariant. Here the action of the symmetric group $S_{r+\ell'}$ (respectively S_r) is on the tensor factors involving W (respectively E').*

For $r = n\ell$, the map $\tau_{n\ell}$ realizes the isomorphism (6.2). Furthermore,

Lemma 6.2. *Let $\mathbf{w} \in T^{n\ell+\ell'}(W)$ and $\mathbf{v}' \in T^{n\ell}(E')$. Then*

$$\tau_{n\ell,\ell'}(s_{B|A}(\mathbf{w}) \otimes \mathbf{v}') \in \left(\mathcal{C}^{n\ell,A}\right)^{\mathrm{SL}(E)}.$$

By slight abuse of notation, we view from now on $\tau_{n\ell,\ell'}$ as a map of $T^{n\ell+\ell'}(W)$ by setting

$$\tau_{n\ell,\ell'}(\mathbf{w}) := \tau_{n\ell,\ell'}(\mathbf{w} \otimes (u'_1 \otimes \cdots \otimes u'_\ell)^n).$$

We let $V^{[k]}$ ($W^{[k]}$) be the space of harmonic k-tensors in V (W); i.e., the tensors which are annihilated by all the contractions C_{ij}. We let $\mathcal{E}^k(V) \subset T^k(V)$ be the orthogonal complement of the harmonic tensors. Thus $\mathcal{E}^k(V)$is the sum of the images of the insertion maps $E_{ij}(g_V^*) : T^{k-2}(V) \to T^k(V), 1 \leq i < j \leq k$ with the metric g_V^* of V. Similarly, we define $\mathcal{E}^k(W) \subset T^k(W)$. Note $\mathbb{S}_{[\lambda]}(W) \subset \mathbb{S}_{[\lambda]}(V)$. However note, that if $\bar{\mathbf{w}} \in T^{\ell'}(W)$ is a nonzero tensor in the orthogonal complement of $T^{[\ell']}(W)$ (i.e., spanned by tensors in the image of the inclusion with the metric for W), then $\bar{\mathbf{w}}$ does *not* necessarily lie in the orthogonal complement in $T^{[\ell']}(V)$ (since the metric of V is different).

Proposition 6.3. *Let B again be the given filling of the Young diagram associated to $\ell\omega_n$ and A be a filling for λ.*

(i) *Let $\mathbf{w} \in \mathbb{S}_{B|A}(W)$. Then $\tau_{n\ell,\ell'}(\mathbf{w})$ defines a cocycle in $\mathcal{C}^{n\ell,\ell'}$. More precisely, we obtain a map*

$$\mathbb{S}_{B|A}(W) \to H^{n\ell}(\mathfrak{n}_P, \mathbb{S}_A(V))^{\mathrm{SL}(E)}.$$

(ii) *Let $n \leq \left[\frac{\dim W}{2}\right]$ and let $\mathbf{w} \in \mathbb{S}_{[B|A]}(W)$. Then the cohomology class*

$$[\tau_{n\ell,\ell'}(\mathbf{w})] \in H^{n\ell}(\mathfrak{n}_P, \mathbb{S}_{[A]}(V))^{\mathrm{SL}(E)}$$

does not vanish.Thus we obtain an embedding

$$\mathbb{S}_{[B|A]}(W) \hookrightarrow H^{n\ell}(\mathfrak{n}_P, \mathbb{S}_{[A]}(V))^{\mathrm{SL}(E)}.$$

(iii) *Let $\mathbf{w} \in \mathbb{S}_{B|A}(W) \cap \mathcal{E}^{n\ell+\ell'}(W)$ be in the orthogonal complement of $\mathbb{S}_{[B|A]}(W)$ inside $\mathbb{S}_{B|A}(W)$. Then*

$$[\pi_{[A]} \circ \tau_{n\ell,\ell'}(\mathbf{w})] = 0$$

in $H^{n\ell}(\mathfrak{n}_P, \mathbb{S}_{[A]}(V))$. Here $\pi_{[A]}$ is the natural projection from $H^\bullet(\mathfrak{n}_P, \mathbb{S}_A(V))$ to $H^\bullet(\mathfrak{n}_P, \mathbb{S}_{[A]}(V))$ induced by the orthogonal projection $\mathbb{S}_A(V) \to \mathbb{S}_{[A]}(V)$. In particular, for $\mathbf{w} \in \mathbb{S}_{B|A}(W)$, we have

$$[\pi_{[A]} \circ \tau_{n\ell,\ell'}(\mathbf{w})] = [\tau_{n\ell,\ell'}(\pi_{[B|A]}(\mathbf{w}))].$$

Remark 6.4. The action of the torus A on the image of $\tau_{n\ell,\ell'}$ has weights $\alpha_{i_1}^{-n\ell}, \ldots, \alpha_{i_k}^{-n\ell}$.

Proof. We give a rough sketch. (ii) follows from Kostant's theorem [15], which explicitly decomposes $H^\bullet(\mathfrak{n}_P, \mathbb{S}_{[\lambda]}(V))$ as a module of L. Namely, in degree $n\ell$ representations of highest weight of the form $w(\rho+\lambda)-\rho$ occur, where w is an element of the Weyl group for L of length $n\ell$. Here ρ is the half sum of the positive roots as usual. In our case, we have $w(\rho+\lambda)-\rho = \ell\varpi_n + \lambda$ for $w = w_1\cdots w_n$ with $w_i = s_{\ell+i-1}\cdots s_i$, where s_k is the associated reflections along the root α_k.

(iii) of course also follows in principle from Kostant, but for that one would need to determine the complete list of representations which occur in $H^{n\ell}(\mathfrak{n}_P, \mathbb{S}_{[\lambda]}(V))$, which is tedious. For later use, we record the primitives below.

(i) follows from (ii) and (iii), but we briefly indicate a direct proof. First note that the action of $\sigma \in S_{\ell'}$ on the coefficients $T^{\ell'}(V)$ commutes with the differentiation d: $d\circ(1\otimes\sigma\otimes 1) = (1\otimes\sigma\otimes 1)\circ d$. Furthermore, in the first factor $T^{n\ell}(W)$, $\tau_{n\ell,\ell'}$ factors through c_B, the column anti-symmetrizer for Young tableau B, that is, $\tau_{n\ell,\ell'}\circ(c_B\otimes 1) = \tau_{n\ell,\ell'}$. Combining this with Lemma 6.1 gives $\tau_{n\ell,\ell'}\circ(c_{B|A}) = (1\otimes c_A)\circ\tau_{n\ell,\ell'}$ on $T^{n\ell+\ell'}(W)$.

Therefore it suffices to show that $\tau_{n\ell,\ell'}(r_{B|A}(\mathbf{w}))$ is closed. Indeed, we have

$$d\left(\tau_{n\ell,\ell'}(s_{B|A}(\mathbf{w}))\right) = d\left((1\otimes c_A)\circ\tau_{n\ell,\ell'}(r_{B|A}(\mathbf{w}))\right) = (1\otimes c_A)\circ d(\tau_{n\ell,\ell'}(r_{B|A}(\mathbf{w}))).$$

Furthermore, it suffices to establish closedness for $n=1$. Indeed, if the Young diagram A arises from the partition $(\ell'_1,\ell'_2,\dots,\ell'_n)$ of ℓ', we write $\mathbf{w} = \mathbf{w}_1\otimes\cdots\otimes\mathbf{w}_n \in T^{n\ell}(W)$ with $\mathbf{w}_i \in T^{\ell}(W)$ and $\bar{\mathbf{w}} = \bar{\mathbf{w}}_1\otimes\cdots\otimes\bar{\mathbf{w}}_n$ with $\bar{\mathbf{w}}_i \in T^{\ell'_i}(W)$. We then have a natural product decomposition

$$\tau_{n\ell,\ell'}(\mathbf{w}\otimes\bar{\mathbf{w}}) = \tau_{\ell,\ell'_1}(\mathbf{w}_1\otimes\bar{\mathbf{w}}_1)\wedge\cdots\wedge\tau_{\ell,\ell'_n}(\mathbf{w}_n\otimes\bar{\mathbf{w}}_n), \tag{6.4}$$

for which d acts as a derivation. Now it is not hard to show that applying the differentiation d to the element $\tau_{\ell,\ell'}(\mathbf{w})$ with $\mathbf{w}\in \mathrm{Sym}^{\ell+\ell'}(W)$ gives rise to a map

$$\begin{aligned} \mathrm{Sym}^{\ell+\ell'}(W) &\to \bigoplus_{i=1}^{\ell} \bigwedge^{\ell+1}(W\otimes E')\otimes(E'_i\otimes T^{\ell'-1}(W)) \\ &= \bigoplus_{i=1}^{\ell}\bigoplus_{C} \mathbb{S}_C(W)\otimes\mathbb{S}_{C'}(E')\otimes(E'_i\otimes T^{\ell'-1}(W)). \end{aligned} \tag{6.5}$$

Here $E'_i = \mathbb{C}u'_i$, and the sum extends over all Young diagrams C of size $\ell+1$, which have at least 2 rows (otherwise the dual diagram C' would have at least $\ell+1$ rows, which is impossible as $\dim E' = \ell$). By the Littlewood-Richardson rule we now see that in the decomposition of $\mathbb{S}_C(W)\otimes T^{\ell'-1}(W)$ into irreducibles only Young diagrams with at least 2 rows can occur. Hence $\mathrm{Sym}^{\ell+\ell'}(W)$ does not occur on the right hand side of (6.5), and the map vanishes identically. This proves (i).

To prove (iii), we record the primitives. It suffices to show that for any $\mathbf{w}\in T^{n\ell+\ell-2}$, the form $\pi_{[A]}\circ\tau_{n\ell,\ell'}(s_{B|A}(E_{i,j}(g^*_W)(\mathbf{w})))$ is exact. For this, it suffices to show that $\tau_{n\ell,\ell'}(r_{B|A}E_{i,j}(g^*_W)(\mathbf{w})))$ is exact up to terms involving the inclusion of the metric g^*_V into the coefficient system. The product decomposition (6.4) reduces the claim to the cases of $n=1$ (in case the metric g^*_W occurs in one factor for (6.4)) or $n=2$ (if g^*_W occurs in two factors). We sketch the case $n=1$ by giving the primitive exactly leaving $n = 2$ to the reader. We need a bit of notation. Let $s \leq \ell$. For an $(\ell-1)$-tensor $\mathbf{w} = w_1 \otimes w_{s-1} \otimes w_{s+1} \otimes w_\ell \in T^{\ell-1}(W)$, we set

$$\tau_{\ell-1}^{(s)}(\mathbf{w}) = (w_1 \otimes u'_1) \wedge \cdots \wedge (w_{s-1} \otimes u'_{s-1}) \wedge (w_{s+1} \otimes u'_{s+1}) \wedge \cdots \wedge (w_\ell \otimes u'_\ell) \in \bigwedge^{\ell-1}(W \otimes E').$$

Similarly, we define for a $(\ell-2)$-tensor $\mathbf{w} \in T^{\ell-2}(W)$, the element $\tau_{\ell-2}^{(s_1,s_2)}(\mathbf{w}) \in \bigwedge^{\ell-2}(W \otimes E')$, for $1 \leq s_1 < s_2 \leq \ell$ by omitting u'_{s_1} and u'_{s_2}. For $\mathbf{w} = w_1 \otimes \cdots w_b \in T^b(W)$, we let $s_b(\mathbf{w}) = \frac{1}{b!} \sum_{\sigma \in S_b} w_{\sigma^{-1}(1)} \otimes \cdots \otimes w_{\sigma^{-1}(b)}$. Finally, we consider $\mathbf{w} = w_1 \otimes \cdots \otimes w_{\ell+\ell'-2}$, a $(\ell+\ell'-2)$-tensor. We write P_a for the set of multi-indices of size a inside $\{1, \ldots, \ell+\ell'+2\}$, consisting of the (ordered) sets $I = \{i_1, \ldots, i_a\} \subset \{1, \ldots, \ell+\ell'+2\}$ of size a, we write $\mathbf{w}_I = w_{i_1} \otimes \cdots \otimes w_{i_a}$. We also let $\overline{I}$ be the complement of I in $\{1, \ldots, \ell+\ell'+2\}$. Then

$$\tfrac{(\ell+\ell')!}{\ell!\ell'!} \tau_{\ell,\ell'}\big(s_{B|A}\big(E_{1,2}(g_W^*)\mathbf{w}\big)\big) - \sum_{K \in P_\ell} \tau_\ell(s_B(\mathbf{w}_K)) \otimes s_A(g_V^* \otimes \mathbf{w}_{\overline{K}})$$

is exact with primitive

$$\frac{2}{\ell(\ell-1)} \sum_{I \in P_{\ell-2}} \sum_{1 \leq s_1 < s_2 \leq \ell} (-1)^{s_1+s_2} (u'_{s_1} \wedge u'_{s_2}) \wedge \tau_{\ell-2}^{(s_1,s_2)}(s_{\ell-2}(\mathbf{w}_I)) \otimes s_A(\mathbf{w}_{\overline{I}})$$
$$+ \frac{2}{\ell} \sum_{J \in P_{\ell-1}} \sum_{s=1}^{\ell} (-1)^s \tau_{\ell-1}^{(s)}(s_{\ell-1}(\mathbf{w}_J)) \otimes s_A(u'_s \otimes \mathbf{w}_{\overline{J}}).$$

□

6.2. **The map ι_P.** We now assume again that all objects are defined over $\mathbb{R}$.

We construct the map $\iota_P : C_W^\bullet \hookrightarrow A_P^\bullet$ of complexes mentioned in the introduction.

We let U and U' be two complex representations of G and $T : U' \to U$ be G-intertwiner. We let $\mathcal{C}^\bullet(\mathfrak{n}_P, U) = (\bigwedge^\bullet \mathfrak{n}_P^*) \otimes U)$ be the complex computing the nilpotent cohomology $\mathcal{H}^s(\mathfrak{n}_P, U)$, and we let $\mathcal{C}^\bullet_{\text{closed}}(\mathfrak{n}_P, U)$ be the subspace of cocycles in $\mathcal{C}^\bullet(\mathfrak{n}_P, U)$.

Lemma 6.5. *Define a map*

$$\eta^{r,s} : \Big[\bigwedge^r(\mathfrak{p}_M^*) \otimes \Big((\bigwedge^s \mathfrak{n}_P^*) \otimes U'\Big)\Big]^{K_P} \to \Big[\bigwedge^{r+s}(\mathfrak{p}_M^* \oplus \mathfrak{n}_P^*) \otimes U\Big]^{K_P}$$

by

$$\eta^{r,s}(\omega^{(r)} \otimes (\omega^{(s)} \otimes u') = (\omega^{(r)} \wedge \omega^{(s)}) \otimes T(u').$$

Then $\eta^{r,s}$ induces a map of relative Lie algebra complexes

$$\eta : \mathcal{C}^\bullet\left(\mathfrak{m}, \mathfrak{k}_P; \mathcal{C}^s_{closed}(\mathfrak{n}_P, U')\right) \longrightarrow \mathcal{C}^{\bullet+s}\left(\mathfrak{p}, \mathfrak{k}_P; U\right)$$

and the induced map in cohomology factors through $H^\bullet\left(\mathfrak{m}, \mathfrak{k}_P; H^s(\mathfrak{n}_P, U')\right)$.

Proof. This is essentially in [13], Lemma 2.6, see also [22], section 2, together with the standard spectral sequences argument in this context. Note that Harder actually considers instead of cocycles in $\mathcal{C}(\mathfrak{n}_P, U')$ the nilpotent cohomology group $\mathcal{H}^s(\mathfrak{n}_P, U)$ realized as subspace in $\mathcal{C}(\mathfrak{n}_P, U')$ by harmonic forms as discussed in section 6.1. □

Definition 6.6. We define the map ι_P on $C_W^{j,r,k}$ as follows. In fact, it is defined on the underlying tensor spaces without taking the group invariants. First we set ι_P to be zero if $k < n\ell$. If $k \geq n\ell$ we split the two tensor factors

$$T^k(\mathbb{C}^n)^* = T^{n\ell}(\mathbb{C}^n)^* \otimes T^{k-n\ell}(\mathbb{C}^n)^* \quad \text{and} \quad T^k(W_\mathbb{C}) = T^{n\ell}(W_\mathbb{C}) \otimes T^{k-n\ell}(W_\mathbb{C}).$$

We define ι_P on tensors which are decomposable relative to these two splittings. We let $u_1 = \theta_1 \wedge \cdots \wedge \theta_n$ be the standard generator of $U = \bigwedge^n(\mathbb{C}^n)^*$ (with the twisted K'-action). Let $u_1^j \otimes x \otimes f \otimes \omega \otimes w$ be a single tensor component of an element in $C_W^{j,r,k}$ and assume that $k \geq n\ell$. Assume that x and w are decomposable, that is

$$x = x_1 \otimes x_2 \in T^{n\ell}(\mathbb{C}^n)^* \otimes T^{k-n\ell}(\mathbb{C}^n)^* \quad \text{and} \quad w = w_1 \otimes w_2 \in T^{n\ell}(W_\mathbb{C}) \otimes T^{k-n\ell}(W_\mathbb{C}).$$

Then we define

$$\begin{aligned} &\iota_P(u_1^j \otimes x \otimes f \otimes \omega \otimes w) \\ &\quad = (-1)^{n\ell(\frac{(q-\ell)(n-1)}{2}+1)} \eta^{r,n\ell}\Big((u_1^j \otimes s(B)^*(x_1)) \otimes x_2 \otimes f \otimes \omega \otimes \tau_{n\ell}(w_1) \otimes w_2 \Big) \\ &\quad \in T^{j+\ell}(U) \otimes T^{k-n\ell}(\mathbb{C}^n)^* \otimes \mathcal{W}_{n,W} \otimes \bigwedge^r (\mathfrak{p}_W^*)_\mathbb{C} \otimes \bigwedge^{n\ell} (\mathfrak{n}_W^*)_\mathbb{C} \otimes T^{k-n\ell}(W_\mathbb{C}). \end{aligned}$$

Note here that by Lemma 3.2, we see that $\mathbb{S}_B(\mathbb{C}^n)^* = s(B)^* T^{n\ell}(\mathbb{C}^n)^* \simeq T^\ell(U)[0]$ and therefore $u_1^j \otimes s(B)^*(x_1)$ lies in $T^{j+\ell}(U)[-\frac{p-q}{2}]$ and is zero if and only if $s(B)^*(x_1) = 0$.

Proposition 6.7. *ι_P is a map of complexes*

$$\iota_P : C_W^{j,r,k} \to A_P^{j+\ell, r+n\ell, k-n\ell}.$$

Proof. In view of Lemma 6.5, it suffices to show that the map on $C_W^{j,r,k}$ to

$$\mathcal{C}^r\left(\mathfrak{m}, \mathfrak{k}_P; \mathcal{C}^{n\ell}(\mathfrak{n}_P, T^{k-n\ell}(W_\mathbb{C})) \otimes T^{j+\ell}(U)[-\tfrac{p-q}{2}] \otimes \mathcal{W}_{n,W}\right) \tag{6.6}$$

induced by

$$u_1^j \otimes x \otimes f \otimes \omega \otimes w \mapsto (u_1^j \otimes s(B)^*(x_1)) \otimes x_2 \otimes f \otimes \omega \otimes \tau_{n\ell}(w_1) \otimes w_2 \tag{6.7}$$

gives a cocycle for the nilpotent $\mathfrak{n}_P$-complex. Going through the proof of Theorem 6.3(i), we see that the composition of the $\mathfrak{n}_P$-differential with (6.7) factors when viewed as a map on $T^k(W_\mathbb{C})$ through representations $\mathbb{S}_C(W_\mathbb{C})$ with C having at least $n+1$ rows. But now by Lemma 5.3 such representations do not occur in $C_W^{j,r,k}$. □

The reader easily checks from the definition that ι_P satisfies the following properties.

Lemma 6.8. (1) *ι_P is a $[T(U) \otimes \mathcal{W}_{n,W} \otimes \bigwedge \mathfrak{p}_W^*]^{K' \times K_W}$-module homomorphism. That is,*

$$\iota_P(\varphi^W_{j',r',0} \cdot \varphi^W_{j,r,k}) = \varphi^W_{j',r',0} \cdot \iota_P(\varphi^W_{j,r,k})$$

for $\varphi^W_{j',r',0} \in C_W^{j',r',0}$ and $\varphi^W_{j,r,k} \in C_W^{j,r,k}$.

(2) *$\iota_P(\varphi^W_{j,r,k})$ is zero if $k < n\ell$.*

(3) *Suppose $\varphi^W_{j,r,k} \in C_W^{j,r,k}$ with $k \geq n\ell$ and $\varphi^W_{j',r',\ell'} \in C_W^{j',r',\ell'}$. Then*

$$\iota_P(\varphi^W_{j,r,k} \cdot \varphi^W_{j',r',\ell'}) = \iota_P(\varphi^W_{j,r,k}) \cdot \varphi^W_{j',r',\ell'}.$$

(4) *Let* $x \in T^{n\ell}(\mathbb{C}^n)^*$ *and* $w \in T^{n\ell}(W_{\mathbb{C}})$. *Then*

$$\iota_P(1_U \otimes x \otimes 1_{\mathcal{F}} \otimes 1_{\mathfrak{p}_W^*} \otimes w) = x(\varepsilon_B)(u_\ell \otimes 1_{\mathbb{C}^n} \otimes 1_{\mathcal{F}} \otimes 1_{\mathfrak{p}_W^*} \otimes \tau_{n\ell}(w) \otimes 1_{T(V_{\mathbb{C}})}).$$

Proposition 6.9. *Let* $k = n\ell + \ell'$ *as above. Let* λ *be a dominant weight of* $\mathrm{GL}_n(\mathbb{C})$, *and we let* A *be a standard filling of the associated Young diagram* $D(\lambda)$. *We let* $B|A$ *be the associated filling for the weight* $\ell\varpi_n + \lambda$, *see section 3.*

(i) *Then the preimage of* $A_P^{j+\ell,r+n\ell,A}$ *under* ι_P *lies in* $C_W^{j,r,B|A}$*;i.e.,*

$$\iota_P^{-1}\left(A_P^{j+\ell,r+n\ell,A}\right) = C_W^{j,r,B|A}.$$

Moreover, if $\iota_P(\varphi') = \varphi$ *for* $\varphi' \in C_W^{j,r,n\ell+\ell'}$ *and* $\varphi \in A_P^{j+\ell,r+n\ell,\ell'}$, *then*

$$\pi_A(\varphi) = \iota_P(\pi_{B|A}(\varphi')).$$

Here $\pi_{B|A}$ *is the projection from* $C_W^{j,r,n\ell+\ell'}$ *to* $C_W^{j,r,B|A}$, *see* (5.12), *and* π_A *is the one from* $A_P^{j+\ell,r+n\ell,\ell'}$ *to* $A_P^{j+\ell,r+n\ell,A}$.

(ii) *Let* $\varphi \in A_P^{j+\ell,r+n\ell,[A]}$ *a closed form such that* $\iota_P(\varphi') = \varphi$ *for some* $\varphi' \in C_W^{j,r,B|A}$. *Let* $\pi_{[B|A]}$ *be the projection from* $C_W^{j,r,B|A}$ *to* $C_W^{j,r,[B|A]}$. *Then the cohomology class* $[\varphi]$ *satisfies*

$$[\varphi] = [\iota_P(\pi_{[B|A]}(\varphi'))].$$

Proof. (i) We first observe that ι_P is invariant under $s(B)$ in the $T^{n\ell}(W)$-factor and also $s(B^*)$-invariant in the $T^{n\ell}(\mathbb{C}^n)^*$-factor, that is,

$$\begin{aligned}\iota_P &= \iota_P \circ (1_U \otimes 1_{T^{n\ell}(\mathbb{C}^n)^*} \otimes 1_{T^{\ell'}(\mathbb{C}^n)^*} \otimes 1_{\mathcal{W}} \otimes 1_{\mathfrak{p}_W^*} \otimes s(B) \otimes 1_{T^{\ell'}(W)})\\ &= \iota_P \circ (1_U \otimes s(B^*) \otimes 1_{T^{\ell'}(\mathbb{C}^n)^*} \otimes 1_{\mathcal{W}} \otimes 1_{\mathfrak{p}_W^*} \otimes 1_{T^{n\ell}(W)} \otimes 1_{T^{\ell'}(W)}).\end{aligned}$$

Taking the $S_{\ell'}$-invariance into account, we see that ι_P maps

(6.8)
$$\left[T^j(U) \otimes \mathbb{S}_B(\mathbb{C}^n)^* \otimes \mathbb{S}_A(\mathbb{C}^n)^* \otimes \mathcal{W}_{n,W} \otimes \bigwedge\nolimits^r(\mathfrak{p}_W^*)_{\mathbb{C}} \otimes \mathbb{S}_B(W_{\mathbb{C}}) \otimes \mathbb{S}_A(W_{\mathbb{C}})\right]^{K' \times K_W}$$

to $A^{j+2\ell,r+n\ell,A}$. But now

Lemma 6.10.

(6.9)
$$\left[T^j(U) \otimes \mathbb{S}_B(\mathbb{C}^n)^* \otimes \mathbb{S}_A(\mathbb{C}^n)^* \otimes \mathcal{W}_{n,W} \otimes \bigwedge\nolimits^r(\mathfrak{p}_W^*) \otimes \mathbb{S}_B(W_{\mathbb{C}}) \otimes \mathbb{S}_A(W_{\mathbb{C}})\right]^{K' \times K_W} = C_W^{j,r,B|A}.$$

Proof. In (6.9), we first observe $\mathbb{S}_B(\mathbb{C}^n)^* \otimes \mathbb{S}_A(\mathbb{C}^n)^* = \mathbb{S}_{B|A}(\mathbb{C}^n)^*$ as subspaces of $T^{n\ell+\ell'}(\mathbb{C}^n)$, see Corollary 3.4. But then by Schur-Weyl theory, see Lemma 5.3 or Remark 5.4, we can now replace $\mathbb{S}_B(W_{\mathbb{C}}) \otimes \mathbb{S}_A(W_{\mathbb{C}})$ with its subspace $\mathbb{S}_{B|A}(W_{\mathbb{C}})$ in (6.9), that is, the left hand side in (6.9) is equal to $C_W^{j,r,B|A}$. □

From this we obtain Proposition 6.9(i). Proposition 6.9(ii) follows from Proposition 6.3(iii) and Lemma 6.5. □

7. Special Schwartz forms

Again, in this section, V will denote a real quadratic space of dimension m and signature (p,q).

7.1. **Construction of the special Schwartz forms.** We recall the construction in [9] of the special Schwartz forms $\varphi_{nq,\ell'}$, $\varphi_{nq,\lambda}$, and $\varphi_{nq,[\lambda]}$, which define cocycles in $C_V^{\bullet,\ell'}$, $C_V^{\bullet,\lambda}$, and $C_V^{\bullet,[\lambda]}$ respectively. It will be more convenient to use the model $C_V^{\bullet}$ consisting of homomorphisms on $T^{\ell'}(\mathbb{C}^n)$ (and its subspaces $\mathbb{S}_{t(\lambda)}(\mathbb{C}^n)$), see (5.7) and Remark 5.4. We will initially use the Schrödinger model $\mathcal{S}(V^n)$ of the Weil representation.

In [9], we construct for $n \leq p$ a family of Schwartz forms $\varphi_{nq,\ell'}$ on V^n such that $\varphi_{nq,\ell'} \in C_V^{q,nq,\ell'}$. So

$$\begin{aligned}(7.1)\qquad \varphi_{nq,\ell'} &\in \left[\mathrm{Hom}\left(T^{\ell'}(\mathbb{C}^n), T^q(U)\otimes\mathcal{S}(V^n)\otimes\mathcal{A}^{nq}(D)\otimes T^{\ell'}(V_{\mathbb{C}})\right)\right]^{K'\times G\times S_{\ell'}}\\ &\simeq \left[\mathrm{Hom}\left(T^{\ell'}(\mathbb{C}^n), T^q(U)\otimes\mathcal{S}(V^n)\otimes {\bigwedge}^{nq}(\mathfrak{p}_{\mathbb{C}}^*)\otimes T^{\ell'}(V_{\mathbb{C}})\right)\right]^{K'\times K\times S_{\ell'}}.\end{aligned}$$

These Schwartz forms are generalizations of the Schwartz forms considered by Kudla and Millson [16, 17, 18]. Under the isomorphism in (7.1), the standard Gaussian $\varphi_0(\mathbf{x}) = 1\otimes e^{-\pi tr(\mathbf{x},\mathbf{x})_{z_0}} \in [T^0(U)\otimes\mathcal{S}(V^n)]^{K'\times K}$ corresponds to

$$\varphi_0(\mathbf{x},z) = 1\otimes e^{-\pi tr(\mathbf{x},\mathbf{x})_z} \in \left[T^0(U)\otimes\mathcal{S}(V^n)\otimes C^\infty(D)\right]^{K'\times G}.$$

Definition 7.1. Let $n \leq p$. The form $\varphi_{nq,0}$ with trivial coefficients is given by applying the operator

$$\mathcal{D} = \frac{1}{2^{nq/2}} A\left(u_1^q\right)\otimes\prod_{i=1}^{n}\prod_{\mu=p+1}^{p+q}\left[\sum_{\alpha=1}^{p}\left(x_{\alpha i} - \frac{1}{2\pi}\frac{\partial}{\partial x_{\alpha i}}\right)\otimes A(\omega_{\alpha\mu})\right]$$

to φ_0:

$$\varphi_{nq,0} = \mathcal{D}(\varphi_0) \in C_V^{q,nq,0} = \left[T^q(U)\otimes\mathcal{S}(V^n)\otimes {\bigwedge}^{nq}(\mathfrak{p}_{\mathbb{C}}^*)\right]^{K'\times K}.$$

Here as before $A(\cdot)$ denotes left multiplication and u_1 is the generator of $U = {\bigwedge}^n(\mathbb{C}^n)^*$. Furthermore, Theorem 3.1 of [16] implies that $\varphi_{nq,0}$ is indeed K'-invariant.

For $T(V_{\mathbb{C}})$, we define for $1 \leq i \leq n$ another K-invariant differential operator $\mathcal{D}_i'$ which acts on

$$(7.2)\qquad \mathcal{S}(V^n)\otimes {\bigwedge}^{\bullet}(\mathfrak{p}_{\mathbb{C}}^*)\otimes T(V_{\mathbb{C}})$$

by

$$(7.3)\qquad \mathcal{D}_i' = \frac{1}{2}\sum_{\alpha=1}^{p}\left(x_{\alpha i} - \frac{1}{2\pi}\frac{\partial}{\partial x_{\alpha i}}\right)\otimes 1\otimes A(e_\alpha).$$

Let $I = (i_1,\dots,i_{\ell'}) \in \{1,\dots,n\}^{\ell'}$ be a multi-index of length ℓ' and write

$$(7.4)\qquad \varepsilon_I = \varepsilon_{i_1}\otimes\cdots\otimes\varepsilon_{i_{\ell'}}$$

for the corresponding standard basis element of $T^{\ell'}(\mathbb{C}^n)$. Then for $\varepsilon_I \in T^{\ell'}(\mathbb{C}^n)$, we define an operator

$$\mathcal{T}_{\ell'}(\varepsilon_I) = \mathcal{D}'_{i_1} \circ \cdots \circ \mathcal{D}'_{i_{\ell'}} \tag{7.5}$$

extend $\mathcal{T}_{\ell'}$ linearly to $T^{\ell'}(\mathbb{C}^n)$.

Definition 7.2. Define

$$\varphi_{nq,\ell'} \in C_V^{q,nq,\ell'} = \operatorname{Hom}_{\mathbb{C}}\Big(T^{\ell'}(\mathbb{C}^n), T^q(U) \otimes \mathcal{S}(V^n) \otimes \bigwedge\nolimits^{nq}(\mathfrak{p}_{\mathbb{C}}^*) \otimes T^{\ell'}(V_{\mathbb{C}})\Big)^{K'\otimes K \otimes S_{\ell'}}$$

by

$$\varphi_{nq,\ell'}(w) = \mathcal{T}_{\ell'}(w)\varphi_{nq,0}$$

for $w \in T^{\ell'}(\mathbb{C}^n)$. We put $\varphi_{nq,\ell'} = 0$ for $\ell' < 0$. Here the $S_{\ell'}$-invariance of $\varphi_{nq,\ell'}$ is shown in Proposition 5.2 in [9], while the K'-invariance is Theorem 5.6 in [9].

Using the projections $\pi_{t(\lambda)}$ and $\pi_{[t(\lambda)]}$, see (5.12) and (5.14), we can therefore make the following definitions.

Definition 7.3. For any standard filling $t(\lambda)$ of $D(\lambda)$, we define

$$\varphi_{nq,t(\lambda)} = \pi_{t(\lambda)}\varphi_{nq,\ell'} \in C_V^{q,nq,\lambda},$$
$$\varphi_{nq,[t(\lambda)]} = \pi_{[t(\lambda)]}\varphi_{nq,\ell'} \in C_V^{q,nq,[\lambda]}.$$

We write $\varphi_{nq,\lambda}$ and $\varphi_{nq,[\lambda]}$, if we do not want to specify the standard filling.

Proposition 7.4 (Theorem 5.7 [9])**.** *The form $\varphi_{nq,\ell'}$ is closed. That is, for $w \in T^{\ell'}(\mathbb{C}^n)$ and $\mathbf{x} \in V^n$, the differential form*

$$\varphi_{nq,\ell'}(w)(\mathbf{x}) \in \Big[A^{nq}\Big(D; T^{\ell'}(V_{\mathbb{C}})\Big)\Big]^G$$

is closed.

7.2. **Explicit formulas.** We give more explicit formulas for $\varphi_{nq,\ell'}$ in the various models of the Weil representation.

7.2.1. *Schrödinger model.* We introduce multi-indices $\underline{\alpha_i} = (\alpha_{i1}, \cdots, \alpha_{iq})$ of length q (typically) with $1 \leq i \leq n$ and $\underline{\beta} = (\beta_1, \cdots, \beta_{\ell'})$ of length ℓ' (typically) with values in $\{1, \ldots, p\}$ (typically). Note that we suppressed their length from the notation. We also write $\alpha = (\alpha_{ij})$ for the $n \times q$ matrix of indices. With I as above, we then define

$$\begin{aligned}
\omega_{\underline{\alpha_i}} &= \omega_{\alpha_{i1}p+1} \wedge \cdots \wedge \omega_{\alpha_{iq}p+q} \\
\omega_{\alpha} &= \omega_{\underline{\alpha_1}} \wedge \cdots \wedge \omega_{\underline{\alpha_n}} \\
\mathcal{H}_{\underline{\alpha_i}} &= \mathcal{H}_{\alpha_{i1}i} \circ \cdots \circ \mathcal{H}_{\alpha_{iq}i}, \\
\mathcal{H}_{\alpha} &= \mathcal{H}_{\underline{\alpha_1}} \circ \cdots \circ \mathcal{H}_{\underline{\alpha_n}} \\
\mathcal{H}_{\underline{\beta},I} &= \mathcal{H}_{\beta_1 i_1} \circ \cdots \circ \mathcal{H}_{\beta_{\ell'} i_{\ell'}} \\
e_{\underline{\beta}} &= e_{\beta_1} \otimes \cdots \otimes e_{\beta_{\ell'}}
\end{aligned} \tag{7.6}$$

Let $1 \leq \gamma \leq p$ and $1 \leq j \leq n$. For I, α, and $\underline{\beta}$ fixed, let

$$\delta_{\gamma j} = \#\{k;\, \alpha_{kj} = \gamma\} + \#\{k;\, (\beta_k, i_k) = (\gamma, j)\}. \tag{7.7}$$

This defines a $p \times n$ matrix $\Delta_{\alpha,\underline{\beta},I} = \Delta_{\alpha,\underline{\beta},I;+}$ and Schwartz functions $\varphi_{\Delta_{\alpha,\underline{\beta},I}}$ as in Definition 4.4.

Lemma 7.5. *The Schwartz form $\varphi_{nq,\ell'}(\varepsilon_I)$ is given by*

$$\varphi_{nq,\ell'}(\varepsilon_I) = \frac{1}{2^{nq/2+\ell'}} \sum_{\alpha,\underline{\beta}} u_1^q \otimes \varphi_{\Delta_{\alpha,\underline{\beta},I}} \otimes \omega_\alpha \otimes e_{\underline{\beta}}.$$

Proof. With the above notation we have

(7.8)

$$\begin{aligned}\varphi_{nq,\ell'}(\varepsilon_I) &= \frac{1}{2^{nq/2+\ell'}} \sum_{\substack{\underline{\alpha_1},\dots,\underline{\alpha_n} \\ \underline{\beta}}} u_1^q \otimes ((\mathcal{H}_{\underline{\alpha_1}} \circ \cdots \circ \mathcal{H}_{\underline{\alpha_n}} \circ \mathcal{H}_{\underline{\beta},I})\varphi_0) \otimes (\omega_{\underline{\alpha_1}} \wedge \cdots \wedge \omega_{\underline{\alpha_q}}) \otimes e_{\underline{\beta}} \\ &= \frac{1}{2^{nq/2+\ell'}} \sum_{\alpha,\underline{\beta}} u_1^q \otimes (\mathcal{H}_\alpha \circ \mathcal{H}_{\underline{\beta},I})\varphi_0 \otimes \omega_\alpha \otimes e_{\underline{\beta}}.\end{aligned}$$

But now we easily see

$$\left(\mathcal{H}_\alpha \circ \mathcal{H}_{\underline{\beta},I}\right) \varphi_0(\mathbf{x}) = \prod_{\gamma=1}^{p} \prod_{j=1}^{n} \widetilde{H}_{\delta_{\gamma,j}}(x_{\gamma j})\varphi_0(\mathbf{x}), \tag{7.9}$$

which gives the assertion. □

7.2.2. *Mixed model.* We now describe the Schwartz form $\varphi_{nq,\ell'}$ in the mixed model. We describe this in terms of the individual components $\varphi_{\Delta_{\alpha,\underline{\beta},I}}$ described in the Schrödinger model. From Lemma 4.7, Lemma 4.8, and Proposition 4.9 we see

Lemma 7.6.

$$\widehat{\varphi^V_{\Delta_{\alpha,\underline{\beta},I}}} \begin{pmatrix} \xi \\ \mathbf{x}_W \\ u' \end{pmatrix} = \varphi^W_{\Delta'_{\alpha,\underline{\beta},I}}(\mathbf{x}_W)\widehat{\varphi^E_{\Delta''_{\alpha,\underline{\beta},I}}}(\xi, u').$$

Note that $\varphi^W_{\Delta'_{\alpha,\underline{\beta},I}}$ only depends on the indices α_{ij}, β_j such that $\alpha_{ij}, \beta_j \geq \ell + 1$, while $\widehat{\varphi^E_{\Delta''_{\alpha,\underline{\beta},I}}}$ only depends on the indices α_{ij}, β_j such that $\alpha_{ij}, \beta_j \leq \ell$. In particular, if all *$\alpha_{ij}, \beta_j \geq \ell + 1$, then*

$$\widehat{\varphi^V_{\Delta_{\alpha,\underline{\beta},I}}} \begin{pmatrix} \xi \\ \mathbf{x}_W \\ 0 \end{pmatrix} = \varphi^W_{\Delta'_{\alpha,\underline{\beta},I}}(\mathbf{x}_W)\varphi^E_0(\xi, 0).$$

On the other hand, if one *of the α_{ij}, β_j is less or equal to ℓ, then*

$$\widehat{\varphi}^E_{\Delta''_{\alpha,\underline{\beta},I}}(0,0) = \widehat{\varphi^V_{\Delta_{\alpha,\underline{\beta},I}}} \begin{pmatrix} 0 \\ \mathbf{x}_W \\ 0 \end{pmatrix} = 0.$$

7.2.3. *Fock model.* In the Fock model, the form $\varphi_{nq,\ell'}$ looks particularly simple. We have

Lemma 7.7.

$$\varphi_{nq,\ell'}(\varepsilon_I) = \frac{1}{2^{nq/2+\ell'}}\left(\frac{1}{2\pi i}\right)^{nq+\ell'} \sum_{\substack{\underline{\alpha_1},\ldots,\underline{\alpha_n}\\ \underline{\beta}}} u_1^q \otimes z_{\underline{\alpha_1},1}\cdots z_{\underline{\alpha_n},n}\cdot z_{\underline{\beta},I}\otimes\Big(\omega_{\underline{\alpha_1}}\wedge\cdots\wedge\omega_{\underline{\alpha_q}}\Big)\otimes e_{\underline{\beta}}.$$

Here we use the notational conventions in (7.6) *and in addition*

$$z_{\underline{\alpha_j},j} = z_{\alpha_{j1}j}\cdots z_{\alpha_{jq}j}, \qquad z_{\underline{\beta},I} = z_{\beta_1 i_1}\cdots z_{\beta_{\ell'} i_{\ell'}}. \tag{7.10}$$

7.3. **The forms $\varphi_{0,k}$.** We now define another class of special forms. We will only do this in the Fock model.

Definition 7.8. We define $\varphi_{0,k} \in \operatorname{Hom}\big(T^k(\mathbb{C}^n); T^0(U)\otimes\mathcal{F}_{n,V}\otimes T^k(V_{\mathbb{C}})\big)$ by

$$\varphi_{0,k}(\varepsilon_I) = \frac{1}{2^k}\left(\frac{1}{2\pi i}\right)^k \sum_{\underline{\beta}} 1\otimes z_{\underline{\beta},I}\otimes e_{\underline{\beta}}. \tag{7.11}$$

Remark 7.9. The element $\varphi_{0,k}$ is the image of the operator $\mathcal{T}_k$ (see (7.5)) applied to the Gaussian φ_0 under the intertwiner from the Schrödinger to the Fock model. Also note that $\varphi_{0,k}$ is *not* closed, hence they do not define cocycles.

We also leave the proof of the following lemma to the reader. It follows (in large part) from Remark 7.9 and the corresponding properties of $\varphi_{nq,\ell'}$.

Lemma 7.10.

$$\varphi_{0,k} \in [T^0(U)\otimes T^k(\mathbb{C}^n)^*\otimes\mathcal{F}_{n,V}\otimes T^k(V_{\mathbb{C}})]^{K'\times K\times S_k},$$

i.e.,

$$\varphi_{0,k}\in C_V^{0,0,k}.$$

From Lemma 7.7, we immediately see

Lemma 7.11.

$$\varphi_{nq,\ell'} = \varphi_{nq,0}\cdot\varphi_{0,\ell'}$$

and

$$\varphi_{0,k_1}\cdot\varphi_{0,k_2} = \varphi_{0,k_1+k_2},$$

where the multiplication is the one in $C_V^{\bullet}$.

Remark 7.12. This kind of product decomposition for $\varphi_{nq,\ell'}$ and $\varphi_{0,k}$ in Lemma 7.11 only holds in the Fock model. In the Schrödinger model this only makes sense in terms of the operators $\mathcal{D}$ and $\mathcal{T}_{\ell'}$ of Definition 7.1 and Definition 7.2 respectively.

We apply the projection $\pi_{t(\lambda)}$, see (5.12), to define $\varphi_{0,t(\lambda)}$:

Definition 7.13.

$$\varphi_{0,t(\lambda)} := \pi_{t(\lambda)}\varphi_{0,k}\in C_V^{0,0,t(\lambda)}.$$

The following product formula will be important later.

Proposition 7.14. *Let $A = t(\lambda)$ be a filling of the Young diagram associated to λ and let $B = B_{n,\ell}$ be the filling of the $n \times \ell$ rectangular Young diagram introduced in section 3. Then*

$$\varphi^W_{0,B} \cdot \varphi^W_{0,A} = \varphi^W_{0,B|A}.$$

The proposition will follow from the next two lemmas.

Lemma 7.15. *Both $\varphi^W_{0,B} \cdot \varphi^W_{0,A}$ and $\varphi^W_{0,B|A}$ are elements of*

$$C_W^{0,B|A,0} = \left[T^0(U) \otimes \mathbb{S}_{B|A}(\mathbb{C}^n)^* \otimes \mathcal{F}_{n,W} \otimes \mathbb{S}_{B|A}(W_{\mathbb{C}})\right]^{K' \times K_W}.$$

Proof. Since $\mathbb{S}_B(\mathbb{C}^n)^* \otimes \mathbb{S}_A(\mathbb{C}^n)^* = \mathbb{S}_{B|A}(\mathbb{C}^n)^*$ as subspaces of $T^{n\ell+\ell'}(\mathbb{C}^n)$, see Corollary 3.4, the claim follows in the same way as Lemma 6.10. □

Lemma 7.16.

$$(\varphi^W_{0,B} \cdot \varphi^W_{0,A})(s(B)\varepsilon_B \otimes s(A)\varepsilon_A) = \varphi^W_{0,B|A}(s(B)\varepsilon_B \otimes s(A)\varepsilon_A).$$

Proof. This is a little calculation using Lemma 3.3 and Lemma 7.11. Indeed, we have

$$\begin{aligned}
\left(\varphi^W_{0,B} \cdot \varphi^W_{0,A}\right)(s(B)\varepsilon_B \otimes s(A)\varepsilon_A) &= \left(\varphi^W_{0,n\ell} \cdot \varphi^W_{0,\ell'}\right)(s(B)\varepsilon_B \otimes s(A)\varepsilon_A) \\
&= \varphi^W_{0,n\ell+\ell'}(s(B)\varepsilon_B \otimes s(A)\varepsilon_A) = c(A,B)\varphi^W_{0,n\ell+\ell'}\left(s(B|A)\varepsilon_{B|A}\right) \\
&= c(A,B)\varphi^W_{0,B|A}\left(s(B|A)\varepsilon_{B|A}\right) = \varphi^W_{0,B|A}(s(B)\varepsilon_B \otimes s(A)\varepsilon_A).
\end{aligned}$$

□

Now we can prove Proposition 7.14. By Lemma 7.15 we see that $\varphi^W_{0,B} \cdot \varphi^W_{0,A}$ and $\varphi^W_{0,B|A}$ are $U(n)$-equivariant homomorphisms from $\mathbb{S}_{B|A}(\mathbb{C}^n)^*$ to $T^0(U) \otimes \mathcal{F}_{n,W} \otimes \mathbb{S}_{B|A}(W_{\mathbb{C}})$. By Lemma 7.16 they agree on the highest weight vector (see Lemma 3.3), hence coincide.

8. Local Restriction

We retain the notation from the previous sections. In this section, we will give formulas for the restrictions $r_P^{\mathcal{W}}$ and r_P of $\varphi_{nq,\ell'}$. Finally, we will establish Theorem 1.3, the local restriction formula.

Proposition 8.1. *We have*

$$(r_{\underline{P}}^{\mathcal{W}} \varphi^V_{nq,\ell'})(\varepsilon_I) = \frac{1}{2^{nq/2+\ell'}} \sum_{\alpha',\underline{\beta}'} u_1^q \otimes \varphi^W_{\Delta'_{\alpha',\underline{\beta}',I}} \otimes \omega_{\alpha'} \otimes e_{\underline{\beta}'}.$$

Here $\varepsilon_I = \varepsilon_{i_1} \otimes \cdots \otimes \varepsilon_{i_\ell} \in T^\ell(\mathbb{C}^n)$, α' and $\underline{\beta}'$ are the same indices as before with

$$\ell + 1 \leq \alpha'_{ij}, \beta'_j \leq p.$$

Loosely speaking, $r_P^{\mathcal{W}}\left(\varphi^V_{nq,\ell'}\right)$ is obtained from $\varphi^V_{nq,\ell'}$ by "throwing away" all the indices less or equal to ℓ. In particular,

$$r_{\underline{P}}^{\mathcal{W}} \varphi^V_{nq,\ell'} = 0$$

if $n > p - \ell$.

Proof. This follows from Lemma 7.5, the formula for $\varphi_{nq,\ell}$ in the Schrödinger model, and from Lemma 7.6. For the last statement, we observe that $\omega_{\alpha'}$ is in the nq-exterior power of a $(p-\ell)q$-dimensional space. ☐

The local restriction looks particularly simple in the Fock model. We have

Proposition 8.2.

$$r_P^{\mathcal{W}}\left(\varphi^V_{nq,\ell'}(\varepsilon_I)\right)$$
$$= \frac{1}{2^{nq/2+\ell'}}\left(\frac{1}{2\pi i}\right)^{nq+\ell'} \sum_{\substack{\underline{\alpha'_1},\dots,\underline{\alpha'_n}\\ \underline{\beta'}}} u_1^q \otimes z_{\underline{\alpha'_1}}\cdots z_{\underline{\alpha'_n}}\cdot z_{\underline{\beta'},I}\otimes\left(\omega_{\underline{\alpha'_1}}\wedge\cdots\wedge\omega_{\underline{\alpha'_q}}\right)\otimes e_{\underline{\beta'}}.$$

Here $\underline{\alpha'_j}$ and $\underline{\beta'}$ are as before in Proposition 8.1.

Proof. This follows immediately either from Proposition 8.1 and applying the intertwiner to the Fock model or also from Proposition 4.12 and Lemma 7.7. ☐

Proposition 8.3. *For the restriction of $\varphi^V_{nq,\ell'}$, we have*

$$r_P\varphi^V_{nq,\ell'} = \left(1_U\otimes 1_{\mathbb{C}^n}\otimes r_P^{\mathcal{W}}\otimes\sigma^*\otimes 1_V\right)\varphi^V_{nq,\ell'}.$$

Analogous statements hold for $\varphi^V_{nq,\lambda}$ and $\varphi^V_{nq,[\lambda]}$.

Proof. By Definition 5.5, the restriction $r_P: C^\bullet_V \to A^\bullet_P$ is given by $1_U\otimes 1_{\mathbb{C}^n}\otimes r_P^{\mathcal{W}}\otimes(\iota^*\circ\sigma^*)\otimes 1_V$. Then the theorem follows from Proposition 8.2 and Lemma 2.2, in particular (2.49): The components of $\sigma^*\varphi^V_{nq,\ell'}$ involving $\mathfrak{a}^*$ already become annihilated under $r_P^{\mathcal{W}}$, so that ι^* acts trivially on $\sigma^* r_P^{\mathcal{W}}\varphi^V_{nq,\ell'}$. ☐

We define

$$\varphi_{P,n\ell} = \frac{1}{2^{n\ell}}\left(\frac{1}{2\pi i}\right)^{n\ell}\sum_{\underline{\gamma_1},\dots,\underline{\gamma_n}} u_1^\ell\otimes z_{\underline{\gamma_1},1}\cdots z_{\underline{\gamma_n},n}\otimes(\nu_{\underline{\gamma_1}}\wedge\cdots\wedge\nu_{\underline{\gamma_n}}). \tag{8.1}$$

Here $\underline{\gamma_j} = (\gamma_{jm-\ell+1},\dots,\gamma_{jm})$ is a multi-index of length ℓ such that $\ell+1\le\gamma_{ji}\le p$, and $z_{\underline{\gamma_j},j}$ as in (7.10). Furthermore, we have set

$$\nu_{\underline{\gamma_j}} = \nu_{\gamma_{jm-\ell+1}\ell}\wedge\cdots\wedge\nu_{\gamma_{jm}1}\in\bigwedge^\ell(\mathfrak{n}_W^*). \tag{8.2}$$

We have

Lemma 8.4.

$$\iota_P(\varphi^W_{0,B}) = \iota_P(\varphi^W_{0,n\ell}) = (-1)^{n\ell(\frac{(q-\ell)(n-1)}{2}+1)}\varphi_{P,n\ell}.$$

Proof. First note that by Proposition 6.9 we have $\iota_P(\varphi^W_{0,B}) = \iota_P(\varphi^W_{0,n\ell})$. We let $\underline{\beta_1},\dots,\underline{\beta_n}$ be n indices of length ℓ with $\ell+1\le\beta_{ji}\le p$. For the corresponding elements $e_{\underline{\beta_j}}\in T^\ell(W)$, we easily see

(8.3)

$$\sum_{\underline{\beta_1},\dots,\underline{\beta_n}}(z_{\underline{\beta_1}1}\cdots z_{\underline{\beta_n}n})\otimes\tau_{n\ell}(e_{\underline{\beta_1}}\otimes\cdots\otimes e_{\underline{\beta_n}}) = \sum_{\underline{\beta_1},\dots,\underline{\beta_n}}(z_{\underline{\beta_1}1}\cdots z_{\underline{\beta_n}n})\otimes(\nu_{\underline{\beta_1}}\wedge\cdots\wedge\nu_{\underline{\beta_n}})$$

with $\nu_{\underline{\beta_j}}$ as in (8.2). With that, we conclude

(8.4)
$$\iota_P(\varphi^W_{0,B}) = (-1)^{n\ell(\frac{(q-\ell)(n-1)}{2}+1)} \frac{1}{2^{n\ell}} \left(\frac{1}{2\pi i}\right)^{n\ell} \sum_{\underline{\beta_1},\dots,\underline{\beta_n}} u_1^\ell \otimes (z_{\underline{\beta_1}1} \cdots z_{\underline{\beta_n}n}) \otimes (\nu_{\underline{\beta_1}} \wedge \cdots \wedge \nu_{\underline{\beta_n}})$$
$$= (-1)^{n\ell(\frac{(q-\ell)(n-1)}{2}+1)} \varphi_{P,n\ell}$$

by (8.1). ☐

Now Theorem 1.3 easily follows:

Theorem 8.5. *Let A be a standard filling of Young diagram with ℓ' boxes and let $B_{n,\ell}$ be the standard tableau associated to the n by ℓ rectangle as in section 3. Then*
$$r_P(\varphi^V_{nq,\ell'}) = \iota_P(\varphi^W_{n(q-\ell),n\ell+\ell'}),$$
$$r_P(\varphi^V_{nq,A}) = \iota_P(\varphi^W_{n(q-\ell),B|A}).$$
Furthermore, for the form $\varphi^V_{nq,[A]}$ with harmonic coefficients, we have
$$[r_P(\varphi^V_{nq,[A]})] = [\iota_P(\varphi^W_{n(q-\ell),[B|A]})].$$

Proof. We first note
$$r_P\varphi^V_{nq,\ell'} = (-1)^{n\ell(\frac{(q-\ell)(n-1)}{2}+1)} \varphi^W_{n(q-\ell),0} \cdot \varphi_{P,n\ell} \cdot \varphi^W_{0,\ell'}.$$
Here we view $\varphi^W_{n(q-\ell),0} \in A_P^{q-\ell,n(q-\ell),0}$ and $\varphi^W_{0,\ell'} \in A_P^{0,0,\ell'}$ in the natural fashion. The analogous statements hold for $\varphi^V_{nq,A}$ and $\varphi^V_{nq,[A]}$. Indeed, this follows immediately from Proposition 8.2 and

$$\sigma^*\omega_{\underline{\alpha'_j}} = (-1)^\ell \frac{1}{2^{\ell/2}} \omega_{\alpha'_{j1}p+1} \wedge \cdots \wedge \omega_{\alpha'_{jq-\ell}m-\ell} \wedge \nu_{\alpha'_{jq-\ell+1}\ell} \wedge \cdots \wedge \nu_{\alpha'_{jq}1}, \tag{8.5}$$

which follows from Lemma 2.2. The sign arises from 'sorting' $\sigma^*\left(\omega_{\underline{\alpha'_1}} \wedge \cdots \wedge \omega_{\underline{\alpha'_q}}\right)$ according to (8.5) into $\omega_{\alpha'_\bullet}$'s (which lie in $\mathfrak{p}_W$) and $\nu_{\alpha'_\bullet}$'s (which lie in $\mathfrak{n}^*_W$). From this and Lemma 8.4 we conclude
$$r_P(\varphi^V_{nq,\ell'}) = \iota_P(\varphi^W_{n(q-\ell),0} \cdot \varphi^W_{0,n\ell} \cdot \varphi^W_{0,\ell'}) = \iota_P(\varphi^W_{n(q-\ell),n\ell+\ell'}).$$
By $S_{\ell'}$-equivariance of ι_P we also obtain
$$r_P(\varphi^V_{nq,A}) = \iota_P(\varphi^W_{n(q-\ell),0} \cdot \varphi^W_{0,B} \cdot \varphi^W_{0,A}) = \varphi^W_{n(q-\ell),B|A}$$
since $\varphi^W_{0,B} \cdot \varphi^W_{0,A} = \varphi^W_{0,B|A}$ (see Proposition 7.14) and $\varphi^W_{n(q-\ell),B|A} = \varphi_{n(q-\ell),0} \cdot \varphi^W_{0,B|A}$ (see Lemma 7.11).

The cohomology statement now follows from Proposition 6.9(ii). ☐

Corollary 8.6. *We have $[r_P(\varphi^V_{nq,[\lambda]})] = 0$ for $n > \min\left(p, \left[\frac{m}{2}\right]\right) - \ell$ (if $\ell \geq 2$) and $n > p-1$ or $n > m-2-i(\lambda)$ (if $\ell = 1$).*

Proof. The Schur functor $\mathbb{S}_{[B|A]}(W_{\mathbb{C}})$ vanishes in this range. ☐

On the other hand, we have

Corollary 8.7. *Let P be a (real) parabolic subgroup as above such that the associated space W is positive definite. Assume*

$$i(\lambda) \leq n \leq \begin{cases} \left[\frac{p-q}{2}\right] & \text{if } q \geq 2 \\ p-1-i(\lambda) & \text{if } q = 1. \end{cases}$$

Then

$$[r_P(\varphi^V_{nq,[\lambda]})] \neq 0.$$

9. Global complexes, theta series, and the global restriction of $\theta_{\varphi_{nq,\ell'}}$

In this section, we return to the global situation and assume that V, W, E etc. are $\mathbb{Q}$-vector spaces. Furthermore, $\underline{P}$ is a standard $\mathbb{Q}$-parabolic subgroup and $P = \underline{P}_0(\mathbb{R})$ for its real points etc. All the 'local' notions (over $\mathbb{R}$) of the previous sections carry over naturally to this situation, and we make use of the already established notation.

9.1. **Global complexes and theta series.**

9.1.1. *Global complexes.* We first define "global" versions of the "local" complexes $C^\bullet$ of forms on $X = \Gamma\backslash D$, $A^\bullet_P$ of forms on $e'(\underline{P}) = \Gamma_P\backslash e(\underline{P})$. We set

$$C^\infty(\Gamma', j, \lambda) := C^\infty\left(\Gamma'\backslash G'; T^j(U) \otimes \mathbb{S}_\lambda(\mathbb{C}^n)^*\right)^{K'} \tag{9.1}$$

for Γ' an (appropriate) arithmetic subgroup of $\mathrm{Sp}(n,\mathbb{Z})$. Note that we can identify $C^\infty(\Gamma', j, \lambda)$ in the usual way with the space of vector-valued C^∞-functions on the Siegel upper half space of genus n, transforming like a Siegel modular form of type $\det^{j/2} \otimes \mathbb{S}_\lambda(\mathbb{C}^n)$. Furthermore, we denote by $Mod(\Gamma', j, \lambda)$ the space of holomorphic Siegel modular forms of this type. We let

$$\begin{aligned} \widetilde{C}_V^{j,r,\lambda} &= C^\infty(\Gamma', j, \lambda) \otimes \left[\mathcal{A}^r(D) \otimes \mathbb{S}_\lambda(V_\mathbb{C})\right]^\Gamma, \\ &\simeq C^\infty(\Gamma', j, \lambda) \otimes \left[\bigwedge\nolimits^r(\mathfrak{p}^*_\mathbb{C}) \otimes \mathbb{S}_\lambda(V_\mathbb{C}) \otimes C^\infty(\Gamma\backslash G\right]^K \end{aligned} \tag{9.2}$$

and

$$\begin{aligned} \widetilde{A}_P^{j,r,\lambda} &= C^\infty(\Gamma', j, \lambda) \otimes \left[\mathcal{A}^r(e'(P) \otimes \mathbb{S}_\lambda(V_\mathbb{C})\right]^{\Gamma_P} \\ &\simeq C^\infty(\Gamma', j, \lambda) \otimes \left[\bigwedge\nolimits^r(\mathfrak{n} \oplus \mathfrak{p}_M)^*_\mathbb{C} \otimes \mathbb{S}_\lambda(V_\mathbb{C}) \otimes C^\infty(\Gamma_P\backslash P)\right]^{K_P} \end{aligned} \tag{9.3}$$

We then define $\widetilde{C}_V^{j,r,[\lambda]}$ and $\widetilde{A}_P^{j,r,[\lambda]}$ as in the local case by harmonic projection onto $\mathbb{S}_{[\lambda]}(V_\mathbb{C})$. The local map ι_P induces a global intertwining map of complexes

$$\tilde{\iota}_P : \widetilde{C}_W^{j-\ell,r,\ell\varpi_n+\lambda} \to \widetilde{A}_P^{j,n\ell+r,\lambda}. \tag{9.4}$$

by lifting functions on $\Gamma_W\backslash \mathrm{SO}_0(W_\mathbb{R})$ to $\Gamma_M\backslash M$. This induces a map on cohomology

$$\begin{aligned} &\tilde{\iota}_P : C^\infty(\Gamma', j, \lambda) \otimes H^{n(q-\ell)}(X_W, \mathbb{S}_{[\ell\varpi_n+\lambda]}(W_\mathbb{C})) \\ &\hookrightarrow C^\infty(\Gamma', j, \lambda) \otimes H^{n(q-\ell)}(X_M, H^{n\ell}(\mathfrak{n}, \mathbb{S}_{[\lambda]}(V_\mathbb{C}))) \\ &\hookrightarrow C^\infty(\Gamma', j, \lambda) \otimes H^{nq}(e'(P), \mathbb{S}_{[\lambda]}(V_\mathbb{C})). \end{aligned} \tag{9.5}$$

We also introduce

$$\overline{C}_V^{j,r,\lambda} = C^\infty\left(\Gamma'\backslash G'; T^j(U)\otimes \mathbb{S}_\lambda(\mathbb{C}^n)^*\right)^{K'} \otimes \mathcal{A}^r(\overline{X}, \mathbb{S}_\lambda(\mathcal{V}_\mathbb{C})), \tag{9.6}$$

the complex associated to the differential forms on the compactification $\overline{X}$ with values in $\mathbb{S}_\lambda(\mathcal{V}_\mathbb{C})$, the local system associated to $\mathbb{S}_\lambda(V_\mathbb{C})$. We then have a restriction map

$$\tilde{r}_P : \overline{C}_V^\bullet \to \widetilde{A}_P^\bullet \tag{9.7}$$

induced by the inclusion $e'(P) \hookrightarrow \overline{X}$.

9.1.2. *Theta series.* Using the Schrödinger model $\mathcal{S}(V_\mathbb{R}^n)$ of the Weil representation, we now introduce for $\varphi \in C_V^{j,r,\lambda}$, its theta series $\theta(\varphi)$ as follows. We fix $h \in (L^\#)^n$ in the dual lattice once and for all and assume that $\Gamma \subset \underline{G}(\mathbb{Z})$ stabilizes $L^\#$. We set $\mathcal{L} = \mathcal{L}_V = L^n + h$. For $g' \in G'$, we then define for $z \in D$, the theta series

$$\theta_{\mathcal{L}_V}(g', z, \varphi) = \sum_{\mathbf{x}\in\mathcal{L}_V} \omega(g')\varphi(\mathbf{x}, z). \tag{9.8}$$

We easily see that the series is Γ-invariant as Γ stabilizes $\mathcal{L}_V$. Thus $\theta_{\mathcal{L}_V}$ descends to a closed differential nq-form on the locally symmetric space $X = \Gamma\backslash D$. More precisely, by the standard theta machinery, we have

$$\theta_{\mathcal{L}_V}(\varphi) \in \widetilde{C}_V^{j,r,\lambda} \tag{9.9}$$

for some congruence subgroup $\Gamma' \subseteq \mathrm{Sp}(n,\mathbb{Z})$. Summarizing, the theta distribution $\theta_{\mathcal{L}_V}$ associated to $\mathcal{L}$ gives rise to a $G' \times G$ intertwining map of complexes

$$\theta_{\mathcal{L}_V} : C_V^\bullet \longrightarrow \widetilde{C}_V^\bullet. \tag{9.10}$$

Remark 9.1. The main point of [9] is that for the Schwartz forms $\varphi_{nq,[\lambda]}$ one has

$$[\theta_{\mathcal{L}_V}(\varphi_{nq,[\lambda]})] \in Mod(\Gamma', j, \lambda) \otimes H^{nq}(X, \mathbb{S}_\lambda(\mathcal{V}_\mathbb{C})),$$

and the Fourier coefficients are Poincaré dual classes of special cycles with nontrivial local coefficients.

For a similar theta intertwiner for A_P, we note that A_P involves the Weil representation for $W = E^\perp/E$. Recall (see Proposition 4.2 and Definition 4.3) that we can extend the action of $\mathrm{O}(W_\mathbb{R})$ on $S(W_\mathbb{R}^n)$ to P such that the Weil representation intertwining map $r_P^{\mathcal{W}}$ becomes an MN-intertwiner. In particular, N and M_P' act trivially on $S(W_\mathbb{R}^n)$. We let $\mathcal{L}_W$ be a linear combination of delta functions of (cosets of) lattices in W^n, which is stabilized by Γ_P, that is by Γ_W. Recall that we defined Γ_W as the image of Γ_P when acting on $E^\perp/E$. It contains $\Gamma \cap \mathrm{SO}_0(W_\mathbb{R})$ as a finite subgroup of finite index. Applying the theta distribution asociated to $\mathcal{L}_W$ we obtain an intertwining map

$$\theta_{\mathcal{L}_W} : A_P^\bullet \to \widetilde{A}_P^\bullet. \tag{9.11}$$

Furthermore, $\theta_{\mathcal{L}_W}$ commutes with ι_P:

$$\theta_{\mathcal{L}_W} \circ \iota_P = \tilde{\iota}_P \circ \theta_{\mathcal{L}_W}. \tag{9.12}$$

More general, we let

$$A_P^{\bullet,\mathcal{L}_W,\Gamma_W} = \{\varphi \in A_P^\bullet;\ \theta_{\mathcal{L}_W}(\varphi) \text{ is } \Gamma_W\text{-invariant}\}. \tag{9.13}$$

and obtain a map $\theta_{\mathcal{L}_W} : A_P^{\bullet;\mathcal{L}_W,\Gamma_W} \to \widetilde{A}_P^{\bullet}$ as before.

We will be interested in a particular $\mathcal{L}_W$, which naturally arises from $\mathcal{L}_V$ as follows. We can write

$$\mathcal{L} \cap E^{\perp} = \coprod_k \left(L_{E,k}^n + h_{E,k}\right) \oplus \left(L_{W,k}^n + h_{W,k}\right) \tag{9.14}$$

for certain lattices $L_{E,k} \subset E$ and $L_{W,k} \subset W$ respectively and vectors $h_{E,k} \in (L_{E,k}^{\#})^n$ and $h_{W,k} \in (L_{W,k}^{\#})^n$. Considering $\mathcal{L} \cap E^{\perp}$ modulo E, we then set

$$\widehat{\mathcal{L}}_W = \sum_k \det(L_{E,k})^{-n} \left(L_{W,k}^n + h_{W,k}\right), \tag{9.15}$$

viewed as a linear combination of delta functions on subsets of W. Here the determinant is taken with respect to the basis $u_1, \dots, u_\ell$ of E. For this definition, it is crucial to view $W = E^{\perp}/E$ as a subquotient of V (and not as a subspace). Namely, $L_{W,k}^n + h_{W,k}$ would only arise in $W \cap L$ (when W viewed as subspace of V) if $h_{E,k} \in L_{E,k}^N$.

Remark 9.2. The definition of $\widehat{\mathcal{L}}_W$ and its notation become more transparent (but a bit less explicit) if one changes to the adelic setting. Adelically, $\mathcal{L}$ corresponds to the characteristic function $\chi_{\mathcal{L}_V}$ of the image of $\mathcal{L}_V$ inside $V(\mathbb{A}_f)$, where $\mathbb{A}_f$ denotes the finite adeles. Then in this setting, $\widehat{\mathcal{L}}_W$ corresponds to the partial Fourier transform of $\chi_{\mathcal{L}_V}$ with respect to $E(\mathbb{A}_f)$ when restricted to W. From this perspective, the assignment $\mathcal{L} \to \widehat{\mathcal{L}}_W$ is the analogue at the finite places of the local restriction map $r_{\underline{P}}$ at the infinite place.

Theorem 9.3. *The theta series $\theta_{\mathcal{L}_V}(r_{\underline{P}}\varphi_{nq,\ell'})$, $\theta_{\mathcal{L}_V}(r_{\underline{P}}\varphi_{nq,\lambda})$, $\theta_{\mathcal{L}_V}(\varphi_{nq,[\lambda]})$ extend to $\overline{X}$, that is,*

$$\theta_{\mathcal{L}_V}(\varphi_{nq,\bullet}) \in \overline{C}_V^{q,nq,\bullet}.$$

Moreover, for a standard rational parabolic $\underline{P}$, its restriction $\tilde{r}_P$ to the boundary component $e'(\underline{P})$ is given by

$$\tilde{r}_P\left(\theta_{\mathcal{L}_V}(\varphi_{nq,\bullet})\right) = \theta_{\widehat{\mathcal{L}}_W}(r_P\varphi_{nq,\bullet}).$$

In particular, $r_P(\varphi_{nq,\bullet}) \in A_P^{\bullet,\widehat{\mathcal{L}}_W,\Gamma_W}$.

Combining this Theorem 8.5, we obtain

Corollary 9.4.

$$\tilde{r}_P(\theta_{\mathcal{L}_V}(\varphi_{nq,\ell'}^V)) = \widetilde{\iota}_P(\theta_{\widehat{\mathcal{L}}_W}(\varphi_{n(q-\ell),n\ell+\ell'}^W)), \qquad \tilde{r}_P(\theta_{\mathcal{L}_V}(\varphi_{nq,\lambda]}^V)) = \widetilde{\iota}_P(\theta_{\widehat{\mathcal{L}}_W}(\varphi_{n(q-\ell),\ell\varphi_n+\lambda}^W)),$$

and

$$[\tilde{r}_P(\theta_{\mathcal{L}_V}(\varphi_{nq,[\lambda]}^V))] = [\widetilde{\iota}_P(\theta_{\widehat{\mathcal{L}}_W}(\varphi_{n(q-\ell),[\ell\varphi_n+\lambda]}^W))].$$

Remark 9.5. The proof of Theorem 9.3 actually shows that $\theta_{\mathcal{L}_V}(\varphi_{nq,\ell'}^V)$ is exponentially decreasing in the direction of $e'(P)$ if $n > p - \ell$. In particular, $\theta_{\mathcal{L}_V}(\varphi_{nq,\ell'}^V)$ is exponentially decreasing for $n = p$.

Remark 9.6. More generally, the proof also shows that $\theta_{\mathcal{L}_V}(\varphi^V_{nq,[\lambda]})$ is "almost" a special differential form in the sense of weighted cohomology, see [12]. Namely, $\tilde{r}_P(\theta_{\mathcal{L}_V}(\varphi^V_{nq,[\lambda]}))$ is N_P-invariant and while $\theta_{\mathcal{L}_V}(\varphi^V_{nq,[\lambda]})$ is in any neighborhood of $e'(P)$ in $\overline{X}$ not the pull-up of the restriction via the geodesic retraction, the difference is rapidly decreasing.

Proof of Theorem 9.3. It suffices to show this for $\varphi^V_{nq,\ell'}$. For $g \in G$ and $g' \in G'$, we let

$$\theta^V_{\alpha,\underline{\beta},I}(g',g) = \sum_{\mathbf{x}\in L^n+h} \omega_V(g')\varphi^V_{\Delta_{\alpha,\underline{\beta},I}}(g^{-1}\mathbf{x}) \otimes g^*\omega_\alpha \otimes ge_{\underline{\beta}}. \tag{9.16}$$

be the theta series associated to one fixed component of $\varphi^V_{nq,\ell'}$. For the purposes of studying the restriction to $e'(\underline{P})$, we can assume $g'=1$ (since it intertwines with the restriction) and also $g = a(\mathbf{t}) \in A$ (since g varies in a Siegel set and by Lemma 4.1). Then by Poisson summation we have

Lemma 9.7. *Let $a(\mathbf{t}) \in A$. Then*

$$\theta^V_{\alpha,\underline{\beta},I}(a(\mathbf{t})) = \sum_k \det(L_{E,k})^{-n} \sum_{\mathbf{x}_W\in L^n_{W,k}+h_{W,k}} \sum_{\substack{\xi\in(L^{\#}_{E,k})^n \\ u'\in L^n_{E',k}+h_{E',k}}} e\left(2\pi i(\xi,h_{E,k})\right)$$
$$\times |\mathbf{t}|^n \widehat{\varphi}^E_{\Delta''_{\alpha,\underline{\beta},I}}(\widetilde{\mathbf{t}}(\xi^t+u'),\widetilde{\mathbf{t}}u')\varphi^W_{\Delta'_{\alpha,\underline{\beta},I}}(\mathbf{x}_W) \otimes a(\mathbf{t})^*\sigma^*\omega_\alpha \otimes a(\mathbf{t})e_{\underline{\beta}}.$$

Proof. This follows directly from the formulas given in Lemma 4.1. □

Write $\lambda_i = \alpha_i(a(\mathbf{t}))$ for the value of the rational root α_i for $a(\mathbf{t})$. Then, in the non-split case (in particular, $W \neq 0$)

$$t_i = \prod_{j=i}^{r+1-j} \lambda_j. \tag{9.17}$$

Lemma 9.8. *Assume that at least one of the α_{kj} and β_k is less or equal than ℓ. Then*

$$r_{\underline{P}}\theta^V_{\alpha,\underline{\beta},I} = 0.$$

Proof. By Lemma 9.7 and (9.17), we clearly see that each term in $\theta^V_{\alpha,\underline{\beta},I}(a(\mathbf{t})$ is rapidly decreasing as $\lambda_i \to \infty$ for a nontrivial root α_i for $\underline{P}$ unless both $\xi = u = 0$. But by Lemma 7.6, we have

$$\widehat{\varphi}^E_{\Delta''_{\alpha,\underline{\beta},I}}(0,0) = \widehat{\varphi}^V_{\Delta_{\alpha,\underline{\beta},I}}\begin{pmatrix} 0 \\ \mathbf{x}_W \\ 0 \end{pmatrix} = 0. \tag{9.18}$$

in that case.

In the split case of $\mathrm{O}(p,p)$, everything goes through as above if $\alpha_p = \alpha_{p-1} = 1$. Otherwise $W=0$, and the pullback $\theta_{\mathcal{L}_V}(\varphi^V_{nq,\ell'})$ to $e'(P)$ already vanishes. In this case we have $t_p = \sqrt{\lambda_p/\lambda_{p-1}}$ and $t_{p-1} = \sqrt{\lambda_p\lambda_{p-1}}$. If exactly either α_p or α_{p-1} is trivial, then $\theta_{\mathcal{L}_V}(\varphi^V_{nq,\ell'})$ is exponentially decreasing. If both are nontrivial, then considering $\lambda_{p-1}\lambda_p = t_p \to \infty$, we see that $\theta_{\mathcal{L}_V}(\varphi^V_{nq,\ell'})$ is only bounded near $e'(P)$. □

Now for the remainder of the proof of Theorem 9.3, assume that

$$\alpha_{kj}, \beta_k \geq \ell + 1. \tag{9.19}$$

Again, each term in Lemma 9.7 is rapidly decreasing unless $\xi = u = 0$. So it suffices to consider

$$\widehat{a(\mathbf{t})\varphi^V_{\Delta_{\alpha,\underline{\beta},I}}} \begin{pmatrix} 0 \\ \mathbf{x}_W \\ 0 \end{pmatrix} = |\mathbf{t}| \varphi^W_{\Delta'_{\alpha,\underline{\beta},I}}(\mathbf{x}_W) \otimes a(\mathbf{t})^* \sigma^* \omega_\alpha \otimes a(\mathbf{t}) e_{\underline{\beta}}. \tag{9.20}$$

Now $a(\mathbf{t})e_{\underline{\beta}} = e_\beta$ by (9.19). We have

$$\sigma^* \omega_{\underline{\alpha_j}} = \frac{(-1)^\ell}{2^{\ell/2}} \omega_{\alpha_{j1} p+1} \wedge \cdots \wedge \omega_{\alpha_{jq-\ell} m-\ell} \wedge \nu_{\alpha_{jq-\ell+1} \ell} \wedge \cdots \wedge \nu_{\alpha_{jq} 1}, \tag{9.21}$$

and A acts trivially on the $\omega_\bullet$'s, while for the $\nu_\bullet$'s we have

$$a(\mathbf{t})^* \nu_{ji} = \frac{db_{ji}}{t_i}, \tag{9.22}$$

where $1 \leq i \leq \ell$ and $\ell + 1 \leq j \leq m - \ell$. Here b_{ji} is the coordinate of $W \otimes E$ for $e_j \otimes u_i$ and t_i is the parameter in $a(t_1, \dots, t_i, \dots, t_\ell) \in A$. We obtain

$$\begin{aligned} |\mathbf{t}| a(\mathbf{t})^* \sigma^* \omega_\alpha = \frac{(-1)^{n\ell}}{2^{n\ell/2}} & |\mathbf{t}| \omega_{\alpha_{1\,1} p+1} \wedge \cdots \wedge \omega_{\alpha_{q-\ell\,1} m-\ell} \wedge \frac{db_{\alpha_{q-\ell+1\,1} \ell}}{t_\ell} \wedge \cdots \wedge \frac{db_{\alpha_{q\,1} 1}}{t_1} \\ & \wedge \cdots \\ & \wedge \omega_{\alpha_{1\,n} p+1} \wedge \cdots \wedge \omega_{\alpha_{q-\ell\,n} m-\ell} \wedge \frac{db_{\alpha_{q-\ell+1\,n} \ell}}{t_\ell} \wedge \cdots \wedge \frac{db_{\alpha_{q\,n} 1}}{t_1} \\ = \frac{(-1)^{n\ell}}{2^{n\ell/2}} & \omega_{\alpha_{1\,1} p+1} \wedge \cdots \wedge \omega_{\alpha_{q-\ell\,1} m-\ell} \wedge db_{\alpha_{q-\ell+1\,1} \ell} \wedge \cdots \wedge db_{\alpha_{q\,1} 1} \\ & \wedge \cdots \\ & \wedge \omega_{\alpha_{1\,n} p+1} \wedge \cdots \wedge \omega_{\alpha_{q-\ell\,n} m-\ell} \wedge db_{\alpha_{q-\ell+1\,n} \ell} \wedge \cdots \wedge db_{\alpha_{q\,n} 1} \end{aligned}$$

This shows for (9.20) we have

$$\widehat{a(\mathbf{t})\varphi^V_{\Delta_{\alpha,\underline{\beta},I}}} \begin{pmatrix} 0 \\ \mathbf{x}_W \\ 0 \end{pmatrix} = r_{\underline{P}} \varphi^V_{\Delta_{\alpha,\underline{\beta},I}}(\mathbf{x}_W) \tag{9.23}$$

independent of $\mathbf{t}$. This completes the proof of Theorem 9.3. □

9.2. **Nonvanishing.** We now prove Theorem 1.5.

By the hypotheses we can find a rational parabolic $\underline{P}$ such that $\dim E = \ell = q$, so W is positive definite and X_W is a point. Then by Theorem 9.3,

$$\begin{aligned} [\tilde{r}_P \theta_{\mathcal{L}_V}(\tau, \varphi^V_{q,[\lambda]})] &= \tilde{\iota}_P [\theta_{\widehat{\mathcal{L}}_W}(\tau, \varphi^W_{0,[\ell\varpi_n+\lambda]})] \\ &\in Mod(\Gamma', m/2, \lambda) \otimes \tilde{\iota}_P \left(H^0(X_W, \mathbb{S}_{[\ell\varpi_n+\lambda]}(W_\mathbb{C})) \right) \\ &\simeq Mod(\Gamma', m/2, \lambda) \otimes \tau_{nq,\ell'} \left(\mathbb{S}_{[\ell\varpi_n+\lambda]}(W_\mathbb{C}) \right) \\ &\simeq Mod(\Gamma', m/2, \lambda) \otimes \mathbb{S}_{[\ell\varpi_n+\lambda]}(W_\mathbb{C}). \end{aligned} \tag{9.24}$$

So in this case $\tilde{\iota}_P$ is an embedding. Hence the restriction to $e'(P)$ vanishes if and only if the positive definite theta series $\theta_{\widehat{\mathcal{L}}_W}(\tau, \varphi^W_{0,[\ell\varpi_n+\lambda]})$ vanishes. Furthermore, the restriction of the class $[\theta_{\mathcal{L}_V}(\tau, \varphi^V_{q,[\lambda]})]$ cannot arise from an invariant form on D, since in that case one would need to obtain the trivial representation in the coefficients.

To obtain the nonvanishing, we first observe

Lemma 9.9. *Given* $\varphi^W_{0,[\ell\varpi_n+\lambda]}$ *as above, then there exists a coset of a lattice* $\mathcal{L}_W$ *which we can take to be contained* $\widehat{\mathcal{L}}_W$ *such that*

$$\theta_{\mathcal{L}_W}(\tau, \varphi^W_{0,[\ell\varpi_n+\lambda]}) \neq 0.$$

Proof. We give a very simple argument which we learned from E. Freitag and R. Schulze-Pillot. We can assume $V = \mathbb{Q}^m$ with the standard inner product. First find a vector $h \in \frac{1}{N_1}(\mathbb{Z}^m)^n$ with $N_1 \in \mathbb{Z}$ such that $\varphi^W_{0,[\ell\varpi_n+\lambda]}(h) \neq 0$. Then pick a lattice $L = N_1N_2\mathbb{Z}^m$ such that $\| \sum_{x \in L^n} \varphi^W_{0,[\ell\varpi_n+\lambda]}(x)\| < \|\varphi^W_{0,[\ell\varpi_n+\lambda]}(h)\|$. Such a $N_2 \in \mathbb{Z}$ exists as $\varphi^W_{0,[\ell\varpi_n+\lambda]}$ is a Schwartz function. Then the theta series associated to $\varphi^W_{0,[\ell\varpi_n+\lambda]}$ for $\mathcal{L}_W = L^n + h$ does not vanish. □

From this data then, we now can find a $\mathcal{L}'_V$ contained in $\mathcal{L}_V$ such that $\widehat{\mathcal{L}'}_W = \mathcal{L}_W$ with $\theta_{\mathcal{L}_W}(\tau, \varphi^W_{0,[\ell\varpi_n+\lambda]}) \neq 0$. Replace Γ with $\Gamma \cap \operatorname{Stab} L'$. Then $[\tilde{r}_P \theta_{\mathcal{L}_V}(\tau, \varphi^V_{q,[\lambda]})] \neq 0$. This proves Theorem 1.5.

One feature of our method to establish non-vanishing is that we retain some control over the cover X', since this reduces to the very concrete question of non-vanishing of positive definite theta series. An easy example for this is the following.

Example 9.10. Consider the integral quadratic form given by

$$y_1y'_1 + \cdots + y_qy'_q + 2x_1^2 + \cdots + 2x_k^2$$

with $y_i, y'_i, x_j \in \mathbb{Z}$. So $L = \mathbb{Z}^m$ with $m = 2q + k$. Assume $k \geq q$. Note $L^\# \subset \frac{1}{4}\mathbb{Z}^m$. We let Γ be the subgroup in $\operatorname{Stab}(L)$ which stabilizes $L^\#/L$. Then

$$H^q(\Gamma, \mathbb{Z}) \neq 0.$$

Using our method this follows from the non-vanishing of the theta series $\sum_{\mathbf{x} \in \mathbb{Z}^k + (\frac{1}{4},\dots,\frac{1}{4})} x_1 \cdots x_q e^{4\pi i (\sum x_i^2)\tau}$.

References

[1] N. Bergeron, *Produits dans la cohomologie des variétés arithmétiques : quelques calculs sur les séries theta*, preprint.

[2] H. Boerner, *Representations of Groups*, Elsevier, 1970.

[3] A. Borel and L. Ji, *Compactifications of symmetric and locally symmetric spaces*, Birkäuser, 2006.

[4] A. Borel and J.-P. Serre, *Corners and arithmetic groups*, Comment. Math. Helv. **48** (1973), 436-491.

[5] A, Borel and N. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, second edition, Mathematical Surveys and Monographs **67**, AMS, 2000.

[6] N. Bourbaki, *Groupes et Algèbres de Lie, Chapters 4,5,6*, Hermann, 1981.

[7] W. Fulton and J. Harris, *Representation Theory, A First Course*, Graduate Texts in Mathematics **129**, Springer, 1991.

[8] J. Funke, *Special cohomology classes for the Weil representation*, to appear in the Proceedings of the Conference on Automorphic Forms and Automorphic L-Functions, RIMS Kyoto, 2008.

[9] J. Funke and J. Millson, *Cycles with local coefficients for orthogonal groups and vector-valued Siegel modular forms*, American J. Math. **128** (2006), 899-948.

[10] J. Funke and J. Millson, *On a theorem of Hirzebruch and Zagier*, in preparation.

[11] R. Goodman and N. R. Wallach, *Representations and Invariants of the Classical Groups*, Encyclopedia of Mathematics and its Applications **68**, Cambridge University Press, 1998.
[12] M. Goresky, G. Harder, and R. MacPherson, *Weighted cohomology*, Invent. Math. **116** (1994),139-213.
[13] G. Harder, *On the cohomology of discrete arithmetically defined groups*, In: Proc. of the Int. Colloq. on Discrete Subgroups of Lie Groups and Appl. to Moduli (Bombay 1973), Oxford 1975, 129 - 160.
[14] F. Hirzebruch and D. Zagier, *Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus*, Inv. Math. **36** (1976), 57-113.
[15] B. Kostant, *Lie algebra cohomology and the generalized Borel-Weil theorem*, Ann. of Math. **74** (1961), 329 - 387.
[16] S. Kudla and J. Millson, *The Theta Correspondence and Harmonic Forms I*, Math. Ann. **274** (1986), 353-378.
[17] S. Kudla and J. Millson, *The Theta Correspondence and Harmonic Forms II*, Math. Ann. **277** (1987), 267-314.
[18] S. Kudla and J. Millson, *Intersection numbers of cycles on locally symmetric spaces and Fourier coefficients of holomorphic modular forms in several complex variables*, IHES Pub. **71** (1990), 121-172.
[19] N.N. Lebedev, Special functions and their applications, Dover, 1972.
[20] J.-S. Li, *Nonvanishing theorems for the cohomology of certain arithmetic quotients*, J. Reine Angew. Math. **428** (1992), 177-217.
[21] J. Millson, *The cohomology with local coefficients of compact hyperbolic manifolds*, Proceedings of the International Conference on Algebraic Groups and Arithmetic Subgroups, Mumbai (2001), Narosa Publishing House, international distribution by the AMS.
[22] J. Schwermer, *Kohomologie arithmetisch definierter Gruppen und Eisensteinreihen*, Lecture Notes in Math. **988**, Springer, 1983.
[23] T. Shintani, *On the construction of holomorphic cusp forms of half integral weight*, Nagoya math. J. **58** (1975), 83-126.l
[24] B. Speh and T. N. Venkataramana, *Construction of Some Generalised Modular Symbols*, Pure and Applied Mathematics Quarterly **1** (2005), 737-754.
[25] G. van der Geer, *Hilbert modular surfaces*, Ergebnisse der Math. und ihrer Grenzgebiete (3), vol. 16, Springer, 1988.

Department of Mathematical Sciences, University of Durham, Science Laboratories, South Rd, Durham DH1 3LE, United Kingdom
E-mail address: jens.funke@durham.ac.uk

Department of Mathematics, University of Maryland, College Park, MD 20742, USA
E-mail address: jjm@math.umd.edu