Instanton bundles

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A *mathematical instanton* E on $\mathbb{P}^3$ is a vector bundle, or more generally a torsion free sheaf, which satisfies $H^1(E(-2)) = H^2(E(-2)) = 0$ and which is trivial on some line.

Our talk is motivated by an old conjecture saying that moduli space of rank 2 mathematical instantons on $\mathbb{P}^3$ is smooth and irreducible. We will show connections between several different constructions of moduli spaces of framed instantons on $\mathbb{P}^3$ and we will try to study their properties and use them to study non-framed instantons.

One of the constructions of the moduli space of framed instantons uses the moduli space of Gieseker δ-semistable framed modules constructed by D. Huybrechts and M. Lehn. Unfortunately, the moduli space of framed instantons on $\mathbb{P}^3$ cannot be constructed as a subset of such a moduli space on $\mathbb{P}^3$. Nevertheless it exists and it can be constructed using the Huybrechts-Lehn moduli spaces on the blow up of $\mathbb{P}^3$ along a fixed line. This also helps to show another interpretation of the moduli space of framed instantons as a subscheme of the scheme of rational curves on the relative moduli space of framed bundles on a $\mathbb{P}^2$-bundle.

Another construction was recently provided by I. Frenkel and M. Jardim, who generalized the ADHM construction and showed a set-theoretical bijection between framed instantons on $\mathbb{P}^3$ and stable complex ADHM data satisfying ADHM equations. This can be generalized to give another construction of the moduli space of framed instantons given by an analogue of the hyper-Kähler quotient construction. We show that deformation theory of framed instantons is governed by this construction.

We show that the moduli spaces of framed instantons are smooth for charge $c \leq 2$. Then we provide a few counter-examples to some conjectures by I. Frenkel and M. Jardim. The first example shows that these moduli spaces for charge $c \geq 3$ are in general singular. We use it to show that the parameter space for rank 8 symplectic mathematical instantons with $c_2 = 6$ is singular which suggests that the original conjecture in the rank 2 case can be false for $c_2 = 9$.

In the second example we show that one cannot omit vanishing of $H^2(E(-2))$ in the definition of instantons, contradicting another conjecture of Frenkel and Jardim.

Finally we define perverse instantons, which are generalizations of mathematical instantons but are in general objects in the derived category. Such objects are necessary for the study of the analogue of the natural map from partial Gieseker to partial Donaldson-Uhlenbeck compactification of moduli space of framed instantons. This is the first example where such a map was constructed for moduli spaces of sheaves on a 3-dimensional manifold.