ANALYTIC TORSION AND COHOMOLOGY OF HYPERBOLIC 3-MANIFOLDS

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1. Introduction

In this talk we discuss the connection between the Ray-Singer analytic torsion of hyperbolic 3-manifolds and the torsion of the integer cohomology of arithmetic hyperbolic 3-manifolds.

1. Analytic torsion. Let X be a compact Riemannian manifold of dimension n. Let $\rho\colon \pi_1(X) \to \mathrm{GL}(V_\rho)$ be a finite-dimensional representation of the fundamental group of X and let $E_\rho \to X$ be the associated flat vector bundle. Then the Ray-Singer analytic torsion $T_X(\rho)$ attached to ρ is defined as follows. Pick a Hermitian fiber metric h in E_ρ and let

$$\Delta_p(\rho)\colon \Lambda^p(X, E_\rho) \to \Lambda^p(X, E_\rho)$$

be the Laplacian on E_ρ-valued p-forms w.r.t. the hyperbolic metric g on X and the fibre metric h in E_ρ. Then $\Delta_p(\rho)$ is a non-negative self-adjoint operator whose spectrum consists of eigenvalues $0 \le \lambda_1 \le \lambda_2 \le \cdots$ of finite multiplicities. Let

$$\zeta_p(s;\rho) = \sum_{\lambda_i > 0} \lambda_i^{-s}, \quad \mathrm{Re}(s) > n/2.$$

be the zeta function of $\Delta_p(\rho)$. It is well known that $\zeta_p(s;\rho)$ admits a meromorphic extension to $s \in \mathbb{C}$ which is regular at $s = 0$. Then the regularized determinant $\det \Delta_p(\rho)$ of $\Delta_p(\rho)$ is defined as

$$\det \Delta_p(\chi) = \exp\left(-\frac{d}{ds} \zeta_p(s;\rho)\big|_{s=0} \right).$$

The analytic torsion is defined as the following weighted product of regularized determinants

$$T_X(\rho; h) = \prod_{p=0}^{n} \left(\det \Delta_p(\rho) \right)^{(-1)^p p/2}.$$

By definition it depends on h. However, if n is odd and ρ is acyclic, i.e., $H^*(X, E_\rho) = 0$, then $T_X(\rho; h)$ is independent of h (see [Mu1]) and we denote it by $T_X(\rho)$.

The representation ρ is called unimodular, if $|\det \rho(\gamma)| = 1$, $\forall \gamma \in \Gamma$. Let ρ be a unimodular, acyclic representation. Then the Reidemeister tosion $\tau_X(\rho)$ is defined [Mu1, section 1]. It is defined combinatorially in terms of a smooth triangulation of X and we have $T_X(\rho) = \tau_X(\rho)$ [Ch], [Mu2], [Mu1].

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2. Hyperbolic 3-manifolds. Let X be a compact oriented 3-dimensional hyperbolic manifold. Then there exists a discrete, torsion free, co-compact subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$ such that $X = \Gamma\backslash\mathbb{H}^3$, where $\mathbb{H}^3 = \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ is the 3-dimensional hyperbolic space.

For $m \in \mathbb{N}$ let $\rho_m\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(S^m(\mathbb{C}^2))$ be the standard irreducible representation of dimension $m+1$ acting in the space homogeneous polynomials $S^m(\mathbb{C}^2)$ of degree m. By restriction of ρ_m to Γ we obtain a representation of Γ which we continue to denote by ρ_m. It follows from [BW, Theorem 6.7, Chapt. VII] that ρ_m is acyclic. Since $\mathrm{SL}(2,\mathbb{C})$ is semisimple, it follows that $\det \rho_m(g) = 1$ for all $g \in \mathrm{SL}(2,\mathbb{C})$. Therefore the Reidemeister torsion $\tau_X(\rho_m)$ of X with respect to $\rho_m|_\Gamma$ is well defined. Our main result determines the asymptotic bahavior of $\tau_X(\rho_m)$ as $m \to \infty$.

Theorem 1. *Let X be a closed, oriented hyperbolic 3-manifold $\Gamma\backslash\mathbb{H}^3$. Then*

$$-\log \tau_X(\rho_m) = \frac{1}{\pi}\mathrm{vol}(X)m^2 + O(m)$$

as $m \to \infty$.

3. Arithmetic groups.

Let $F \subset \mathbb{C}$ be an imaginary quadratic field. Let $\mathcal{H} = \mathcal{H}(a,b;F)$ be a quaternion algebra over F, $a, b \in F^\times$. Then $\mathcal{H}$ splits over $\mathbb{C}$

$$\varphi\colon \mathcal{H} \otimes_F \mathbb{C} \cong M(2,\mathbb{C}).$$

Let $\mathfrak{R}$ be an order in $\mathcal{H}$ and let $\mathfrak{R}^1 = \{x \in \mathfrak{R}\colon N(x) = 1\}$. Let $\Gamma = \varphi(\mathfrak{R}^1)$. Then Γ is a lattice in $\mathrm{SL}(2,\mathbb{C})$. Moreover Γ is co-compact, if and only if $\mathcal{H}$ is a skew field. The norm 1 elements of $\mathcal{H}$ act by conjugation on the trace zero elements. In this way we get a Γ-invariant lattice $\Lambda \subset S^2(\mathbb{C}^2)$. Taking symmetric powers, it induces a Γ-invariant lattice in all even symmetric powers $S^{2m}(\mathbb{C}^2)$. So the integer cohomology $H^*(\Gamma\backslash\mathbb{H}^3, E_{2m,\mathbb{Z}})$ is defined. These are finite abelian groups. Denote by $|H^p(\Gamma\backslash\mathbb{H}^3, E_{2m,\mathbb{Z}})|$ the order of $H^p(\Gamma\backslash\mathbb{H}^3, E_{2m,\mathbb{Z}})$. Then we have

$$\tau_{\Gamma\backslash\mathbb{H}^3}(\rho_{2m}) = \prod_{p=1}^{3} |H^p(\Gamma\backslash\mathbb{H}^3, E_{2m,\mathbb{Z}})|^{(-1)^{(p+1)}}. \tag{1}$$

Combining this result with Theorem 1, we get

Theorem 2. *Let Γ be a co-compact, arithmetic lattice. Then*

$$\sum_{p=1}^{3}(-1)^p \log |H^p(\Gamma\backslash\mathbb{H}^3, E_{2m,\mathbb{Z}})| = \frac{4}{\pi}\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)m^2 + O(m)$$

as $m \to \infty$.

4. Ruelle zeta function. The proof of Thoerem 1 is based on the study of the twisted

Ruelle zeta function $R(s,\rho)$ attached to a finite-dimensional representation ρ of Γ. In a half-plane $\operatorname{Re}(s) \gg 0$ it is defined by the following infinite product

$$R(s,\rho) = \prod_{\substack{[\gamma]\neq 1 \\ prime}} \det\left(\mathrm{I} - \rho(\gamma)e^{-s\ell(\gamma)}\right),$$

where the product runs over all non-trivial prime conjugacy classes in Γ and $\ell(\gamma)$ denotes the length of the corresponding closed geodesic. It admits a meromomorphic extension to $s \in \mathbb{C}$ [Fr2, p.181]. It follows from the main result of [Wo] that $R(s,\rho_m)$ is holomorphic at $s = 0$ and

$$|R(0,\rho_m)| = T_{\Gamma\backslash\mathbb{H}^3}(\rho_m)^2. \tag{2}$$

The corresponding result for unitary representations ρ was proved by Fried [Fr1]. Now the proof of Theorem 1 is reduced to the study of the asymptotic behavior of $|R(0,\rho_m)|$ as $m \to \infty$. The volume appears through the functional equation satisfied by $R(s,\rho_m)$.

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