THE CANONICAL GENUS OF A CLASSICAL AND VIRTUAL KNOT

ALEXANDER STOIMENOW, VLADIMIR TCHERNOV AND ALINA VDOVINA

Abstract. A diagram D of the knot defines the corresponding Gauss Diagram G_D. However, not all Gauss diagrams correspond to the ordinary knot diagrams. The Gauss diagrams that do not correspond to ordinary knot diagrams are called virtual.

From a Gauss diagram G we construct closed surfaces F_G and S_G in two different ways, and we show that if the Gauss diagram corresponds to an ordinary knot diagram D, then their genus is the genus of the canonical Seifert surface associated to D. Using these constructions we introduce the virtual canonical genus invariant of a knot and improve previous estimates on the number of alternating knots of given genus and given crossing number.

1. Virtual knot diagrams and virtual knot genus

1.1. *Diagrams of Knots and Gauss Diagrams.* Knots (smooth embeddings of S^1 into $\mathbb{R}^3$) are usually presented by knot diagrams that are generic immersions of S^1 into the $\mathbb{R}^2$-plane enhanced by information on overpasses and underpasses at the double points. A generic immersion of a circle into the plane is characterized by its *Gauss diagram* that consists of the circle with the preimages of each double point of the immersion connected by a chord. To incorporate the information on over- and underpasses the chords are oriented from the upper branch to the lower one. Furthermore each chord is equipped with the sign of the corresponding double point.

A Gauss diagram is usually considered up to orientation preserving homeomorphism of the underlying circle. The Gauss diagram defines (up to isotopy of S^2) a knot diagram on S^2, i.e. a knot diagram embedded into S^2 via an embedding $\mathbb{R}^2 \to S^2$. A knot diagram on $\mathbb{R}^2$ can be recovered from a knot diagram on S^2 modulo a finite ambiguity (that involves the choice of which connected component of the diagram contains the point at infinity), but the underlying knot itself is recovered uniquely up to isotopy.

Not every picture that looks like a Gauss diagram is really a Gauss diagram of some knot (diagram). The Gauss diagrams that come from knot diagrams will be called *realizable.* A *virtual knot diagram* is a generic immersion of a circle into the $\mathbb{R}^2$-plane with double points divided into real crossings and virtual crossings, with the real crossings enhanced by information on over- and underpasses (as for the classical knot diagrams). At a virtual crossing the crossing branches are not divided into an overpass and an underpass. The *Gauss diagram of a virtual knot* is constructed in the same way as for a classical knot, but all the virtual crossings are disregarded.

1.2. It is well-known that when a knot changes by a generic isotopy, its diagram undergoes a sequence of *Reidemeister moves* shown in Figure 1. A diagram of a virtual knot can undergo the same Reidemeister moves as well as the moves shown in Figure 2. The additional moves are the *virtual moves*. A virtual knot is a class of virtual knot diagrams consisting of diagrams that can be transformed to each other by sequence of Reidemeister and virtual moves. Such a sequence of moves is called a *virtual isotopy*. (Observe that the moves shown in Figure 3 are prohibited. If one allows these moves then the theory of virtual knots becomes trivial and every virtual knot can be unknotted.)

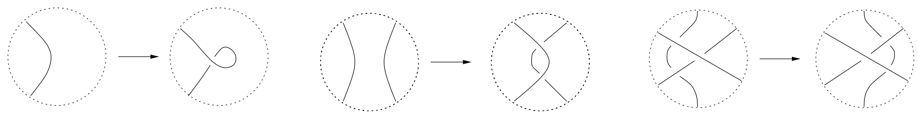


FIGURE 1

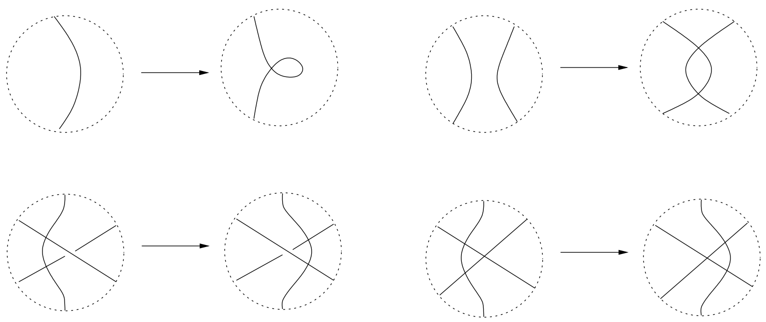


FIGURE 2

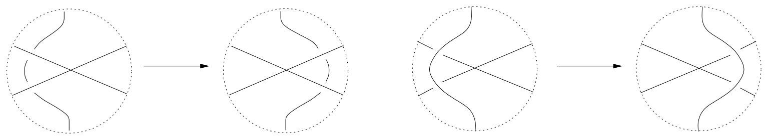


FIGURE 3

Theorem 1.3 (M. Goussarov, M. Polyak and O. Viro [3]). *A Gauss diagram defines a virtual knot up to virtual moves.*

1.4. *Virtual canonical genus.* A Seifert surface F_K of a knot K is an oriented surface with one boundary component embedded into $\mathbb{R}^3$ so that the boundary component of F_K equipped with the natural orientation is the knot K. The *genus* $g(K)$ *of a knot* K is the classical knot invariant that is equal to the minimum of genus $g(F_K)$ of Seifert surfaces for K.

For each planar diagram $P(K)$ of a knot K equipped with the information about over and underpasses there is *a canonical Seifert surface* $F_{P(K)}$ associated with this diagram (see for example [8], p. 120). The minimal genus of all canonical Seifert surfaces for K is called the *canonical genus* $g_c(K)$ *for* K. It was shown by Moriah [6] that the difference between $g(K)$ and $g_c(K)$ can be arbitrarily large. Namely for any $n \in \mathbb{N}$ there exist a knot K such that $g_c(K) - g(K) > n$. (Results of similar spirit were later obtained by Kawauchi [4] and one of the authors [10]. See also [5].)

For a chord diagram G put F'_G to be an orientable surface with boundary obtained as follows. Take a thin annulus, and for each chord of G take a thin strip and glue its ends to the annulus in the places that correspond to the end points of the chord so that the obtained surface is orientable.

Put F_G to be the closed orientable surface obtained by gluing all the boundary components of F'_G with the 2-disks, and put *the genus* $g(G)$ *of the chord diagram* to be the genus $g(F_G)$ of F_G.

Theorem 1.5. *Let $P(K)$ be a planar diagram of a knot K, let $G(P(K))$ be the Gauss diagram of $P(K)$. Then $g(F_G) = g(F_{P(K)})$.*

1.6. *Proof of Theorem 1.5.* The Euler characteristic $\chi(F_{P_K})$ of F_{P_K} is the number of Seifert circles minus the number of crossings of P_K.

The Euler characteristic $\chi(F_G)$ of F_G is the number of disks that were glued to the boundary components of F'_G minus the number of chords in G. Or equivalently $\chi(F_G)$ is the number of boundary components of F'_G minus the number of crossings of $P(K)$.

Geometrical considerations show that there is a natural bijection between the boundary components of F'_G different from the outer boundary of the annulus and the Seifert circles appearing in the construction of the canonical Seifert surfaces for $P(K)$.

Thus $\chi(F_G) = \chi(F_{P_K}) - 1$ (F_{P_K} has one boundary component) and hence $g(F_G) = g(F_{P_K})$. $\square$

Definition 1.7. For a virtual knot K put *the virtual canonical genus* $g_{vc}(K)$ *of* K to be the minimum over all the Gauss diagrams G that realize K of $g(F_G)$

The following corollary follows immediately from Theorem 1.5.

Corollary 1.8. *If a virtual knot K admits a planar diagram without virtual crossings that realizes the isotopy class of an ordinary knot $\tilde{K}$, then $g_c(\tilde{K}) \geq g_{vc}(K)$.*

Theorem 1.9. *F_G hs minimal genus among all surfaces, on which G can be embedded.*

1.10. *Proof of Theorem 1.9.* If M is a surface of minimal genus, on which G can be embedded, then all components of $M \setminus G$ must be disks. But then, M can be obtained by gluing disks into a band-thickening of G, and hence $\chi(M) = \chi(F_G)$.

We remark the following combinatorial description of $g(F_G)$. If we denote by $s_n \in S_n$ the permutation of n elements sending $i \in \{1, \dots, n\}$ to $i \bmod n+1$, then each Gauss diagram G of n chords defines a fix-point free involution $s_G \in S_n$ up to conjugating with a power of s_n. Then the number of discs to glue in to obtain F_G is equal to the number of cycles $c(s_n \circ s_G)$ of $s_n \circ s_G$, and hence

$$g(F_G) = \frac{n - c(s_n \circ s_G) + 1}{2}.$$

There is yet another way of associating surfaces (and genera) to Gauss diagrams, namely the following:

Definition 1.11. We indicate the enspoints of every chord of G by the same letter x, but with the opposite powers 1 and -1; different letters correspond to different chords. So, on the circle we have a cyclic word W in some group alphabet. Let S_G be the surface defined by the word W, i.e. the surface obtained by identifying of corresponded sides respecting orientation from a disk, on the boundary of which W is written.

Again this genus coincides with the previous one(s).

Theorem 1.12. $\chi(S_G) = \chi(F_G)$. *In particular, if G is realizable as $G(P(K))$ for a diagram $P(K)$ of a knot K, then $g(S_G) = g(F_G) = g(F_{P(K)})$.*

1.13. *Proof of Theorem 1.12.* The boundary of the disk from definition 1.11 becomes a graph on the surface S_G. The number of edges is the same as the number of chords of G and the number of vertices is the same with the number of Seifert cycles. So, we finish the proof by applying Euler's formula.

2. On the number of alternating knots of given genus

In [9] the canonical genus was used to give estimates on the number of alternating knots of given genus and given crossing number.

Definition 2.1. A chord a in a Gauss diagram G is called isolated, if its endpoints occur as

$$\dots aa^{-1}\dots$$

when denoting the endpoints of the chords of the diagram in cyclic order by a letter and its inverse for each chord, or more generally there is no other chord b such that the cyclic notation is of the type

$$\dots a\dots b^{-1}\dots a^{-1}\dots b\dots$$

If G is realizable by a knot diagram D, then the crossing in D corresponding to a is called nugatory. Two chords a and b in a Gauss diagram are called parallel, if their endpoints occur as

$$\dots ab^{-1}\dots ba^{-1}\dots$$

in this cyclic order notation. Three chords a, b and c are called parallel, if a and b, and b and c are parallel, i.e. we have

$$\dots abc\dots c^{-1}b^{-1}a^{-1}\dots$$

The basic notion introduced in [9] was a weaker version of deleting parallel chords, which does not spoil realizability. For its definition, which we rephrase here slightly in more Gauss diagrammatic terms, we need the flype move of [7].

Definition 2.2. A knot diagram is called $\bar{t}_2$ irreducible, if after any sequence of flypes its Gauss diagram has not triple of parallel chords.

Theorem 2.3. *(*[9, theorem 3.1]*) The set $\mathcal{D}_g$ of $\bar{t}_2$ irreducible alternating diagrams with no nugatory crossings and genus g is finite. We call elements of $\mathcal{D}_g$ 'generating diagrams' of genus g.*

This result allows to make the following definition.

Definition 2.4. Let for a knot diagram D, $c(D)$ be its crossing number, and $d(D)$ be the minimal chord number of a Gauss diagram obtained by deleting a chord in a pair of parallel chords from the Gauss diagrams of all flyped versions of D. (This is the same as the number of $\sim$-equivalence classes of crossings under the equivalence relation $a \sim b$ iff a and b can be made to be parallel chords in the Gauss diagram after some sequence of flypes on D.)

Set

$$c_g := \max\{\, c(D) \,:\, D \in \mathcal{D}_g \,\},$$

$$d_{g,e} := \max\{\, d(D) \,:\, D \in \mathcal{D}_g,\, c(D) \text{ even} \,\},$$

$$d_{g,o} := \max\{\, d(D) \,:\, D \in \mathcal{D}_g,\, c(D) \text{ odd} \,\},$$

and

$$d_g := \max(d_{g,e}, d_{g,o})\,.$$

Note, that clearly

$$d_g \le c_g \le 2d_g\,.$$

A consequence of the result of [7] is an asymptotical bound for the number $a_{n,g}$ of alternating knots of n crossings and genus g. (Here for two sequences of positive integers $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ we write $a_n = O_n(b_n)$ iff $\limsup_{n\to\infty} \frac{a_n}{b_n} < \infty$, and $a_n \asymp_n b_n$ if $a_n = O_n(b_n)$ and $b_n = O_n(a_n)$.)

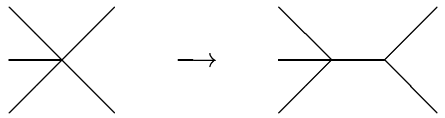
Corollary 2.5. *$a_{n,g} = O_n(n^{d_g-1})$. More exactly, $a_{n,g} \asymp_n n^{d_{g,e}-1}$ for n even and $a_{n,g} \asymp_n n^{d_{g,o}-1}$ for n odd.*

This motivates the quest for some good upper bound on d_g. The bound on d_g given in [9] was obtained by estiamting c_g, and this estiamte was exponential in g. Here we improve the bound on d_g (and hence c_g) to the conjecturally best possible.

Theorem 2.6. *A Gauss diagram with no two parallel chords of genus g has at most $6g-3$ chords.*

For the proof we need a small graph-theoretical definition.

Definition 2.7. An anticontraction is an operation on a graph of the following type



It makes a vertex of valence $v \ge 4$ into two vertices $p_{1,2}$ with valences $v_{1,2} < v$ with $v_1 + v_2 = v + 2$, such that $p_{1,2}$ are connected by an edge.

2.8. *Proof of Theorem 2.6.* Consider the Gauss diagram of a generating diagram of genus g and remove a chord in each pair of parallel chords, obtaining a (not necessarily realizable) Gauss diagram G of c_g chords. In this Gauss diagram there is no pair of parallel chords. We can also assume that in this Gauss diagram there is no isolated chord.

If you consider a $2c_g$-gon with chords indicating how to glue its edges according to definition 1.11, you obtain by Theorem 1.12 a surface S_G of genus g with a graph

$\hat{G}$ on it. As G has no pair of parallel chords and no isolated chord, the graph $\hat{G}$ has no vertices of valence 1 or 2. Moreover, the number of edges of $\hat{G}$ is the number of chords of G.

If now $\hat{G}$ has a vertex of valence $n \geq 4$, then it can be made into $n-2$ vertices of valence 3 by anticontractions, each one of which inserts one edge. Repeat this procedure until you obtain a trivalent graph $\tilde{G}$ on S_G. (Neither $\tilde{G}$ needs to be unique, nor coming from a realizable Gauss diagram, but neither of this is relevant.)

It suffices to see that $\tilde{G}$ has $6g-3$ edges. As each edge has valence 3, for the number of edges e and the number of vertices v in $\tilde{G}$ we must have $e = 3k$ and $v = 2k = 2e/3$ for some $k \in \mathbb{N}$. The Euler characteristic of the surface S_G is then $v - e + 1$, because of the one face of the triangulation given by $\tilde{G}$, and on the other hand $2 - 2g$, so that

$$v - e + 1 = -\frac{e}{3} + 1 \,=\, 2 - 2g\,,$$

whence $e = 6g - 3$, as desired.

The following is a straightforward consequence from the work of [9] described above (note, that the bound $6g-3$ is superadditive in g, so that considering prime diagrams suffices).

Corollary 2.9. *$d_g \leq 6g-3$ and $c_g \leq 12g-6$. In particular, the number of alternating genus g knots of n crossings is $O_n(n^{6g-4})$ as $n \to \infty$.*

The work done in [1] can be used to calculate the explicit values for c_3 and d_3.

Proposition 2.10. *$c_3 = 23$ and $d_3 = 15$. In particular, the number of alternating genus 3 knots of n crossings grows as n^{14}.*

2.11. *Proof of Proposition 2.10.* Consider the Gauss diagram G of a generating diagram of genus 3, realizing c_3 or d_3, and remove in it from each pair of parallel chords one of them, obtaining a diagram G'. (This preserves the genus, even if it spoils realizability).

We say that a word W' of a free group on a set X' is a *orientable word* if each element of X' that occurs in W' exactly twice, once with exponent $+1$ once with exponent -1.

A quadratic word W is *irredundant* if there is no pair of distinct, noninverse letters x, y which appear in W only in subwords (xy) and $(xy)^{-1}$.

It is known (see, for example [2]), that the maximal length of the genus g cyclically reduced orientable irredandant word is $12g-6$, and every genus g word without cancellations and reductions can be obtained from some word of maximal length by substituting some letters (and their inverses) by unit.

In the language of chord diagrams it means that each Gauss diagram of genus g with no isolated chord and no pair of parallel chords can be obtained by deleting chords from the subset of such diagrams, which enjoy the additional property that the endpoints of each chord a participate in a cycle of the type

$$\dots ab^{-1} \dots bc^{-1} \dots ca^{-1} \dots \quad \text{or} \quad \dots ba^{-1} \dots cb^{-1} \dots ac^{-1} \dots$$

for (the endpoints of) some other chords b and c, when denoting the endpoints of the chords of the diagram in cyclic order. These diagrams all have $6g-3$ chords.

Such diagrams were enumerated in [1], and the explicit list $\mathcal{L}_g$ was generated by computer for $g \leq 3$.

To calculate c_3 and d_3 we proceeded as follows. First we factored out the elements in $\mathcal{L}_3$ by inversion. Then we considered diagrams obtained by just retaining or doubling chords in diagrams in $\mathcal{L}_3$. We generated by computer all such diagrams with decreasing crossing number (from 30 downward), and tested realizability.

The first crossing number where we found realizable diagrams was 23. The existence of such a diagram already shows that $d_3 = d_{3,o} = 15$. (As in [1] flypes were not considered, some of the diagrams obtained may contain chords at different positions, which can be made parallel after flypes, and so the realizable diagrams had again to be subjected to a test for $\bar{t}_2$ irreducibility.) Then we found such diagrams also for crossing number 22, ensuring also $d_{3,e} = 15$.

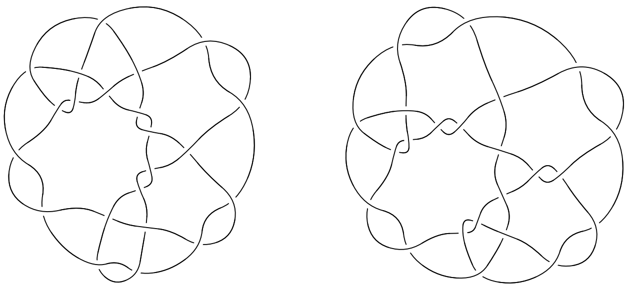


FIGURE 4. Two $\bar{t}_2$ irreducible diagrams of genus 3 of maximal even and odd crossing number.

Then we needed to check that, with the additional option for each chord to be deleted from a diagram in $\mathcal{L}_3$, there are no realizable diagrams of more than 23 crossings. This establishes that indeed $c_3 = 23$. (There were, in fact, some realizable 24 crossing diagrams, but they all were not $\bar{t}_2$ irreducible.)

We conclude by several questions. (Here we say that a diagram D realizes c_g resp. d_g if $D \in \mathcal{D}_g$ and $c(D) = c_g$ resp. $d(D) = d_g$.)

Question 2.12. *Is always $d_g = 6g - 3$ and $c_g = 10g - 7$?*

Question 2.13. *Is for any $g > 1$, d_g attained by both even and odd crossing number diagrams? Do the maximal even and odd crossing numbers of such diagrams differ by ± 1?*

Question 2.14. *Are for any $g > 1$ some (any?) alternating genus g diagrams realizing c_g special (i.e., they are also positive)?*

Question 2.15. *Is some (any?) knot attaining c_g also attaining d_g (which does not happen for $g = 1$)?*

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Max-Planck-Institut für Mathematik, P. O. Box 7280, D-53072 Bonn, Germany

E-mail address: `{alex,chernov,vdovina}@mpim-bonn.mpg.de`