On the relation between upper central quotients and lower central series of a group

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1. Introduction

A group H gives rise to an upper central series $1 = Z_0H \leq Z_1H \leq \cdots$ and a lower central series $H = \gamma_1 H \geq \gamma_2 H \geq \cdots$. In this article we consider a normal subgroup $N \trianglelefteq H$ contained in Z_cH, and study the influence of the quotient $G = H/N$ on the lower central group $\gamma_{c+1}H$. An old result of R. Baer [1][18] states that $\gamma_{c+1}H$ is finite whenever the quotient G is finite. We develop Baer's techniques to obtain the following three results.

A. For any finite G we give an upper bound on the order of $\gamma_{c+1}H$. (A previous paper [9] gave a bound on $|\gamma_{c+1}H|$ when G is finite nilpotent; the present result incorporates a small improvement in this case. Several authors have given bounds when $c = 1$. In particular, there are papers by J.A. Green [17], J. Wiegold [29] [30], W. Gaschütz *et. al.* [15], and M.R. Jones [19] [20] [21]. The case $c = 1$ is also studied in [11] where the results obtained are slightly sharper than that got by specialising our general bound to $c = 1$.)

B. For any finite G we give an upper bound on the exponent of $\gamma_{c+1}H$. (For $c = 1$ this provides a generalisation of a result of A. Lubotzky and A. Mann [23] on the exponent of the Schur multiplier $\mathcal{M}(G)$ of a powerful p-group G; it also sharpens a bound of M.R. Jones [21] on the exponent of the Schur multiplier of a prime-power group. Furthermore, for $c \geq 1$ our bound yields a generalisation and sharpening of an estimate, given in [7], on the exponent of the c-nilpotent Baer invariant $M^{(c)}(G)$. This improvement for $c \geq 1$ has the following practical implication. An electronically down-loadable appendix to the paper [12] contains a MAGMA computer program for calculating a number of homotopy-theoretic constructions. In particular, it contains a function for computing $M^{(c)}(G)$ which requires, as input data, a finite presentation of a finite group G together with any positive integer q divisible by e^c where e denotes the exponent of $M^{(c)}(G)$. The improved estimate for e helps in choosing a suitable value for q.)

C. For G equal to a dihedral group, or quaternion group, or extra-special group we list all possible groups that can arise as $\gamma_{c+1}H$. (This extends work of N.D. Gupta and M.R.R. Moghaddam [16] which handles the dihedral 2-groups. It also extends work of D. MacHale and P.Ó'Murchú [26], and J. Burns *et. al.* [8] which treats all groups G of order at most 30 for $c = 1$, and all groups G of order at most 16 for $c = 2$.)

A precise statement of results A-C is provided in Section 2. Their proofs are given in Sections 4-6 respectively. The proofs involve three techniques with which the reader may not be too familiar. The first is the use of a nonabelian tensor product of groups. The second is the use of a Schur multiplier $\mathcal{M}(N, G)$ of a group G relative to a normal subgroup N. The third is the use of Baer invariants of a group. Relevant details of these techniques are recalled, and developed, in Section 3. Some results in Section 3 (in particular Propositions 5, 8 and 9) may be of independent interest.

2. Statement of results

Let a group G be presented as the quotient of a free group F by a normal subgroup R. We state our results in terms of the Baer invariants

$$M^{(c)}(G) = \frac{R \cap \gamma_{c+1}F}{\gamma_{c+1}(R, F)}\ , \qquad c \geq 1,$$

and related invariants

$$\gamma^*_{c+1}(G) = \frac{\gamma_{c+1}F}{\gamma_{c+1}(R,F)}$$

of the group G, where $\gamma_1(R,F) = R$, $\gamma_{c+1}(R,F) = [\gamma_c(R,F),F]$ and $\gamma_{c+1}F = \gamma_{c+1}(F,F)$. It was shown by R. Baer [1] (see also [14] [25]) that these invariants are, up to group isomorphism, independent of the choice of free presentation of G. Note that there are canonical actions of G on $M^{(c)}(G)$ and $\gamma^*_{c+1}(G)$ given by conjugation in $F/\gamma_{c+1}(R,F)$.

If G is of the form $G \cong H/N$ with N a normal subgroup of H contained in Z_cH, then it is routine [6] to establish the existence of the canonical short exact sequence

$$A \rightarrowtail \gamma^*_{c+1}(G) \twoheadrightarrow \gamma_{c+1}H$$

where A is a submodule of the $\mathbf{Z}G$-module $M^{(c)}(G)$. We thus have inequalities (in which $\leq$ can be taken to mean 'divides')

$$|\gamma_{c+1}H| \leq |\gamma^*_{c+1}(G)| = |M^{(c)}(G)||\gamma_{c+1}G| \ , \tag{1}$$

$$\exp(\gamma_{c+1}H) \leq \exp(\gamma^*_{c+1}(G)) \ , \tag{2}$$

involving the orders and exponents of groups. Since $M^{(c)}(G)$ is a subgroup of $\gamma^*_{c+1}(G)$, we also have an inequality

$$\exp(M^{(c)}(G)) \leq \exp(\gamma^*_{c+1}(G)) \ . \tag{3}$$

Furthermore, a group K arises as $\gamma_{c+1}H$ for some H if and only if

$$K \cong \frac{\gamma^*_{c+1}(G)}{A} \tag{4}$$

with A a submodule of $M^{(c)}(G)$.

Observations (1), (2) and (4) allow us to state results A-C in terms of the invariant $\gamma^*_{c+1}(G)$. For the statement of result A we let $\chi_c(d)$ denote the number of elements in a basis of the free abelian group $\gamma_cF/\gamma_{c+1}F$ with F the free group on d generators. (There is a well-known formula for $\chi_c(d)$ due to E. Witt [27]. Let $\mu(m)$ be the Möbius function, defined for all positive integers by $\mu(1) = 1$, $\mu(p) = -1$ if p is a prime, $\mu(p^k) = 0$ for $k > 1$, and $\mu(ab) = \mu(a)\mu(b)$ if a and b are coprime integers. Witt's formula is

$$\chi_c(d) = (1/c)\sum_{m|c} \mu(m) d^{(c/m)}$$

where m runs through all divisors of c. Thus, for instance, $\chi_2(d) = (d^2 - d)/2$, $\chi_3(d) = (d^3 - d)/3$, $\chi_4(d) = (d^4 - d^2)/4$.)

For an arbitrary finite abelian p-group A we define the integer

$$\Lambda_c(A) = e_1\chi_{c+1}(d_1) + \sum_{j=2}^{k} e_j\{\chi_{c+1}(d_1 + \cdots + d_j) - \chi_{c+1}(d_1 + \cdots + d_{j-1})\}$$

where the parameters d_j, e_j, k are determined by expressing A uniquely in the form

$$A \cong (C_{p^{e_1}})^{d_1} \times (C_{p^{e_2}})^{d_2} \times \cdots \times (C_{p^{e_f}})^{d_k}$$

with $e_1 > e_2 > \cdots > e_k > 1$.

For an arbitrary finite d-generator p-group P we define the integer

$$\Psi_c(P) = m_c d + m_{c-1}d^2 + \cdots + m_1 d^c$$

where the terms of the lower central series of P have orders $|\gamma_j(P)| = p^{m_j}$.

Note that an arbitrary finite group G has a smallest term L in its lower central series, namely the unique group $L = \gamma_r G$ that satisfies $\gamma_r G = \gamma_{r+1} G$. Suppose that P is a d-generator p-Sylow subgroup of G with Frattini subgroup $\Phi(P) = [P,P]P^p$, that $P/(P \cap [G,G])$ is a δ-generator group, that $(L \cap P)/(L \cap \Phi(P))$ has order p^t, that $L \cap P$ has order p^β, and that $[L \cap P, P]$ has order $p^{\beta'}$. We use these various parameters to define the integer

$$\Theta_c(G,L,P) = \beta + (\omega - \beta')(1 + \delta + \delta^2 + \cdots + \delta^{c-1}) \ ,$$

where

$$\omega = d\beta - (1/2)t(t+1) \ .$$

Theorem A. *Let G be a finite group whose order has prime factors $p_1, p_2, \cdots, p_n$. Let L be the smallest term in the lower central series of G. The quotient G/L is nilpotent and thus a direct product*

$$G/L \cong S_1 \times S_2 \times \cdots \times S_n$$

with S_i a (possibly trivial) p_i-group. For each i let P_i be some p_i-Sylow subgroup of G. Then

$$|\gamma^*_{c+1}(G)| \le \prod_{i=1}^{n} p_i^{\Lambda_c(S_i^{ab}) + \Psi_c(S_i) + \Theta_c(G,L,P_i)} \ .$$

The bound is attained, for instance, when G is abelian.

Note that if G is perfect then $\Lambda_c(S_i^{ab}) = 0$, $\Psi_c(S_i) = 0$ and $\Theta_c(G,L,P_i) = d_i(2\beta_i - d_i - 1)/2 + \alpha_i$ where $p_i^{\beta_i}$ is the order of a d_i-generator p_i-Sylow subgroup P_i, and $p_i^{\alpha_i}$ is the order of the abelianisation P_i^{ab}. If, at the other extreme, G is nilpotent then we have $\Theta_c(G,L,P_i) = 0$. If G is abelian then $\Theta_c(G,L,P_i) = 0$ and $\Psi_c(S_i) = 0$.

The bound in Theorem A can be sharpened by involving the relative Schur multiplier $\mathcal{M}(L,G)$ whose definition is recalled in Section 3. More precisely, in the definition of $\Theta_c(G,L,P)$ we can re-define $\omega = \mu + \beta'$ where the pth primary component of the abelian group $\mathcal{M}(L,G)$ has order $|\mathcal{M}(L,G)_p| = p^\mu$. For example, if $|L|$ is coprime to $|G|/\exp(L)$, then the relative multiplier is trivial (see Proposition 7(ii)) and we can take $\Theta_c(G,L,P_i) = \beta_i$ for $c \ge 1$.

Before stating result B let us recall that A. Lubotzky and A. Mann [23] defined a p-group P to be *powerful* if: $p \ge 3$ and $[P,P] \subset P^p$; or $p = 2$ and $[P,P] \subset P^4$ (where P^i is the subgroup of P generated by all ith powers). In other words, P is powerful if $p \ge 3$ and P/P^p is abelian, or if $p = 2$ and P/P^4 is abelian. They proved a number of results about powerful groups P, one of which states that the exponent $\exp(M^{(1)}(P))$ of the Schur multiplier divides the exponent of P. We shall generalise this. Our generalisation implies, for instance, that $\exp(M^{(c)}(P))$ divides $\exp(P)$ for all $c \ge 1$ and all P in a certain class C_p of p-groups; the class C_p consists of those p-groups P satisfying $[[P^{p^{i-1}}, P], P] \subset P^{p^i}$ for $1 \le i \le e$ where $\exp(P) = p^e$. It is shown in [23] that if P is powerful then $[P^{p^{i-1}}, P] \subset P^{p^i}$. Hence the class C_p contains all powerful p-groups.

Given a normal subgroup $N \trianglelefteq G$ of some group G, we say that the *pair* (N,G) has *nilpotency class* k if $\gamma_{k+1}(N,G) = 1$ and $\gamma_k(N,G) \ne 1$. For a real number r we let $[r]$ denote the smallest integer n such that $n \ge r$.

Let N be a normal subgroup of a finite p-group P and suppose that N has exponent p^e. We define the integer

$$\Omega(N,P) = [k_1/2] + [k_2/2] + \cdots + [k_e/2]$$

where k_j denotes the nilpotency class of the pair $(N^{p^{j-1}}/N^{p^j},\ P/N^{p^j})$ for $1 \le j \le e$. For N equal to the trivial group we set $\Omega(1,P) = 0$. Note that $\Omega(N,P) \le [k/2]e$ where k is the nilpotency class of P.

Theorem B. *(i) Let G be a finite group whose order has prime factors $p_1, p_2, \cdots, p_n$. Let L be the smallest term in the lower central series of G. The quotient G/L is thus a direct product*

$$G/L \cong S_1 \times S_2 \times \cdots \times S_n$$

with S_i a (possibly trivial) p_i-group. For each i let P_i be a p_i-Sylow subgroup of G. Suppose that $L \cap P_i/[L \cap P_i, P_i]$ has exponent $p_i^{n_i}$. Suppose that the p_i-primary component of G^{ab} has exponent p^{e_i} with $e_i \geq 0$, and set $m_i = \min(\Omega(L \cap P_i, P_i), e_i)$. Then, for each $c \geq 1$,

$$\exp(\gamma^*_{c+1}(G)) \quad \textit{divides} \quad \prod_{i=1}^{n} p_i^{\Omega(L\cap P_i, P_i)+\Omega(S_i,S_i)+n_i+(c-1)m_i} .$$

The bound is attained if G is abelian.
(ii) Suppose that a p-group P satisfies $[[P^{p^{i-1}}, P], P] \subset P^{p^i}$ for all $1 \leq i \leq e$ where p^e is the exponent of P. Then $\Omega(P,P) = e$.

Note that, by inequality (3), $\exp(M^{(c)}(G))$ divides $\exp(\gamma^*_{c+1}(G))$ for any group G. Thus, for an arbitrary finite p-group P of class k and exponent p^e, Theorem B(i) implies that $\exp(M^{(c)}(P))$ divides $p^{[k/2]e}$; this sharpens the bound $\exp(M^{(c)}(P)) \leq p^{(k-1)e}$ of Corollary 2.6 in [21] (for $c = 1$) and Theorem 6 in [7] (for $c \geq 1$). Theorem B(ii) implies that $\exp(\gamma^*_{c+1}(P))$ divides $\exp(P)$ if, for example, P is a p-group with P/P^p of nilpotency class 2 and P^p contained in the second centre $Z_2(P)$.

The bound in Theorem B(i) can be sharpened by re-defining m_i to be $m_i = \min(\epsilon_i, e_i)$ where $p_i^{\epsilon_i}$ and p^{e_i} are the exponents of the p_i-primary components of $\mathcal{M}(L, G)$ and G^{ab} respectively (*cf.* Proposition 7 in Section 3). The bound is clearly independent of c if G is finite nilpotent, or if G is perfect. We do not know whether the bound can be made independent of c for arbitrary finite groups.

For the statement of result C we let $D_n =< a, b \mid a^2 = b^n = (ab)^2 = 1 >$ denote the dihedral group of order $2m$, and $Q_n =< a, b \mid a^2 = b^n = (ab)^2 >$ denote the quaternion group of order $4n$. Recall [3] that a p-group E is said to be *extra-special* if its commutator subgroup $[E, E]$, its Frattini subgroup $\Phi(E)$, and its centre $Z_1(E)$ coincide and have order p. The extra-special groups have order p^{2k+1} for $k \geq 1$, with precisely two extra-special p-groups for each k (see [3]). We let $E(p, k)$ denote an arbitrary extra-special p-group of order p^{2k+1}; we let $E(p, k)^+$ and $E(p, k)^-$ denote the extra-special p-groups of order p^{2k+1} and exponents p and p^2 respectively. For $c \geq 1$ we have the following.

Theorem C. *(i) For each $n \geq 2$ we have*

$$\gamma^*_{c+1}(D_n) \cong \begin{cases} C_n & \text{odd } n, \\ C_n \times (C_2)^{\chi_{c+1}(2)-1} & \text{even } n. \end{cases}$$

The generator $b \in D_n$ acts trivially on $\gamma^*_{c+1}(D_n)$; the generator $a \in D_n$ acts trivially on elements of order two, and inverts the elements of the cyclic summand C_n.
(ii) For each $n \geq 2$ we have

$$\gamma^*_{c+1}(Q_n) \cong \gamma^*_{c+1}(D_n) .$$

The generators $a, b \in Q_n$ act as in (i).
(iii) For each $k \geq 2$ we have

$$\gamma^*_{c+1}(E(p,k)) \cong (C_p)^{\chi_{c+1}(2k)}.$$

The group $E(p, k)$ acts trivially on $\gamma^*_{c+1}(E(p, k))$.
(iii)$'$ For $p \geq 3$ and some $1 \leq r \leq 2^c$ we have

$$\gamma^*_{c+1}(E(p,1)^+) \cong (C_p)^{\chi_{c+1}(2)+r},$$

$$\gamma^*_{c+1}(E(p,1)^-) \cong (C_p)^{\chi_{c+1}(2)}.$$

Note that the corresponding Baer invariants $M^{(c)}(G)$ are easily obtained applying the formula $M^{(c)}(G) = \ker(\gamma^*_{c+1}(G) \to \gamma_{c+1}(G))$ to the precise details of the isomorphisms given in the proof of Theorem C. This extends the computations on dihedral 2-groups given in [16]. (We remark that there is a slip in the statement of the main theorem in [16]; the statement is correct for $\gamma^*_{c+1}(D_{2^n})$ but incorrect for $M^{(c)}(D_{2^n})$.)

The precise value of r in Theorem C(iii)$'$ needs further investigation. The computer program listed in [12] yields the following results for the Burnside group $B(2,3) = E(3,1)^+$ of exponent 3 on two generators.

c	$\gamma^*_{c+1}(B(2,3))$	r
1	$(C_3)^3$	2
2	$(C_3)^5$	3
3	$(C_3)^9$	6
4	$(C_3)^{15}$	9
5	$(C_3)^{27}$	18

3. Preliminaries

The tensor product of nonabelian groups is a convenient setting for performing commutator calculations. Its functorial properties make it especially suited to the task of relating commutator calculations in a group to those in a homomorphic image of the group. We begin this section by recalling and developing relevant details on this tensor product. We then recall details on a Schur multiplier $\mathcal{M}(N,G)$ defined for pairs of groups. By a *pair* of groups (N,G) we simply mean a group G with normal subgroup N. The advantage of working with pairs is that any finite group G can be expressed as an extension

$$(L,G) \rightarrowtail (G,G) \twoheadrightarrow (G/L, G/L)$$

of a 'perfect' pair (L,G) by a 'nilpotent' pair $(G/L,G/L)$. Various simplifications apply when dealing with the Schur multiplier of perfect or nilpotent pairs. We end the section with some details on Baer invariants.

Suppose given two groups G and H which act on each other via group actions $G \times H \to H, (g,h) \mapsto {}^g h$ and $H \times G \to G, (h,g) \mapsto {}^h g$. Furthermore, suppose that each group acts on itself by conjugation, ${}^x y = xyx^{-1}$. (In keeping with this notation, our convention for commutators is $[x,y] = xyx^{-1}y^{-1}$.) The tensor product $G \otimes H$ is defined [5] [4] to be the group generated by symbols $g \otimes h$ subject to the relations

$$gg' \otimes h = ({}^g g' \otimes {}^g h)\,(g \otimes h)\,,$$

$$g \otimes hh' = (g \otimes h)\,({}^h g \otimes {}^h h')\,,$$

for $g, g' \in G, h, h' \in H$. The actions are said to be *compatible* if

$$^{({}^g h)}g' = {}^g({}^h({}^{g^{-1}}g'))\,, \quad {}^{({}^h g)}h' = {}^h({}^g({}^{h^{-1}}h'))$$

for all $g, g' \in G, h, h' \in H$.

Proposition 1. *[5] Suppose that G and H act compatibly on each other.*
(i) For all $g, g' \in G, h, h' \in H$ the following identities hold in $G \otimes H$:

$$g' \otimes ({}^{g}h)\, h^{-1} = {}^{g'}(g \otimes h)\, (g \otimes h)^{-1}\,,$$

$$g\, ({}^{h}g^{-1}) \otimes h' = (g \otimes h)^{h'}\, (g \otimes h)^{-1}\,,$$

$$g\, ({}^{h}g^{-1}) \otimes ({}^{g'}h')\, h'^{-1} = [g \otimes h, g' \otimes h']\,.$$

(ii) There is a homomorphism $\partial_G\colon G \otimes H \to G, g \otimes h \mapsto g\, {}^{h}g^{-1}$.
(iii) There is a 'diagonal' action of G on $G \otimes H$ given by ${}^{g'}(g \otimes h) = ({}^{g'}g \otimes {}^{g'}h)$.
(iv) There is an isomorphism $G \otimes H \overset{\cong}{\to} H \otimes G,\ g \otimes h \mapsto h \otimes g$.
(v) If ${}^{g}h = h$, ${}^{h}g = g$ for all $g \in G, h \in H$ then $G \otimes H \cong G^{ab} \otimes H^{ab}$, where the right-hand side of the isomorphism denotes the usual tensor product of abelian groups.

For each pair of groups (N, G) we can form the tensor product $N \otimes G$ in which all actions are taken to be conjugation in G. Since conjugation yields compatible actions there is a diagonal action of G on $N \otimes G$. The tensor product $N \otimes G$ acts on G by conjugation in G via the homomorphism $\partial_N\colon N \otimes G \to N$. We can thus construct the triple tensor product $(N \otimes G) \otimes G$. One readily checks that the construction preserves 'compatibility of actions', and that it can therefore be iterated to form the $(c+1)$-fold tensor product

$$\otimes^{c+1}(N, G) = (\cdots((N \otimes G) \otimes G) \otimes \cdots \otimes G) \quad c \geq 1,$$

involving c copies of G and one copy of N.

Proposition 2. *[13] Let G be a d-generator p-group with normal subgroup N. Suppose that $\gamma_i(N, G) = p^{m_i}$ for $i \geq 1$, $m_i \geq 0$. Then, for any $c \geq 1$, we have*

$$|\otimes^{c+1}(N, G)| \leq p^{m_c d + m_{c-1} d^2 + \cdots + m_1 d^c}\,.$$

Lemma 3. *Let $G_3 \hookrightarrow G_2 \twoheadrightarrow G_1$, $H_3 \hookrightarrow H_2 \twoheadrightarrow H_1$ be two short exact sequences of groups. Suppose that G_i and H_i act compatibly on one another for $1 \leq i \leq 3$, and that the homomorphisms preserve actions. Then there is an exact sequence of homomorphisms*

$$(G_3 \otimes H_2) \overline{\times} (G_2 \otimes H_3) \longrightarrow G_2 \otimes H_2 \longrightarrow G_1 \otimes H_1 \longrightarrow 1$$

in which $\overline{\times}$ denotes a semi-direct product whose details need not be specified.

Proof. The lemma is a routine adaption of Proposition 9 in [4]. $\square$

Lemma 4. *Let N be a normal subgroup of G for which the commutator $[n, [n, g]]$ is trivial for all $g \in G, n \in N$. In the tensor product $N \otimes G$, with G and H acting by conjugation, the following identity holds for all $g \in G, n \in N$ and all integers $t \geq 2$:*

$$n^t \otimes g = (n \otimes g)^t\, (n \otimes [n, g]^{t(t-1)/2})\,.$$

Proof. The case $N = G$ is proved in [2]. The proof of the more general case is analogous; it can also be derived directly using Proposition 1(i). $\square$

Recall that a pair (N, G) is said to be *nilpotent* of *class* k if $\gamma_{k+1}(N, G) = 1$ and $\gamma_k(N, G) \neq 1$. Also recall that $[k/2]$ denotes the smallest integer n such that $n \geq k/2$.

Proposition 5. *Let G be a group with normal subgroup N. Suppose that N has prime-power exponent p^e and that the pair (N,G) has nilpotency class $\leq k$. Then, for any $c \geq 1$, we have*

$$\exp(\otimes^{c+1}(N,G)) \quad \textit{divides} \quad p^{[k/2]e}.$$

Proof. For $t = p^e$ the binomial coefficient $\binom{t}{2}$ is divisible by t when $p \geq 3$, and divisible by $t/2$ when $p = 2$. Thus Lemma 4 proves the proposition for $k = 2, c = 1$ (since for $p = 2$, $t = p^e$ and $\gamma_3(N,G) = 1$ the identity

$$n \otimes [n,g]^{t/2} = n \otimes [n^{t/2}, g]$$

holds for all $g \in G, n \in N$; but $[n^{t/2}, g] = 1$ because $n^{t/2}$ has order at most 2.)

Let us now consider $k = 2$ and some $c \geq 2$. Then

$$\otimes^{c+1}(N,G) = \otimes^c(N \otimes G, G)$$

and $N \otimes G$ acts trivially on G. The triviality of this action implies the identity

$$(\cdots((n \otimes g)^t \otimes g_1) \otimes \cdots \otimes g_c) = (\cdots((n \otimes g) \otimes g_1) \otimes \cdots \otimes g_c)^t$$

in $\otimes^{c+1}(N,G)$. Hence $\exp(\otimes^{c+1}(N,G))$ divides $\exp(N \otimes G)$ and the proposition is proved for $k = 2, c \geq 1$.

Suppose now that the proposition has been proved for some c and all $k < k_0$. Suppose $\gamma_{k_0+1}(N,G) = 1$. Lemma 3 implies an exact sequence

$$(\gamma_{k_0-1}(N,G) \otimes G)\overline{\times}(N \otimes \gamma_{k_0-1}(N,G)) \to N \otimes G \to \frac{N}{\gamma_{k_0-1}(N,G)} \otimes \frac{G}{\gamma_{k_0-1}(N,G)}\,.$$

Working in $N \otimes G$, the image of $\gamma_{k_0-1}(N,G) \otimes G$ contains the image of $N \otimes \gamma_{k_0-1}(N,G)$ by virtue of the identity

$$m \otimes [n,g] = ([n,g] \otimes m)^{-1}$$

which follows from Proposition 1(i) for all $g \in G, m, n \in N$. We thus have an exact sequence

$$\gamma_{k_0-1}(N,G) \otimes G \to N \otimes G \to \frac{N}{\gamma_{k_0-1}(N,G)} \otimes \frac{G}{\gamma_{k_0-1}(N,G)}\,.$$

By applying Lemma 3 to this sequence, and invoking a similar identity, we obtain the exact sequence

$$(\gamma_{k_0-1}(N,G) \otimes G) \otimes G \to (N \otimes G) \otimes G \to (\frac{N}{\gamma_{k_0-1}(N,G)} \otimes \frac{G}{\gamma_{k_0-1}(N,G)}) \otimes \frac{G}{\gamma_{k_0-1}(N,G)}\,.$$

Repetition of the process yields an exact sequence

$$\otimes^{c+1}(\gamma_{k_0-1}(N,G), G) \to \otimes^{c+1}(N,G) \to \otimes^{c+1}(\frac{N}{\gamma_{k_0-1}(N,G)}, \frac{G}{\gamma_{k_0-1}(N,G)})$$

from which we deduce that $\exp(\otimes^{c+1}(N,G)) \leq p^{[(k_0-2)/2]}p^e = p^{[k_0/2]}$. By induction, the proposition is proved for all $c, k \geq 1$. $\square$

Following J.-L.Loday [22] we say that a pair of groups (N,G) is *perfect* if $N = [N,G]$.

Proposition 6. *Let (N, G) be any perfect pair of groups and set $M = \ker(\partial_N : N \otimes G \to N)$. Then M is abelian and, for each $c \geq 1$, there is an exact sequence*

$$\otimes^{c+1}(M, G^{ab}) \to \otimes^{c+2}(N, G) \to \otimes^{c+1}(N, G) \to 1$$

where $\otimes^{c+1}(M, G^{ab})$ is the usual iterated tensor product of abelian groups.

Proof. Let $H_3 \hookrightarrow H_2 \twoheadrightarrow H_1$ be a short exact sequence of groups, and let G be a group such that G and H_i act compatibly on each other for $1 \leq i \leq 3$ with the homomorphisms preserving the actions. Then H_3 acts trivially on G via H_2. Suppose that the action of G on H_2 restricts to a trivial action of G on H_3. Then Proposition 1(v) and Lemma 3 imply an exact sequence

$$H_3^{ab} \otimes G^{ab} \to H_2 \otimes G \to H_1 \otimes G \to 1. \tag{5}$$

A perfect pair of groups (N, G) gives rise to a short exact sequence $\ker(\partial_N) \hookrightarrow N \otimes G \overset{\partial}{\twoheadrightarrow} N$. The identity

$${}^{h}(n \otimes g) = ([n, g] \otimes h)^{-1} (n \otimes g)$$

which holds in $N \otimes G$ (see Proposition 1) for all $g, h \in G, n \in N$ implies that G acts trivially on $\ker(\partial_N)$. So (5) implies an exact sequence

$$M \otimes G^{ab} \to (N \otimes G) \otimes G \xrightarrow{\partial \otimes 1} N \otimes G \to 1\,.$$

Note that the diagonal action of G on $M \otimes G^{ab}$ is trivial, and hence G acts trivially on $\ker(\partial \otimes 1)$. Thus a second application of (5) yields the exact sequence $\ker(\partial \otimes 1) \otimes G^{ab} \to \otimes^4(N, G) \to \otimes^3(N, G) \to 1$. From this we derive the exact sequence $\otimes^3(M, G^{ab}) \to \otimes^4(N, G) \to \otimes^3(N, G) \to 1$. The proposition follows from a repetition of this argument. □

Given a pair of groups (N, G) we denote by $\Delta(N, G)$ the subgroup of $N \otimes G$ generated by the elements $n \otimes n$ for $n \in N$. This is a normal subgroup and following [5] we define the *exterior product*

$$N \wedge G = N \otimes G / \Delta(N, G)\,.$$

The homomorphism $\partial_N : N \otimes G \to N$ clearly induces a homomorphism $\partial_N : N \wedge G \to N$. The identity

$$[g, n] \otimes [g', n'] = [(g \otimes n), (g' \otimes n')]\,,$$

of Proposition 1(i) implies an isomorphism $N \wedge G \cong N \otimes G$ in the case of perfect pairs.

Definition. [10] The *Schur multiplier* of a pair of groups (N, G) is the group $\mathcal{M}(N, G)$ defined by

$$\mathcal{M}(N, G) = \ker(\partial_N : N \wedge G \to N)\,.$$

If the pair is perfect then, equivalently,

$$\mathcal{M}(N, G) = \ker(\partial_N : N \otimes G \to N)\,.$$

Proposition 7. *[10] Let G be a finite group with normal subgroup $N \trianglelefteq G$.*
(i) Then $\mathcal{M}(N, G)$ is a finite abelian group with exponent e dividing the order of G.
(ii) Let e' denote the exponent of N. Then, in fact, ee' divides the order of G and e divides the order of N.
(iii) Let K be any subgroup of G such that each $g \in G$ can be expressed (not necessarily uniquely) as a product $g = nk$ with $n \in N, k \in K$. Then e^2 divides $|N| \times |K|$.
(iv) Suppose that P is a p-Sylow subgroup of G. Let $\mathcal{M}(P \cap N, P)_p$ denote the p-component of the multiplier, and $\iota : \mathcal{M}(P \cap N, P) \to \mathcal{M}(N, G)$ the homomorphism induced by inclusion. Then

$$\mathcal{M}(P \cap N, P) \cong \mathcal{M}(N, G)_p \oplus \ker(\iota)\,.$$

Proposition 8. *Let G be a δ-generator p-group with normal subgroup $N \trianglelefteq G$ of order $|N| = p^\beta$. Suppose that $|N/(N \cap \Phi(G))| = p^t$ where $\Phi(G) = [G,G]G^p$. Then*

$$|N \wedge G| \le p^{\delta\beta - \frac{t(t+1)}{2}} .$$

Proof. Proposition 2 implies that $|N \otimes G| \le p^{\delta\beta}$. There is a commutative diagram of group homomorphisms

$$\begin{tikzcd}
 & \Gamma(\frac{N}{[N,G]}) \arrow[r, "\Delta'"] \arrow[dd] & N \otimes G \arrow[r] \arrow[d] & N \wedge G \arrow[r] & 1 \\
 & & G \otimes G \arrow[d, two heads] & & \\
1 \arrow[r] & \Gamma(\frac{G}{\Phi(G)}) \arrow[r, "\Delta"] & \frac{G}{\Phi(G)} \otimes \frac{G}{\Phi(G)} \arrow[r] & \frac{G}{\Phi(G)} \wedge \frac{G}{\Phi(G)} \arrow[r] & 1
\end{tikzcd}$$

in which the rows (but not the columns) are exact [5]. The abelian group $\Gamma(A)$, defined for any additive abelian group A, is J.H.C. Whitehead's universal quadratic construction; it is generated (as an abelian group) by symbols $\gamma(a)$ for $a \in A$ subject to the relations

$$\gamma(-a) = \gamma(a)$$

$$\gamma(a+b+c) + \gamma(a) + \gamma(b) + \gamma(c) = \gamma(a+b) + \gamma(a+c) + \gamma(b+c)$$

for $a, b, c \in A$. The homomorphism Δ is defined on generators by $\Delta(\gamma(x)) = x \otimes x$ for $x \in G/\Phi(G)$. The image of $\Gamma(G/[N,G])$ in $G/\Phi(G) \otimes G/\Phi(G)$ is an elementary abelian group of rank $t(t+1)/2$. Hence the exactness of the top row implies

$$|N \wedge G| = \frac{|N \otimes G|}{|\Delta'(\Gamma(N/[N,G]))|} \le \frac{p^{\delta\beta}}{p^{t(t+1)/2}} .$$

This proves the proposition. □

Proposition 9. *Let N be a normal subgroup of a group G. If N has exponent p^e then*

$$\exp(N \wedge G) \quad \textit{divides} \quad p^{[k_1/2]+[k_2/2]+\cdots+[k_e/2]}$$

where k_i denotes the nilpotency class of the pair $(N^{p^{i-1}}/N^{p^i},\ G/N^{p^i})$ for $1 \le i \le e$.

Proof. Let K, M be normal subgroups of G with $K \le M$. Using the identity $m \otimes k = (k \otimes m)^{-1}$ which holds in $M \wedge G$ for all $k \in K, m \in M$, one readily develops the short exact sequence

$$K \wedge G \to M \wedge G \to M/K \wedge G/K \to 1 \tag{6}$$

from Lemma 3. Now (6) yields the exact sequences

$$N^{p^i} \wedge G \to N^{p^{i-1}} \wedge G \to N^{p^{i-1}}/N^{p^i} \wedge G/N^{p^i} \to 1$$

for $i \ge 1$. Hence

$$\exp(N \wedge G) \le \prod_{i=1}^{e} \exp(N^{p^{i-1}}/N^{p^i} \wedge G/N^{p^i}) .$$

Proposition 5 implies $\exp(N^{p^{i-1}}/N^{p^i} \wedge G/N^{p^i}) \le p^{[k_i/2]}$. □

Tensor products are related to Baer invariants by the following slight generalisation of a result of A.-S.T. Lue [24] (*cf.* [6]).

Proposition 10. *[24] For any group G with normal subgroup $N \trianglelefteq G$, and for $c \geq 1$, there is an exact sequence*

$$\otimes^{c+1}(N,G) \to \gamma^*_{c+1}(G) \to \gamma^*_{c+1}(G/N) \to 1\,.$$

It is convenient to set

$$\overline{\gamma}^*_{c+1}(N,G) = \ker(\gamma^*_{c+1}(G) \to \gamma^*_{c+1}(G/N))\,.$$

Note that $\gamma^*_{c+1}(G) = \overline{\gamma}^*_{c+1}(G,G)$. (The bar is intended to suggest that $\overline{\gamma}^*_{c+1}(N,G)$ is a quotient of some functor $\gamma^*_{c+1}(N,G)$. For example, we can take $\gamma^*_2(N,G) = N \wedge G$ [6]. Proposition 9 and the following Proposition 11 could be subsumed under a single result concerning $\gamma^*_{c+1}(N,G)$.)

Proposition 11. *Suppose that N is a normal subgroup of a group G. If N has exponent p^e then*

$$\exp(\overline{\gamma}^*_{c+1}(N,G)) \quad \textit{divides} \quad p^{[k_1/2]+[k_2/2]+\cdots+[k_e/2]}$$

where k_i denotes the nilpotency class of the pair $(N^{p^{i-1}}/N^{p^i},\, G/N^{p^i})$ for $1 \leq i \leq e$.

Proof. The proof is analogous to that of Proposition 9, but with (6) replaced by the exact sequence

$$\overline{\gamma}^*_{c+1}(K,G) \to \overline{\gamma}^*_{c+1}(M,G) \to \overline{\gamma}^*_{c+1}(M/K,G/K) \to 1\,.$$

We leave the verification of the exactness of this (canonical) sequence to the reader. □

The *upper epicentral series* of an arbitrary group G was introduced in [6]. This is a family of characteristic subgroups $1 = Z^*_0(G) \leq Z^*_1(G) \leq Z^*_2(G) \leq \cdots$ with various useful properties such as those listed in the next proposition. Part (i) of the following proposition can be taken as the definition of $Z^*_c(G)$.

Proposition 12. *[6] Let $c \geq 1$.*
(i) $Z^*_c(G)$ is the smallest normal subgroup of G, contained in $Z_c(G)$, such that the quotient $G/Z^*_c(G)$ is isomorphic to H/Z_cH for some group H.
(ii) $Z^*_{c+1}(G)$ contains $Z^*_c(G)$.
(iii) $Z^*_c(G) = 1$ if and only if there exists an isomorphism $G \cong H/Z_cH$ for some group H.
(iv) Let N be a normal subgroup of G. Then $N \leq Z^*_c(G)$ if and only if the quotient homomorphism $G \twoheadrightarrow G/N$ induces an isomorphism $\gamma^*_{c+1}(G) \xrightarrow{\cong} \gamma^*_{c+1}(G/N)$.

Let A be a d-generator abelian group with generators $a_1, \cdots, a_d$. Let A_i denote the cyclic subgroup of A generated by a_i. Let $\mathcal{L}(d)$ denote the set of basic commutators on the d symbols a_i. To each basic commutator $\lambda = [a_{i_1}, \cdots, a_{i_k}]$ of weight k we associate the k-fold tensor product of abelian groups $T(\lambda) = A_{i_1} \otimes \cdots \otimes A_{i_k}$. Thus T is a cyclic group of order equal to the highest common factor of the orders of the A_{i_j}. It is explained in [9] that the invariant $\gamma^*_{c+1}(A)$ is isomorphic to a direct sum of cyclic groups

$$\gamma^*_{c+1}(A) \cong \oplus_{\lambda \in \mathcal{L}(d)} T(\lambda)\,.$$

The following proposition is an immediate corollary to this isomorphism. An alternative derivation of the proposition can be found in [28].

Proposition 13. *Let A be a direct product of cyclic groups*

$$A = (C_{n_1})^{d_1} \times (C_{n_2})^{d_2} \times \cdots \times (C_{n_k})^{d_k}$$

with each n_i divisible by n_{i+1}. Then

$$\gamma^*_{c+1}(A) \cong (C_{n_1})^{\chi_{c+1}(d_1)} \times \prod_{j=2}^{k} (C_{n_j})^{\{\chi_{c+1}(d_1+\cdots+d_j) - \chi_{c+1}(d_1+\cdots+d_{j-1})\}}\,.$$

Proposition 14. *[9] Let $G = S_1 \times S_2 \times \cdots \times S_k$ be a direct product of groups whose abelianisations S_i^{ab} have finite, and mutually coprime, orders. Then, for each $c \geq 1$, there is an isomorphism*

$$\gamma_{c+1}^*(G) \cong \gamma_{c+1}^*(S_1) \times \cdots \times \gamma_{c+1}^*(S_k)\,.$$

Proposition 15. *Let N be a non-trivial normal subgroup of a p-group G. Let K denote the kernel of the canonical surjection $\gamma_{c+1}^*(G) \twoheadrightarrow \gamma_{c+1}^*(G/N)$.*
(i) K is non-trivial if and only if there exists some group H for which $H/Z_cH \cong G$.
(ii) If $p \geq 3$ and $N \subset G^p \cap Z_2G$, or if $p = 2$ and $N \subset G^{p^2} \cap Z_2G$, then K is contained in the Frattini subgroup of $\gamma_{c+1}^*(G)$.
(iii) If N is a proper subgroup of a cyclic normal subgroup in G, and if $N \subset Z_2G$, then K is contained in the Frattini subgroup of $\gamma_{c+1}^*(G)$.

Proof. Proposition 12 implies (i).

Proposition 10 implies that K is generated by the image of tensors of the form $(\cdots((n \otimes g_1) \otimes g_2) \otimes \cdots \otimes g_c)$. The hypothesis of (ii) with Lemma 4 implies that the canonical image in $\otimes^{c+1}(G,G)$ of each such tensor lies in the subgroup $\otimes^{c+1}(G,G)^p$ generated by pth powers of tensors. The hypothesis of (iii) implies that the image lies in the subgroup generated by pth powers of tensors together with tensors of the form $(\cdots((g \otimes g^t) \otimes g_2) \otimes \cdots \otimes g_c)$. In both cases K lies in the Frattini subgroup of the p-group $\gamma_{c+1}^*(G)$. $\square$

4. Proof of Theorem A

Let G, S_i, P_i, L be as in the statement of Theorem A. For each prime p_i let $\otimes^{c+1}(L,G)_{p_i}$ denote some p_i-Sylow subgroup of $\otimes^{c+1}(L,G)$, and set

$$\Lambda_c^i = \log_{p_i} |\gamma_{c+1}^*(S_i^{ab})|\,,$$

$$\Psi_c^i = \log_{p_i} |\otimes^{c+1}([S_i,S_i],S_i)|\,,$$

$$\Theta_c^i = \log_{p_i} |\otimes^{c+1}(L,G)_{p_i}|\,.$$

Propositions 10 and 14 imply exact sequences

$$\otimes^{c+1}(L,G) \to \gamma_{c+1}^*(G) \to \prod_{i=1}^{n} \gamma_{c+1}^*(S_i) \to 1\,,$$

$$\otimes^{c+1}([S_i,S_i],S_i) \to \gamma_{c+1}^*(S_i) \to \gamma_{c+1}^*(S_i^{ab}) \to 1\,.$$

Hence

$$|\gamma_{c+1}^*(G)| \leq \prod_{i=1}^{n} p_i^{\Lambda_c^i + \Psi_c^i + \Theta_c^i}\,.$$

To complete the proof we must find appropriate upper bounds $\Lambda_c(S_i^{ab})$, $\Psi_c(S_i)$, $\Theta_c(G,L,P_i)$ for Λ_c^i, Ψ_c^i, Θ_c^i.

Proposition 13 furnishes the appropriate formula for $\Lambda_c(S_i^{ab})$. Proposition 2 provides the appropriate formula for $\Psi_c(S_i)$. Suppose that P_i is a d_i-generator group, that $P_i/(P_i \cap [G,G])$ is a δ_i-generator group, that $(L \cap P_i)/(L \cap \Phi(P_i))$ has order p^{t_i}, that $L \cap P_i$ has order $p_i^{\beta_i}$, and that $[L \cap P_i, P_i]$ has order $p^{\beta_i'}$. Set $M = \mathcal{M}(L,G) = \ker(L \otimes G \twoheadrightarrow L)$ and let M_{p_i} denote the p_i-primary component of M. Since the pair (L,G) is perfect, Proposition 6 implies

$$|\otimes^{c+1}(L,G)| \leq |\otimes^{c}(M,G^{ab})| \times |\otimes^{c-1}(M,G^{ab})| \times \cdots \times |\otimes^{2}(M,G^{ab})| \times |M| \times |L|\,,$$

and thus

$$|\otimes^{c+1}(L,G)_{p_i}| \le |\otimes^{c}(M_{p_i},G^{ab})| \times |\otimes^{c-1}(M_{p_i},G^{ab})| \times \cdots \times |\otimes^{2}(M_{p_i},G^{ab})| \times |M_{p_i}| \times |L\cap P_i| \,.$$

Proposition 7(iv) implies that $M_{p_i} \subset \mathcal{M}(L\cap P_i, P_i)$. Since $|L\cap P_i| = p_i^{\beta_i}$, Proposition 8 implies that

$$|\mathcal{M}(L\cap P_i,P_i)| \times |[L\cap P_i,P_i]| \le p_i^{d_i\beta_i - \frac{t_i(t_i+1)}{2}} \,.$$

Setting $\omega_i = d_i\beta_i - (1/2)t_i(t_i+1)$, we have

$$|M_{p_i}| \le |\mathcal{M}(L\cap P_i,P_i)| \le p_i^{\omega_i-\beta_i'}$$

and hence

$$|\otimes^{c+1}(L,G)_{p_i}| \le p_i^{(\omega_i-\beta_i')\delta_i^c} p_i^{(\omega_i-\beta_i')\delta_i^{(c-1)}} \cdots p_i^{(\omega_i-\beta_i')\delta_i} p_i^{(\omega_i-\beta_i')} p_i^{\beta_i} \,.$$

This yields the appropriate formula for $\Theta_c(G,L,P_i)$.

5. Proof of Theorem B

Let G,S_i,P_i,L be as in the statement of Theorem B. Let $\overline{\gamma}^*_{c+1}(L,G)_{p_i}$ denote a p_i-Sylow subgroup of the group $\overline{\gamma}^*_{c+1}(L,G)$. The short exact sequence $\overline{\gamma}^*_{c+1}(L,G) \hookrightarrow \gamma^*_{c+1}(G) \twoheadrightarrow \gamma^*_{c+1}(G/L)$ with Proposition 14 implies

$$\exp(\gamma^*_{c+1}(G)) \le \prod_{i=1}^{n} \exp(\overline{\gamma}^*_{c+1}(L,G)_{p_i}) \exp(\gamma^*_{c+1}(S_i)) \,.$$

We apply Propositions 6 and 10 and the surjection $\otimes^{c+1}(L,G) \twoheadrightarrow \overline{\gamma}^*_{c+1}(L,G)$ to obtain

$$\exp(\gamma^*_{c+1}(G)) \le \prod_{i=1}^{n} \exp((\mathcal{M}(L,G)\otimes G^{ab})_{p_i})^{c-1} \exp((L\otimes G)_{p_i}) \exp(\gamma^*_{c+1}(S_i)) \,.$$

Proposition 7(iv) yields

$$\exp(\gamma^*_{c+1}(G)) \le \prod_{i=1}^{n} \exp(\mathcal{M}(L\cap P_i,P_i)\otimes G^{ab})^{c-1} \exp((L\otimes G)_{p_i}) \exp(\gamma^*_{c+1}(S_i)) \,.$$

The exact sequence

$$(L\cap P_i)\wedge P_i \to (L\otimes G)_{p_i} \to L\cap P_i/[L\cap P_i,P_i] \to 1$$

is readily derived, and yields

$$\exp(\gamma^*_{c+1}(G)) \le \prod_{i=1}^{n} \exp(\mathcal{M}(L\cap P_i,P_i)\otimes G^{ab})^{c-1} \exp((L\cap P_i)\wedge P_i) \exp(\frac{L\cap P_i}{[L\cap P_i,P_i]}) \exp(\gamma^*_{c+1}(S_i)) \,.$$

The bound of Theorem B(i) now follows from Propositions 9 and 11. (We take $N = G = S_i$ in Proposition 11.)

To prove Theorem B(ii) it suffices to note that the condition $[[P^{p^{i-1}},P],P] \subset P^{p^i}$ is equivalent to saying that the pair $(P^{p^{i-1}}/P^{p^i},\ P/P^{p^i})$ has nilpotency class at most 2.

6. Proof of Theorem C

Consider the dihedral group $D_n =< a,b \mid a^2 = b^n = (ab) >$ with $n = 2^r m$ where $m \ge 1$ is odd. The smallest term of the lower central series of D_n is $L = \gamma_r(D_n) \cong C_m$. Proposition 7(ii) implies that the relative multiplier $\mathcal{M}(L,G)$ is trivial. Proposition 6 therefore implies an isomorphism $\otimes^{c+1}(L,G)\cong L\otimes G$ for all $c\geq 1$. So Proposition 10 yields a commutative diagram

$$\begin{tikzcd}
 & \mathcal{M}(L,D_n)=1 \arrow[r] \arrow[d, tail] & M^{(c)}(D_n) \arrow[r, "\cong"] \arrow[d, tail] & M^{(c)}(D_{2^r}) \arrow[r] \arrow[d, tail] & L/\gamma_{c+1}(L,D_n)=1 \\
 & L\otimes D_n \arrow[r] \arrow[d, two heads] & \gamma^*_{c+1}(D_n) \arrow[r] \arrow[d, two heads] & \gamma^*_{c+1}(D_{2^r}) \arrow[r] \arrow[d, two heads] & 1 \\
1 \arrow[r] & C_m \arrow[r] & \gamma_{c+1}D_n \arrow[r] & \gamma_{c+1}D_{2^r} \arrow[r] & 1
\end{tikzcd}$$

in which the rows are exact and the columns are short exact. From this, and the isomorphism $\gamma_{c+1}D_n\cong C_m\times\gamma_{c+1}D_{2^r}$, we derive the isomorphism

$$\gamma^*_{c+1}(D_n)\cong C_m\times\gamma^*_{c+1}(D_{2^r})\,.$$

A description of $\gamma^*_{c+1}(D_{2^r})$ is given in [16]. Alternatively, the description can be re-obtained as follows. The central extensions $C_2\hookrightarrow D_{2^r}\twoheadrightarrow D_{2^{r-1}}$ and repeated applications of Proposition 15(i) imply that $|\gamma^*_{c+1}(D_{2^r})|\geq|\gamma^*_{c+1}(C_2\times C_2)|\times 2^{r-1}$. These central extensions together with repeated applications of Proposition 15(iii) imply that $\gamma^*_{c+1}(D_{2^r})$ has the same number of generators as $\gamma^*_{c+1}(C_2\times C_2)$, namely one generator for each basic commutator on two generators a,b. Theorem B implies that $\exp(\gamma^*_{c+1}(D_{2^r}))\leq 2^r$. So, to obtain the isomorphism $\gamma^*_{c+1}(D_{2^r})\cong C_{2^r}\oplus(C_2)^{\chi_{c+1}(2)-1}$ it suffices to verify that at least all but one of the generators are of order 2, and that the generators a,b act as stated in the theorem. (Note that the invariant $\gamma^*_{c+1}(G)$ is abelian precisely when $\gamma_{c+1}(G)$ acts trivially on it.) This verification, which we leave to the reader, yields the desired description of $\gamma^*_{c+1}(D_{2^r})$ and completes the proof of part (i) of the theorem.

To prove part (ii) we note that the quaternion group Q_n is not of the form $H/Z(H)$ for any group H with centre $Z(H)$ [3]. Proposition 12(ii)(iii) thus implies that the cth term $Z^*_c(Q_n)$ of the upper epicentral series of Q_n contains the centre $Z(Q_n)\cong C_2$. Since $Q_n/Z(Q_n)\cong D_n$, the isomorphism $\gamma^*_{c+1}(Q_n)\cong\gamma^*_{c+1}(D_n)$ follows from Proposition 12(iv).

To prove part (iii) we note that an extraspecial group $E(p,k)$, $k\geq 2$, is not of the form $H/Z(H)$ for any group H with centre $Z(H)$ [3]. Arguing as in the previous paragraph, we see that $\gamma^*_{c+1}(E(p,k))\cong\gamma^*_{c+1}(E(p,k)/Z(E(p,k)))\cong\gamma^*_{c+1}(C_p\times C_p)$. Proposition 13 completes the proof of part (iii). This argument also holds for $E(p,1)^-$, $p\geq 3$.

To obtain our partial description of $\gamma^*_{c+1}(E(p,1)^+)$, $p\geq 3$, we first remark that $Z^*_1(G)$ is trivial for the group $G=E(p,1)^+$ [3]. Letting $Z=Z(G)$ denote the centre of this group, Proposition 10 and Proposition 1(v) yield an exact sequence $\otimes^{c+1}(Z,G)=Z\otimes G^{ab}\otimes\cdots\otimes G^{ab}\to\gamma^*_{c+1}(G)\xrightarrow{\phi}\gamma^*_{c+1}(C_p\times C_p)\to 1$. The group $\otimes^{c+1}(Z,G)$ is elementary abelian of rank 2^c, and Proposition 12(iv) implies that ϕ has non-trivial kernel. Theorem B implies that $\exp(\gamma^*_{c+1}(G))=p$. The commutator subgroup $[G,G]=Z$ acts trivially on $\gamma^*_{c+1}(G)$, and so $\gamma^*_{c+1}(G)$ is abelian. Hence $\gamma^*_{c+1}(G)$ is elementary abelian of rank at most $\chi_{c+1}(2)+2^c$, and at least $\chi_{c+1}(2)+1$.

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