Higher K-Theory of Modules Over EI-Categories

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Introduction

Let $\mathcal{C}$ be an EI- category (i.e. a small category in which every endomorphism is an isomorphism), R a commutative ring with dentity. An $R\mathcal{C}$- module is a contravariant functor from $\mathcal{C}$ to the category of R-modules. For all $n \geq 0$, let $K_n(R\mathcal{C})$ be the (Quillen) K_n of the category $\mathbf{P}(R\mathcal{C})$ of finitely generated projective $R\mathcal{C}$- modules (see §1 for definitions).

The significance of the study of K- theory of $R\mathcal{C}$- modules lies mainly in the fact that several geometric invariants take values in the K- groups associated with $R\mathcal{C}$ where $\mathcal{C}$ is an apropriately defined EI- category and R could be $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, etc. For example, if π is a finite group, $\mathcal{C} = \text{orb}(\pi)$ the orbit category of π, (a finite EI-category), X a $\pi - CW$ complex with round structure, then the equivariant Reidemester torsion takes values in $Wh(\mathbb{Q}\text{orb}(\pi))$ where $Wh(\mathbb{Q}\text{orb}(\pi))$ is the quotient of $K_1(\mathbb{Q}\ \text{orb}(\pi))$ by its subgroup of "trivial units"-see [16].

Now, let $\mathcal{C}$ be a finite EI- category, R the ring of integers in a number field F. We show in §2 that for all $n \geq 0$, $K_n(R\mathcal{C})$ is a finitely generated Abelian group and that $SK_n(R\mathcal{C})$ is a finite group, while in §3 we show that for all $n \geq 1$, $G_n(R\mathcal{C}) = K_n$ of the category of finitely generated $R\mathcal{C}$- modules is a finitely generated Abelian group and that $SG_n(R\mathcal{C}) = 0$.

In §3 we consider Cartan maps and show that if k is a field of characteristic p, and $\mathcal{C}$ a finite EI- category, then for all $n \geq 0$, the Cartan map $K_n(k\mathcal{C}) \to G_n(k\mathcal{C})$ induces an isomorphism

$$\mathbb{Z}\left(\frac{1}{p}\right) \otimes K_n(k\,\mathcal{C}) \simeq \mathbb{Z}\left(\frac{1}{p}\right) \otimes G_n(k\,\mathcal{C}).$$

We then discuss some consequences of this result for integers in number fields.

In a final section, we discuss module structures on $K_n(R\mathcal{C})$, $G_n(R\mathcal{C})'$ as modules over $G_o(R,\mathcal{C}) := K_o$ of the the category $\mathbf{P}_R(R\mathcal{C})$ of finitely generated $R\mathcal{C}$ -modules M such that $M(X)$ is a projective R -module for all $X \in \text{orb}\,\mathcal{C})$ as well as modules over the Burnside ring of $\mathcal{C}$ (see §5).

Notes on Notation. For a finite group π, we write $\text{con}(\pi)$ for the set of conjugacy classes of π and $\overline{\gamma}$ for the conjugacy class of γ if $\gamma \leq \pi$. If $\mathcal{C}$ is a category, X, Y $\mathcal{C}$-objects, we write $\mathcal{C}(X,Y)$ for the set of $\mathcal{C}$- morphisms from X to Y, and $Is(\mathcal{C})$ for the set of isomorphism classes of $\mathcal{C}$- objects.

If R is a commutative ring with identity, then for any set S, we write RS for the free R- module on S. In particular, we write $R\mathcal{C}(X,Y)$ for the free R- module on the set $\mathcal{C}(X,Y)$,

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For any exact category $\mathcal{E}$, we write $K_n(\mathcal{E})$ for $\pi_{n+1}(BQ\mathcal{E})$ (see [17]). If A is a ring with identity, we write $\mathbf{P}(A)$ for the category of finitely generated projective A-modules. If A is Noetherian, we write $\mathbf{M}(A)$ for the category of finitely generated A- modules. We denote $K_n(\mathbf{P}(A)$ by $K_n(A)$, $K_n(\mathbf{M}(A))$ by $G_n(A)$. Also $K_n(A) = \pi_n(BCL^+(A))$ in the notation of the plus construction of Quillen (see [4]).

§1

Modules over EI-categories

In this section, we briefly discuss modules over EI- categories and the associated K- theories, thus developing some necessary notations.

Definition 1.1. An EI-category $\mathcal{C}$ is a small category in which every endomorphism is an isomorphism. $\mathcal{C}$ is said to be finite if the set $Is(\mathcal{C})$ of isomorphism classes of $\mathcal{C}$-objects is finite and for any two $\mathcal{C}$- objects X, Y, the set $\mathcal{C}(X, Y)$ of $\mathcal{C}$-morphism from X to Y is finite

Examples.

(i) Let π be a finite group. Let ob$\mathcal{C} = \{\pi/\gamma | \gamma \leq \pi\}$ and morphisms be π- maps. Then $\mathcal{C}$ is a finite EI-category called the orbit category of π and denoted orb(π). Here $\mathcal{C}(\pi/\gamma, \pi/\gamma) \simeq \text{Aut}(\pi/\gamma) \approx N_\pi(\gamma)/\gamma$ where $N_\pi(\gamma)$ is the normaliser of γ in π. (see [12]). We shall denote the group $N_\pi(\gamma)/\gamma$ by $\overline{N_\pi(\gamma)}$.

(ii) Suppose that π be a Lie group. $ob\mathcal{C} = \{\pi/\gamma | \gamma$ compact subgroup of $\pi\}$ is also called the orbit category of π and denoted orb(π).

(iii) Let π be a Lie group. Let ob$\mathcal{C} = \{\pi/\gamma | \gamma$ compact subgroup of $\pi\}$ and for $\pi/\gamma, \pi/\gamma' \in$ ob$\mathcal{C}$, let $\mathcal{C}(\pi/\gamma, \pi/\gamma')$ be the set of homotopy classes of π- maps. Then $\mathcal{C}$ is an EI- category called the discrete orbit category of π and denoted by orb$/(\pi)$.

Note. For further examples, see [16].

1.2.

Let R be a commutative ring with identity, $\mathcal{C}$ an EI-category. An $R\mathcal{C}$- module as a contravariant functor $\mathcal{C} \to R-\textbf{mod}$

An ob$\mathcal{C}$- set is a functor N from $\mathcal{C}$ to the category of sets. Alternatively, an ob$\mathcal{C}$-set could be visualized as a pair (N, β) where N is a set and $\beta : N \to$ ob $\mathcal{C}$ is a set map. Then $N = \{\beta^{-1}(X) | X \in \text{ob}\ \mathcal{C}\}$.

An ob $\mathcal{C}$- set (N, β) is said to be finite if N is a finite set. If S is an (N, β)-subset of an $R\mathcal{C}$- module M, define span$S$ as the smallest $R\mathcal{C}$- submodule of M containing S. Say that M is finitely generated if $M =$ span S for some finite ob $\mathcal{C}$-subset S of M.

If R is a Noetherian ring and $\mathcal{C}$ a finite EI-category, let $\mathbf{M}(R\mathcal{C})$ be the category of finitely generated $R\mathcal{C}$- modules. Then $\mathbf{M}(R\mathcal{C})$ is an exact category in the sense of Quillen [17], see [16].

An $R\mathcal{C}$- module P is said to be projective if any exact sequence of $R\mathcal{C}$- modules $0 \to M' \to M \to P \to 0$ splits or equivalently if Hom $_{R\mathcal{C}}(P, -)$ is exact.

Let $\mathbf{P}(R\mathcal{C})$ be the category of finitely generated projective $R\mathcal{C}$- modules. Then $\mathbf{P}(R\mathcal{C})$ is also exact. We write $K_n(R\mathcal{C})$ for $K_n(\mathbf{P}(R\mathcal{C}))$.

Finally, let R be a commutative ring with identity, $\mathcal{C}$ a finite EI- category, $\mathbf{P}_R(R\mathcal{C})$ the category of finitely generated $R\mathcal{C}$- modules such that for each $X \in$ ob $\mathcal{C}$, $M(X)$ is projective as R- module. Then $\mathbf{P}_R(R\mathcal{C})$ is an exact category and we write $G_n(R, \mathcal{C})$ for $K_n(\mathbf{P}_R(R\mathcal{C}))$. Note that if R is regular, then $G_n(R, \mathcal{C}) \cong G_n(R\mathcal{C})$.

1.3.

Let $\mathcal{C}_1, \mathcal{C}_2$ be EI- categories, R a commutative ring with identity, B a functor $\mathcal{C}_1 \to \mathcal{C}_2$. Let $R\mathcal{C}_2(??, B(?))$ be the $R\mathcal{C}_1 - R\mathcal{C}_2$- bimodule: $\mathcal{C}_1 \times \mathcal{C}_2 \to R-$mod given by $(X_1, X_2) \to R\mathcal{C}_2(X_2, BX_1)$. Define $R\mathcal{C}_2 - R\mathcal{C}_1$ bimodule analogously. Now define induction functor $\text{ind}_B : \mathcal{C}_1 - \text{mod} \to R\mathcal{C}_2 - \text{mod}$ given by $M \to M \underset{R\mathcal{C}_1}{\otimes} R\mathcal{C}_2(??, B(?))$

Also define a restriction functor $\text{res}_B : R\mathcal{C}_2 - \text{mod} \to R\mathcal{C}_1 - \text{mod}$ by $N \to N \circ B$.

1.4.

A homomorphism $R_1 \to R_2$ of commutative rings with identity induces a functor $B : R_1 - \text{mod} \to R_2 - \text{mod} : N \to R_2 \underset{R_1}{\otimes} N$. So, if $\mathcal{C}$ is an EI-category, we have an induced functor $R_1\mathcal{C} - \text{mod} \to R_2\mathcal{C} - \text{mod} : M \to B \circ M$ where $(B \cdot M)(X) = B(M(X))$.

We also have an induced exact functor $\mathbf{P}(R_1\mathcal{C}) \to \mathbf{P_2}(R_2\mathcal{C}) : M \to B \circ M$ and hence a homomorphism $K_n(R_1\mathcal{C}) \to K_n(R_2\mathcal{C})$.

Now, suppose that R is a Dedekind domain with quotient fields F, and $R \hookrightarrow F$ the inclusion map, it follows from above that we have group homomorphisms $K_n(R\mathcal{C}) \to K_n(F\mathcal{C})$ and $G_n(R\mathcal{C}) \to G_n(F\mathcal{C})$.

Now define $SK_n(R\mathcal{C}) :=$ Kernel of $K_n(R\mathcal{C}) \to K_n(F\mathcal{C})$ and $SG_n(R\mathcal{C}) :=$ Kernel of $G_n(R\mathcal{C}) \to G_n(F\mathcal{C})$.

§2

$K_n(RC), SK_n(RC)$

The aim of this section is to prove the following:

Theorem 2.1.

Let R be the ring of integers in a number field F, C any finite EI- category. Then for all $n \geq 1$,

(i) $K_n(RC)$ is a finitely generated Abelian group.

(ii) $SK_n(RC)$ is a finite group.

The proof of 2.1 will depend on the following splitting result

Theorem 2.2.

Let R be a commutative ring with identity, C a finite EI- category. Then

$$K_n(RC) \cong \bigoplus_{\overline{X} \in Is(C)} K_n(R(Aut(X)))$$

Proof.

We give a sketch of the proof of 2.2. Details can be found in [16].

Step I: For $X \in \text{ob}C$, define the "splitting functor" $S_X : RC - \mathbf{mod} \to R(Aut(X)) - \mathbf{mod}$ by $S_X(M) = M(X)/M'(X)$ where $M'(X)$ is the R- submodule of $M(X)$ generated by the images of the R- homomorphisms $M(f) : M(Y) \to M(X)$ induced by all non-isomorphisms $f : X \to Y$.

Step II: Define the 'extension functor' $E_X : R(Aut(X)) - \mathbf{mod} \to RC - \mathbf{mod}$ by

$$(E_X(M) = M \bigotimes_{RAut(X)} RC(?, X)$$

Step III. For $\overline{U} \in Is(C)$, the objects $X \in \overline{U}$ constitute a full subcategory of C which we denote by $C(U)$. Now define

$$\text{split}K_n(RC) := \bigoplus_{\overline{U} \in Is(C)} K_n(RC(U))$$

Step IV. For each $U \in \text{ob}C$, define the functor $\hat{E}_U : RC(U) - \mathbf{mod} \to RC\text{-}\mathbf{mod}$ by

$$\hat{E}_U(M) = M \underset{RC(U)}{\otimes} RC(?, ??)$$

This induces a functor $\mathbf{P}(RC(U)) \to \mathbf{P}(RC)$ and a homomorphism $K_n(\hat{E}_U) : K_n(RC(U)) \to K_n(RC)$ and hence a homomorphism

$$E_n(RC) = \oplus K_n(\hat{E}_U) : \oplus_{\overline{U} \in IsC} K_n(RC(U)) \to K_n(RC)$$

that is, a homomorphism

$$E_n(RC) : \text{split}K_n(RC) \to K_n(RC)$$

Step V: For any $U \in$ ob $(\mathcal{C})$, define a functor $\hat{S}_U : R\mathcal{C} - \mathbf{mod} \to R\mathcal{C}(U) - \mathbf{mod}$ by $\hat{S}_U(M) = M \bigotimes_{R\mathcal{C}} B$ for the $R\mathcal{C} - R\mathcal{C}(U)$ bimodule B given by $B(X,Y) = R\mathcal{C}(X,Y)$ if $Y \in \overline{U}$ and $B(X,Y) = \{0\}$ if $Y \notin \overline{U}$, where X runs through ob $\mathcal{C}(U)$, and $Y \in$ ob$(\mathcal{C})$. Then each $\hat{S}_U$ induces a homomorphism $K_n(\hat{S}_U) : K_n(R\mathcal{C}) \to K_n(R\mathcal{C}(U))$ and hence a homomorphism

$$S_n(R\mathcal{C}) : K_n(R\mathcal{C}) \to \bigoplus_{\overline{U} \in Is(\mathcal{C})} K_n(R\mathcal{C}(U))$$

i.e.

$$S_n(R\mathcal{C}) : K_n(R\mathcal{C}) \to \mathrm{split}K_n(R\mathcal{C}).$$

Step VI:

$$E_n(R\mathcal{C}) : K_n(R\mathcal{C}) \to \bigoplus_{\overline{U} \in Is(\mathcal{C})} K_n R\mathcal{C}(U)$$

and

$$S_n(R\mathcal{C}) : \bigoplus_{\overline{U} \in Is(\mathcal{C})} K_n(R\mathcal{C}(U)) \to K_n(R\mathcal{C})$$

are isomorphisms, one the inverse of the other.

Step VII:

$$K_n(R\mathcal{C}(U)) \simeq K_n(R(Aut(X)))$$

(for any $X \in \overline{U}$), via the equivalence of categories $Aut(X)' \to \mathcal{C}(U)$ where for any group π, π' is the groupoid with one object π and morphisms left translations $\lg : \pi \hookrightarrow \pi : h \hookrightarrow \underline{gh}$. □

We also reed the following

Theorem 2.3. *Let $\mathcal{C}$ be a finite EI- category, R the ring of integers in a number field F. Then for any $\mathcal{C}$- object $X, K_n(R(AutX))$ is finitely generated Abelian group for all $n \geq 1$.*

Proof. Write A for $R(Aut(x))$ and put $K_{n,m}(A) := \pi_n(BGL_m^+(A))$. Then $K_n(A) = \lim_{m\to\infty} K_{n,m}(A)$. Now, $BE_n^+(A)$ is the universal covering space of $BGL_m^+(A)$ (since $BE(A)$ is the coverig space of $BGL(A)$ with respect to the subgroup $E(A)$ of $GL(A)$ generated by elementary matrices):

So

$$\pi_n(BE_m^+(A)) \simeq \pi_n(BGL_m^+(A))$$

Now, by the stability result of Suslin (see [19]), $K_{n,m}(A) \simeq K_{n,m+1}(A)$ if $m \geq (2n+2, n+3)$ since A as an R- order satisfies the stable range condition SR_3 (see [1]). So $K_n(A) \simeq \pi_n(BE_m^+(A))$ for $m \geq (2n+2, n+3)$. Now, for $m \geq 3$, $E_m(A)$ is an Arithmetic group since $SL_n(A)/E_n(A)$ is a finite group. Hence by Borel-Serre, [6], $H_n(E_m(A))$ is finitely generated. Now, for all $m \geq 2$ $H_n(E_m(A) = H_n(BE_m(A)) = H_n(BE_m^+(A))$ by Quillen's plus construction since $BE_m(A) \to BE_m^+(A)$ is acyclic. Moreover, $BE_m^+(A)$ is a simply connected H- space for $m \geq 3$. Also by [18] 9.6.16, $\pi_n(BE_m^+(A))$ is finitely generated if and only if $H_n(BE_m^+(A))$ is finitely generated. Hence $K_n(A) \simeq \pi_n(BE_m^+(A))$ is finitely generated. □

Proof of 2.1.

(i) follows from 2.2 and 2.3 since $K_n(R\mathcal{C})$ is a finite direct sum of finitely generated groups $K_n(R\,Aut(X))\,\overline{X} \in Is(\mathcal{C})$.

(ii) First note that we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 \longrightarrow & SK_n(R\mathcal{C}) & \longrightarrow & K_n(R\mathcal{C}) & \longrightarrow & K_n(F\mathcal{C}) \\
 & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \approx} \\
0 \longrightarrow & \bigoplus\limits_{\overline{X}\in Is\mathcal{C}} SK_n(RAut(X)) & \longrightarrow & \bigoplus\limits_{\overline{X}\in Is\mathcal{C}} K_n(R(Aut(X))) & \longrightarrow & \bigoplus\limits_{\overline{X}\in Is(\mathcal{C})} K_n(FAut(X))
\end{array}$$

Now by [13], 3.2, $SK_n(R(Aut(X)))$ is a finite group. Hence $\bigoplus\limits_{\overline{X}\in Is\mathcal{C}} SK_n(RAut(X))$ being a direct sum of finite groups is finite. That $SK_n(R\mathcal{C})$ is finite now follows from the fact that α is injective.

Remarks 2.4. Let π be a finite group and $\mathcal{C} = \mathrm{orb}(\pi)$ the orbit category of π (see E.g 1.1[1]).

It is well known that there is one-one correspondence between $Is(\mathcal{C})$ and the conjugacy classes $con(\pi)$ of π , i.e. $\pi/\gamma \simeq \pi/\gamma'$, if γ is conjugate to γ'. It is also well-known that $\mathcal{C}(\pi/\gamma, \pi/\gamma' = Aut(\pi/\gamma) = N_\pi(\gamma)/\gamma := \overline{N_\pi \gamma}$ where $N_\pi\gamma$ is the normaliser of γ in π.

So, for any commutative ring R with identity, $K_n(R\,\mathrm{orb}(\pi)) = \bigoplus\limits_{\gamma\in con(\pi)} K_n R\overline{N_\pi(\gamma)}$.

§3

$G_n(RC), SG_n(RC)$

The aim of this section is to prove the following

Theorem 3.1. *Let R be the ring of integers in a number field F, C a finite EI-category. Then for all $n \geq 1$*
(i) $G_n(RC)$ is a finitely generated Abelian group
(ii) $SG_n(RC) = 0$.

The proof of 2.1 depends on the following splitting result for G_n of RC.

Theorem 3.2. *Let R be a commutative Noetherian ring with identity, C any finite EI- category. Then for all $n \geq 1$*

$$G_n(RC) \cong \bigoplus_{\overline{X} \in Is(C)} G_n(R(Aut(X))).$$

Proof. We sketch the proof of 2.2 and refer the reader to [16] for missing details.

Step I: For each $X \in \text{ob}\, C$, define $Res_X : RC$-**mod** $\to (R\,Aut(X))$ **-mod** $Res_X(M) = M(X)$. Then an RC- module M is finitely generated iff $Res_X(M)$ is finitely generated for all Xin ob C (see [16]).

Moreover, Res_X induces an exact functor $\mathbf{M}(RC) \to \mathbf{M}(R(Aut(X)))$ which also induces for all $n \geq 0$ homomorphisms $G_n(RC) \to G_n(R(Aut(X)))$ and hence homomorphism $Res : G_n(RC) \to \bigoplus_{\overline{X} \in Is(C)} G_n(R(Aut(X)))$. We write split$G_n(RC)$ for $\bigoplus_{\overline{X} \in IsC} G_n(R(Aut(X)))$.

Step II: For $X \in \text{ob}\, C$, define a functor $I_X : R(Aut(X)) \to RC$ by

$$I_X(M) = \begin{cases} M \underset{RAut(X)}{\otimes} RC(Y,X) & \text{if } \overline{Y} = \overline{X} \\ 0 & \text{if } \overline{Y} \neq \overline{X} \end{cases}$$

Then we have an induced homomorphism

$$I : \text{split}\, G_n(RC) = \bigoplus_{\overline{X} \in IsC} G_n(R(Aut(X))) \to G_n(RC).$$

Step III: Res and I are isomorphisms inverse to each other. □

Theorem 3.3. *Let R be the ring of integers in a number field F, C a finite EI-category. Then for any $X \in ob\, C$ and all $n \geq 1$ $G_n(R(Aut(X)))$ is a finitely generated Abelian group.*

Proof. We provide a sketch of proof here. Details can be found in Kuku [10].

Put $\Lambda = R(Aut(X))$ and note that Λ is an R- order in the semi-simple F- algebra $F(Aut(X))$. Let Γ be a maximal order containing Λ, and let $\alpha_n : G_n(\Gamma) \to G_n(\Lambda)$ be the homomorphism induced by the functor $\mathbf{M}(\Gamma) \to \mathbf{M}(\Lambda)$ given by restriction of scalors. As proved in [10] 1.3 (i) and (iii) for all $n \geq 1$ (a) $\alpha_{2n-1} : G_{2n-1}(\Gamma) \to G_{2n-1}(\Lambda)$ has finite kernel and cokernel (b) $\alpha_{2n} : G_{2n}(\Gamma) \to G_{2n}(\Lambda)$ is injective with finite cokernel. The conclusion that $G_n(\Lambda)$ is finitely generated follows as in the proof of [10]1.3 (iii). □

Proof of 3.1.

(i) Follows from 3.2 and 3.3 since split $G_n(RC)$ as a finite direct sum of finitely generated Abelian groups is finitely generated.

(ii) First note that the following diagram is commutative with exact rows:

$$\begin{array}{ccccccc}
0 \longrightarrow & SG_n(RC) & \longrightarrow & G_n(RC) & \longrightarrow & G_n(FC) \\
& \downarrow \alpha & & \downarrow \simeq\beta & & \downarrow \simeq\gamma \\
0 \longrightarrow & \bigoplus_{\overline{X} \in IsC} SG_n(Aut(X)) & \longrightarrow & \bigoplus_{\overline{X} \in IsC} G_n(RAut(X)) & \longrightarrow & \bigoplus_{\overline{X} \in IsC} G_n(F(AutX))
\end{array}$$

Since β is an isomorphism, α is injective. Now each $SG_n(RAut(X)) = 0$ by [15] theorem 1. Hence $SG_n(RC) = 0$.

§4

Cartan maps

Let R be a commutative Noetherian ring, $\mathcal{C}$ an EI- category. Then, for all $n \geq 0$, the inclusion functor $\mathbf{P}(R\mathcal{C}) \to \mathbf{M}(R\mathcal{C})$ induces a homomorphism $K_n(R\mathcal{C}) \to G_n(R\mathcal{C})$ called Cartan maps.

The aim of this section is to prove the following

Theorem 4.1. *Let k be a field of characteristic p, $\mathcal{C}$ a finite EI- category. Then for all $n \geq 0$, the Cartan homomorphism $K_n(k\mathcal{C}) \to G_n(k\mathcal{C})$ induce isomorphism*

$$\mathbb{Z}\left(\frac{1}{p}\right) \otimes K_n(k\mathcal{C}) \cong \mathbb{Z}\left(\frac{1}{p}\right) \otimes G_n(k\mathcal{C})$$

Proof. By 2.2

$$K_n(k\mathcal{C}) \cong \bigoplus_{\overline{X} \in Is(\mathcal{C})} K_n(kAut(X))$$

and by 3.2 $G_n(k\mathcal{C}) \cong \bigoplus_{\overline{X} \in Is\mathcal{C}} G_n(kAut(X))$. Now it was proved by Dress/Kuku (see [7]) via the theory of Mackey functors that for any finite group π the Cartan map $K_n(k\pi) \to G_n(k\pi)$ induces an isomorphism

$$\mathbb{Z}\left(\frac{1}{p}\right) \bigotimes K_n(k\pi) \simeq \mathbb{Z}\left(\frac{1}{p}\right) \bigotimes G_n(k\pi).$$

Hence $\mathbb{Z}\left(\frac{1}{p}\right) \bigotimes K_n(kAut(X)) \simeq \mathbb{Z}\left(\frac{1}{p}\right) \bigotimes G_n(k(Aut(X)))$ for all $X \in \text{ob}(\mathcal{C})$.

Hence

$$\begin{aligned}
\mathbb{Z}\left(\frac{1}{p}\right) \bigotimes K_n(k\mathcal{C}) &\simeq \mathbb{Z}\left(\frac{1}{p}\right) \bigotimes \Big(\bigoplus_{\overline{X} \in Is\mathcal{C}} K_n(kAut(X))\Big) \\
&\simeq \bigoplus_{\overline{X} \in Is(\mathcal{C})} \left(\mathbb{Z}\left(\frac{1}{p}\right) \bigotimes K_n(kAut(X))\right) \simeq \bigoplus_{\overline{X} \in Is\mathcal{C}} \left(\mathbb{Z}\left(\frac{1}{p}\right) \bigotimes G_n(kAutX)\right) \\
&\simeq \mathbb{Z}\left(\frac{1}{p}\right) \bigotimes \Big(\bigoplus_{\overline{X} \in Is\mathcal{C}} G_n(kAut(X))\Big) \\
&\simeq \mathbb{Z}\left(\frac{1}{p}\right) \bigotimes G_n(k\mathcal{C}).
\end{aligned}$$

Corollary 4.2. *Let R be the ring of integers in a number field F, $\mathbf{m}$ a prime ideal of R lying over a rational prime p. Then for all $n \geq 1$*

(a) the Cartan map $K_n((R/\mathbf{m})\mathcal{C}) \to G_n((R/\mathbf{m})\mathcal{C})$ is surjective

(b) $K_{2n}(R/\mathbf{m})\mathcal{C})$ is a finite p- group.

Proof. Since $\mathbf{m}$ lies above a rational prime p, $R/\mathbf{m}$ is a finite field of characteristic p. Hence by 4.1, $K_n((R/\mathbf{m})\mathcal{C}) \stackrel{\alpha_n}{\to} G_n((R/\mathbf{m})\mathcal{C})$ is an isomorphism mod p-torsion for all $n \geq 0$. Now $G_n((R/\mathbf{m})\mathcal{C}) \simeq \bigoplus_{\overline{X} \in Is\mathcal{C}} G_n((R/\mathbf{m})AutX)$ by 3.2. Also

$G_n((R/\mathbf{m})(Aut(X))$ is a finite group since $((R/\mathbf{m})Aut(X))$ is a finite ring. (G_n of a finite ring is finite see[8]). So $G_n((R/\mathbf{m})\mathcal{C})$ is a finite group.

Also $G_{2n}((R/\mathbf{m})\mathcal{C}) = \bigoplus_{\overline{X}\in Is\mathcal{C}} G_{2n}((R/\mathbf{m})Aut(X)) = 0$ since each $G_{2n}((R/\mathbf{m})Aut(X)) = 0$ see [8]). So, Coker $\alpha_{2n} = 0$ i.e. α_{2n} is surjective.

Now each $G_{2n-1}((R/\mathbf{m})Aut(X))$ has order relatively prime to p by [10]1.1. Hence $G_{2n-1}((R/\mathbf{m})\mathcal{C})$ is finite of order relatively prime to p. Now $|\text{Coker}\,\alpha_{2n-1}|$ is a power of p and devides $|G_{2n-1}((R/\mathbf{m})\pi)$ which is $\equiv 1\text{mod}(p)$ and this is possible if and only if $\text{Coker}\,\alpha_{2n-1} = 0$. Hence $\text{Coker}\,\alpha_n = 0\ \forall n \geq 1$. i.e. α_n is surjective $\forall n \geq 1$.

(ii) Since $G_{2n}((R/\mathbf{m})\mathcal{C}) = 0$, we have $\text{Ker}\,\alpha_{2n} = K_{2n}((R/\mathbf{m})\mathcal{C})$. Now, $K_{2n}((R/\mathbf{m})\mathcal{C}) \simeq \bigoplus_{\overline{X}\in Is\mathcal{C}} K_{2n}((R/\mathbf{m})Aut(X))$ is a finite group, since each $((R/\mathbf{m})Aut(X))$ is a finite ring and K_n of a finite ring is finite by [13].1.1. So $\text{Ker}\,\alpha_{2n} = K_{2n}(R/\mathbf{m})\mathcal{C})$ is a finite p- group.

§5

PAIRINGS and module STRUCTURES

5.1. Let $\mathcal{E}, \mathcal{E}_1, \mathcal{E}_2$ be three exact categories and $\mathcal{E}_1 \times \mathcal{E}_2$ the product category. An exact pairing $\mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E} : (M_1, M_2) \to M_1 \circ M_2$ is a covariant functor from $\mathcal{E}_1 \times \mathcal{E}_2$ to $\mathcal{E}$ such that $\mathcal{E}_1 \times \mathcal{E}_2((M_1, M_2), (M_1', M_2')) = \mathcal{E}_1(M_1, M_1') \times \mathcal{E}_2(M_2, M_2') \to \mathcal{E}(M_1 \circ M_2, M_1' \circ M_2')$ is bi-additive and bi-exact, that is for a fixed M_2, the functor $\mathcal{E}_1 \to \mathcal{E}$ given by $M_1 \mapsto M_1 \circ M_2$ is additive and exact and for fixed M_1, the functor $\mathcal{E}_2 \to \mathcal{E} : M_2 \to M_1 \circ M_2$ is additive and exact. It follows from [21], that such a pairing gives rise to a K-theoretic cup product $K_i(\mathcal{E}_j) \times K_j(\mathcal{E}_2) \to K_{i+j}(\mathcal{E})$, and in particular to natural pairing $K_0(\mathcal{E}_1) \circ K_n(\mathcal{E}_2) \to K_n(\mathcal{E})$ which could be defined as follows:

Any object $M_1 \in \mathcal{E}$ induces an exact functor $M_1 : \mathcal{E}_2 \to \mathcal{E} : M_2 \to M_1 \circ M_2$ and hence a map $K_n(M_1) : K_n(\mathcal{E}_2) \to K_n(\mathcal{E})$. If $M_1' \to M_1 \to M_1''$ is an exact sequence in $\mathcal{E}_1$, then we have an exact sequence of exact functors $M_1' \to M_1^* \to M_1''^*$ from $\mathcal{E}_2$ to $\mathcal{E}$ such that for each object $M_2 \in \mathcal{E}_2$, the sequence $M_1'(M_2) \to M_1^*(M_2) \to M_1''^*(M_2)$ is exact in $\mathcal{E}$ and hence by a result of Quillen [17], induces the relation $K_n(M_1'^*) + K_n(M_1''^*) = K_n(M_1^*)$. So, the map $M_1 \to K_n(M_1) \in \operatorname{Hom}(K_n(\mathcal{E}_2), K_n(\mathcal{E}))$ induces a homomorphism $K_0(\mathcal{E}_1) \to \operatorname{Hom}(K_n(\mathcal{E}), K_n(\mathcal{E}))$ and hence a pairing $K_0(\mathcal{E}_1) \times K_n(\mathcal{E}) \to K_n(\mathcal{E})$. We could obtain a similar pairing $K_n(\mathcal{E}_1) \times K_0(\mathcal{E}_2) \to K_n(\mathcal{E})$.

If $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$ and the pairing $\mathcal{E} \times \mathcal{E}$ is naturally associative (and commutative) then the associated pairing $K_0(\mathcal{E}) \times K_0(\mathcal{E}) \to K_0(\mathcal{E})$ turns $K_0(\mathcal{E})$ into an associative (and commutative ring which may not contain the identity). Suppose that there is a pairing $\mathcal{E} \circ \mathcal{E}_1 \to \mathcal{E}_1$ which is naturally associative with respect to the pairing $\mathcal{E} \circ \mathcal{E} \to \mathcal{E}$, then the pairing $K_0(\mathcal{E}) \times K_n(\mathcal{E}_1) \to K_n(\mathcal{E}_1)$ turns $K_n(\mathcal{E}_1)$ into a $K_0(\mathcal{E})$ -module which may or may not be unitary. However, if $\mathcal{E}$ contains a natural unit i.e. an object E s.t. $E \circ M = M \circ E$ are naturally isomorphic to M for each $\mathcal{E}$ -object M, then the pairing $K_0(\mathcal{E}) \times K_n(\mathcal{C}_1) \to K_n(\mathcal{E}_1)$ turns $K_n(\mathcal{E}_1)$ into a unitary $K_0(\mathcal{E})$ -module.

5.2. We now apply the above to the following situation. Let R be a commutative ring with identity, $\mathcal{C}$ a finite EI-category.

(i) Let $\mathcal{E} = \mathbf{P}_R(R\mathcal{C})$ be the category of finitely generated $R\mathcal{C}$ -modules such that for all $X \in ob(\mathcal{C})$. $M(X)$ is projective as an R-module. So, $\mathbf{P}_R(R\mathcal{C})$ is an exact category on which we have a pairing

(1) $$\otimes : \mathbf{P}_R(R\mathcal{C}) \times \mathbf{P}_R(R\mathcal{C}) \to \mathbf{P}_R(R\mathcal{C})$$

If we take $\mathcal{E}_1 = \mathbf{P}(R\mathcal{C})$, then the pairing

(2) $$\otimes : \mathbf{P}_R(R\mathcal{C}) \times \mathbf{P}(R\mathcal{C}) \to \mathbf{P}(R\mathcal{C})$$

is naturally associative with respect to the pairing (I) and so $K_n(R\mathcal{C})$ is a unitary $(K_0(\mathbf{P}_R(R\mathcal{C})) = G_0(R,\mathcal{C})$ -module. Also, $G_n(R,\mathcal{C})$ is a $G_0(R,\mathcal{C})$ -module.

5.3. Let $\mathcal{C}$ be a finite EI -category and $\mathbb{Z}(Is(\mathcal{C}))$ the free Abelian group on $Is(\mathcal{C})$. Note that $\mathbb{Z}(Is(\mathcal{C})) = \bigoplus_{Is(\mathcal{C})} \mathbb{Z}$. If $\mathbb{Z}^{(Is(\mathcal{C}))}$ is the ring of $\mathbb{Z}$-valued functions on $Is\mathcal{C}$, we can identify each element of $\mathbb{Z}(Is(\mathcal{C}))$ as a function $Is(\mathcal{C}) \to \mathbb{Z}$ via an injective map $\beta : \mathbb{Z}(Is(\mathcal{C})) \to \mathbb{Z}^{(Is(\mathcal{C}))}$ given by $\beta(X)(Y) = |\mathcal{C}(Y, X)|$ for $X, Y \in ob\mathcal{C}$. Moreover β identifies $\mathbb{Z}(Is(\mathcal{C}))$ as a subring of $\mathbb{Z}^{(Is(\mathbb{Z}))}$. Call $\mathbb{Z}(Is(\mathcal{C}))$ the Burnside ring of $\mathcal{C}$ and denote this ring by $\Omega(\mathcal{C})$. Note that if $\mathcal{C} = \mathrm{orb}(\pi)$, π a finite group, then $\mathbb{Z}(Is(\mathcal{C}))$ is the well-known Burnside ring of π which is denoted by $\Omega(\pi)$.

4.4. If R is a commutative ring with identity and $\mathcal{C}$ a finite EI -category, let $\mathbf{F}(R\mathcal{C})$ be the category of finitely generated free $R\mathcal{C}$ -modules. Then for all $n \geq 1$, the inclusion functor $\mathbf{F}(R\mathcal{C}) \to \mathbf{P}(R\mathcal{C})$ induces an isomorphism $K_n(\mathbf{F}(R\mathcal{C})) \simeq K_n(R\mathcal{C})$ and $K_0(\mathbf{F}(R\mathcal{C})) \simeq \mathbb{Z}(Is\mathcal{C})$ see [16] 10.42. Now by the discussion in 4.1, the pairing $K_0(\mathbf{F}(R\mathcal{C})) \times K_n(\mathbf{P}(R\mathcal{C})) \to K_n(\mathbf{P}(R\mathcal{C}))$ makes $K_n(R\mathcal{C})$ a unitary module over the Burnside ring $\mathbb{Z}(Is(\mathcal{C})) \simeq K_0(\mathbf{F}(R\mathcal{C}))$.

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