On complex analytic compactifications on $\mathbb{C}^3$ (II)

by

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Introduction. Let (X,Y) be a smooth projective compactification of $\mathbb{C}^3$ such that Y has at most isolated singularities. Then X is a Fano 3-fold of index r $(1 \leq r \leq 4)$ with the second Betti number $b_2(X) = 1$ and Y is a hyperplane section of X. In particular, Y is normal. In case of $r \geq 2$, such a (X,Y) is determined [1]. In case of $r = 1$, I proved in [3] that if such a (X,Y) exists, then $(X,Y) \cong (V_{22}, H_{22})$, where V_{22} is a Fano 3-fold of degree 22 in $\mathbb{P}^{13}$ and H_{22} is a singular K-3 surface which is rational (see also [11]). However, unfortunately, in case of $r = 1$, such a (X,Y) does not exist. The purpose of this paper is to prove it (see § 3).

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§ 1. Determination of boundary

Let $(X,Y) = (V_{22},H_{22})$ be as in Introduction. Let us denote the singular locus by Sing Y . We put

$$S := \{y \in \text{Sing } Y;\ Y \text{ is not a rational double point}\} .$$

Since Y is a singular K-3 surface with $h^1(\mathcal{O}_Y) = 0$, S consists of exactly one point $\{x\}$ with $p_g(x) = 1$ by Umezu [13]. We put

$$\text{Sing } Y - \{x\} := \{y_1,\ldots,y_k\} \quad (k \geq 0) .$$

Let $\pi : \tilde{Y} \longrightarrow Y$ be the minimal resolution of the singularities of Y , and put $\pi^{-1}(x) = \Gamma$, $\pi^{-1}(\{y_1,\ldots,y_k\}) = B$, and $s := b_2(\Gamma)$ (the numbers of the irreducible components of Γ). Let $K_{\tilde{Y}}$ be the canonical divisor on $\tilde{Y}$. Then we have

$$-K_{\tilde{Y}} = Z ,$$

where Z is the fundamental cycle of the singular point x associated with the resolution $(\tilde{Y},\pi)$ (see [7], [2]). Since X is a smooth 3-fold, Sing Y consists of hyper-surface singularities. By Laufer [7], we have

$$-3 \leq Z^2 \leq -1 ,$$

where $Z^2 = (Z\cdot Z)_{\tilde{Y}}$.

By Noether formula, we have

Lemma 1. $(Z^2, b_2(\tilde{Y})) = (-1,11, (-2.12)$ or (-3.12) ,

where $b_2(Y) = \dim_{\mathbb{R}} H^2(Y;\mathbb{R})$.

Let U_0 (resp. U_i $1\leq i\leq k$) be a contractible neighbourhood of x (resp. y_i) in Y . We may assume that U_0, U_i $(1\leq i\leq k)$ are disjoint. We put $U = \bigcup_{i=0}^{k} U_i$ and $\partial U = \bigcup_{i=0}^{k} \partial U_i$, where ∂U_i is the boundary of U_i . Then we have the excat sequence of Poincare homomorphism P_2 :

$$0 \longrightarrow H^2(Y;\mathbb{Z}) \xrightarrow{P_2} H_2(Y;\mathbb{Z}) \longrightarrow H_1(\partial U:\mathbb{Z}) \longrightarrow 0 .$$

By Lemma 2.5 in Peternell-Schneider [11], we have

$$H_1(\partial U:\mathbb{Z}) \cong \mathbb{Z}_{22} .$$

In particular, we have

Lemma 2 ([11]). $H_1(\partial U_0:\mathbb{Z}) \cong \mathbb{Z}_{22}$ and $H_1(\partial U_i:\mathbb{Z}) \cong 0$ for $1\leq i\leq k$, namely, y_i's are all E_8-singularities.

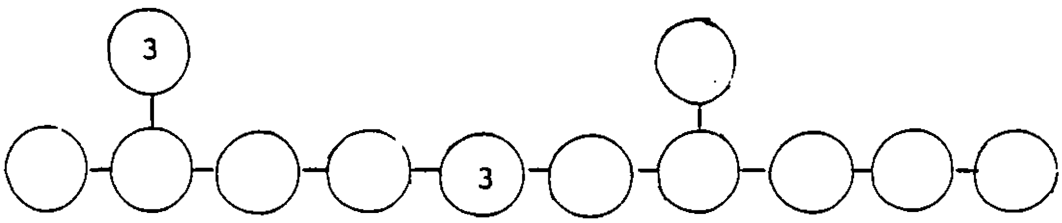
By Lemma 2, we have $b_2(B) = 8k$. Since $b_2(Y) = b_2(Y) + b_2(\Gamma) + b_2(B) = 1 + s + 8k$, by Lemma 1, we have $k \leq 1$, in particular,

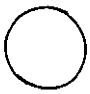
$$(Z^2,s) = \begin{cases} (-1,2),\ (-2,3) \text{ or } (-3,4) \text{ if } k = 1 \\ (-1,10),\ (-2,11) \text{ or } (-3,12) \text{ if } k = 0 \end{cases} .$$

Using the classification table of minimally elliptic singularities due to Laufer [7], we can determine the possible type of singularities of Y as Table 1 below.

In each case of Table 1, calculating the homology group $H_1(\partial U_0 : \mathbb{Z})$ according to Mumford [10], we have

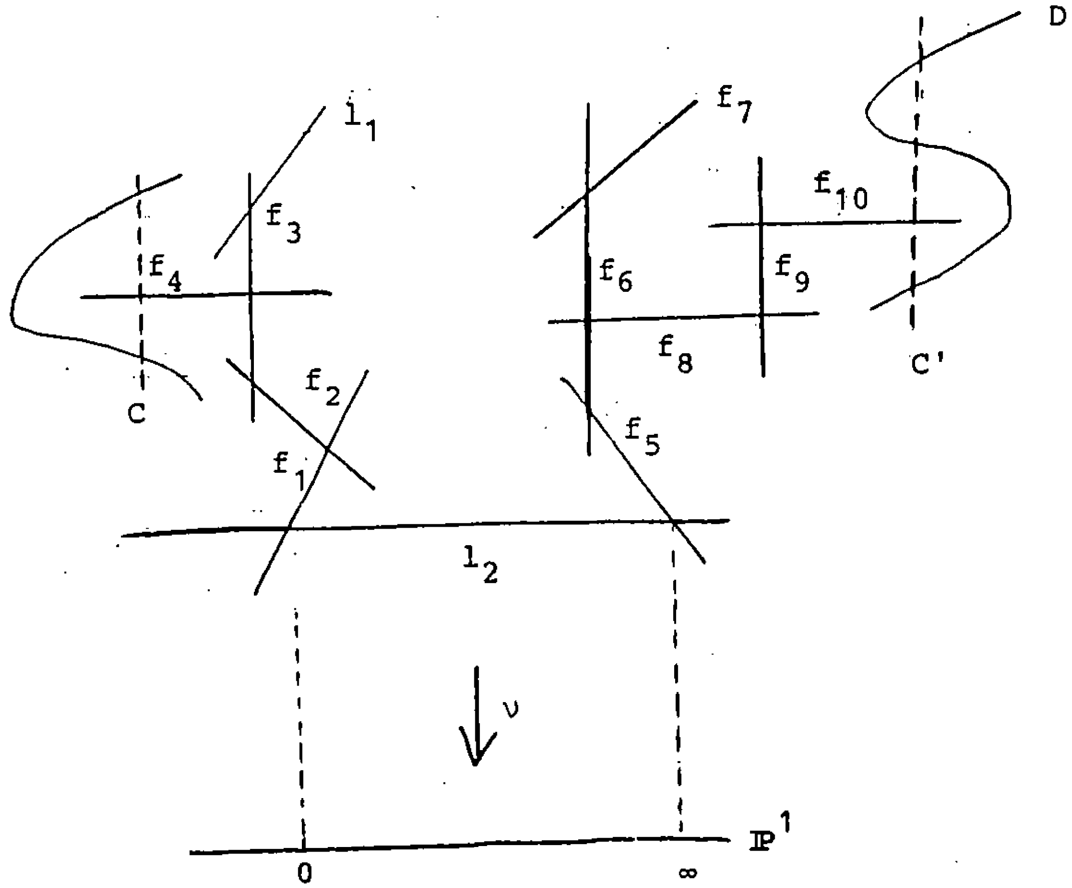
Lemma 3. Sing Y consists of exactly one point x of the type



where we denote by  (resp. 

) a smooth rational curve with self-intersection number -2 (resp. -3).

Let $(X,Y) = (V_{22},H_{22})$, Sing Y be as above. Then the minimal resolution $\tilde{Y}$ is obtained from $\mathbb{P}^2$ by 12 times blowing ups. $\tilde{Y}$ can be represented as a ruled surface $\nu : \tilde{Y} \longrightarrow \mathbb{P}^1$ over $\mathbb{P}^1$ (see Figure 1),



(Figure 1)

where $\nu^{-1}(0) = C \cup f_1 \cup f_2 \cup f_3 \cup f_4 \cup l_1$,
$\nu^{-1}(\infty) = \bigcup_{i=5}^{10} f_i \cup C'$ are singular fibers, and l_2 is a section.
Moreover, we have

(i) $(l_1 \cdot l_1)_{\tilde{Y}} = (l_2 \cdot l_2)_{\tilde{Y}} = -3$, $(f_i \cdot f_i)_{\tilde{Y}} = -2$,

$(C \cdot C)_{\tilde{Y}} = (C' \cdot C')_{\tilde{Y}} = -1 \quad (1 \leq i \leq 10)$

(ii) $(\tilde{D} \cdot C)_{\tilde{Y}} = 2$, $(\tilde{D} \cdot C')_{\tilde{Y}} = 3$, where $D = \pi(\tilde{D})$ is a canonical hyperplane section such that $\mathrm{Pic}\, X \cong \mathbb{Z} \cdot \mathcal{O}_Y(D)$, in particular, $\deg D = (D \cdot D)_Y = 22$.

(iii) $Z = f_4 + 2f_3 + 2f_2 + 2f_1 + l_1 + 2l_2 +$

$3f_5 + 4f_6 + 2f_7 + 3f_8 + 2f_9 + f_{10}$

(iv) $Z^2 = -3$, hence the multiplicity $m(\mathcal{O}_{Y,x})$ of the local ring $\mathcal{O}_{Y,x}$ at x is equal to 3 by Laufer [7].

Lemma 4.

(1) there is no line in X through the point x .

(2) $C_0 := \pi(C) \hookrightarrow Y$ is unique conic on X through the point x .

Proof. Since $m(\mathcal{O}_{Y,x}) = 3$ and Y is a hyperplane section of X , a line or a conic through the point x must be contained in Y . Since $(C\cdot\tilde{D})_{\tilde{Y}} = (C_0\cdot D)_Y = 2$ and D is a hyperplane section, C_0 is a conic on X . Let F be a line or a conic on X through the point x , hence $F \subset Y$, in particular, the proper transform $\tilde{F}$ of F in $\tilde{Y}$ must be the exceptional curve of the first kind. We can write $\tilde{D}$ as follows:

$$\tilde{D} = 2C + 4f_4 + 6f_3 + 2l_1 + 6f_2 + 6f_1 + 6l_2$$
$$+ 12f_5 + 18f_6 + 9f_7 + 15f_8 + 12f_9$$
$$+ 9f_{10} + 6C' .$$

By assumption, $(\widetilde{D}\cdot\widetilde{F})_{\widetilde{Y}} = 1$ or 2 . This implies the assertions (i) and (ii).

Q.E.D.

§ 2. Triple projection from a point

Let $(X,Y) = (V_{22},H_{22})$, Sing $Y = \{x\}$ be as in § 1. Let H be a sufficiently general hyperplane section of X . Let us consider the linear system $|H-3x|$ on X . Since $m(\mathcal{O}_{Y,x}) = 3$, $Y \in |H-3x|$. Let $\sigma_1 : X_1 \longrightarrow X = V_{22}$ be the blowing up at the point x , and put $\sigma_1^{-1}(x) := E_1 \cong \mathbb{P}^2$. Since $-K_X = H$ and $Y \in |H-3x|$, $-K_{X_1} = \sigma_1^*H-2E_1$, $Y_1 = \sigma_1^*H-3E_1$, where Y_1 is the proper transform of Y in X_1 . By the adjunction formula, we have

$$\begin{aligned} K_{Y_1} &= K_{X_1}\big|_{Y_1} + Y_1\big|_{Y_1} \\ &= -(Y_1+E_1)\big|_{Y_1} + Y_1\big|_{Y_1} \\ &= -E_1\big|_{Y_1} \quad . \end{aligned} \tag{2.1}$$

Lemma 5. $H^i(X_1,\mathcal{O}_{X_1}(\sigma_1^*H-3E_1)) = 0$ for $i > 0$ and $\dim H^0(X_1,\mathcal{O}_{X_1}(\sigma_1^*H-3E_1)) = 4$.

Proof. Let us consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{X_1} \longrightarrow \mathcal{O}_{X_1}(Y_1) \longrightarrow \mathcal{O}_{Y_1}(Y_1) \longrightarrow 0 .$$

Since $Y_1 = \sigma_1^* H - 3E_1$ and $H^i(X_1, \mathcal{O}_{X_1}) = 0$ for $i > 0$, we have only to prove $H^i(Y_1, \mathcal{O}_{Y_1}(Y_1)) = 0$ for $i > 0$. By (2.1), we have

$$\mathcal{O}_{Y_1}(Y_1) = \mathcal{O}_{Y_1}(\sigma_1^* H - 3E_1)$$

$$= \mathcal{O}_{Y_1}(D_1 + 3K_{Y_1}) ,$$

where $D_1 = \sigma^* H|_{Y_1}$ is the proper transform of D in Y_1. By Kawamata vanishing theorem [6], we have only to prove that $\mathcal{O}_{Y_1}(D+2K_{Y_1})$ is nef and big on Y_1. Indeed, there exists the birational morphism $\mu_1 : \tilde{Y} \longrightarrow Y_1$ such that $\pi = (\sigma_1|_{Y_1}) \circ \mu_1$. Then $\mu_1^*(D_1 + 2K_{Y_1}) = \tilde{D} - 2Z$. We can easily see that $\tilde{D} - 2Z$ is nef and big on $\tilde{Y}$. Thus $\mathcal{O}_{Y_1}(D_1 + 2K_{Y_1})$ is nef and big.

By Riemann-Roch theorem, $\dim H^0(X_1, \mathcal{O}_{X_1}(\sigma_1^* H - 3E_1)) = 4$. This completes the proof.

Q.E.D.

Corollary. $\dim H^0(Y_1, \mathcal{O}_{Y_1}(Y_1)) = 3$.

By Lemma 5, the linear system $|H-3x|$ defines a rational map

$$\Phi := \Phi_{|H-3x|} : X \dashrightarrow \mathbb{P}^3 .$$

Now, by Corollary, we have

$$3 = \dim H^0(Y_1, \mathcal{O}_{Y_1}(Y_1))$$
$$= \dim H^0(Y_1, \mathcal{O}_{Y_1}(D_1+3K_{Y_1}))$$
$$= \dim H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(\tilde{D}-3Z)) \; .$$

Let $\{g_1, g_2, g_3\}$ be a basis of $H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(\tilde{D}-3Z))$ such that

$$(g_1) = 11C + 10f_4 + 9f_3 + 2l_1 + 6f_2 + 3f_1$$
$$(g_2) = 5C + 4f_4 + 3f_3 + 2f_2 + f_1 + 2f_5$$
$$4f_6 + 2f_7 + 4f_8 + 4f_9 + 4f_{10}$$
$$+ 4C'$$
$$(g_3) = 8C + 7f_4 + 6f_3 + l_1 + 4f_2 +$$
$$2f_1 + f_5 + 2f_6 + f_7 + 2f_8$$
$$+ 2f_9 + 2f_{10} + 2C' \; .$$

Since $2(g_3) = (g_1) + (g_2)$, $g := (g_1 : g_2 : g_3)$ defines a rational map $\tilde{Y} \longrightarrow Q$ of $\tilde{Y}$ onto a conic $Q := \{w_2^2 = w_0 w_1\} \hookrightarrow \mathbb{P}^2(w_0 : w_1 : w_2)$. This implies that $\Phi(Y) = Q \cong \mathbb{P}^1$ and $W = \Phi(X)$ is a quadric hypersurface in $\mathbb{P}^3$. Thus,

Lemma 6. Let $\Phi : X \dashrightarrow \mathbb{P}^3$ be the triple projection from the point x . Then the image $W = \Phi(X)$ is an irreducible quadric hypersurface in $\mathbb{P}^3$ and $Q = \Phi(Y)$ is a smooth hyperplane section.

§ 3. Non-existence of the case $r = 1$

Let X, x, X_1, σ_1, E_1, Y_1 ... be as in § 2.

Let $\Delta \hookrightarrow X$ be a small neighbourhood of x with a coordinate system (z_1, z_2, z_3). By Laufer [7], we may assume

$$\Delta \cap Y = \{z_2 \cdot z_1^2 = z_1^3 z_2 + z_1 z_3^3 + z_1 z_2^4\} ,$$

$$x = (0,0,0) \in \Delta .$$

By an easy calculation, we find that Y_1 has two rational double points, namely, A_4-singularity q_1 and D_6-singularity q_0. Let $\mu_1 : \tilde{Y} \longrightarrow Y_1$ be the projection as in § 2. Then $\mu_1^{-1}(q_1) = f_1 \cup f_2 \cup f_3 \cup f_4$ and $\mu_1^{-1}(q_0) = \bigcup_{j=5}^{10} f_j$. We put $l_i^{(1)} = \mu_1(l_i)$ $(i = 1,2)$, and $C_1 = \mu_1(C)$. Then C_1 is the proper transform of C_0 in X_1, and $E_1 \cdot Y_1 = l_1^{(1)} + l_2^{(1)}$, in particular, C_1 is a smooth rational curve in $Y_1 \subset X_1$ with $q_1 \in C_1$, $q_0 \notin C_1$, and $l_i^{(1)}$ $(i = 1,2)$ are two distinct lines in $E_1 \cong \mathbb{P}^2$.

We will resolve the indeterminancy of the linear systems $L_1 := |\sigma_1^* H - 3E_1|$ on X_1. We remark that L_1 has no fixed component, and the base locus $B_s L_1 = C_1 \cup \{q_1\}$.

Lemma M([9]). Let E be a smooth rational curve in a smooth projective 3-fold X, and S be a surface with only one singularity x of A_{n+1} - type such that $E \subset S \subset X$. Let $\lambda : \tilde{S} \longrightarrow S$ be the minimal resolution, and put $\lambda^{-1}(x) = \bigcup_{j=1}^{n+1} C_j$, where $(C_i \cdot C_{i+1})_{\tilde{S}} = 1$ $(1 \leq i \leq n)$, $(C_i \cdot C_j)_{\tilde{S}} = 0$ if $|i-j| \geq 2$. Let $\tilde{E}$ be the proper transform of E in $\tilde{S}$. Assume that

(i) $N_{\tilde{E}|\tilde{S}} = \mathcal{O}_{\tilde{E}}(-1)$, where $N_{\tilde{E}|\tilde{S}}$ is the normal bundle of $\tilde{E}$ in $\tilde{S}$, and

(ii) $\deg N_{E|X} = -2$, where $N_{E|X}$ is the normal bundle of E in X .

Then we have

(1) $N_{E|X} \cong \mathcal{O}_E \oplus \mathcal{O}_E(-2)$ if $x \in E$ and $(C_j \cdot \tilde{E})_{\tilde{S}} = 1$ ($j = 1$ or $n + 1$)

(2) $N_{E|X} \cong \mathcal{O}_E(-1) \oplus \mathcal{O}_E(-1)$ if $x \notin E$.

Proof. In the proof of Theorem 3.2 in Morrison [9], we have only to replace the conormal bundle $N^*_{\tilde{E}|\tilde{S}} \cong \mathcal{O}_{\tilde{E}}(2)$ with $N^*_{\tilde{E}|\tilde{S}} \cong \mathcal{O}_{\tilde{E}}(1)$. The assertion (2) is easy.

Since $\deg N_{C_1|X_1} = -2$, $q_i \in C_1$, by Lemma M , we have

$$N_{C_1|X_1} \cong \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_1}(-2) .$$

(Step 1). Let $\sigma_2 : X_2 \longrightarrow X_1$ be the blowing up along C_1 and put $C_1' = \sigma_2^{-1}(C_1) \cong \mathbb{F}_2$. Let Y_2, E_2 be the proper transforms of Y_1, E_1 in X_2 respectively. By easy calculation, we find that Y_3 has only the A_3-singularity q_2 and the D_6-singularity q_0 . Then there exists the birational map $\mu_2 : \tilde{Y} \longrightarrow Y_2$ such that $\mu_2^{-1}(q_2) = f_2 \cup f_3 \cup f_4$ and $\mu_2^{-1}(q_0) = \bigcup_{i=5}^{10} f_i$. We put $l_i^{(2)} = \mu_2(l_i)$ $(i = 1,2)$, $f_1^{(2)} := \mu_2(f_1)$, and $C_2 = \mu_2(C)$. Then we have $C_1' \circ Y_2 = f_1^{(2)} + C_2$, $q_2 \in C_2$, $q_0 \notin C_2$. In particular, $f_1^{(2)}$ is a fiber and C_2 is a negative section of $C_1' \cong \mathbb{F}_2$, and $(l_1^{(2)} \cdot l_1^{(2)})_{E_2} = (l_2^{(2)} \cdot l_2^{(2)})_{E_2} = 0$, $(f_1^{(2)} \cdot f_1^{(2)})_{E_2} = -1$. We also have $\deg N_{C_2|X_2} = -2$, hence by Lemma M ,

$$N_{C_2|X_2} \cong \mathcal{O}_{C_2} \oplus \mathcal{O}_{C_2}(-2) .$$

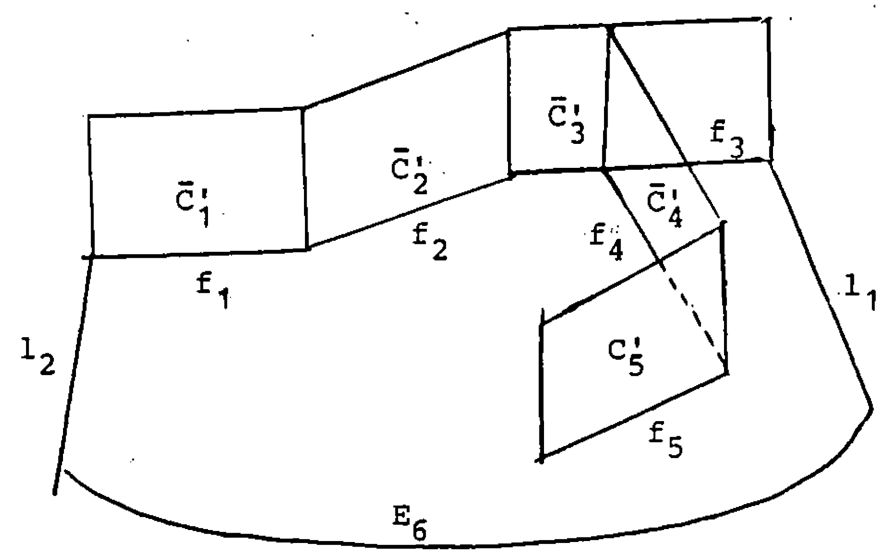
(Step k, k≥2). Let $\sigma_k : X_k \longrightarrow X_{k-1}$ be the blowing up along C_{k-1} , and put $C_{k-1}' = \sigma_k^{-1}(C_{k-1}) \cong \mathbb{F}_2$ for $2 \leq k \leq 5$. Let Y_k, E_k be the proper transforms of Y_{k-1}, E_{k-1} in X_k respectively. Then Y_k has only the A_{5-k}-singularity q_k and the D_6-singularity q_0 , where A_0-singularity q_5 means a smooth point.

There exists the birational morphism $\mu_k : \tilde{Y} \longrightarrow Y_k$ such that $\mu_k^{-1}(q_k) = f_k \cup \quad \cup f_4$ (for $k = 5$, μ_5 is isomorphic), and $\mu_k^{-1}(q_0) = \bigcup_{i=5}^{10} f_i$. We put $l_i^{(k)} = \mu_k(l_i)$ $(i = 1,2)$, $f_{k-1}^{(k)} := \mu_k(f_{k-1})$ and $C_k = \mu_k(C)$. Then we have

$C'_{k-1} \cdot Y_k = f^{(k)}_{k-1} + C_k$, $q_k \notin C_k$. In particular, $f^{(k)}_{k-1}$ is a fiber and C_k is a negative section of $C'_{k-1} \cong \mathbb{F}_2$, and C_k, $f^{(k)}_{k-1}$ are the proper transforms of C_{k-1}, $f^{(k-1)}_{k-2}$ $(0<j\leq k-2)$ in X_k . By Lemma M , we have

$$\begin{cases} N_{C_k|X_k} = \mathcal{O}_{C_k} \oplus \mathcal{O}_{C_k}(-2) & \text{for } 2\leq k\leq 4 \\ N_{C_5|X_5} \cong \mathcal{O}_{C_5}(-1) \oplus \mathcal{O}_{C_5}(-1) . & \end{cases}$$

(Step 6). Let $\sigma_6 : X_6 \longrightarrow X_5$ be the blowing up along C_5 and put $C'_5 = \sigma_6^{-1}(C_5) \cong \mathbb{P}^1 \times \mathbb{P}^1$. Let Y_6, E_6 be the proper transforms of Y_5, E_5 respectively. Then $\tilde{Y}_0 := \tilde{Y}/\bigcup_{j=5}^{10} f_j$ is isomorphic to Y_6 , identifying $\tilde{Y}_0$ with Y_6 , we will use the same notations l_1, l_2, f_1, f_2, ... as in $\tilde{Y}_0$. Let $\bar{C}'_j$ $(1\leq j\leq 4)$ be the proper transform of C'_j in X_6 (see Figure 2) .



(Figure 2)

Let f_5 be a fiber of $C_5' \cong \mathbb{P}^1 \times \mathbb{P}^1$ arising from the blowing up. Then we have

$$(l_2 \cdot l_2)_{E_6} = 0, \quad (f_i \cdot f_i)_{E_6} = -2 \quad (1 \leq i \leq 4)$$

$$(l_1 \cdot l_1)_{E_6} = -2, \quad (f_5 \cdot f_5)_{E_6} = -1 \ .$$

In particular, $C = C_5' \cdot Y_6$ gives another ruling on C_5'. Since

$$Y_6 = \overset{*}{\sigma}_6\overset{*}{\sigma}_5\overset{*}{\sigma}_4\overset{*}{\sigma}_3\overset{*}{\sigma}_2\overset{*}{\sigma}_1 H - 3\overset{*}{\sigma}_6\overset{*}{\sigma}_5\overset{*}{\sigma}_4\overset{*}{\sigma}_3\overset{*}{\sigma}_2 E_1$$

$$- 5C_5' - 4\bar{C}_4' - 3\bar{C}_3' - 2\bar{C}_2' - \bar{C}_1' \ ,$$

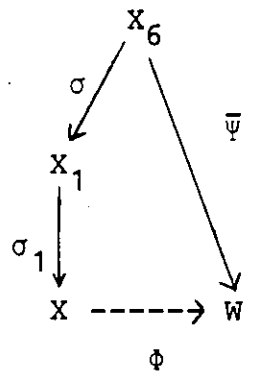
we have

$$\mathcal{O}_{Y_6}(Y_6) = \mathcal{O}_{Y_6}(D + 3K_{Y_6} - 5C - 4f_4 - 3f_3 - 2f_2 - f_1)$$

$$\cong \mathcal{O}_{\tilde{Y}}(\tilde{D} - 3Z - 5C - 4f_4 - 3f_3 - 2f_2 - f_1)$$

$$\cong \mathcal{O}_{\tilde{Y}}(2f) \ ,$$

where f is a general fiber of $\nu : \tilde{Y} \longrightarrow \mathbb{P}^1$ (see § 1, Figure 1). In fact, $\nu : \tilde{Y}_0 \longrightarrow \mathbb{P}^1$ is the morphism defined by the linear system $|\mathcal{O}_{\tilde{Y}}(2f)|$, thus, $\tilde{Y}_0$ can be considered as a ruled surface over a smooth conic $Q \cong \mathbb{P}^1$ in $\mathbb{P}^2$. Therefore $|\mathcal{O}_{Y_6}(Y_6)|$ is free from base points and fixed components, hence, so is $|Y_6| = |\mathcal{O}_{X_6}(Y_6)|$. Let $\bar{\Psi} := \bar{\Psi}_{|Y_6|} : X_6 \longrightarrow W \hookrightarrow \mathbb{P}^3$ be morphism defined by the linear system $|Y_6|$. Then we have the following diagram, which gives the resolution of the indeterminancy of the rational map $\Phi : X \dashrightarrow \mathbb{P}^3$;



$$\sigma = \sigma_2 \circ \dots \circ \sigma_6 \quad .$$

It is easy to see that

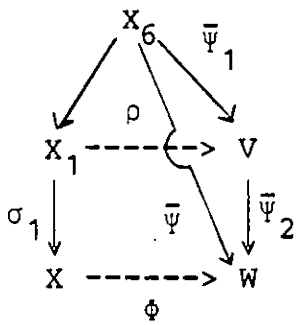
$$\bar{\Psi}(Y_6) = \Phi(Y) = \nu(Y_6) = Q \cong l_2$$

$$\bar{\Psi}\left(\bigcup_{j=1}^{4} \bar{C}_j' \cup C_5' \right) = \bar{\Psi}(f_5) \quad \text{(a line in } \mathbb{P}^3 \text{) .}$$

Since $N_{C_5|X_5} \cong \mathcal{O}_{C_5}(-1) \oplus \mathcal{O}_{C_5}(-1)$, by Reid [12], C_5' can be blowed down along C , and then the blowing downs can be done step-by-step. Finally, we have a smooth projective 3-fold V of $b_2(V) = 2$, the morphisms $\bar{\Psi}_1 : X_6 \longrightarrow V$, $\bar{\Psi}_2' : V \longrightarrow W$ and the birational map $\rho : X_1 \dashrightarrow V$ such that

(i) $\bar{\Psi} = \bar{\Psi}_1 \circ \bar{\Psi}_2$

(ii) $X_1 - C_1 \overset{\rho}{\cong} V - \bar{f}_3$, where $\bar{f}_3 := \bar{\Psi}_1(f_3)$.



Since $-K_{X_1} = Y_1 + E_1$, by (ii) above we have $-K_V = A + \Sigma$, where we put $A = \bar{\Psi}_1(Y_6)$ and $\Sigma := \bar{\Psi}(E_6)$. For a general fiber F of $\bar{\Psi}_2 : V \longrightarrow W$,

$$\deg(K_F) = (K_V \cdot F) = -(\Sigma \cdot F) \leq -1 ,$$

hence $F \cong \mathbb{P}^1$ and $(\Sigma \cdot F) = 2$. Therefore Σ is a meromorphic double section of $\bar{\Psi}_2 : V \longrightarrow W$. Let G be a scheme theoric fiber. Then $(G \cdot \Sigma) = 2$. Taking an account of $V - (A \cup \Sigma) \cong \mathbb{C}^3$, $\bar{\Psi}_2 : V \longrightarrow W$ is a conic bundle over W, and $\bar{\Psi}_2$ is the contraction of an extremal ray on a smooth projective 3-fold V. Therefore W must be smooth by Mori [8]. Since $\deg W = 2$, $W \cong \mathbb{P}^1 \times \mathbb{P}^1$. But this is a contradiction, since $b_2(V) = 2$. This completes the proof.

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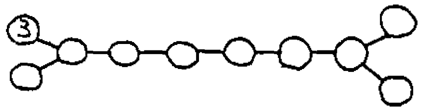
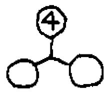
Table 1

(1)  + E_8

(2) 

(3) 

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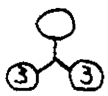
(4) 

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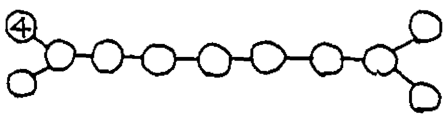
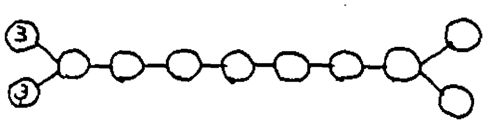
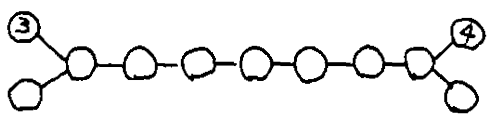
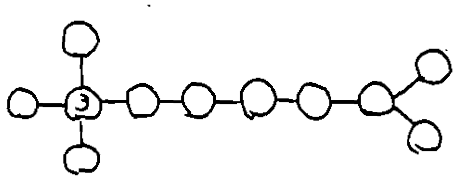
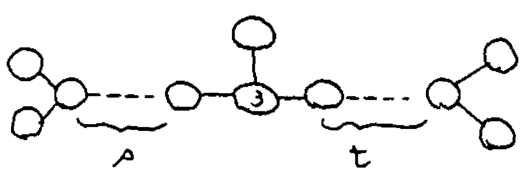
(5) 

(6) 

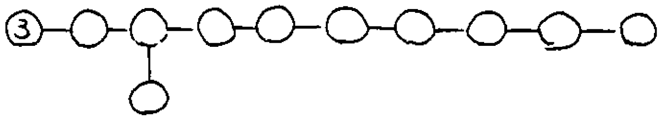
(7) 

(8) 

(9) 

$s+t=5 \quad (s \geq 1,\ t \geq 1)$

(10)



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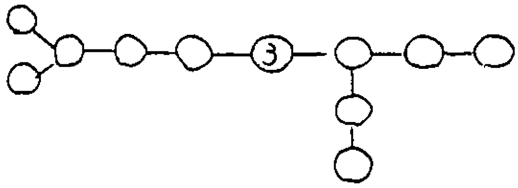

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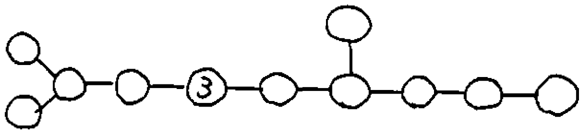

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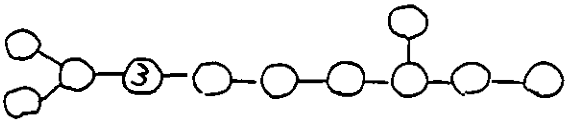

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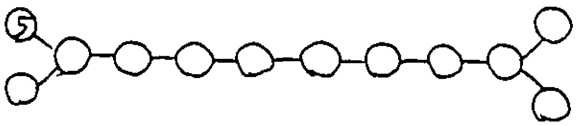

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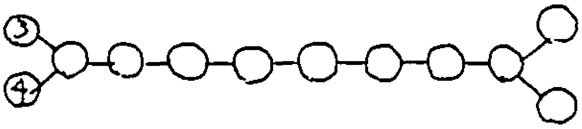

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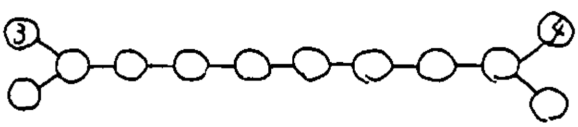

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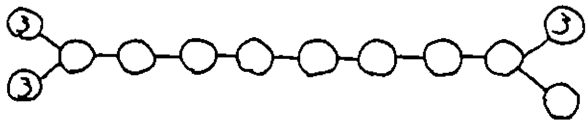

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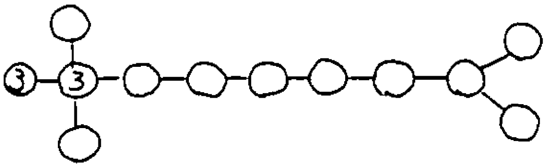

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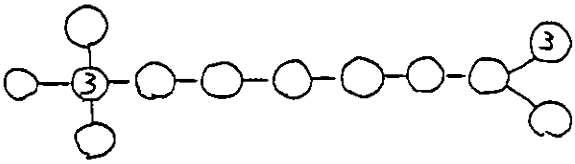

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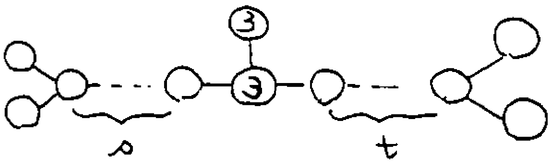
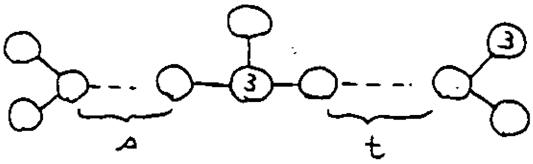


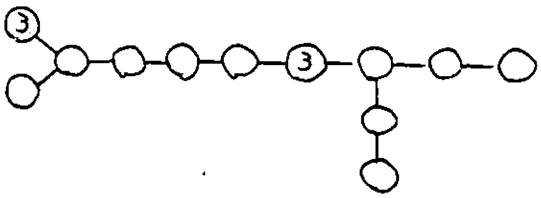
$s + t = 6$, $(s \geq 1, t \geq 1)$

(21)



$s + t = 6$ $(s \geq 1, t \geq 1)$

(22)



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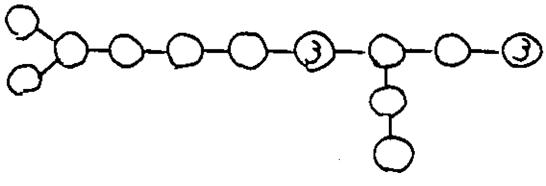

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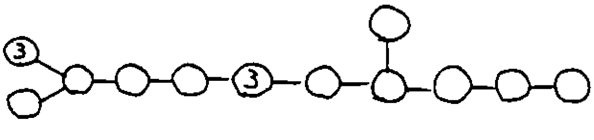

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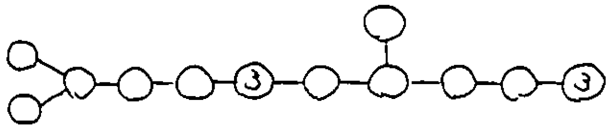

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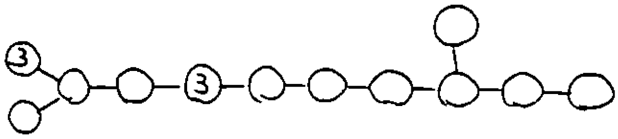

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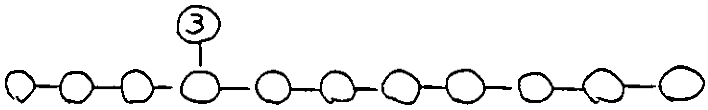

(28)