Tables of 3-Manifolds up to Complexity 6

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1 Introduction.

It is well-known that knot tables turned to be very useful in knot theory as sources of different examples and conjectures. The tables provide also a convenient way to identify knots and make references. They are usually composed in order of increasing crossing numbers, see [Ro76]. In this paper we present similar tables of closed irreducible orientable 3-manifolds. The crucial question here is the choice of filtration in the set $\mathcal{M}$ of all compact manifolds. It would be desirable to have a finite number of 3-manifolds in each term of the filtration, each term containing manifolds that are in some sense simpler than those in the following terms. A useful tool here would be a measure of "complexity" of a 3-manifold. Given such a measure, we might hope to enumerate all "simple" manifolds before moving on to more complicated ones. There are several well-known candidates for such a complexity function. For example, take the Heegaard genus $g(M)$, defined to be the minimal genus over all Heegaard decompositions of M. Other examples include the number of simplices in a minimal triangulation of M and the crossing number in a minimal surgery presentation of M.

Each of these measures has shortcomings. The Heegaard genus is additive with respect to connected sums of 3-manifolds but for $g \geq 1$ there are infinitely many distinct manifolds of Heegaard genus g, and already for $g = 2$ one can hardly expect a simple classification. The same defect has the surgery complexity (because of framings). The number of simplices in a minimal triangulation is not a "natural" measure of complexity because the simplest possible closed manifold, S^3, already would have non-zero complexity, and we would have no chances to get the additivity.

In Section 2 we construct an integral non-negative function $c\colon \mathcal{M} \to Z$, called *complexity function*, which has the following properties:

1. c is additive, that is, $c(M_1 \# M_2) = c(M_1) + c(M_2)$.

2. For any $k \in Z$, there are only finitely many irreducible manifolds $M \in \mathcal{M}$ with complexity $c(M) = k$.

3. $c(M)$ is relatively easy to estimate.

Section 3 is devoted to description of simplification moves on spines.

Section 4 contains a brief description of a computer program that enumerates 3-manifolds, descriptions of tables, and examples. The tables are presented in the Appendix. The table of spines was prepared with a help of M. Ovchinnikov.

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2 What is the Complexity of a 3-Manifold?

Denote by $\Delta^{(1)}$ the complete graph with 4 vertices. Clearly, it is homeomorphic to the 1-dimensional skeleton of the standard 3-simplex.

Definition 2.1 *A compact 2-dimensional polyhedron P is called* almost simple *if the link of each of its points can be embedded in $\Delta^{(1)}$. The points whose links are homeomorphic to $\Delta^{(1)}$ are said to be* vertices *of P.*

Definition 2.2 *The* complexity $c(M)$ *of a compact 3-manifold M equals k if M possesses an almost simple spine with k genuine vertices and has no almost simple spines with a smaller number of vertices.*

It turns out that the notion of complexity introduced above is naturally related to practically all the known methods of presenting manifolds and adequately describes complexity of manifolds in the informal sense of the expression. The following properties of the complexity are proved in [Ma].

Finiteness property. *For any integer k there exists only a finite number of distinct closed irreducible orientable 3-manifolds of complexity k.*

Additivity property. *The complexity of the connected sum of compact 3-manifolds is equal to the sum of their complexities.*

Definition 2.3 *A compact polyhedron P is called* simple *if the link of each of its points is homeomorphic to one of the following 1-dimensional polyhedra:*
(a) a circle;
(b) a circle with a diameter;
(c) a circle with three radii.

Typical neighborhoods of points of a simple polyhedron are shown in Fig. 1. Probably it would be illuminating to present the vertex singularity in different forms, see Fig. 2. The second model is the cone over the 1-dimensional skeleton

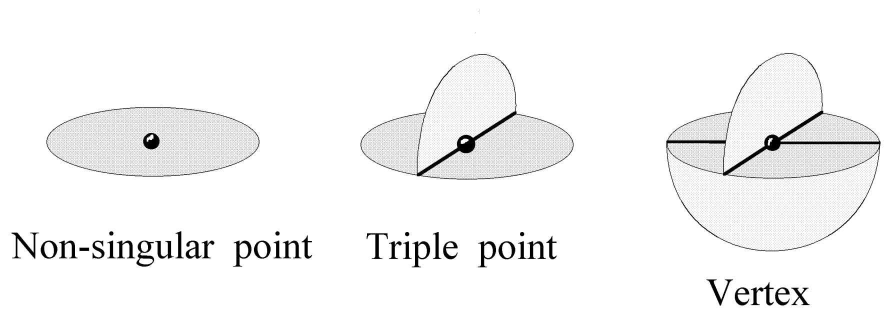


Figure 1: Allowable neighborhoods in a simple polyhedron

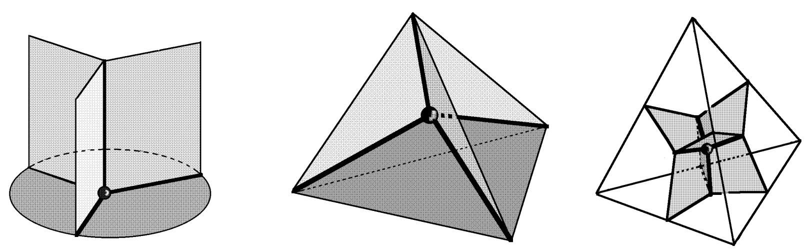


Figure 2: Equivalent ways of looking at vertices

$\Delta^{(1)}$ of a regular ôn. The third model is placed into the regular tetrahedron to emphasize that the singularity is totally symmetric. It can be viewed as the union $\cup \mid \mathrm{lk}(v_i, \Delta') \mid$ of links of vertices of the tetrahedron Δ in the first barycentric subdivision Δ'.

Definition 2.4 *The set of singular points of a simple polyhedron (that is, the union of vertices and triple lines) is called its* singular graph *and is denoted by* SP.

In general, SP is not a graph in the usual sense since it can contain closed triple lines without vertices. If there are no closed triple lines then SP is a regular graph of degree 4, i.e. every vertex of SP is incident to exactly four edges.

Let us describe the structure of simple polyhedra in detail. Each simple polyhedron is naturally stratified. In this stratification each stratum of dimension 2 (*a 2-component*) is a connected component of the set of non-singular points. Strata of dimension 1 consist of open or closed triple lines, and dimension 0

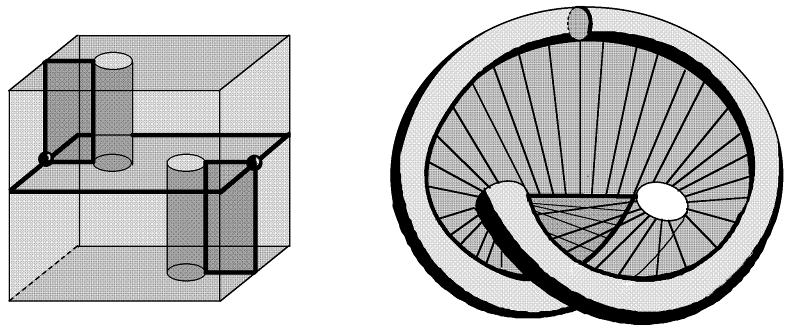


Figure 3: Bing's House and Abalone

strata are vertices. Sometimes it is convenient to imagine vertices as transverse intersection points of triple lines.

It is natural to want each of the strata to consist of cells — that is, we would like P to be cellular. We will make this a demand of our future considerations, as can be seen in the following definition:

Definition 2.5 *A simple polyhedron P is called* special *if*
(1) each 1-stratum of P is an open 1-cell;
(2) each 2-component of P is an open 2-cell.

Remark 2.1 If P is connected and contains at least one vertex, then the condition (1) in the above definition follows from condition (2).

Definition 2.6 *A spine of a 3-manifold is called* simple *or* special *if it is a simple or special polyhedron, respectively.*

Two examples of special spines of the 3-ball are shown in Fig. 3: Bing's House with Two Rooms and the Abalone (a marine mollusk with an oval, somewhat spiral shell). It is known ([Ca65, Ma73] that any homeomorphism between special spines can be extended to a homeomorphism between the corresponding manifolds. It means that a special spine P of a 3-manifold M may serve as a presentation of M. Moreover, M can be reconstructed from a regular neighborhood $N(SP)$ of the singular graph SP of P: starting from $N(SP)$, one can easily reconstruct P by attaching 2-cells to all the circles in $\partial N(SP)$, and then reconstruct M. If M is orientable, then $N(SP)$ can be embedded into R^3. This gives us a very convenient way for presenting 3-manifolds: we simply draw pictures, see Fig. 4.

Theorem 2.1 ([Ma90]) *Suppose M is a compact orientable irreducible 3-manifold with incompressible boundary (possibly empty) and without essential annuli. Then any minimal almost simple spine of M is special, except for the manifolds $S^3, RP^3, L_{3,1}$ having complexity 0.*

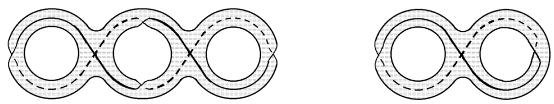


Figure 4: Bing's House with Two Rooms and the Abalone presented as regular neighborhoods of their special spines

It should be noted that the number of vertices of a special spine as a measure of complexity of 3-manifolds was implicitly used by numerous authors. H. Ikeda proved that any simply-connected manifold having a simple spine with ≤ 4 vertices is homeomorphic to S^3 [Ik84]. Together with I. Yoshinobu [IkYo85] he listed all closed 3-manifolds which in our terminology possess complexity ≤ 2. A complete list of all closed orientable irreducible 3-manifolds of complexity ≤ 5 was obtained by means of a computer as early as 1973 by S. Matveev and V. Savvateev [MaSa74]. D. Gillman and P. Laszlo, who were interested only on homology spheres [GiLa83], proved by a computer that among manifolds of complexity ≤ 5 only S^3/P_{120} and S^3 have trivial homology, which actually easy follows by looking through the Matveev and Savvateev list. A list of closed orientable irreducible 3-manifolds of complexity 7 was obtained by M. Ovchinnikov [Ov97]. It consists of about 150 manifolds and is too large to be presented here.

3 Simplification Moves on Spines

We describe here two main types of moves. The moves have the following advantage: it is extremely easy to determine whether or not one can apply them to a given special spine. For a description of other moves see [Ma98].

Definition 3.1 *Let P be a special polyhedron and c a 2-component of P. Then we say that the boundary curve of c has a* counterpass *if it passes along one of the edges of P twice in opposite directions. We say that the boundary curve is* short *if it passes through no more than 3 vertices of P and through each of them no more than once.*

For instance, Bing's House contains two 2-components with boundary curves of length 1 while the boundary curve of the third 2-component has a few counterpasses (see Fig. 3 and Fig. 4).

Proposition 3.1 *Suppose P is a special spine of a 3-manifold M and suppose at least one of the following conditions hold:*

(1) P has a 2-component with a short boundary curve;

(2) M is closed and the boundary curve of one of the 2-components of P has a counterpass.

Then M possesses an almost simple spine with a smaller number of vertices.

Proof. Assume that P has a 2-component c with a short boundary curve. Then a regular neighborhood of $\mathrm{Cl}(c)$ in P may be presented as a lateral surface of a cylinder with $k \leq 3$ wings and the 2-component c as a middle disc. Attach to P a 2-cell parallel to c and make a hole in a lateral face of the cylinder thus obtained, see Fig. 5. Collapsing the resulting polyhedron, we get a new almost simple spine of M. It has a smaller number of vertices, since the attaching of the 2-cell creates k new vertices, and the piercing through the lateral face and collapsing destroys at least four of them if $k > 1$, and at least two if $k = 1$.

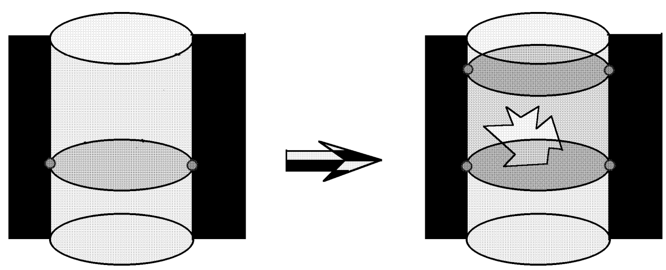


Figure 5: Attaching a new 2-cell and making a hole decreases the number of vertices

Assume now that M is closed and the boundary curve of a 2-component c of P has a counterpass on an edge e. Then there exists a simple closed curve $l \subset \mathrm{Cl}(c)$ that intersects e transversely at exactly one point. It decomposes c into two 2-cells c', c''. Since M is closed, one can easily find a disc $D^2 \subset M$ such that $D^2 \cap P = \partial D^2 = l$ (to construct D^2, one may push l by an isotopy to the boundary of a regular neighborhood of P and span it by a disc in the complementary ball). The polyhedron $P \cup D^2$ is a special spine of the twice punctured M, that is, of M with two removed balls. To get a spine of M, we make a hole in c' or c'' depending on which of them is a common face of these balls. After collapsing we get an almost simple spine of M having a smaller number of vertices.

Remark 3.1 Suppose P has a 2-component such that its boundary curve visits four vertices, and each of them exactly once. If we apply the same trick (glue in a parallel 2-cell and puncture a lateral one), we get another spine of M having the same number of vertices. We used this transformation for recognition of duplicates.

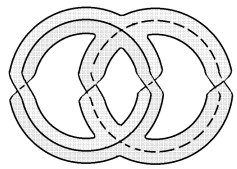


Figure 6: The minimal spine of the complement to figure eight knot has counterpasses

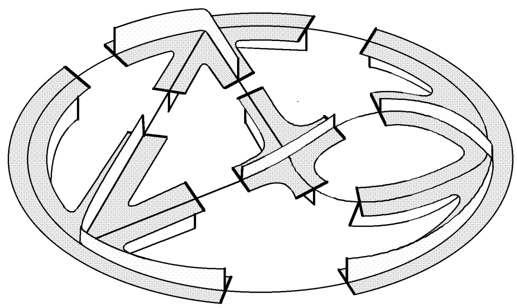


Figure 7: A decomposition of $N(SP)$ into copies of E_V

Remark 3.2 The assumption M is closed in item (2) of Proposition 3.1 can be replaced by the requirement that ∂M consists of spheres. If ∂M contains tori or surfaces of a higher genus, in general the counterpass simplification does not work. For example, the special spine of the complement to the figure eight knot shown in Fig. 6 has counterpasses but can not be simplified since it is minimal.

4 Manifolds of small complexity

The simplification moves proved to be very effective and useful for enumerating of 3-manifolds. For example, the number of special spines with ≤ 6 vertices is very large (several millions) but no more than 200 of them cannot be simplified by the moves. The computer works in the following way. It first looks through all the regular graphs of degree 4 with a given number of vertices. The graphs are considered as work-pieces for singular graphs. The vertices of each graph are replaced by copies of the standard vertex singularity as it is shown in Fig.7.

Then the computer enumerates all possible gluings together of the copies and produces corresponding special spines. Each spine P was tested for the following questions.

1. Is there a short boundary curve?

2. Is there a counterpass?

3. Is the corresponding manifold closed and orientable ?

By a positive answer to one of the first two questions, or a negative answer to the third question, the computer refuses to consider P and goes on to the next spine. In the converse case, the result is printed out. The final composing of tables was made by hand.

The list of all closed orientable irreducible 3-manifolds up to complexity 6 contains 135 manifolds, see the tables and in Appendix. Each manifold is presented by a regular neighborhood of the singular graph of minimal special spine. If the manifold has several minimal spines, we draw all of them. Let us comment the results of the enumeration.

1. All closed orientable irreducible 3-manifolds up to complexity 6 are Seifert manifolds. All the manifolds of complexity ≤ 5 and many manifolds of complexity 6 have finite fundamental groups. They are elliptic, that is, can be presented as quotient spaces of S^3 by free linear actions of finite groups. Groups which can linearly act on S^3 without fixed points are well known (see [Mi57]). They are:

1. the finite cyclic groups;

2. the groups $Q_{4n}, n \geq 2$;

3. the groups $D_{2^k(2n+1)}, k \geq 3, n \geq 1$;

4. the groups P_{24}, P_{48}, P_{120}, and $P'_{8(3^k)}, k \geq 2$;

5. the direct product of any of these groups with a cyclic group of coprime order.

Lower indices show the orders of the groups. Presentations by generators and relations, and abelian quotients of the groups are the following:

1. $Q_{4n} =< x, y: x^2 = (xy)^2 = y^{2n} >$; $Z_2 \oplus Z_2$ if n is even, and Z_4 if it is odd.

2. $D_{2^k(2n+1)} =< x, y : x^{2^k} = 1, y^{2n+1} = 1, xyx^{-1} = y^{-1} >$; Z_{2^k}.

3. $P_{24} =< x, y : x^2 = (xy)^3 = y^3, x^4 = 1 >$; Z_3.

4. $P_{48} =< x, y : x^2 = (xy)^3 = y^4, x^4 = 1 >$; Z_2.

5. $P_{120} =< x, y : x^2 = (xy)^3 = y^5, x^4 = 1 >$; 0.

6. $P'_{8(3^k)} =< x, y, z : x^2 = (xy)^2 = y^2, zxz^{-1} = y, zyz^{-1} = xy, z^{3^k} = 1 >$; Z_{3^k}.

2. The list contains representatives of all the five series of elliptic manifolds. In particular, the manifolds $S^3/P_{24}, S^3/P_{48}$, and the Poincaré homology sphere S^3/P_{120} have complexities 4,5, and 5, respectively. The first manifold with non-abelian fundamental group is S^3/Q_8, where Q_8 is the group of quaternion units. It has complexity 2. More generally, for $2 \leq n \leq 6$ the manifolds S^3/Q_{4n} have complexity n. The simplest manifold of the type $S^3/D_{2^k(2n+1)}$, that is, S^3/D_{24}, has complexity 4 while the simplest manifold of the type $S^3/P'_{8(3^k)}$, that is, S^3/P'_{72}, has complexity 5. There also occur quotient spaces of S^3 by actions of direct products of the above-mentioned groups with cyclic groups of relatively prime orders. The simplest of these (the manifold $S^3/Q_8 \times Z_3$) has complexity 4.

3. All five flat closed orientable 3-manifolds have complexity 6, among them the torus $S^1 \times S^1 \times S^1$ and the Whitehead manifold obtained from S^3 by Dehn surgery on the Whitehead link with trivially framed components. The last two are the only closed orientable irreducible manifolds of complexity ≤ 6 having the first homology group of rank ≥ 2. Recall that the Whitehead manifold coincides with the mapping torus of a homeomorphism $S^1 \times S^1 \to S^1 \times S^1$ that induces multiplication by -1 in $H_1(S^1 \times S^1; Z)$.

4. Among the manifolds of complexity ≤ 6 there is just one non-trivial homology sphere S^3/P_{120}. It has a unique minimal special spine with 5 vertices. The singular graph of the spine is the complete graph on 5 vertices.

5. If the complexity of the lens space $L_{p,q}$ with $p > 2$ does not exceed 6, then it can be calculated by the formula $c(L_{p,q}) = S(p,q) - 3$, where $S(p,q)$ is the sum of all complete quotients in the expansion of p/q as a regular continued fraction. Must probably, the formula holds for all lens spaces, but we know how to prove only the inequality $c(L_{p,q}) \leq S(p,q) - 3$: it follows from the proof of Proposition 4.1. In practice, it is more convenient to calculate $c(L_{p,q})$ by the following empirical rule: if $p > 2q$, then $c(L_{p,q}) = c(L_{p-q,q}) + 1$. For example, $c(L_{33,10}) = c(L_{23,10}) + 1 = c(L_{13,10}) + 2 = c(L_{13,3}) + 2 = c(L_{10,3}) + 3 = c(L_{7,3}) + 4 = c(L_{4,3}) + 5 = c(L_{4,1}) + 5 = c(L_{3,1}) + 6 = 6$ since $c(L_{3,1}) = 0$ (we have used twice that lens spaces $L_{p,q}$ and $L_{p,p-q}$ are homeomorphic). This shows ones again how natural the notion of complexity is. It is interesting to note that not all regular graphs can be realized as singular graphs of minimal special spines of 3-manifolds. Let us try to single out several types of graphs that produce the majority of 3-manifolds up to complexity 6.

Definition 4.1 *A regular graph G of degree 4 is called a* non-closed chain *if it contains two loops, and all the other edges are double. G is a* closed chain *if it has only double edges. At last, G is called a* triangle with a tail *if it is homeomorphic to a wedge of a closed chain with 3 vertices and a non-closed chain such that the common point of the wedge lies in a loop of the non-closed chain. See Fig. 8.*

We will say that a special spine of a closed orientable 3-manifold is *pseudo-*

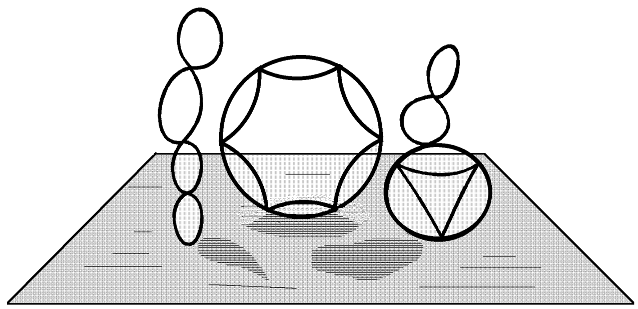


Figure 8: Three useful types of singular graphs: a closed chain, non-closed chain, and a triangle with a tail

minimal if it has no counterpasses and short boundary curves. In particular, any minimal special spine is pseudo-minimal. For brevity we will say that a special spine P is *modeled* on a graph G if G is just the singular graph of P.

Proposition 4.1 (See [Ma75]) *A closed orientable 3-manifold M has a pseudo-minimal special spine modeled on a non-closed chain $\iff$ M is a lens space $L_{p,q}$ with $p > 3$.*

Let us describe a simple method for calculating parameters of a lens space presented by a picture that shows a regular neighborhood of the singular graph of its pseudo-minimal special spine. The method can be easily proved by induction on the number of vertices of the spine. Assign to each double edge and to each loop of the singular graph a letter ℓ or r as shown in Fig. 9. We get a string w of letters that we will consider as a composition of operators $r, \ell\colon Z \oplus Z \to Z \oplus Z$ given by $r(a, b) = (a, a + b)$ and $\ell(a, b) = (a + b, b)$. Then the lens space has parameters $p = m+n, q = m$, where $(m, n) = w(1, 1)$. For example, for the lens space shown in Fig. 9 we have $w = rrrr\ell\ell\ell$, $(m, n) = (4, 17)$, and $(p, q) = (21, 4)$, since by our interpretation of r, ℓ we have

$$(1,1) \xrightarrow{\ell} (2,1) \xrightarrow{\ell} (3,1) \xrightarrow{\ell} (4,1) \xrightarrow{r} (4,5) \xrightarrow{r} (4,9) \xrightarrow{r} (4,13) \xrightarrow{r} (4,17).$$

The same method can be used for construction a pseudo-minimal special spine of a given lens space $L_{p,q}$: one should apply to the pair $(p-q, q)$ operators r^{-1}, ℓ^{-1} until we get $(1, 1)$, and then use the string of letters r, ℓ thus obtained for constructing the spine.

Proposition 4.2 [Ov97] *A closed orientable 3-manifold M has a pseudo-minimal special spine modeled on a triangle with a tail $\iff$ M is an orientable Seifert fibered manifold of the type $(S^2, (2, 1), (2, -1), (n, \beta))$, including the case $n = 1$.*

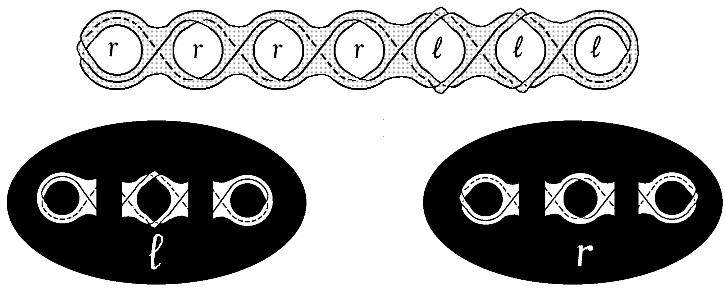


Figure 9: How to write down the developing string for a non-closed chain

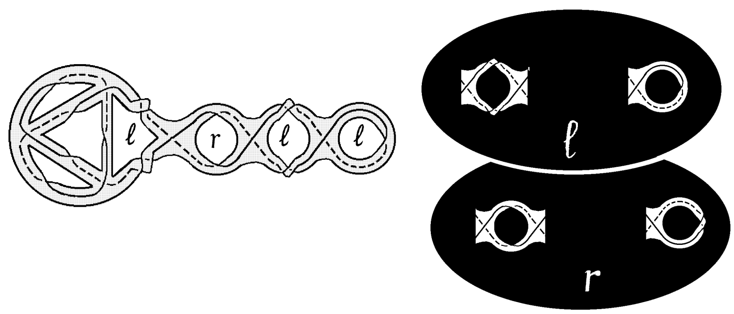


Figure 10: How to write down the developing string for a tail

Let us describe a simple method for calculating parameters (n, β) starting from a pseudo-minimal special spine modeled on a triangle with a tail. The method can be easily proved by induction on the number of vertices of the tail. Assign to the loop and the double edges of the tail, and to the pair of edges adjacent to it, letters ℓ and r as shown in Fig. 10. We get a string w of letters that, as above, can be considered as a composition of operators $r, \ell\colon Z \oplus Z \to Z \oplus Z$ given by $r(a,b) = (a, a+b)$ and $\ell(a,b) = (a+b, b)$. Then $(n, \beta) = w(1,1)$. For example, for the spine shown in Fig. 10 we have $w = \ell r \ell \ell$ and $w(1,1) = (n, \beta) = (7,4)$.

The same method can be used for construction a pseudo-minimal special spine of a given Seifert fibered manifold $(S^2, (2,1), (2,-1), (n, \beta))$: one should recover the string of r, ℓ by transforming $(n, |\beta|)$ into $(1,1)$, and then use it for choosing the correct tail.

Note that for any pair of coprime positive integers (n, β) with $n \geq 1$ the fundamental group of the manifold $M_{n,\beta} = (S^3, (2,1), (2,-1), (n, \beta))$ is finite and has the presentation

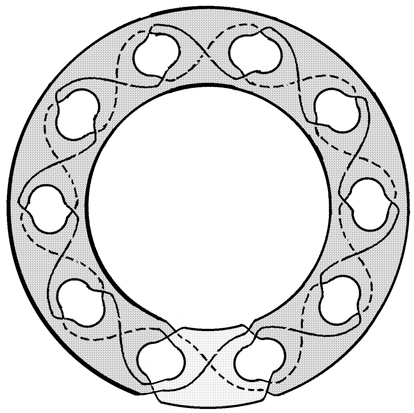


Figure 11: The unique pseudo-minimal special spine modeled on a closed chain with n vertices is a spine of S^3/Q_{4n}

$$< c_1, c_2, c_3, t | c_1^2 = t, c_2^2 = t^{-1}, c_3^n = t^\beta, c_1c_2c_3 = 1 >$$

The order of the homology group $H_1(M_{n,\beta}; Z)$ is equal to 4β. Using this, it is not hard to present $M_{n,\beta}$ as the quotient space of S^3 by the linear action of one of the groups listed above. It turns out that the following is true:

1) if $n > 1$ and β is odd, then $M_{n,\beta} = S^3/Q_{4n} \times Z_\beta$;

2) if $n > 1$ and β is even, then $M_{n,\beta} = S^3/D_{2^{k+2}n} \times Z_{2m+1}$, where k and m can be found from the equality $\beta = 2^k(2m + 1)$;

3) if $n = 1$ then $M_{n,\beta} = L_{4\beta,2\beta+1}$.

If $n = 1$ or $\beta = 1$, the pseudo-minimal special spine of $M_{n,\beta}$ modeled on the triangle with a tail is not minimal. An easy way to see this is to apply the transformation described in Remark 3.1. This is possible since the spine possesses a boundary curve that passes through 4 vertices, and visits each of them exactly once. After the transformation we get a spine that has the same number of vertices but possesses a boundary curve of length 3. Therefore, one can simplify the spine. In the case $n = 1$ we get a spine of the lens space modeled on a non-closed chain with smaller number of vertices. If $\beta = 1$, we get a simple spine of the manifold S^3/Q_{4n}.

Proposition 4.3 [Ma80] *A closed orientable 3-manifold M has a pseudo-minimal special spine modeled on a closed chain with $n \geq 2$ vertices $\iff$ M is S^3/Q_{4n}.*

The following conjectures are motivated by Propositions 4.1 – 4.3 and the results of the computer enumeration.

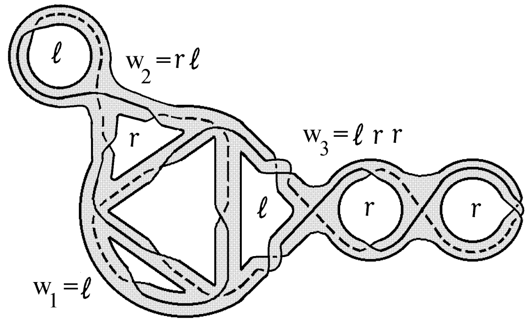


Figure 12: The developing strings are ℓ, $r\ell$, and ℓrr. Thus $M = (S^2, (2,1), (2,3), (4,3), (1,-1))$

Conjecture 4.1 *Any lens space $L_{p,q}$ with $p \geq 3$ has a unique minimal special spine. This spine is modeled on a non-closed chain.*

Conjecture 4.2 *For any $n \geq 2$ the manifold S^3/Q_{4n} has a unique minimal special spine. This spine is modeled on a closed chain with n links.*

Conjecture 4.3 *Manifolds of the type $S^3/Q_{4n} \times Z_\beta, \beta \neq \pm 1$ and $S^3/D_{2^{k+2}n} \times Z_{2m+1}$ have minimal special spine modeled on triangles with a tail.*

Table of minimal spines (see Appendix) shows that the conjectures are true for manifolds of complexity ≤ 6.

One can prove that any pseudo-minimal special spine modeled on a triangle with 3 tails is a spine of a Seifert fibered manifold M over S^2 with 3 exceptional fibers. Let $w_i, 1 \leq i \leq 3$ be the developing rl-strings of the tails. Then $M = (S^2, (n_1, \beta_1), (n_2, \beta_2), (n_3, \beta_3), (1, -1))$, where $(n_i, \beta_i) = w_i(1,1)$ for $1 \leq i \leq 3$. We have inserted the non-exceptional fiber (1,-1) to preserve the symmetry of the expression. Certainly, one may write $M = (S^2, (n_1, \beta_1), (n_2, \beta_2), (n_3, \beta_3 - n_3))$. The formula works also for triangles with < 3 tails, if we adopt the convention that the developing word for the empty tail is ℓ and produces the exceptional fiber of type (2,1). See Fig. 12.

Remark 4.1 Recall that the manifold $M = (S^2, (n_1, \beta_1), (n_2, \beta_2), (n_3, \beta_3), (1, -1))$ with $n_i > 1$ has a finite fundamental group if and only if the triple (n_1, n_2, n_3) is one of the following exceptional triples: $(2,2,n), (2,3,3), (2,3,4)$, and $(2,3,5)$. The following rules my be useful for calculating $\pi_1(M)$, where (n_1, n_2, n_3) is one of exceptional triples and the triple $(\beta_1, \beta_2, \beta_3)$ has the from $(1,1,\beta)$.

a) Let $(n_1, n_2, n_3) = (2,2,n)$, where n is even. Then

$\pi_1(M) = Q_{4n} \times Z_\beta$;

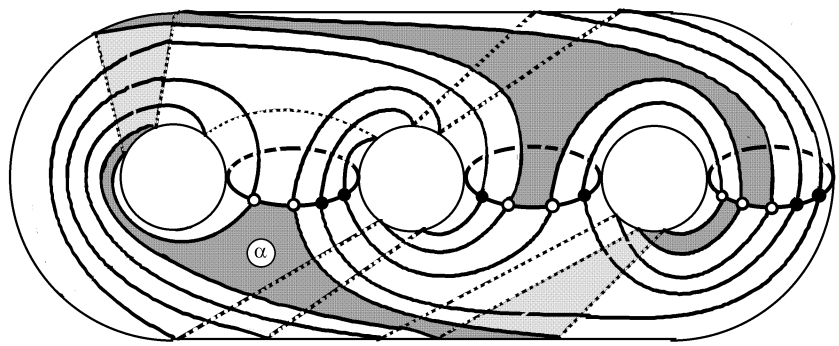


Figure 13: A Heegaard diagram of the manifold $S^3/(P_{24} \times Z_5)$

b) Let $(n_1, n_2, n_3) = (2, 2, n)$, where n is odd. Then
$\pi_1(M) = D_{2^{k+2}n} \times Z_{2m+1}$, where $\beta = 2^k(2m+1)$;

c) Let $(n_1, n_2, n_3) = (2, 3, 3)$. Then
$\pi_1(M) = P'_{8(3^{k+1})} \times Z_{2m+1}$, where $3^k m = 2\beta - 1$ and $m = \pm 1 \bmod 3$;

d) Let $(n_1, n_2, n_3) = (2, 3, 4)$. Then
$\pi_1(M) = P_{48} \times Z_{3\beta-2}$;

e) Let $(n_1, n_2, n_3) = (2, 3, 5)$. Then
$\pi_1(M) = P_{120} \times Z_{6\beta-5}$.

We present an example of using the table for practical recognition of 3-manifolds. Let a 3-manifold M be given by the Heegaard diagram shown on Fig. 13.

Note that the union of the boundary of the handlebody with its meridional discs and with the ones of the complementary handlebody is a special spine of twice punctured M. The spine possesses 13 vertices. An almost special spine of M is obtained from it by puncturing one of the domains into which the meridians of the handlebodies decompose the surface. If we puncture the domain β, then (after collapsing) 7 vertices disappear (they are shown by white dots). Therefore $c(M) \leq 6$. It is easy to calculate that the homology group $H_1(M)$ is isomorphic to the group Z_{15}, while a presentation of the fundamental group $\pi_1(M)$ is of the form

$$\langle a, b, c \mid ab^2 = ac^{-1}bc^{-1}ac^{-1} = ab^{-1}c^2 = 1 \rangle.$$

Adding the relation $c^2 = 1$ yields a presentation of a non-abelian group $\langle b, c \mid b^3 = (bc)^3 = c^2 = 1\rangle$. Therefore $\pi_1(M)$ is non-abelian. One can easily show that M is irreducible. It follows from looking through the table that the only manifold which differs from a lens space and has complexity ≤ 6 and homology group Z_{15} coincides with the quotient space of S^3 by a linear action of the group $P_{24} \times Z_5$.

5 Appendix

5.1 Tables of 3-manifolds up to complexity 6

There are 7 tables that contain all closed orientable irreducible 3-manifolds of complexity c, where $0 \leq c \leq 6$. The number $N(c)$ of manifolds of complexity c can be tabulated as follows:

c	0	1	2	3	4	5	6
$N(c)$	3	2	4	7	14	31	74

Our notation for manifolds is similar to the one for knots: we write c_i for the manifold number i among manifolds of complexity c. S^3/G denotes the quotient space of S^3 by a free linear action of a non-abelien finite group G taken from the list on page 8. $L_{p,q}$ denotes the lens space with parameters p, q. Let $h_A : T \to T$ be a homeomorphism of the torus $T = S^1 \times S^1$ onto itself corresponding to an unimodular integer matrix A of order 2. Then $T \times I/A$ is the mapping torus of h_A. In other words, $T \times I/A$ is obtained from the manifold $T \times I$ by identifying the boundary tori by h_A. Clearly, conjugated matrices produce homeomorphic manifolds.

Recall that the boundary of the orientable I-bundle $K\tilde{\times}I$ over the Klein bottle K is a torus. Choose a coordinate system μ, λ on it such that μ projects onto a non-trivial orientation-preserving circle on K, and λ double covers an orientation-reversing circle on K. Then $K\tilde{\times}I \cup K\tilde{\times}I/A$ denotes the manifold obtained by pasting together two copies of $K\tilde{\times}I$ by the homeomorphism h_A. Finally, $(F, (p_1, q_1), \ldots, (p_k, q_k))$ is the orientable Seifert manifold with the base surface F and k exceptional fibers with unnormalized parameters $(p_i, q_i), 1 \leq i \leq k$. In two cases we use also regular fibers with $p_1 = 1$ in order to describe manifolds fibered into circles without exceptional fibers. We do not present Seifert structures and homology groups of lens spaces since they are well known.

Table 1: Complexity 0

c_i	M
0_1	S^3
0_2	RP^3
0_3	$L_{3,1}$

Table 2: Complexity 1

c_i	M
1_1	$L_{4,1}$
1_2	$L_{5,2}$

Table 3: Complexity 2

c_i	M
2_1	$L_{5,1}$
2_2	$L_{7,2}$
2_3	$L_{8,3}$

c_i	M	Seifert structure	$H_1(M;Z)$
2_4	S^3/Q_8	$(S^2,(2,1),(2,1)(2,-1))$	$Z_2 \oplus Z_2$

Table 4: Complexity 3

c_i	M
3_1	$L_{6,1}$
3_2	$L_{9,2}$
3_3	$L_{10,3}$

c_i	M
3_4	$L_{11,3}$
3_5	$L_{12,5}$
3_6	$L_{13,5}$

c_i	M	Seifert structure	$H_1(M;Z)$
3_7	S^3/Q_{12}	$(S^2,(2,1),(2,1),(3,-2))$	Z_4

Table 5: Complexity 4

c_i	M
4_1	$L_{7,1}$
4_2	$L_{11,2}$
4_3	$L_{13,3}$
4_4	$L_{14,3}$
4_5	$L_{15,4}$

c_i	M
4_6	$L_{16,7}$
4_7	$L_{17,5}$
4_8	$L_{18,5}$
4_9	$L_{19,7}$
4_{10}	$L_{21,8}$

c_i	M	Seifert structure	$H_1(M;Z)$
4_{11}	$S^3/Q_8 \times Z_3$	$(S^2,(2,1),(2,1),(2,1))$	$Z_2 \oplus Z_6$
4_{12}	S^3/Q_{16}	$(S^2,(2,1),(2,1),(4,-3))$	$Z_2 \oplus Z_2$
4_{13}	S^3/D_{24}	$(S^2,(2,1),(2,1),(3,-1))$	Z_8
4_{14}	S^3/P_{24}	$(S^2,(2,1),(3,1),(3,-2))$	Z_3

Table 6: Complexity 5

c_i	M	c_i	M
5_1	$L_{8,1}$	5_{11}	$L_{24,7}$
5_2	$L_{13,2}$	5_{12}	$L_{25,7}$
5_3	$L_{16,3}$	5_{13}	$L_{25,9}$
5_4	$L_{17,3}$	5_{14}	$L_{26,7}$
5_5	$L_{17,4}$	5_{15}	$L_{27,8}$
5_6	$L_{19,4}$	5_{16}	$L_{29,8}$
5_7	$L_{20,9}$	5_{17}	$L_{29,12}$
5_8	$L_{22,5}$	5_{18}	$L_{30,11}$
5_9	$L_{23,5}$	5_{19}	$L_{31,12}$
5_{10}	$L_{23,7}$	5_{20}	$L_{34,13}$

c_i	M	Seifert structure	$H_1(M;Z)$
5_{21}	$S^3/Q_8 \times Z_5$	$(S^2,(2,1),(2,1),(2,3))$	$Z_2 \oplus Z_{10}$
5_{22}	$S^3/Q_{12} \times Z_5$	$(S^2,(2,1),(2,1),(3,2))$	Z_{20}
5_{23}	$S^3/Q_{16} \times Z_3$	$(S^2,(2,1),(2,1),(4,-1))$	$Z_2 \oplus Z_6$
5_{24}	S^3/Q_{20}	$(S^2,(2,1),(2,1),(5,-4))$	Z_4
5_{25}	$S^3/Q_{20} \times Z_3$	$(S^2,(2,1),(2,1),(5,-2))$	Z_{12}
5_{26}	S^3/D_{40}	$(S^2,(2,1),(2,1),(5,-3))$	Z_8
5_{27}	S^3/D_{48}	$(S^2,(2,1),(2,1),(3,1))$	Z_{16}
5_{28}	$S^3/P_{24} \times Z_5$	$(S^2,(2,1),(3,2),(3,-1))$	Z_{15}
5_{29}	S^3/P_{48}	$(S^2,(2,1),(3,1),(4,-3))$	Z_2
5_{30}	S^3/P'_{72}	$(S^2,(2,1),(3,2),(3,-2))$	Z_9
5_{31}	S^3/P_{120}	$(S^2,(2,1),(3,1),(5,-4))$	0

Table 7: Complexity 6

c_i	M	c_i	M	c_i	M	c_i	M
6_1	$L_{9,1}$	6_{10}	$L_{28,5}$	6_{19}	$L_{35,8}$	6_{28}	$L_{41,16}$
6_2	$L_{15,2}$	6_{11}	$L_{29,9}$	6_{20}	$L_{36,11}$	6_{29}	$L_{43,12}$
6_3	$L_{19,3}$	6_{12}	$L_{30,7}$	6_{21}	$L_{37,8}$	6_{30}	$L_{44,13}$
6_4	$L_{20,3}$	6_{13}	$L_{31,7}$	6_{22}	$L_{37,10}$	6_{31}	$L_{45,19}$
6_5	$L_{21,4}$	6_{14}	$L_{31,11}$	6_{23}	$L_{39,14}$	6_{32}	$L_{46,17}$
6_6	$L_{23,4}$	6_{15}	$L_{32,7}$	6_{24}	$L_{39,16}$	6_{33}	$L_{47,13}$
6_7	$L_{24,5}$	6_{16}	$L_{33,7}$	6_{25}	$L_{40,11}$	6_{34}	$L_{49,18}$
6_8	$L_{24,11}$	6_{17}	$L_{33,10}$	6_{26}	$L_{41,11}$	6_{35}	$L_{50,19}$
6_9	$L_{27,5}$	6_{18}	$L_{34,9}$	6_{27}	$L_{41,12}$	6_{36}	$L_{55,21}$

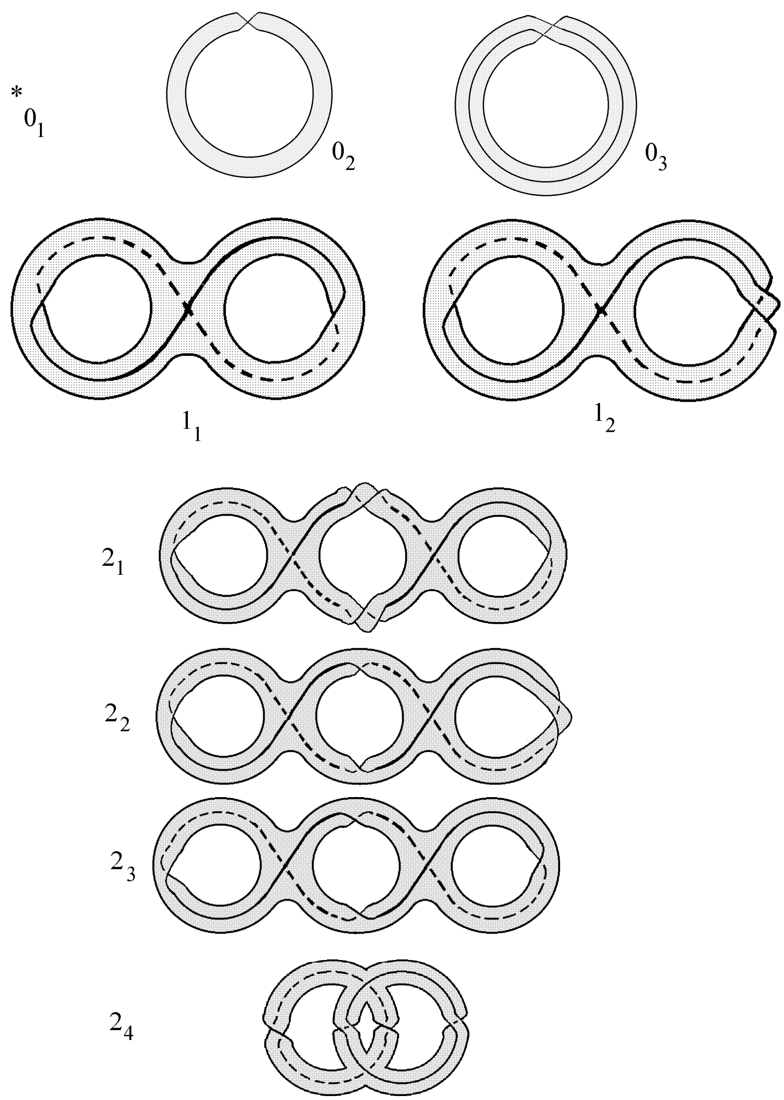
c_i	M	Seifert structure	$H_1(M;Z)$
6_{37}	$S^3/Q_8 \times Z_7$	$(S^2,(2,1),(2,1),(2,5))$	$Z_2 \oplus Z_{14}$
6_{38}	$S^3/Q_{12} \times Z_7$	$(S^2,(2,1),(2,1),(3,4))$	Z_{28}
6_{39}	$S^3/Q_{16} \times Z_5$	$(S^2,(2,1),(2,1),(4,1))$	$Z_2 \oplus Z_{10}$
6_{40}	$S^3/Q_{16} \times Z_7$	$(S^2,(2,1),(2,1),(4,3))$	$Z_2 \oplus Z_{14}$
6_{41}	$S^3/Q_{20} \times Z_7$	$(S^2,(2,1),(2,1),(5,2))$	Z_{28}
6_{42}	S^3/Q_{24}	$(S^2,(2,1),(2,1),(6,-5))$	$Z_2 \oplus Z_2$
6_{43}	$S^3/Q_{28} \times Z_3$	$(S^2,(2,1),(2,1),(7,-4))$	Z_{12}
6_{44}	$S^3/Q_{28} \times Z_5$	$(S^2,(2,1),(2,1),(7,-2))$	Z_{20}
6_{45}	$S^3/Q_{32} \times Z_3$	$(S^2,(2,1),(2,1),(8,-5))$	$Z_2 \oplus Z_6$
6_{46}	$S^3/Q_{32} \times Z_5$	$(S^2,(2,1),(2,1),(8,-3))$	$Z_2 \oplus Z_{10}$
6_{47}	S^3/D_{56}	$(S^2,(2,1),(2,1),(7,-5))$	Z_8
6_{48}	S^3/D_{80}	$(S^2,(2,1),(2,1),(5,-1))$	Z_{16}
6_{49}	S^3/D_{96}	$(S^2,(2,1),(2,1),(3,5))$	Z_{32}
6_{50}	S^3/D_{112}	$(S^2,(2,1),(2,1),(7,-3))$	Z_{16}
6_{51}	S^3/D_{160}	$(S^2,(2,1),(2,1),(5,3))$	Z_{32}
6_{52}	$S^3/P_{24} \times Z_7$	$(S^2,(2,1),(3,1),(3,1))$	Z_{21}
6_{53}	$S^3/P_{24} \times Z_{11}$	$(S^2,(2,1),(3,2),(3,2))$	Z_{33}
6_{54}	$S^3/P_{48} \times Z_5$	$(S^2,(2,1),(3,2),(4,-3))$	Z_{10}
6_{55}	$S^3/P_{48} \times Z_7$	$(S^2,(2,1),(3,1),(4,-1))$	Z_{14}
6_{56}	$S^3/P_{48} \times Z_{11}$	$(S^2,(2,1),(3,2),(4,-1))$	Z_{22}
6_{57}	$S^3/P_{120} \times Z_7$	$(S^2,(2,1),(3,1),(5,-3))$	Z_7
6_{58}	$S^3/P_{120} \times Z_{13}$	$(S^2,(2,1),(3,1),(5,-2))$	Z_{13}
6_{59}	$S^3/P_{120} \times Z_{17}$	$(S^2,(2,1),(3,2),(5,-3))$	Z_{17}
6_{60}	$S^3/P_{120} \times Z_{23}$	$(S^2,(2,1),(3,2),(5,-2))$	Z_{23}
6_{61}	S^3/P'_{216}	$(S^2,(2,1),(3,2),(3,1))$	Z_{27}
6_{62}		$(S^2,(3,2),(3,1),(3,-2))$	$Z_3 \oplus Z_3$
6_{63}		$(S^2,(3,2),(3,2),(3,-2))$	$Z_3 \oplus Z_6$
6_{64}		$(S^2,(3,2),(3,2),(3,-1))$	$Z_3 \oplus Z_9$

Table 7: Complexity 6 (continued)

c_i	M	Seifert structure	$H_1(M;Z)$
6_{65}	$T\times I/\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}$	$(S^2,(2,1),(3,1),(6,-5))$	Z
6_{66}	$T\times I/\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$	$(S^2,(2,1),(4,1),(4,-3))$	$Z_2\oplus Z$
6_{67}	$T\times I/\begin{pmatrix}0 & 1\\ -1 & -1\end{pmatrix}$	$(S^2,(3,1),(3,1),(3,-2))$	$Z_3\oplus Z$
6_{68}	$T\times I/\begin{pmatrix}-1 & 0\\ -1 & -1\end{pmatrix}$	$(K,(1,1))$	$Z_4\oplus Z$
6_{69}	$T\times I/\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$	$(T,(1,1))$	$Z\oplus Z$
6_{70}	$T\times I/\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$	$(S^2,(2,1),(2,-1),(2,1),$ $(2,-1))$	$Z_2\oplus Z_2\oplus Z$
6_{71}	$T\times I/\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$T\times S^1$	$Z\oplus Z\oplus Z$
6_{72}	$K\tilde{\times}I\cup K\tilde{\times}I/\begin{pmatrix}-1 & 0\\ -1 & 1\end{pmatrix}$	$(S^2,(2,1),(2,1),(2,1),$ $(2,-1))$	$Z_2\oplus Z_2\oplus Z_4$
6_{73}	$K\tilde{\times}I\cup K\tilde{\times}I/\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	$(RP^2,(2,1),(2,-1))$	$Z_4\oplus Z_4$
6_{74}	$K\tilde{\times}I\cup K\tilde{\times}I/\begin{pmatrix}-1 & 1\\ -1 & 0\end{pmatrix}$	$(RP^2,(2,1),(2,1))$	$Z_4\oplus Z_4$

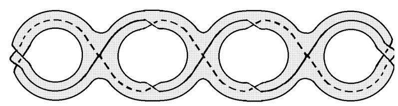
5.2 Minimal spines of 3-manifolds up to complexity 6

For any manifold $c_i, 0 \leq i \leq 6$, we present all minimal almost simple spines. Recall that for $c > 0$ all of them are special. The spines are presented by regular neighborhoods of their singular graphs.

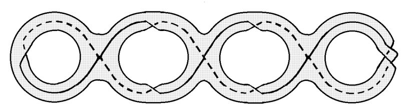


$^*0_1$
0_2
0_3
1_1
1_2
2_1
2_2
2_3
2_4

3_1



3_2



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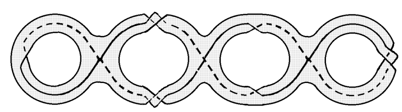

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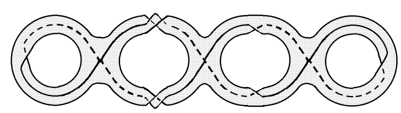

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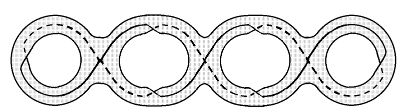
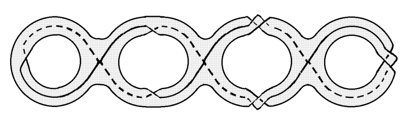


3_6



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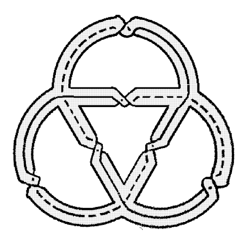

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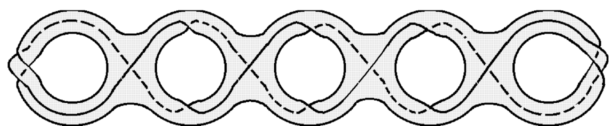

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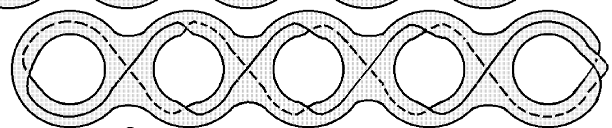

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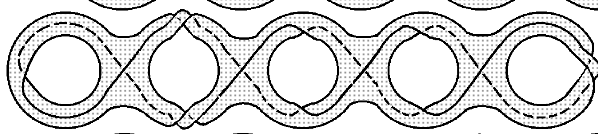

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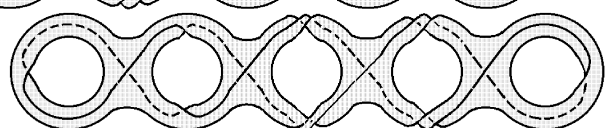

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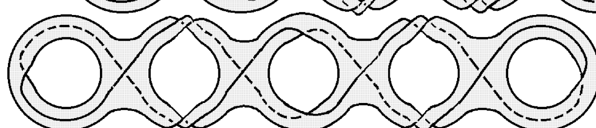

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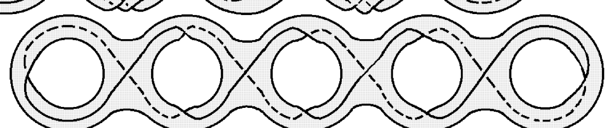

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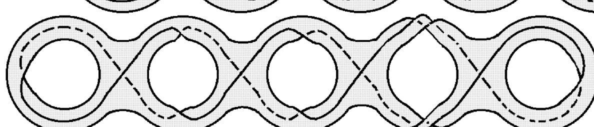

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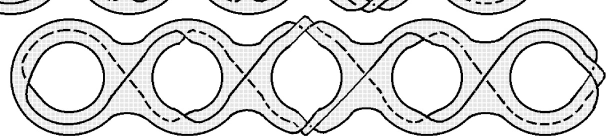
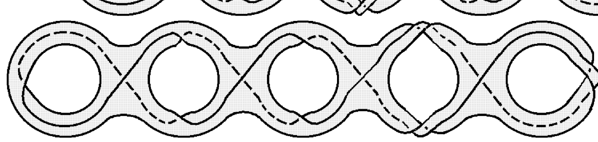
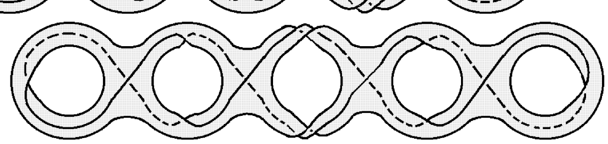
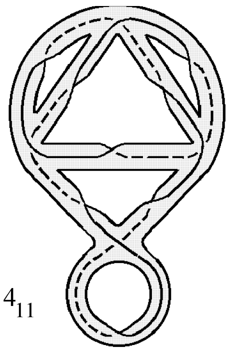


4_9

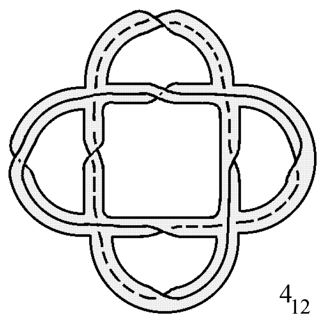


4_{10}

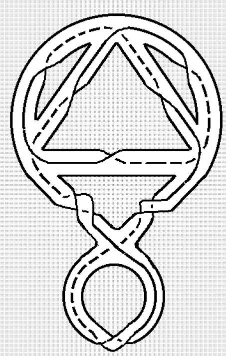


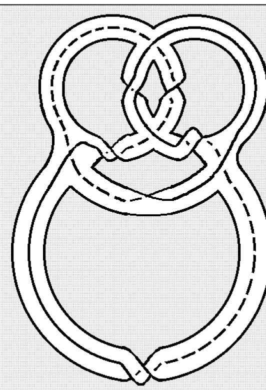


4_{11}

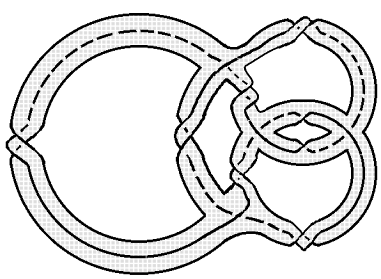


4_{12}





4_{13}



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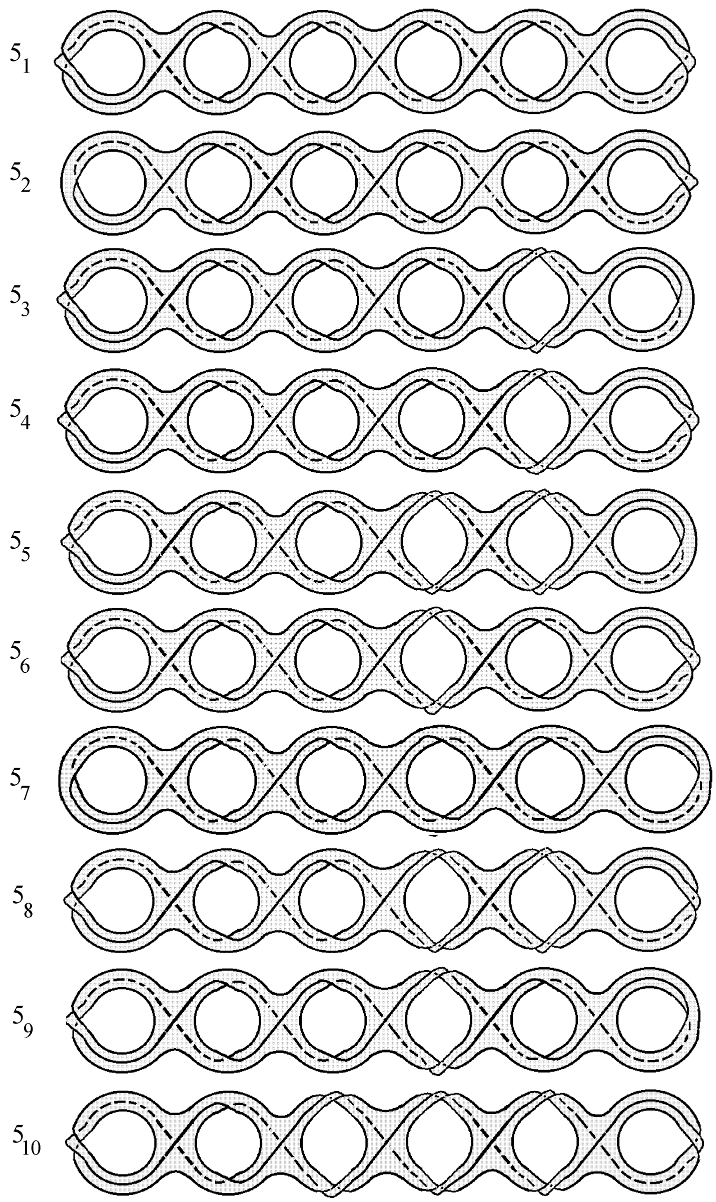
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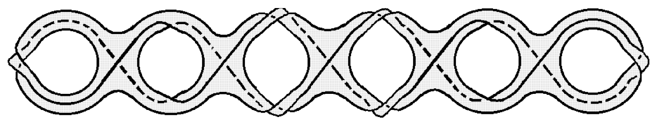

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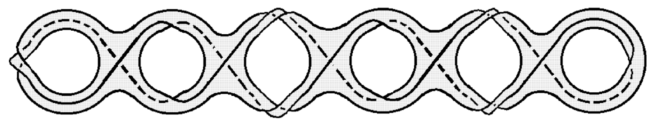

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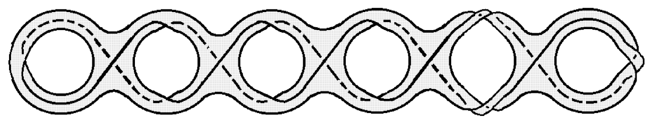

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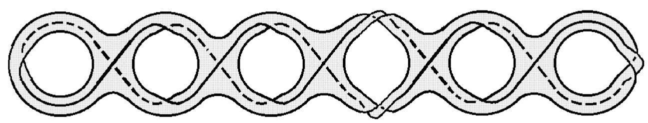

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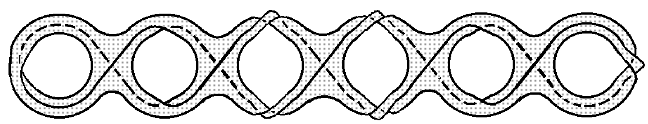

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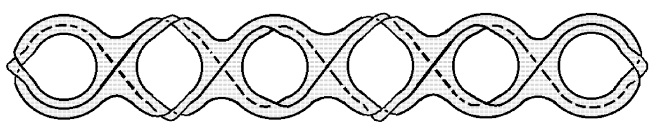

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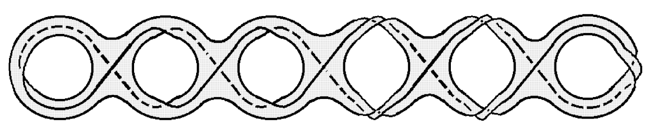

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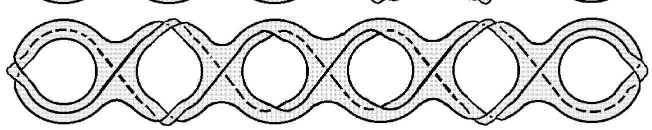

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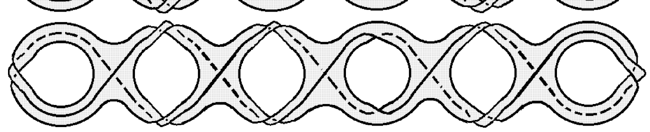

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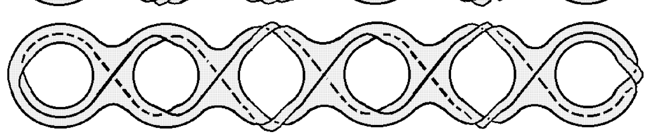

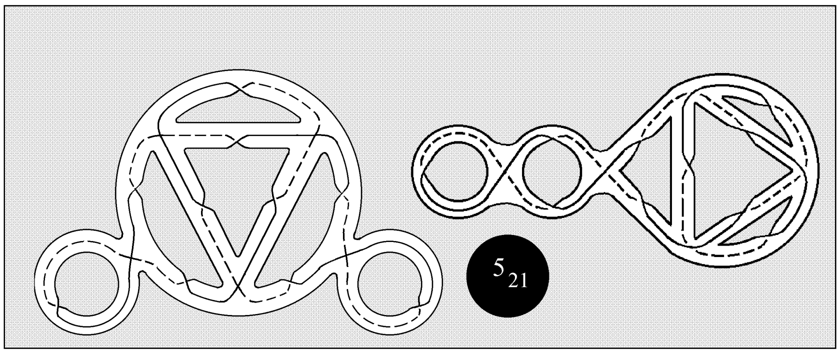
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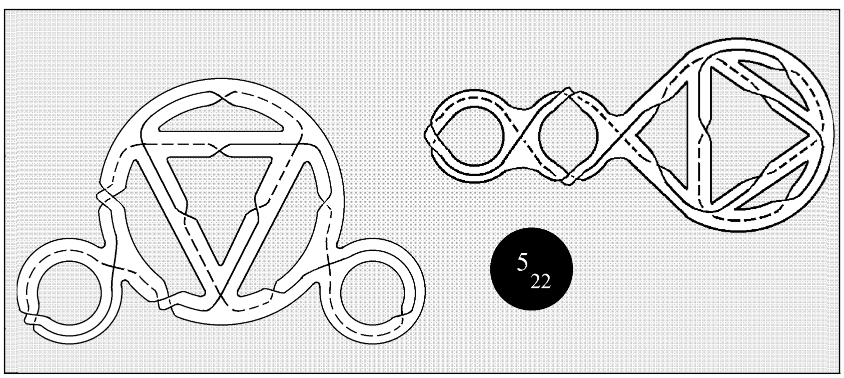
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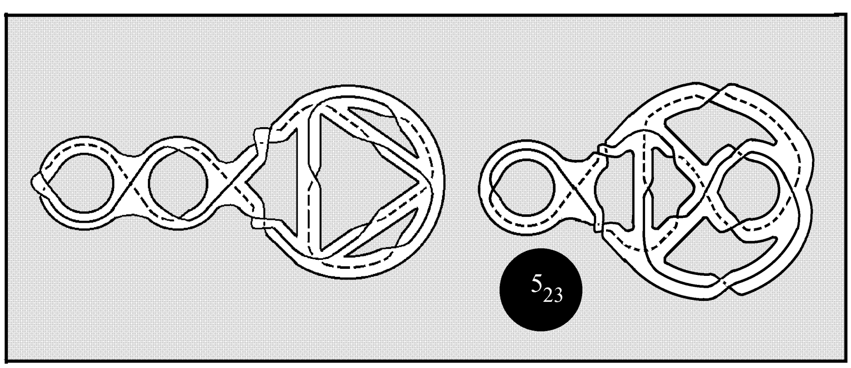
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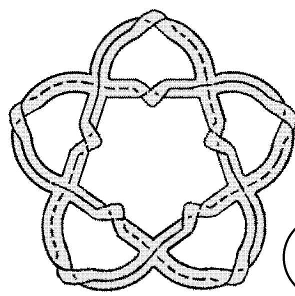

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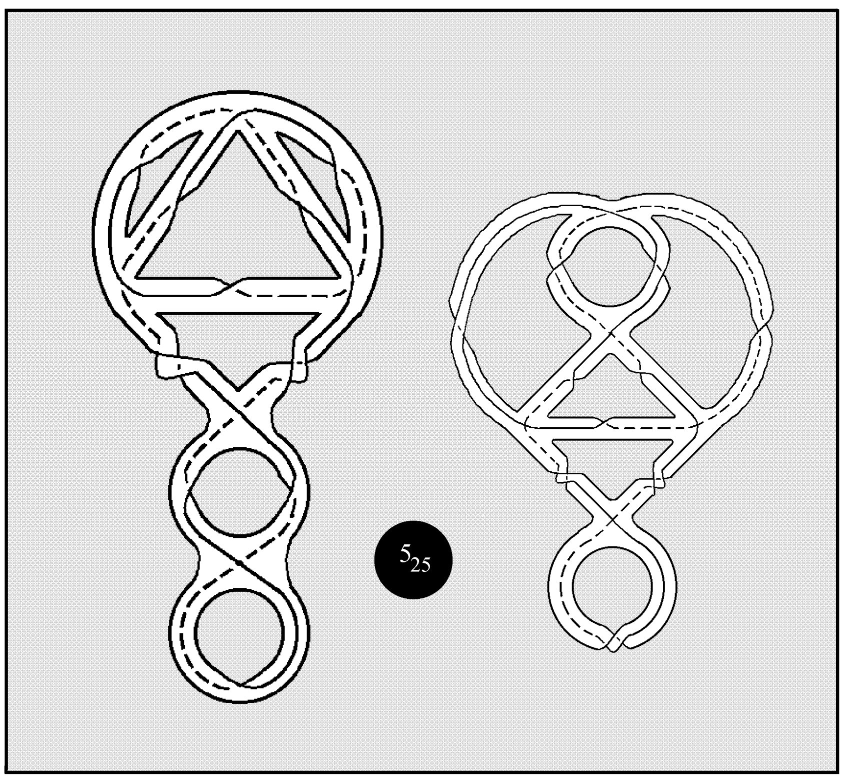

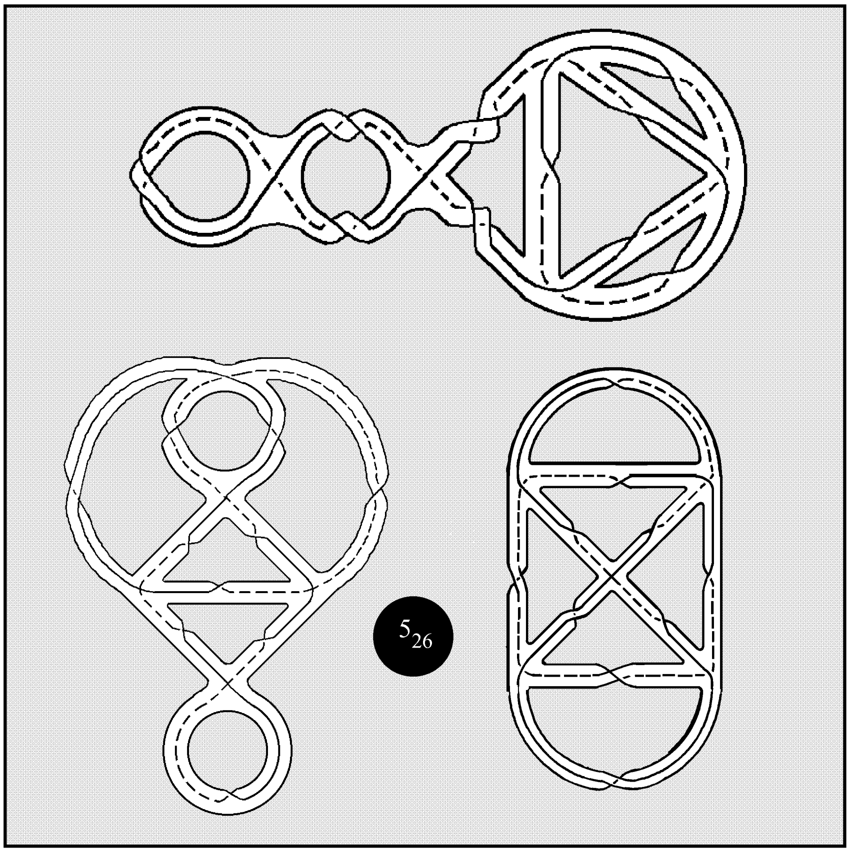
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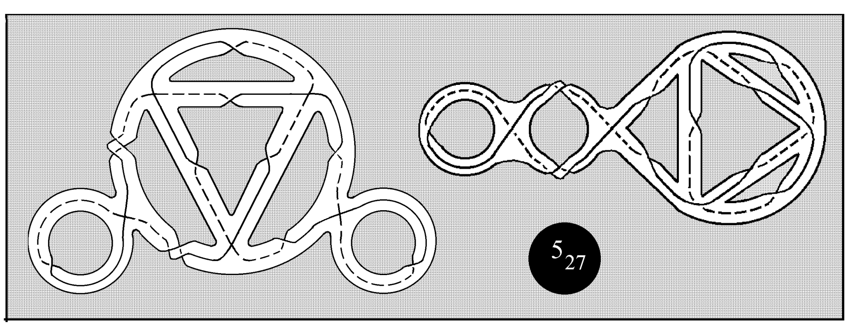
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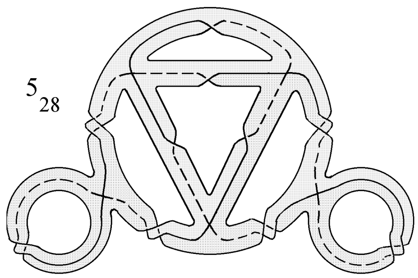

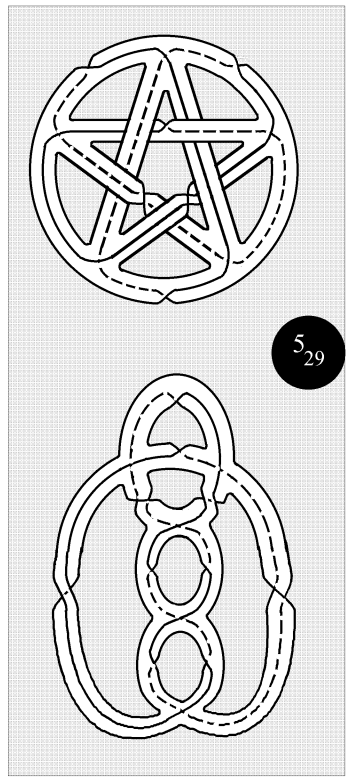

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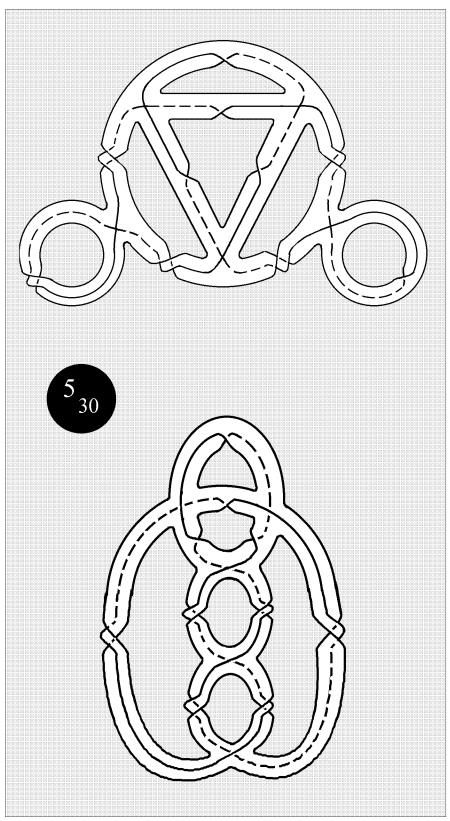

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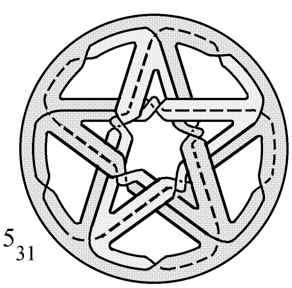

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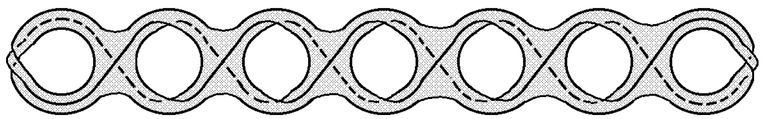

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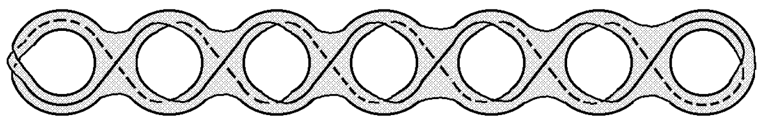

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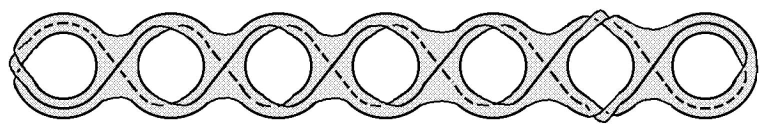

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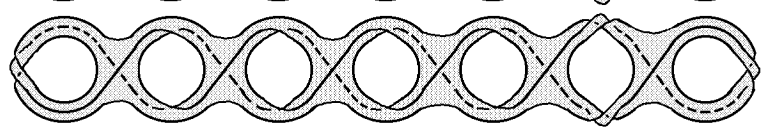

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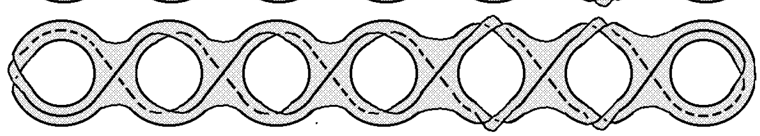

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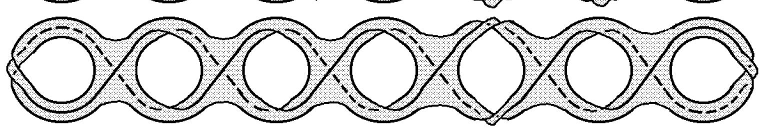

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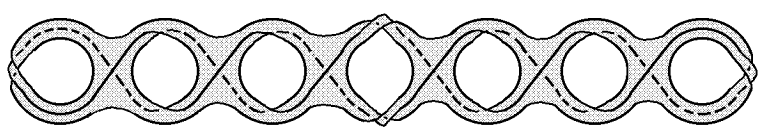

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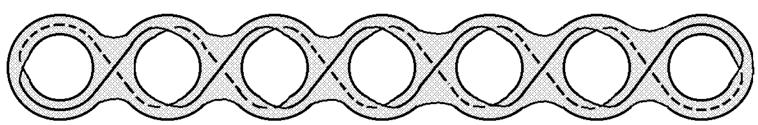

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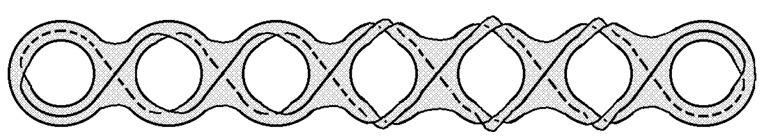

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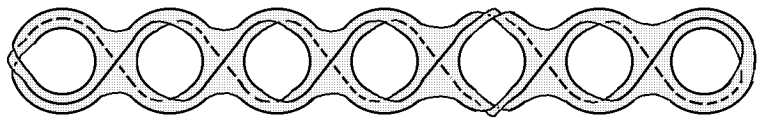

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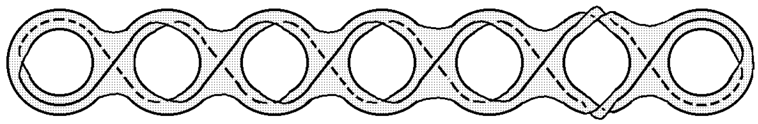

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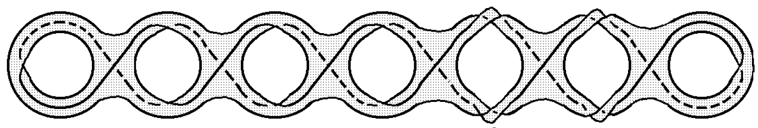

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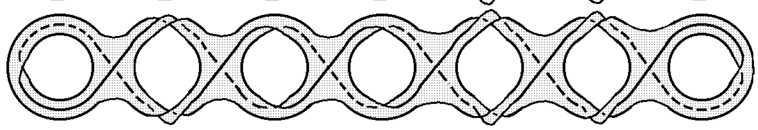

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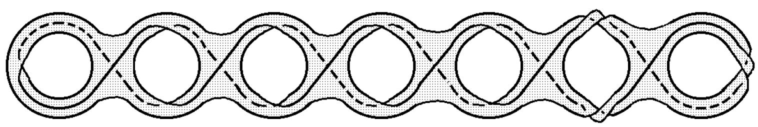

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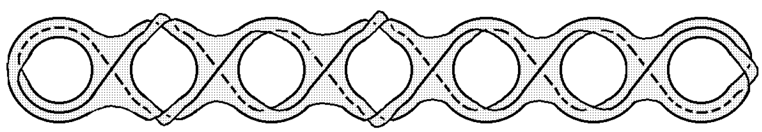

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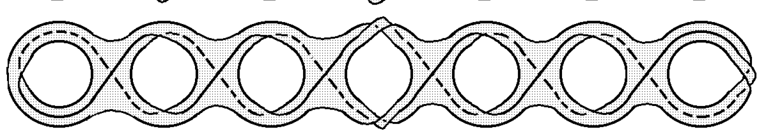

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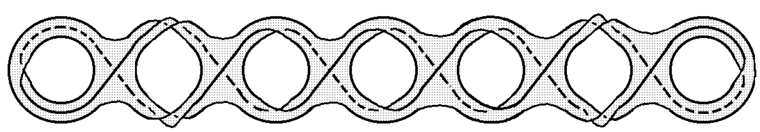

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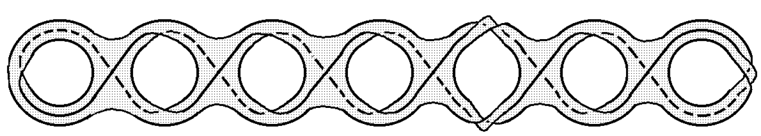

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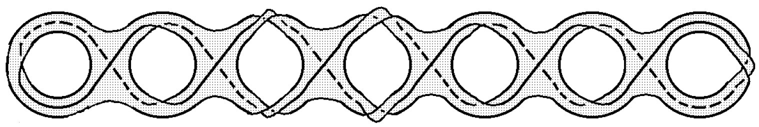

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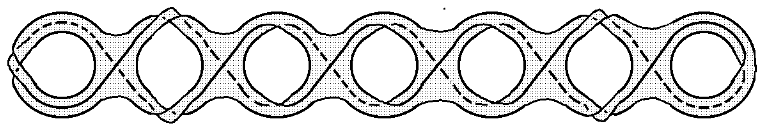

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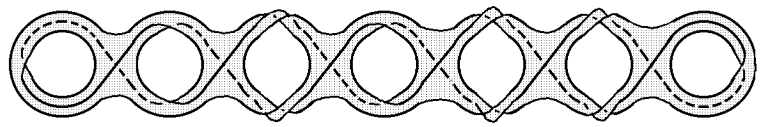

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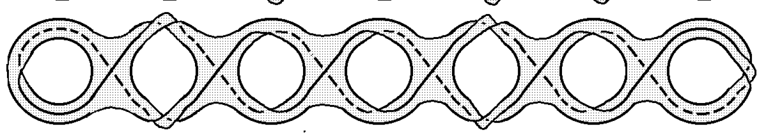

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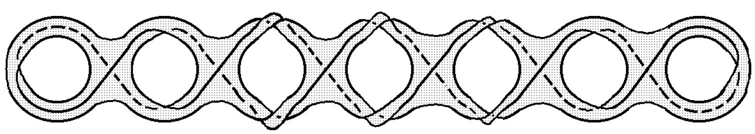

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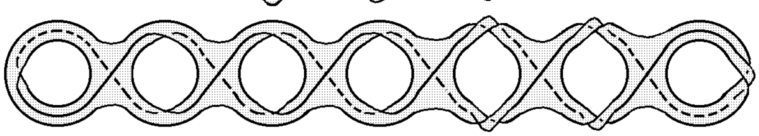

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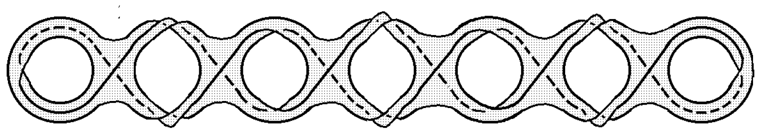

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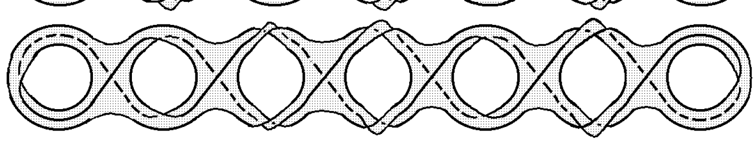

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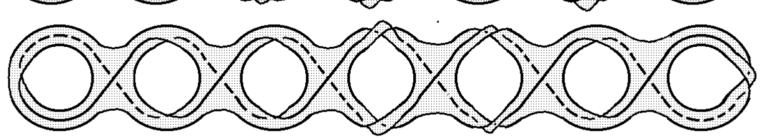

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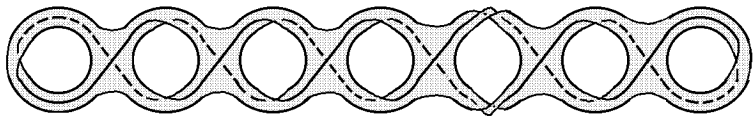

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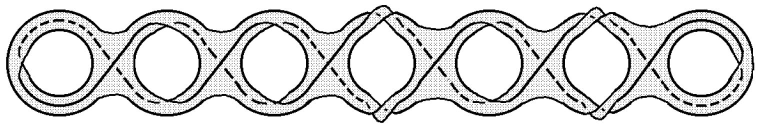

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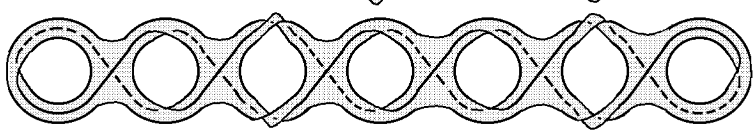

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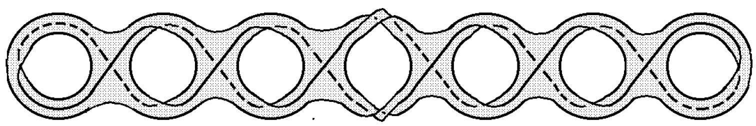

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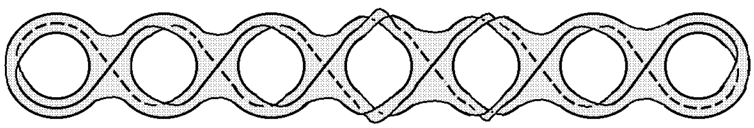

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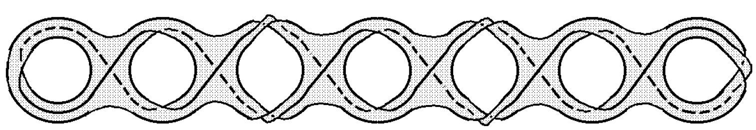

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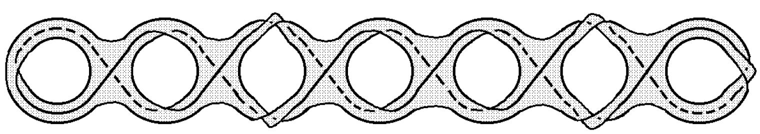

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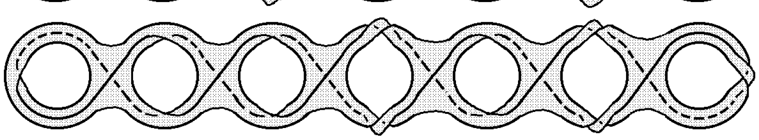

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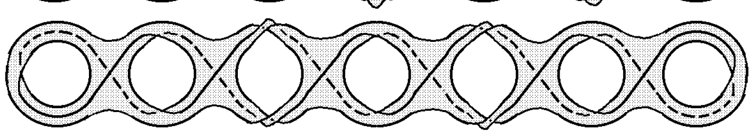

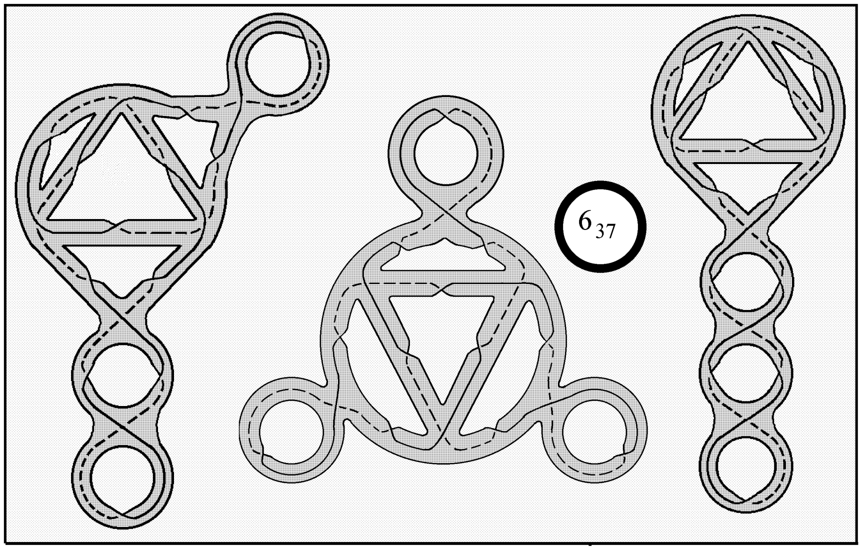

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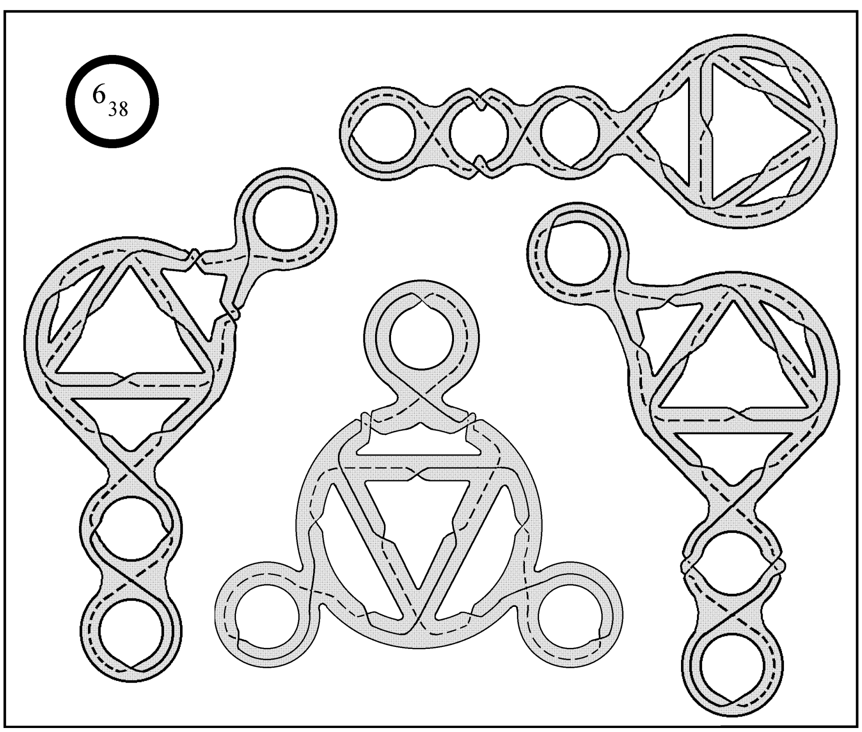

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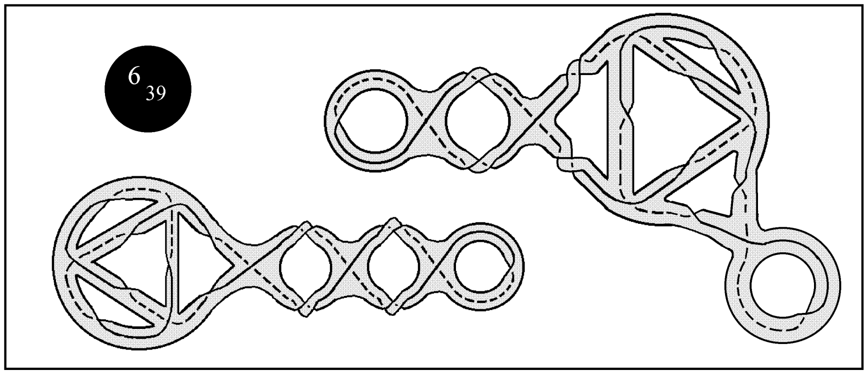
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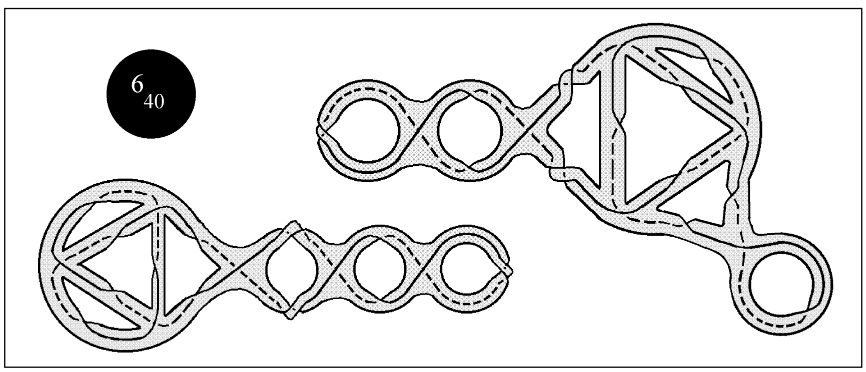
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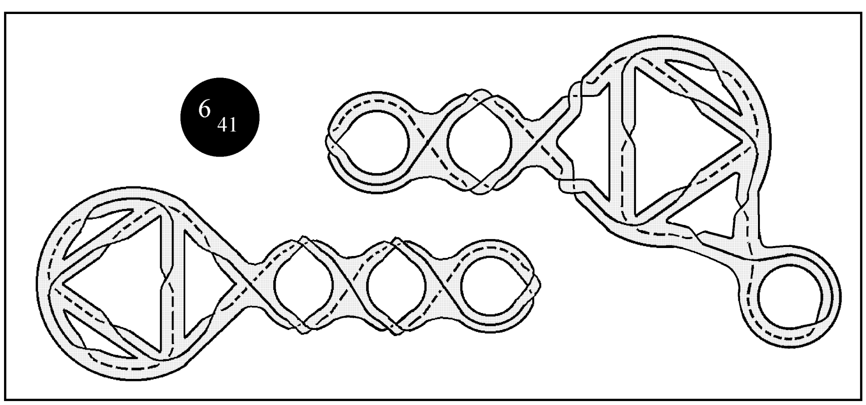
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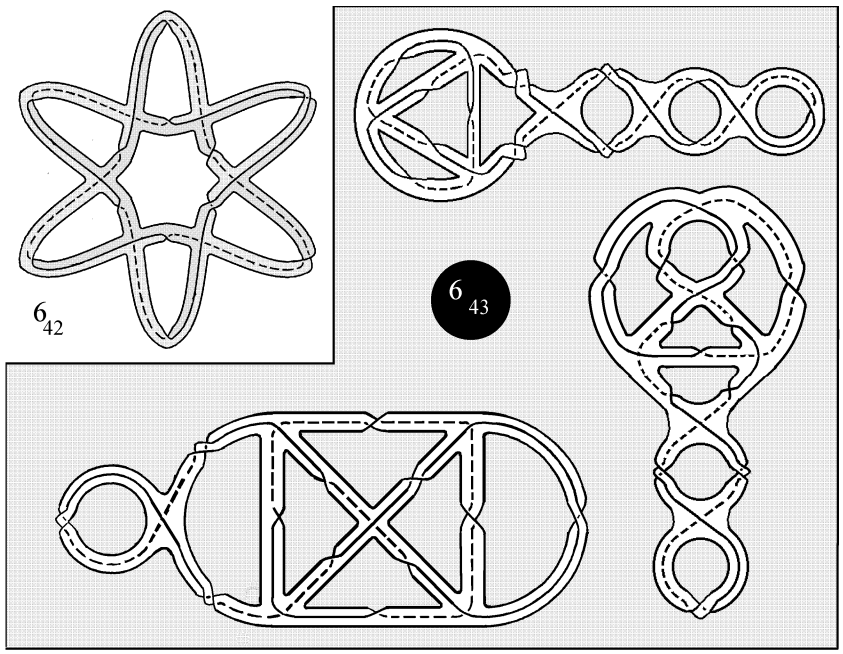
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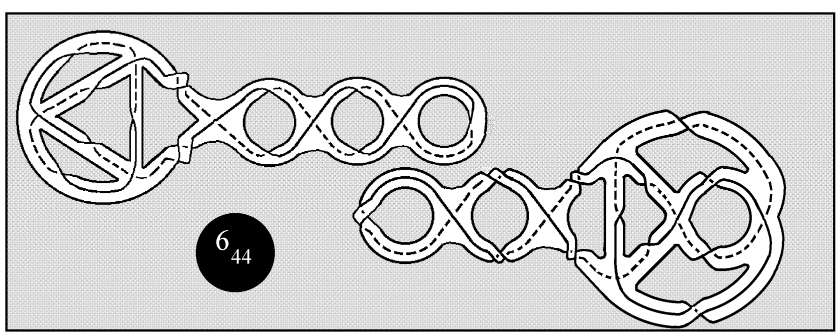
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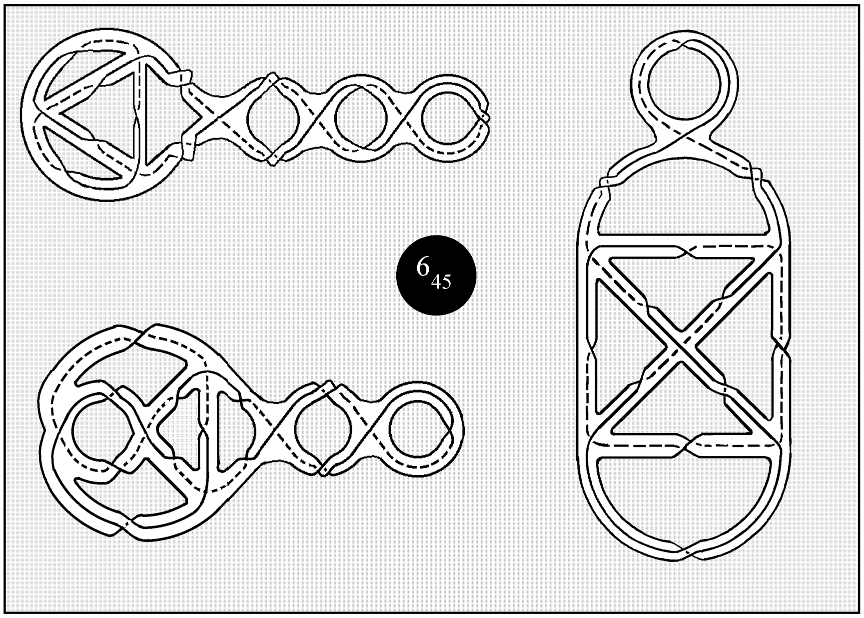

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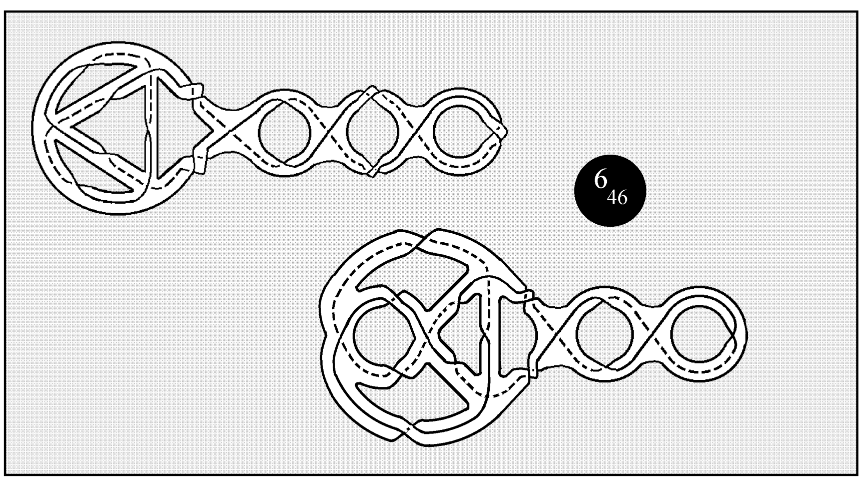

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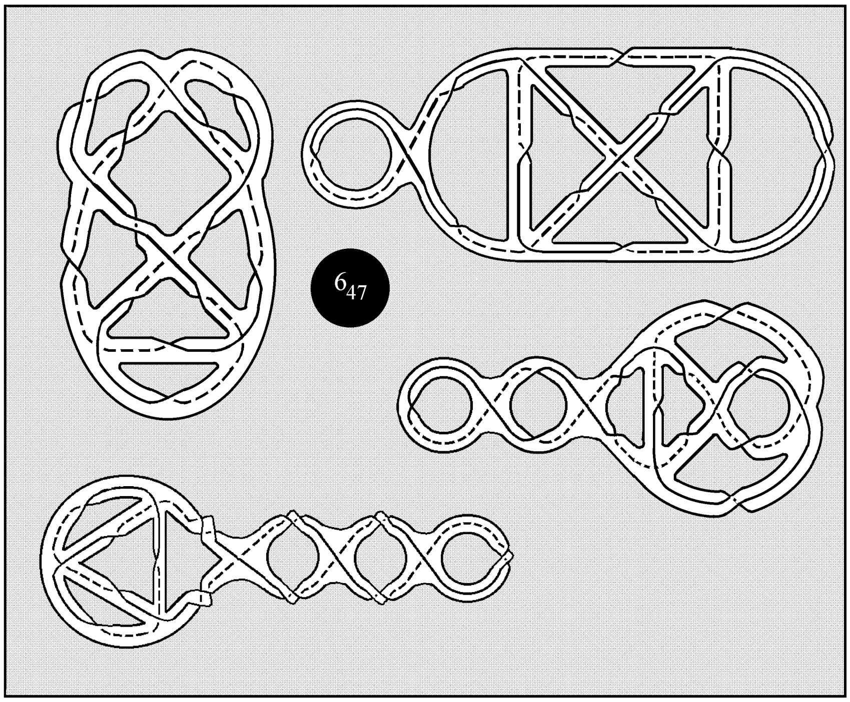
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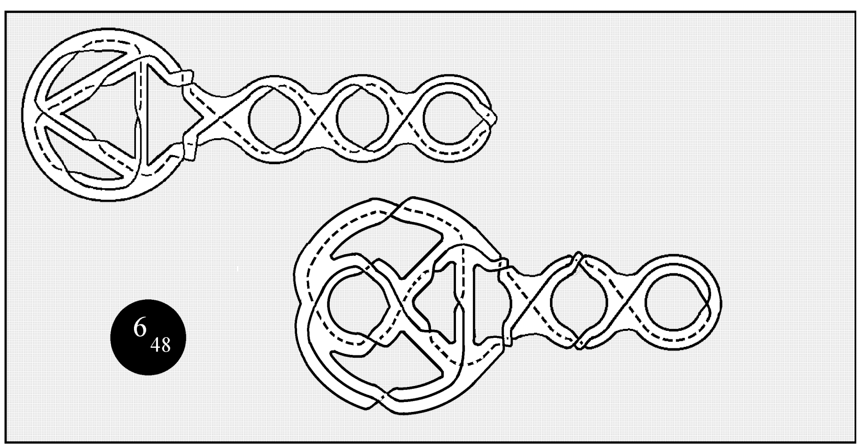
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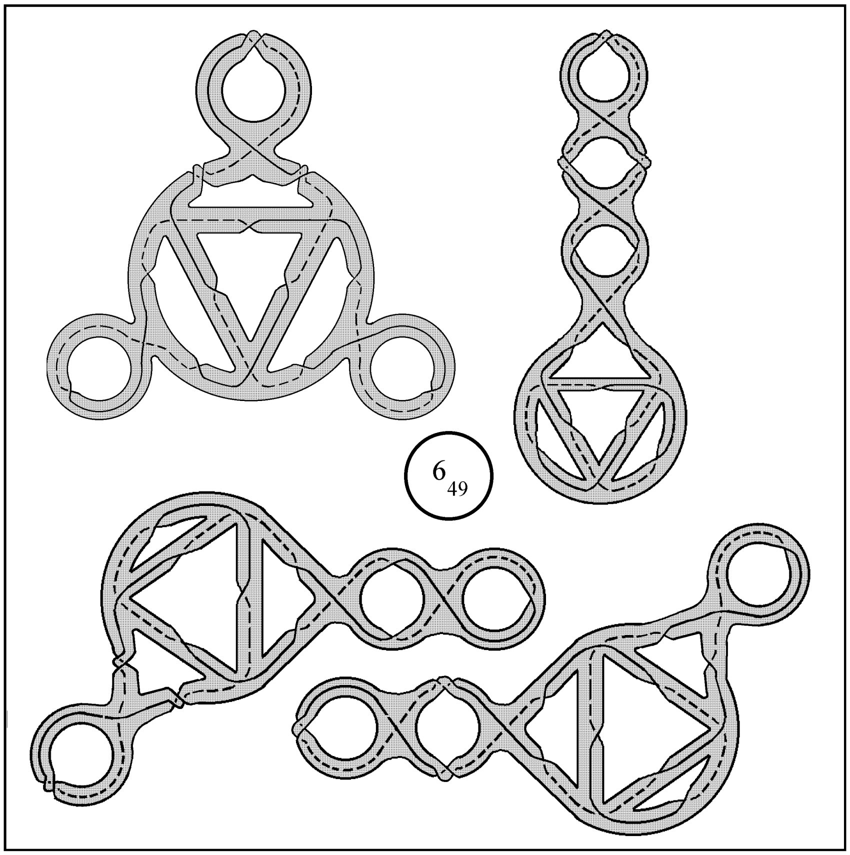
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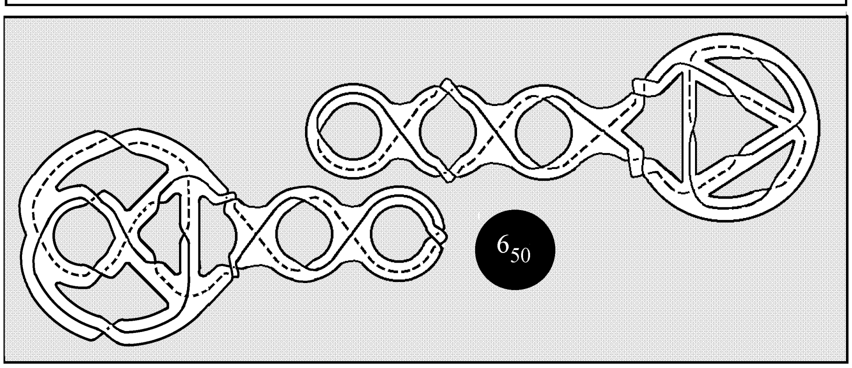
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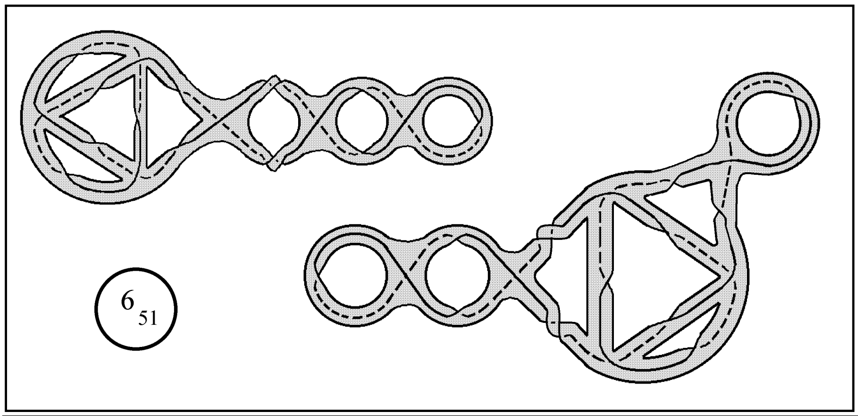

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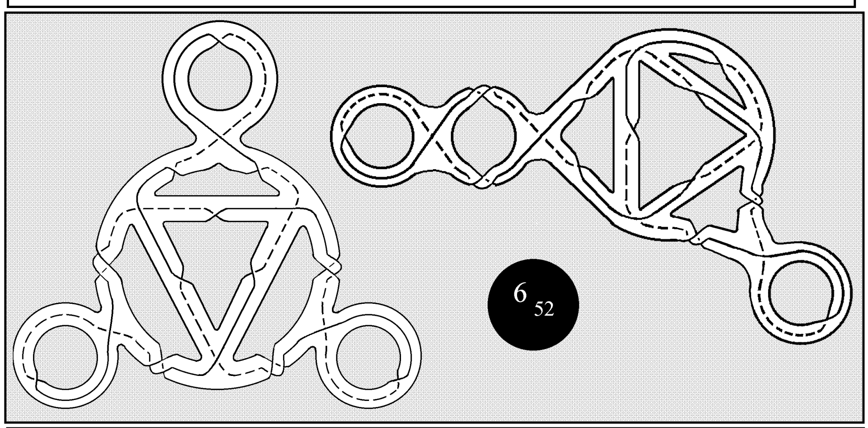

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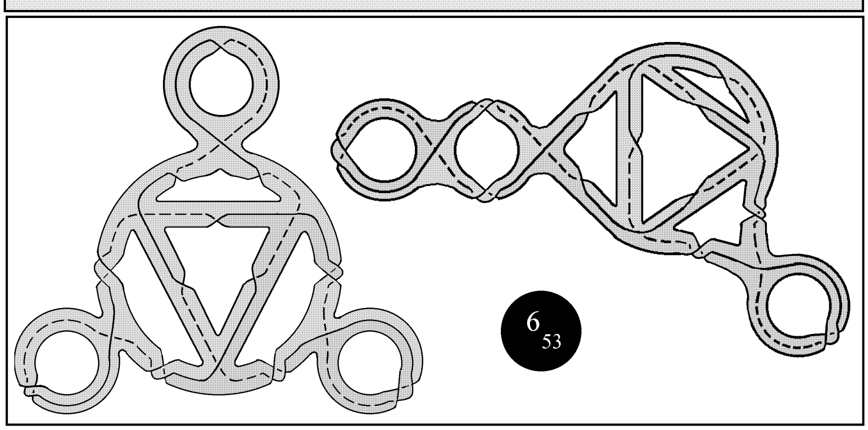

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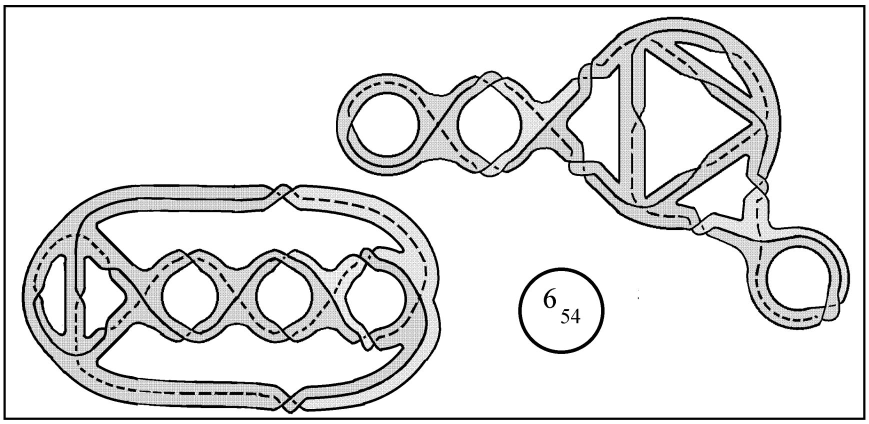
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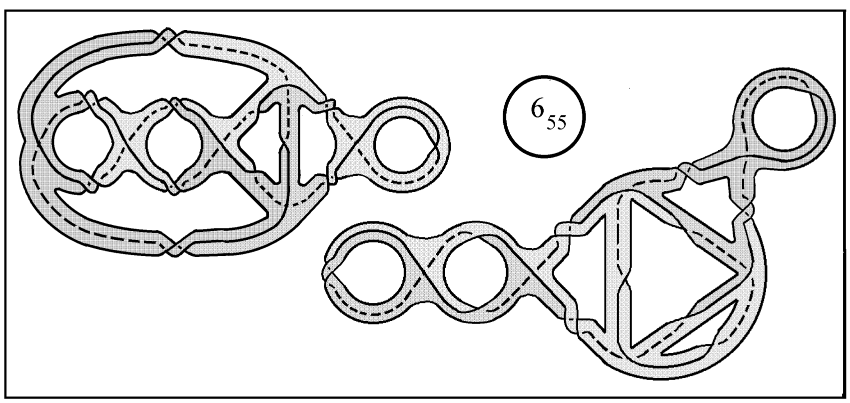
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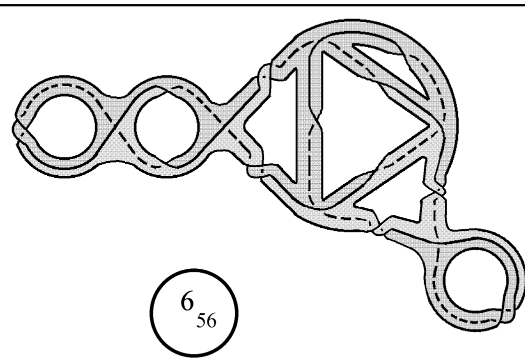
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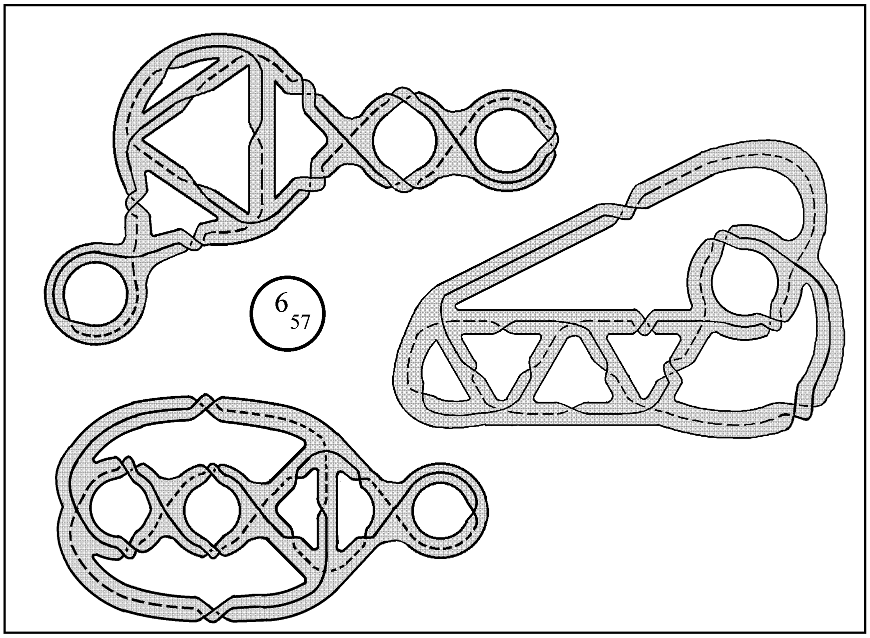
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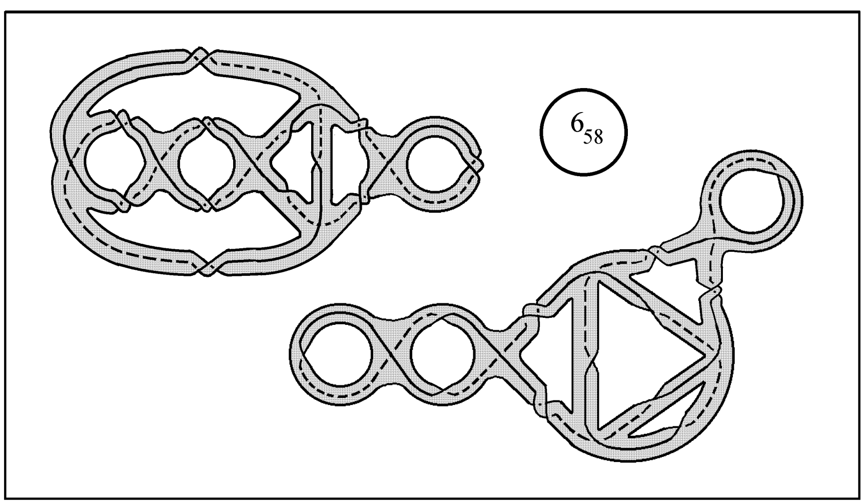
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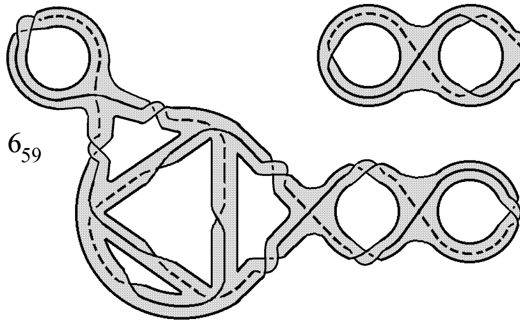

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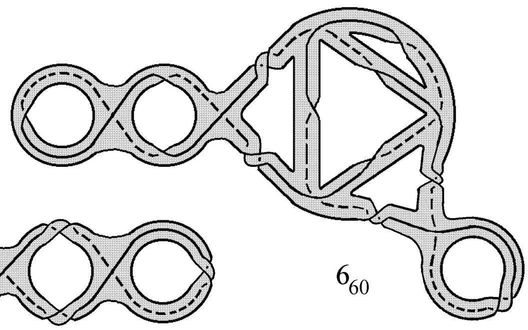

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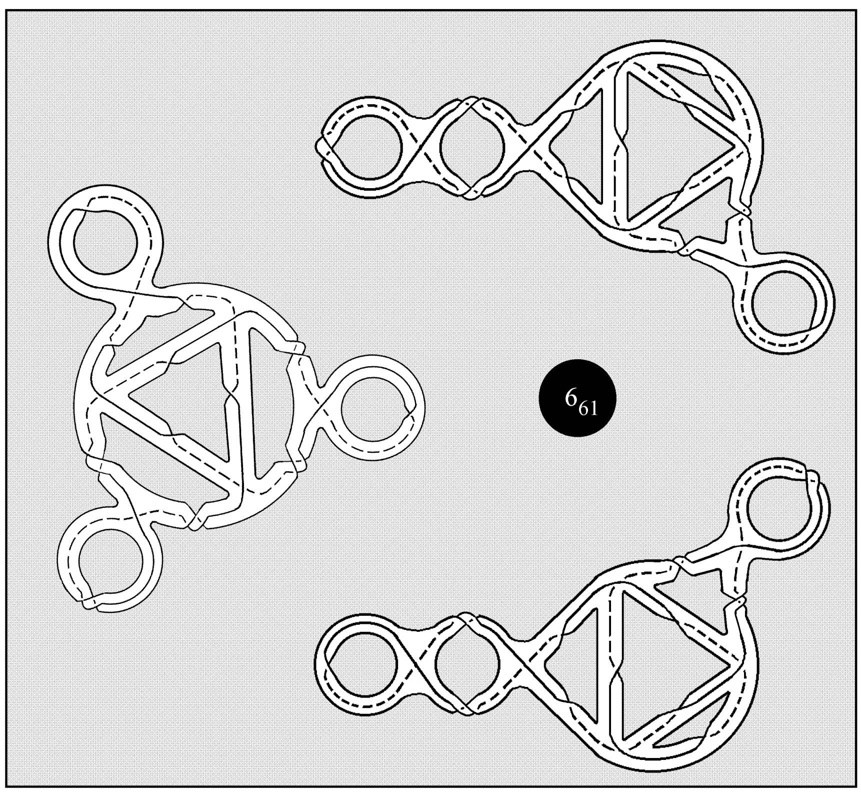

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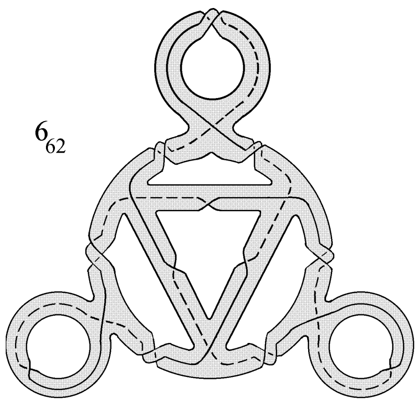

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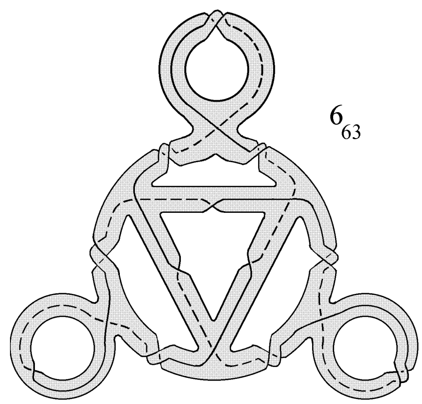

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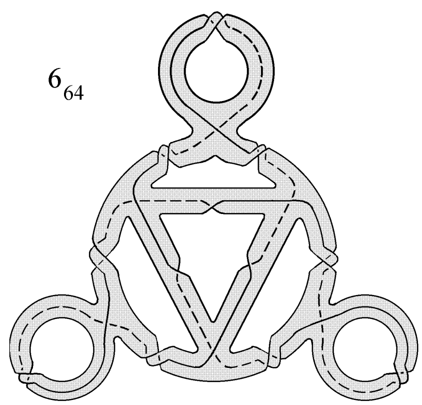

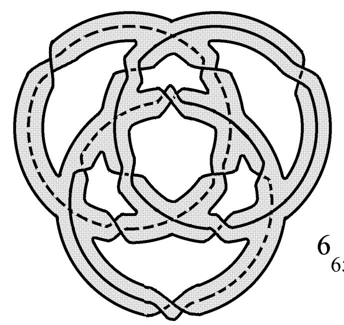

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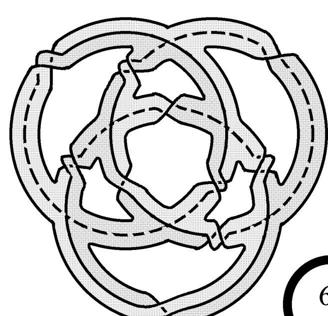

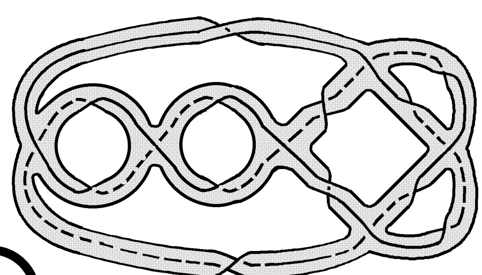

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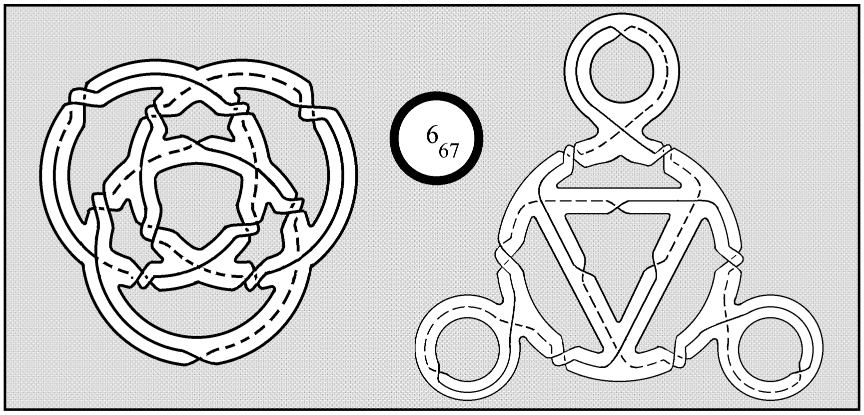

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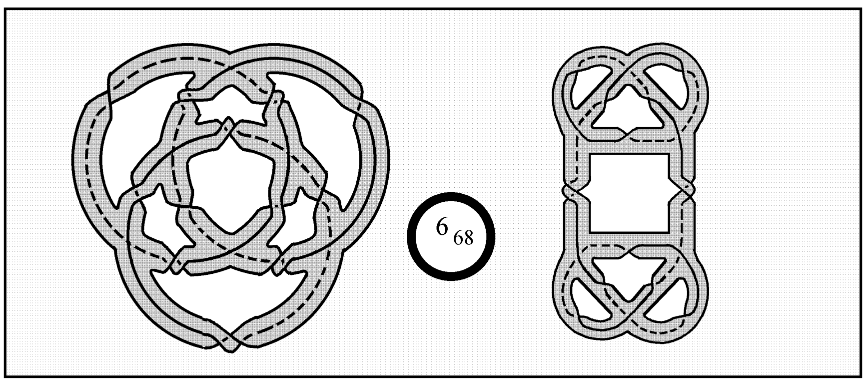

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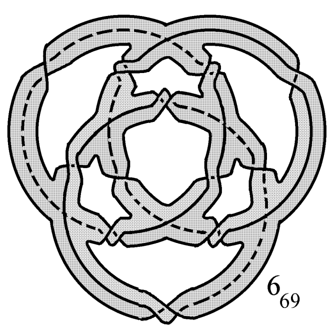

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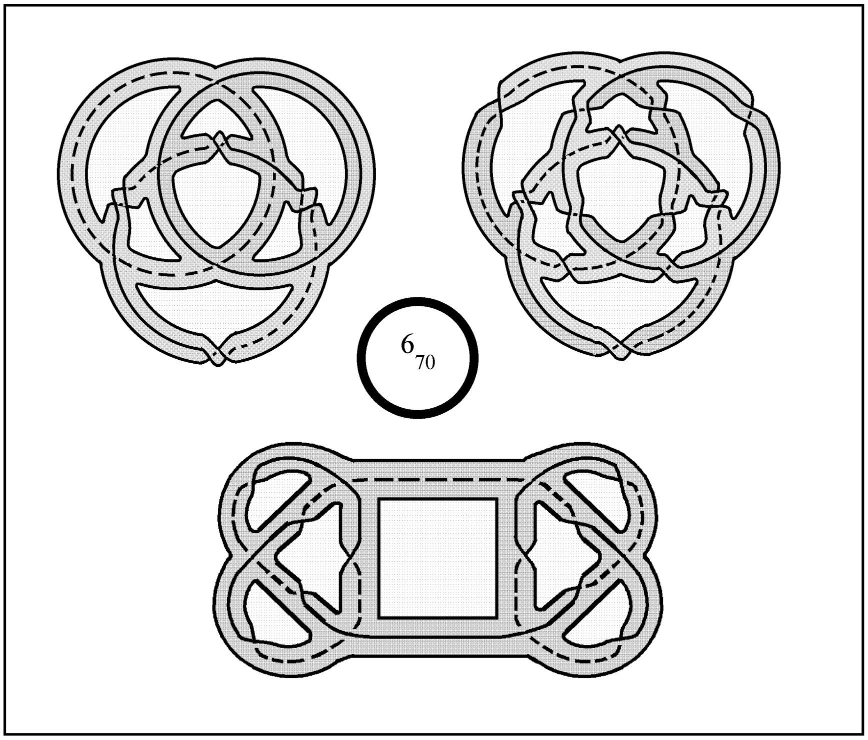

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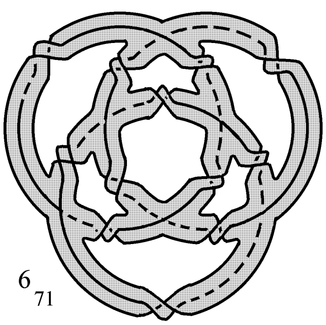

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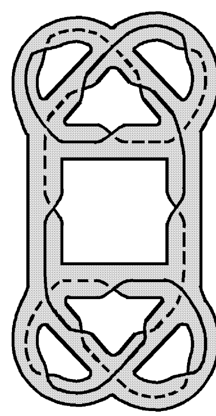

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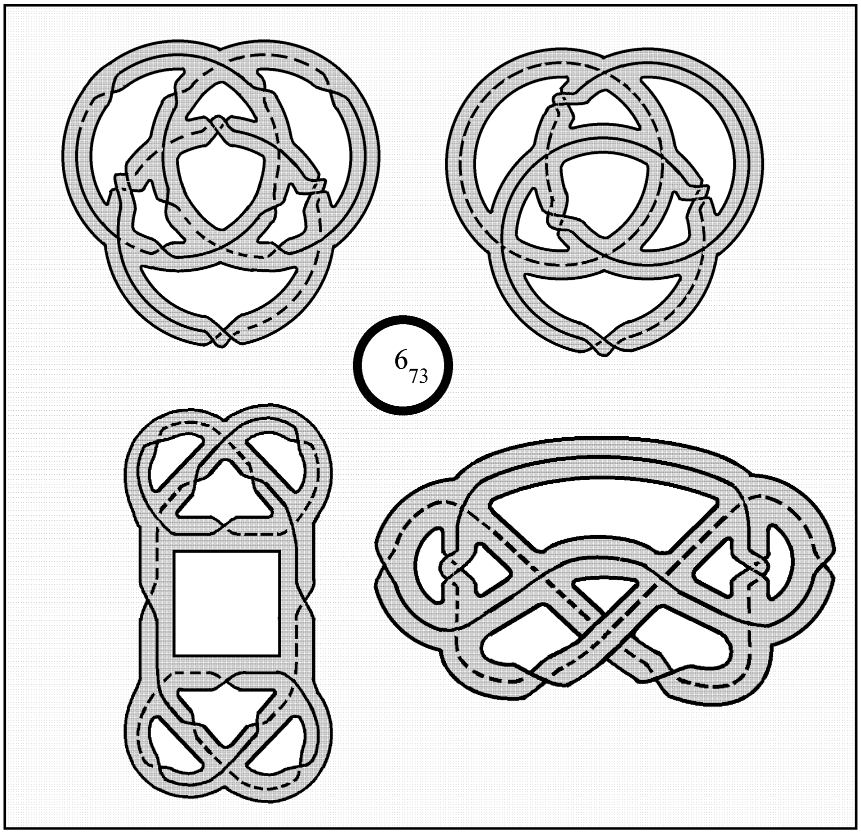
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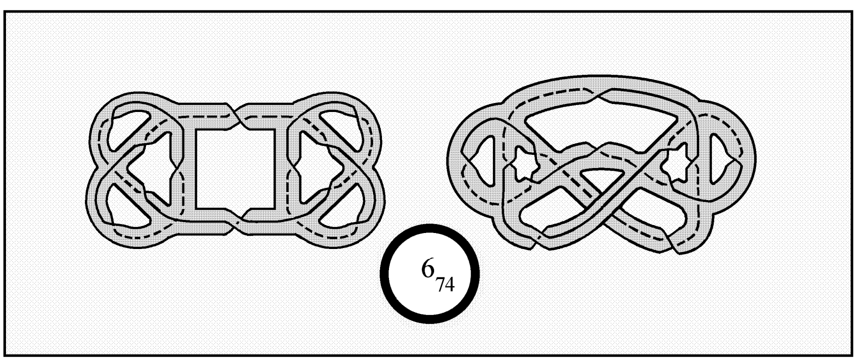
6_{74}

References

[Ca65] B. G. Casler, *An embedding theorem for connected 3-manifolds with boundary*, Proc. Amer. Math. Soc., **16** (1965), 559-566.

[GiLa83] D. Gillman, P. Laszlo, *A computer search for contractible manifolds*, Topology Appl. **16** (1983), 33-41.

[Ik84] H. Ikeda, *Contractible 3-manifolds admitting normal spines with maximal elements*, Kobe J. Math., **1** (1984), 57-66.

[IkYo85] H. Ikeda, I. Yoshinobu, *Invitation to DS-diagrams*, Kobe J., Math., **2** (1985), 169-186.

[Ma73] S. Matveev, *Special spines of piecewise-linear manifolds,* Math. Sb. **92** (1973), N. 2, 282-293 (Russian); English transl. in Math.USSR Sb. 21(1973).

[Ma74] S. Matveev, V. Savvateev , *Three-dimensional manifolds having simple special spines,* Colloq. Math. **32** (1974), 83-97 (Russian).

[Ma75] S. Matveev, *A method of specifying 3-manifolds,* Vestnik Moscov. Univ. **30** (1975), N. 3, 11-20 (Russian); English transl.in Moscow Univ.Math. Bull. **30** (1975), N 3, 7-14.

[Ma80] S. Matveev, *3-manifolds which are constructed on closed chains,* Appl. Math. (Chelyabinsk) 252(1980), 79-83 (Russian).

[Ma90] S. Matveev, *Complexity theory of three-dimensional manifolds,* Acta Appl. Math. **19** (1990), 101-130.

[Ma98] S. Matveev, *Computer Recognition of Tree-Manifolds,* Experimental Mathematics, **7** (1998), N.2, 153-161.

[Mi58] J. Milnor, *Groups which act on S^n without fixed points,* Amer. J., Math., **79** (1967), 623-630.

[Ov97] M. Ovchinnikov, *A table of closed orientable irreducible 3-manifolds of complexity 7*, Chelyabinsk University Preprint, 1997.

[Ro76] D. Rolfsen, *Knots and Links*, Math. Lect. Series, N. 7. Publish or Perish Press (1976), 439 pp.