On Elliptic Surfaces in Characteristic p

by

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Introduction.

The theory of elliptic surfaces over complex numbers has been initiated and developped by K. Kodaira and we have the satisfactory theory. But very little is known about elliptic

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surfaces in positive characteristic. Main difficulty comes from the existence of wild fibres. In this paper we study the question to what extent the theory of elliptic surfaces over $\mathbb{C}$ can be extended or has good analogy in positive characteristic. For example, if S is an elliptic surface with $\kappa(S) = 1$ (that is, the image of a rational map $\Phi_{|mK_S|}$ associated with the m-th canonical system $|mK_S|$ of S is a curve for sufficiently large m) over $\mathbb{C}$, Iitaka [I2] showed that for $m \geq 86$, $\Phi_{|mK_S|}$ gives the original structure of the elliptic surface. In this paper, we shall prove the following:

Theorem. If S is an algebraic elliptic surface defined over an algebraically closed field k of characteristic $p \geq 0$ with $\kappa(S) = 1$, then $\Phi_{|mK_S|}$ gives the unique structure of the elliptic surface for every $m \geq 14$.

Moreover, it will be shown that the number 14 is the best possible if char.k $\neq 2,3$. The difference between our theorem and Iitaka's theorem over $\mathbb{C}$ comes from the fact that we only consider algebraic elliptic surfaces while Iitaka considered all analytic elliptic surfaces. Thus, even in case $k = \mathbb{C}$, our theorem seems new. To prove the above theorem, we need to study elliptic surfaces $f : S \longrightarrow \mathbb{P}^1_k$ with $\chi(\underline{O}_S) = 0$. If such an elliptic surface has multiple fibres $m_i E_i$ ($i = 1, 2, \ldots, \lambda$), their multiplicities and the orders of the normal bundles of E_i should satisfy certain conditions (see Theorem 3.3 below).

Another important fact in the theory of elliptic surfaces over $\mathbb{C}$ is that all multiple fibres are obtained by means of logarithmic transformations (see [K2],I,II). The logarithmic transformation is defined by means of the logarithmic function. Hence, it is non-algebraic. But it is based on the fact that over $\mathbb{C}$, every multiple fibre is reduced to a non-multiple fibre, by taking locally a cyclic covering of a base curve ramified at the point over which the multiple fibre lies, pulling back the elliptic fibration to the covering and taking the normalization. We can therefore ask whether a similar procedure exists in case of positive characteristic. The procedure is divided into two parts. First we reduce a wild fibre to a tame fibre, then a tame fibre to a non-multiple fibre. Our main results in this direction is the following:

Theorem. Let mD be a wild fibre of an elliptic surface $f : S \longrightarrow C$ where D is an ordinary elliptic curve or of type I_n (that is, a cycle of rational curves). Then, there exist an element $\alpha \in H^1(S,\underline{O}_S)$, a covering $\pi_1 : S^{(1)} \longrightarrow S$ associated with α and an elliptic surface $f_1 : S^{(1)} \longrightarrow C^{(1)}$ such that the corresponding multiple fibre in f_1 has a form $m^{(1)}D^{(1)}$ with $pm^{(1)} = m$. By a finite succession of this process, the wild fibre is reduced to a tame fibre.

The more detailed discription can be found in §6 below. The reduction of a tame fibre to a non-multiple fibre will be given in §7. By this theorem, if D is an ordinary elliptic curve or of type I_n, we understand the wild fibre mD well. Namely, in this case, $\pi_1 : S^{(1)} \longrightarrow S$ is a $\mathbb{Z}/p\mathbb{Z}$ étale covering, hence S is obtained by a $\mathbb{Z}/p\mathbb{Z}$ étale quotient. The reason why a wild fibre appears in a $\mathbb{Z}/p^n\mathbb{Z}$ étale quotient can be found in Remark 4.10 below. Certain examples of this type of wild fibres will be found in §8. If D is a supersingular elliptic curve, our result seems a little weak. We do not know whether we can take $\deg \pi_1 = p$ in this case. On the other hand, if D is not of the above types, that is, if $\mathrm{Pic}^0(D) = \mathbb{G}_a$, we cannot directly apply our procedure in §6 to this case and the problem is unsolved. The same difficulty appears when we reduce a tame fibre to a non-multiple fibre, although we know very few examples of tame fibres mD with $\mathrm{Pic}^0(D) = \mathbb{G}_a$ (see [K1]). Note that if the multiple fibre mD with $\mathrm{Pic}^0(D) = \mathbb{G}_a$ is a tame fibre, we have $m = p$, since any torsion point of $\mathbb{G}_a$ is of order p.

It is well-known that over $\mathbb{C}$ m-genera and Kodaira dimensions of surfaces are invariant under smooth deformation (see [I2]). But in positive characteristic, m-genera are not always invariant under smooth deformation or lifting (see Examples 8.7, 8.8 below). However, we have the following:

<u>Theorem</u>. The Kodaira dimension of smooth projective surfaces is invariant under smooth deformation and lifting.

In this paper, we only consider smooth deformation and lifting of a surface over $\mathrm{Spec}(R)$ with discrete valuation ring R, but it is easy to generalize the notion to an arbitrary base space. Our result is valid under such generalization. The theorem says, in particular, if $f : S \longrightarrow C$ is an elliptic surface with $\kappa(S) = 1$ and S' is a smooth deformation or lifting of S, then $\kappa(S') = 1$, hence, if char.k $\neq 2,3$, S' is an elliptic surface $f' : S' \longrightarrow C'$ (in case char.k = 2 or 3, S' may be quasi-elliptic, if we consider equicharacteristic deformation). In §10 we shall show that in this situation, the genera of C and C' are same. We also conjecture, in this situation, $f' : S' \longrightarrow C'$ is a smooth deformation or a lifting of $f: S \longrightarrow C$. Although we will not discuss them, the following two questions are worth while to mention.

<u>Question</u> I. For any elliptic surface $f : S \longrightarrow C$ with $\kappa(S) = 1$,

does there exist a positive integer m_0 such that nm_0-genus P_{nm_0} is invariant under any smooth deformation and lifting of S for any $n \geq 1$?

<u>Question</u> II. Can every elliptic surface $f: S \longrightarrow C$ defined over an algebraically closed field k of positive characteristic with $\kappa(S) = 1$ be lifted to characteristic zero in weak sense (see [O] for the definition of lifting) ?

Finally we give a brief outline of our paper. In §1 we recall basic facts about elliptic surfaces. In §2, to calculate the canonical divisor formula for certain elliptic surfaces, we shall study jumping values of a wild fibre. The results in this section are mainly due to Raynaud [R2]. In §3 we shall study an elliptic surface $f : S \longrightarrow \mathbb{P}^1_k$ with $\chi(\underline{\underline{O}}_S) = 0$ and show that if such a surface exists, then its multiple fibres satisfy certain conditions (Theorem 3.3). In §4 we shall discuss several consequences of Theorem 3.3. We also give an example of an elliptic surface which shows that our number 14 in the first theorem above is the best possible. We also discuss examples of elliptic surfaces obtained by $\mathbb{Z}/p\mathbb{Z}$, μ_p , α_p quotients. In §5, the theorem about the pluricanonical mapping will be proved. In §6 the above mentioned reduction of a wild fibre to a tame fibre will be given. In §7 the process to reduce a tame fibre to a non-multiple fibre will be given. In §8 examples of wild fibres will be given. We shall also give examples of smooth deformation and lifting of elliptic surfaces with wild fibres. In §9 the invariance of Kodaira dimension of a surface under smooth deformation and lifting will be proved. In §10 the invariance of the genus of the base curve of an elliptic surface with $\kappa = 1$ under smooth deformation and lifting will be proved. In Appendix 1, we show the necessary and sufficient condition for an analytic elliptic surface $f : S \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ with $\chi(\underline{\underline{O}}_S) = 0$ to be algebraic. Finally, In Appendix 2, we give the proof of a proposition on the normal form of the action of α_p on a supersingular elliptic curve, which was commented by F. Oort.

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Notation and conventions.

Throughout this paper, we fix an algebraically closed field k of characteristic $p \geq 0$. By an elliptic surface $f : S \longrightarrow C$ we mean that S is a complete smooth surface defined over k, C is a complete smooth curve defined over k, f is a surjective morphism defined over k with connected geometric fibres and almost all closed geometric fibres of f are non-singular elliptic curves. We also assume that any exceptional curve of the first kind is not contained in fibres (this will not be assumed in §9) We shall not distinguish a line bundle from the associated invertible sheaf. Sometimes, a Cartier divisor and the associated invertible sheaf will be identified.

Let X be a complete smooth algebraic variety defined over k. We use the following notation.

$h^i(X,\underline{\underline{F}}) = \dim_k H^i(X,\underline{\underline{F}})$ for any coherent sheaf $\underline{\underline{F}}$ on X.

$\Gamma(U,F)$: the group of sections of F over an open set U of X.

K_X : a canonical divisor (or the canonical bundle) of X.

$\omega_X \cong \underline{\underline{O}}_X(K_X)$: the dualizing sheaf of X.

$P_m(X) = h^0(X,K_X^m)$: the m-genus, $m = 1,2,\ldots\ldots\ldots$

$p_g(X) = P_1(X)$: the geometric genus.

$c_i(X)$: the i-th Chern class of X.

$b_i(X) = \dim_Q H^i_{\text{ét}}(X,Q_l)$, the i-th Betti number.

$g(C)$: the genus of a non-singular curve C.

$[\alpha]$: the largest integer which does not exceed a real number α.

Let D, D' be Cartier divisors on X.

$[D]$: the line bundle associated with D.

$D \sim D'$: linear equivalence.

$D \equiv D'$: algebraic equivalence.

Let mD be a multiple fibre of an elliptic surface.

$\nu = \mathrm{ord}[D]|_D$.

Let R be a discrete valuation ring and $\varphi : X \longrightarrow \mathrm{Spec}(R)$ a proper, smooth and separated morphism of algebraic spaces. By o (resp. η) we mean the closed (resp. generic) point of $\mathrm{Spec}(R)$ and by X_o (resp. X_1) we mean the closed (resp. generic) geometric fibre of φ.

§1. Preliminaries.

In this paper we always assume that an elliptic surface $f : S \longrightarrow C$ is minimal, that is, any fibre of f contains no exceptional curves of the first kind, unless otherwise mentioned. For an elliptic surface, by the Leray spectral sequence we have an exact sequence

(1.1) $$0 \longrightarrow H^1(C,\underline{\underline{O}}_C) \longrightarrow H^1(S,\underline{\underline{O}}_S) \longrightarrow H^0(C,R^1f_*\underline{\underline{O}}_S) \longrightarrow 0.$$

Let $\underline{\underline{T}}$ be the torsion part of $R^1f_*\underline{\underline{O}}_S$. Since C is a non-singular curve, we have

(1.2) $$R^1f_*\underline{\underline{O}}_S/\underline{\underline{T}} \xrightarrow{\simeq} \underline{\underline{O}}_C(\underline{f}),$$

where $\underline{f}$ is a divisor on C. By m_iD_i, $i = 1,2,\ldots\ldots,\lambda$, we denote all the multiple singular fibres of the elliptic surface $f : S \longrightarrow C$. Then we have the following canonical divisor formula

(1.3) $$K_S = f^*(K_C - \underline{f}) + \sum_{i=1}^{\lambda} a_iD_i,$$

where a_i's are integers with $0 \leq a_i \leq m_i - 1$ and

(1.4) $$-\deg \underline{f} = \chi(S,\underline{\underline{O}}_S) + \text{length } \underline{\underline{T}}$$

(see for example [BM],II, Theorem 2). The formula (1.3) implies $c_1(S)^2 = 0$ and by Noether's formula and Igusa's equality ([I] and [Y]), we have

(1.5) $$\chi(S,\underline{\underline{O}}_S) = \frac{1}{12}c_2(S) \geq 0.$$

Put $\nu_i = \operatorname{ord} [D_i]|_{D_i}$. If char.k = 0, then $\nu_i = m_i$. If char.k = $p > 0$, then there exist non-negative integers α_i, $i = 1,2,\ldots\ldots,\lambda$, such that

(1.6) $$m_i = p^{\alpha_i}\nu_i,\ i = 1,2,\ldots\ldots\ldots,\lambda.$$

The following conditions are equivalent.

(1.7) (i) $\underline{\underline{T}}_{p_i} = 0$, $p_i = f(D_i)$, (ii) $h^0(\underline{\underline{O}}_{m_iD_i}) = 1$,

(iii) $a_i = m_i - 1$, (iv) $\nu_i = m_i$.

In this case, the multiple fibre m_iD_i is called a <u>tame</u> fibre. If a multiple fibre is not tame, it is called a <u>wild</u> fibre. A multiple fibre m_iD_i is wild, if and only if one of the following equivalent conditions is satisfied.

(1.8) (i) $\underline{\underline{T}}_{p_i} \neq 0$, $p_i = f(D_i)$, (ii) $h^0(\underline{\underline{O}}_{m_iD_i}) \geq 2$,

(iii) $0 \leq a_i \leq m_i - 2$, (iv) $1 \leq \nu_i \leq m_i - 1$

(see for example [BM] II, Proposition 4). A wild fibre appears only in case of char.k $= p > 0$.

For a multiple fibre m_iD_i, $h^0(\underline{\underline{O}}_{nD_i})$ is a non-decreasing function with respect to positive integers n.

<u>Definition</u> 1.1. A positive integer n is called a <u>jumping value</u> of the multiple fibre m_iD_i, if $h^0(\underline{\underline{O}}_{(n-1)D_i}) < h^0(\underline{\underline{O}}_{nD_i})$.

<u>Lemma</u> 1.2. Assume a multiple fibre m_iD_i is wild. Let l_i be the positive integer such that $h^0(\underline{\underline{O}}_{(l_i-1)D_i}) < h^0(\underline{\underline{O}}_{l_iD_i}) = h^0(\underline{\underline{O}}_{m_iD_i})$. Then we have $a_i + l_i \geq m_i$. Moreover, there exists the jumping value n_i of the multiple fibre with $a_i + n_i = m_i$, $1 \leq n_i \leq m_i$.

<u>Proof</u>. By the Riemann-Roch theorem for a divisor $-rD_i$ and a standard exact sequence, we have

(1.9) $$h^0(\underline{\underline{O}}_{rD_i}) = h^1(O_{rD_i}), \quad r = 1,2,\ldots\ldots\ldots\ldots\ldots$$

and $h^1(\underline{\underline{O}}_{rD_i})$ is a non-decreasing function in r. Let us consider the following exact sequences.

(1.10)
$$\begin{array}{ccccccccc}
H^1(\mathcal{O}_S) & \longrightarrow & H^1(\mathcal{O}_{rD_i}) & \longrightarrow & H^2(\mathcal{O}_S(-rD_i)) & \longrightarrow & H^2(\mathcal{O}_S) & \longrightarrow & 0 \\
\| & & \downarrow \rho_1 & & \downarrow \rho_2 & & \| & & \\
H^1(\mathcal{O}_S) & \longrightarrow & H^1(\mathcal{O}_{(r-1)D_i}) & \longrightarrow & H^2(\mathcal{O}_S(-(r-1)D_i)) & \longrightarrow & H^2(\mathcal{O}_S) & \longrightarrow & 0.
\end{array}$$

Note that ρ_1 and ρ_2 are surjective. If r is not a jumping value, then by (1.9), ρ_1 is isomorphic. Hence ρ_2 is isomorphic. Take a positive integer n_i such that

(1.11) $$h^2(\mathcal{O}_S(-m_iD_i)) = h^2(\mathcal{O}_S(-n_iD_i)) > h^2(\mathcal{O}_S(-(n_i-1)D_i)).$$

Then the number n_i is a jumping value of the multiple fibre. By the Serre duality and (1.3), we have $h^2(\mathcal{O}_S(-m_iD_i)) = h^0(\mathcal{O}_S(K_S+m_iD_i)) = h^0(C,\mathcal{O}_C(K_C-\underset{\sim}{f}+p_i))$, $p_i=f(D_i)$. As m_iD_i is a wild fibre, by (1.4) and (1.8) we have $\deg(K_C-\underset{\sim}{f}+p_i) \geq 2g(C)$. Hence, by (1.11) we have

$$h^2(\mathcal{O}_S(-n_iD_i)) = h^0(C,\mathcal{O}_C(K_C-\underset{\sim}{f})) + 1 = p_g(S) + 1.$$

As we have $h^2(\mathcal{O}_S(-n_iD_i)) = h^0(\mathcal{O}_S(K_S+n_iD_i))$, the canonical divisor formula (1.3) implies

$$a_i + n_i \geq m_i.$$

On the other hand, by (1.11) we have

$$p_g(S) \geq h^2(\mathcal{O}_S(-(n_i-1)D_i)) = h^0(\mathcal{O}_S(K_S+(n_i-1)D_i)),$$

Hence, by (1.3), this implies

$$a_i + n_i - 1 < m_i.$$

q.e.d.

§2. Jumping values of a wild fibre.

In this section, using a theory due to Raynaud [R2], in certain cases we shall calculate the number a_i in the canonical divisor formula (1.3). The present section is essentially due to Raynaud [R2].

Let $f : S \longrightarrow C$ be an elliptic surface and $f^{-1}(p) = mD$ a multiple singular fibre of multiplicity m over a point $p \in C$. For any positive integer n we consider nD as a subscheme $\mathrm{Spec}(\underline{O}_S/O_S(-nD))$. Then the dualizing sheaf ω_n of nD is given by

(2.1) $$\omega_n = \omega_S \otimes \underline{O}_S(nD)|_{nD} \cong \underline{O}_S((n+a))|_{nD}.$$

By (1.9) and the Serre duality, we have

(2.2) $$h^0(\omega_n) = h^1(O_{nD}) = h^0(\underline{O}_{nD}) = h^1(\omega_n).$$

We need the following two lemmas. For the proof we refer the reader to [R2].

Lemma 2.1. ([R2], Corollaire 3.7.6.) i) The dualizing sheaf ω_n is not trivial if and only if $h^0(\omega_n) = h^0(\omega_{n-1})$.

ii) The dualizing sheaf ω_n is trivial if and only if $h^0(\omega_n) = h^0(\omega_{n-1}) + 1$.

Lemma 2.2. ([R2], Lemme 3.7.7.) For the orders of line bundles $[D]|_{(n-1)D}$ and $[D]|_{nD}$, there are only two possibilities:

(i) $\mathrm{ord}([D]|_{nD}) = \mathrm{ord}([D]|_{(n-1)D})$,

(ii) $\mathrm{ord}([D]|_{nD}) = p\,\mathrm{ord}([D]|_{(n-1)D})$.

Moreover, if the case (ii) holds, then the dualizing sheaf ω_n is trivial.

The following lemma is a part of [R2], Lemma 3.7.9. For the reader's convenience we give a proof.

<u>Lemma</u> 2.3. Let $n^{(\ell)}$ be the ℓ-th jumping value of a wild fibre mD. Set $\nu = \mathrm{ord}([D]|_D)$. Then, we have

$$(2.3)\qquad \begin{cases} n^{(1)} = \nu + 1, \\ n^{(2)} = \begin{cases} 2\nu + 1 & \text{if } \mathrm{ord}([D]|_{(\nu+1)D}) = \nu, \\ (p+1)\nu + 1 & \text{if } \mathrm{ord}([D]|_{(\nu+1)D}) = p\nu. \end{cases} \end{cases}$$

Proof. Set $J = \underline{O}_S(-D)$. The conormal sheaf J/J^2 of D in S is of order ν. Therefore, we have

$$(2.4)\qquad \begin{cases} h^0(D, (J/J^2)^r) = h^1(D, (J/J^2)^r) = 0, \quad r = 1,2,\ldots,\nu-1, \\ h^0(D, (J/J^2)^\nu) = h^1(D, (J/J^2)^\nu) = 1. \end{cases}$$

For each positive integer r, we have an exact sequence

$$(2.5)\qquad 0 \longrightarrow (J/J^2)^r \longrightarrow \underline{O}_{(r+1)D} \longrightarrow \underline{O}_{rD} \longrightarrow 0.$$

By (2.4) and (2.5), we infer that $H^0(O_{\nu D})$ is of dimension 1 and consists of constant functions. Therefore, the map $H^0(\underline{O}_{(\nu+1)D}) \longrightarrow H^0(\underline{O}_{\nu D})$ is surjective. By (2.4), we have $h^0(\underline{O}_{(\nu+1)D}) = 2$. Hence, we have $n^{(1)} = \nu + 1$ (see [BM],II). By Lemma 2.2, $\mathrm{ord}([D]|_{(\nu+1)D})$ is ν or $p\nu$. Now, assume $\mathrm{ord}([D]|_{(\nu+1)D}) = \nu$ (resp. $\mathrm{ord}([D]|_{(\nu+1)D}) = p\nu$). since $\nu + 1$ is a jumping value, by (2.2) and Lemma 2.1 $\omega_{\nu+1}$ is trivial. Therefore, by (2.1), we have

$$(2.6)\qquad \nu \mid \nu + 1 + a \quad (\text{resp. } p\nu \mid \nu + 1 + a).$$

If ω_n is trivial for an integer n with $\nu + 1 < n < 2\nu + 1$ (resp. $\nu + 1 < n < (p+1)\nu + 1$), then $\omega_n|_{(\nu+1)D}$ is trivial. Therefore, we have $\nu \mid n + a$ (resp. $p\nu \mid n + a$). Hence, by (2.6) we have $\nu \mid n - 1$ (resp. $p\nu \mid n - \nu - 1$). A contradiction. Thus, ω_n is not trivial for any n with $\nu + 1 < n < 2\nu + 1$ (resp. $\nu + 1 < n < (p+1)\nu + 1$). Therefore, by Lemma 2.2, we have

$ord([D]|_{2\nu D}) = \nu$ (resp. $ord([D]|_{(p+1)\nu D} = p\nu$). If $ord([D]|_{(2\nu+1)D}$ $= \nu$ (resp. $ord([D]|_{(p\nu+\nu+1)D}) = p\nu$), then $\omega_{2\nu+1}$ (resp. $\omega_{p\nu+\nu+1}$) is trivial by (2.1) and (2.6). If $ord([D]|_{(2\nu+1)D}) = p\nu$ (resp. $ord([D]|_{(p\nu+\nu+1)D}) = p^2\nu$), then by Lemma 2.2, $\omega_{2\nu+1}$ (resp. $\omega_{(p+1)\nu+1}$) is trivial. Hence, in both cases, by Lemma 2.1 we have $n^{(2)} = 2\nu + 1$ (resp. $n^{(2)} = (p + 1)\nu + 1$). q.e.d.

By Lemmas 1.2 and 2.3 we infer the following :

<u>Lemma</u> 2.4. Using the above notation, we have the following :

(i) If $h^0(\underline{O}_{mD}) = 2$, then $a + \nu + 1 = m$.

(ii) If $h^0(O_{mD}) = 3$, then $a + \nu + 1 = m$, $a + 2\nu + 1 = m$ or $a + (p + 1)\nu + 1 = m$.

<u>Corollary</u> 2.5 ([BM], II, Corollary to Proposition 4). If $h^1(S, \underline{O}_S) \leq 1$, then we have $a_i + 1 = m_i$ or $a_i + \nu_i + 1 = m_i$.

Proof. By (1.1) and (1.2) we have

$$h^1(S, \underline{O}_S) = h^1(C, \underline{O}_C) + h^0(C, \underline{O}_C(\underline{f})) + h^0(C, \underline{T}).$$

On the other hand, by (2.2) and $R^2f_*O_S = 0$ we have

$$h^0(\underline{O}_{m_iD_i}) = h^1(\underline{O}_{m_iD_i}) = \text{length}(R^1f_*O_S \otimes k(p_i)) = 1 + \text{length}(\underline{T}_{p_i}).$$

Hence, we have $h^0(\underline{O}_{m_i}D_i) \leq 2$. Therefore, by Lemma 2.4, we obtained the desired result. q.e.d.

§3 A necessary condition for algebraicity.

In this section we consider an elliptic surface $f : S \longrightarrow \mathbb{P}^1$ with $\chi(S, \underline{O}_S) = 0$. By Noether's formula, we have $c_2(S) = 0$. Hence from Igusa's formula ([I1]) we infer that $f : S \longrightarrow \mathbb{P}^1$ has no singular fibres except multiple fibres $m_i E_i$, $i = 1,2,\ldots\ldots,\lambda$ with elliptic curves E_i. Put $\nu_i = \operatorname{ord}(\underline{O}_S(E_i)|_{E_i})$.

<u>Definition</u> 3.1. The elliptic surface $f : S \longrightarrow \mathbb{P}^1$ as above is called of type $(m_1, m_2, \ldots\ldots, m_\lambda | \nu_1, \nu_2, \ldots\ldots\ldots, \nu_\lambda)$. In case all multiple fibres are tame, that is $\nu_i = m_i$, $i = 1,2,\ldots\ldots\lambda$, such an elliptic surface is called of type $(m_1, m_2, \ldots\ldots, m_\lambda)$.

<u>Definition</u> 3.2. For a fixed i, $1 \le i \le \lambda$, it is said that $(m_1, m_2, \ldots\ldots, m | \nu_1, \nu_2, \ldots\ldots, \nu_\lambda)$ satisfies condition U_i, if there exist integers $n_1, n_2, \ldots\ldots, n_\lambda$ such that

$$\begin{cases} n_i \equiv 1 \mod \nu_i \\ n_1/m_1 + n_2/m_2 + \cdots\cdots\cdots + n_\lambda/m_\lambda \in \mathbb{Z}. \end{cases}$$

<u>Theorem</u> 3.3. Let $f : S \longrightarrow \mathbb{P}^1$ be an algebraic elliptic surface of type $(m_1, m_2, \ldots\ldots, m_\lambda | \nu_1, \nu_2, \ldots\ldots \nu_\lambda)$. Then $(m_1, m_2, \ldots\ldots, m_\lambda | \nu_1, \nu_2, \ldots\ldots \nu_\lambda)$ satisfies all conditions U_i, $i = 1,2,\ldots\ldots\ldots\lambda$.

In case $k = \mathbb{C}$ we have a more precise result. This will be discussed in Appendix 1.. To prove the theorem, we need the following two lemmas.

<u>Lemma</u> 3.4. For an elliptic surface $g : S \longrightarrow C$, we let $\alpha : S \longrightarrow \operatorname{Alb}(S)$ be an Albanese mapping of S and $\varphi : C \longrightarrow J(C)$ a natural mapping into the Jacobian variety of C. with a suitable choice of base points on S and C. Then, the following conditions are equivalent.

(i) There exists a fibre $f^{-1}(p)$, $p \in C$ such that $\alpha(f^{-1}(p))$ is a point.

(ii) $Alb(S)$ is isomorphic to $J(C)$.

Otherwise, we have $\dim Alb(S) = \dim J(C) + 1$.

Proof. By the universality of the Albanese mapping, choosing suitably φ, we may assume that there exists a surjective homomorphism $\theta : Alb(S) \longrightarrow J(C)$ with $\theta \circ \alpha = \varphi \circ f$.

$$\begin{array}{ccc} S & \xrightarrow{\alpha} & Alb(S) \\ f \downarrow & & \downarrow \theta \\ C & \xrightarrow{\varphi} & J(C) \end{array} \qquad (3.1)$$

If θ is an isomorphism, then by the diagram (3.1), we see that $\alpha(f^{-1}(p))$ is a point for any point p on C. Hence (ii) implies (i).

Now assume (i). If $\dim \alpha(S) = 0$, then we have $\dim Alb(S) = \dim J(C) = 0$. Hence in this case, (i) implies (ii). Assume $\dim \alpha(S) \geq 1$. Suppose that there exists a fibre $f^{-1}(q)$, $q \in C$ such that $\alpha(f^{-1}(q))$ is a curve. Then, there exists a hyperplane section H of $Alb(S)$ such that H intersects $\alpha(f^{-1}(q))$ but does not intersects $\alpha(f^{-1}(p))$. Then, the effective divisor $\alpha^{-1}(H)$ intersects $f^{-1}(q)$ but does not intersect $f^{-1}(p)$. This is a contradiction. Therefore, under our assumption, $\alpha(f^{-1}(q))$ is a point for any $q \in C$. Hence, $\alpha(S)$ is a curve and there exists a morphism from C to $\alpha(S)$. Hence by the universality of $(Alb(S), \alpha)$ and $(J(C), \varphi)$, $Alb(S)$ is isomorphic to $J(C)$.

Finally, assume that (i) does not hold. Take a general point p on C. Then, $\alpha(f^{-1}(p))$ is an elliptic curve E in $Alb(S)$. Using the diagram (3.1), we see that $\theta(E)$

is a point. Hence there exists a one dimensional abelian subvariety B of $\mathrm{Alb}(S)$ such that the following diagram commutes:

$$(3.2)\qquad \begin{array}{ccc} \mathrm{Alb}(S) & \xrightarrow{\pi} & \mathrm{Alb}(S)/_B \\ \theta \downarrow & \swarrow \theta' & \\ J(C) & & \end{array}$$

where θ' is the induced homomorphism which is surjective.
If $\dim (\mathrm{Alb}(S)/_B) = 0$, then $\dim J(C) = 0$, hence, $\dim \mathrm{Alb}(S) = \dim J(C) + 1$. If $\dim (\mathrm{Alb}(S)/_B) \geq 1$, then by the same method as above, $\pi \circ \alpha(S)$ is a curve and there exists a morphism from C to $\pi \circ \alpha(S)$. Therefore, by the universality of the Jacobian variety, there is a homomorphism $\mu : J(C) \longrightarrow \mathrm{Alb}(S)/_B$. Since $\pi \circ \alpha(S)$ generates $\mathrm{Alb}(S)/_B$, μ is surjective. Hence, we have $\dim J(C) = \dim (\mathrm{Alb}(S)/B) = \dim \mathrm{Alb}(S) - 1$. q.e.d.

<u>Lemma</u> 3.5. Let $f : S \longrightarrow \mathbb{P}^1$ be an elliptic surface with $\chi(\mathcal{O}_S) = 0$. Then, we have $\dim \mathrm{Alb}(S) = 1$.

Proof. By Noether's formula, we have

$$0 = 12 \cdot \chi(\mathcal{O}_S) = c_2(S) = 2 - 2b_1(S) + b_2(S).$$

As $b_1(S) = 2\dim \mathrm{Alb}(S)$, we have $\dim \mathrm{Alb}(S) \geq 1$. Hence, by Lemma 3.4, we have the desired result. q.e.d.

Proof of Theorem 3.3.

Let $j : E_i \longrightarrow S$ be the natural closed immersion. Consider the morphism

$$\alpha \circ j \;:\; E_i \longrightarrow S \longrightarrow \mathrm{Alb}(S).$$

By Lemmas 3.4 and 3.5, $\alpha \circ j$ is an isogeny. Hence we have a surjective homorphism

$$j^* \circ \alpha^* : \mathrm{Pic}^0(\mathrm{Alb}(S)) \longrightarrow \mathrm{Pic}^0(S) \longrightarrow \mathrm{Pic}^0(E_i).$$

In particular, we have a surjective homomorphism

$$j^* : \mathrm{Pic}^0(S) \longrightarrow \mathrm{Pic}^0(E_i).$$

Therefore, there exists a divisor L on S such that

(3.3) $$j^*(\underline{O}_S(L)) = \underline{O}_S(-E_i)|_{E_i}, \quad \underline{O}_S(L) \in \mathrm{Pic}^0(S).$$

As $\underline{O}_S(E_i+L)|_{E_i} \simeq \underline{O}_{E_i}$ by (3.3), we have the exact sequence

$$0 \longrightarrow \underline{O}_S(L) \longrightarrow \underline{O}_S(E_i+L) \longrightarrow \underline{O}_{E_i} \longrightarrow 0.$$

Hence we have a long exact sequence

(3.4) $$\begin{aligned} 0 &\longrightarrow H^0(L) \longrightarrow H^0(\underline{O}_S(E_i+L)) \longrightarrow H^0(\underline{O}_{E_i}) \longrightarrow \\ &\longrightarrow H^1(L) \longrightarrow H^1(\underline{O}_S(E_i+L)) \longrightarrow H^1(\underline{O}_{E_i}) \longrightarrow \\ &\longrightarrow H^2(L) \longrightarrow H^2(\underline{O}_S(E_i+L)) \longrightarrow H^2(\underline{O}_{E_i}) \longrightarrow 0, \end{aligned}$$

where $H^0(\underline{O}_{E_i}) \simeq H^1(\underline{O}_{E_i}) \simeq k$. By the Riemann-Roch theorem, we have

(3.5) $$\chi(\underline{O}_S(E_i+L)) = 0.$$

Suppose $H^0(\underline{O}_S(E_i+L)) = H^2(\underline{O}_S(E_i+L)) = 0$. Then by (3.5), we have $H^1(\underline{O}_S(E_i+L)) = 0$. So by (3.4), $h^2(\underline{O}_S(L)) = 1$. By the Serre duality, we have $h^0(\underline{O}_S(K_S-L)) = 1$. By E we denote a general fibre of $f : S \longrightarrow \mathbb{P}^1$. Then, using the canonical divisor formula (1.3), we see that there exists an effective divisor D on S such that

(3.6) $$D \sim mE + \sum_{j=1}^{\lambda} a_j E_j - L,$$

with a suitable integer m. By (3.3), we have $D \cdot E = 0$. Therefore, D consists of components of fibres, that is, there are non-negative integers $n, \alpha_1, \alpha_2, \ldots \alpha_\lambda$ such that

(3.7) $$D = nE + \sum_{j=1}^{\lambda} \alpha_j E_j.$$

By (3.6) and (3.7), we have

$$(3.8)\qquad -L \sim (n-m)E + \sum_{j=1}^{\lambda}(\alpha_j - a_j)E_j.$$

Restricting $\underline{\underline{O}}_S(-L)$ on E_i, we have by (3.3) and (3.8)

$$\underline{\underline{O}}_S(E_i)|_{E_i} \simeq \underline{\underline{O}}_S((\alpha_i - a_i)E_i)|_{E_i}.$$

As $\operatorname{ord} \underline{\underline{O}}_S(E_i)|_{E_i} = \nu_i$, we have

$$\alpha_i - a_i \equiv 1 \mod \nu_i.$$

Let H be a general hyperplane section of S. Then we have $L\cdot H = 0$ and $E\cdot H \neq 0$. Therefore, by (3.8) we have

$$0 = (n-m) + \sum_{j=1}^{\lambda}(\alpha_j - a_j)/m_j.$$

Hence, condition U_i is satisfied.

Next suppose $h^0(\underline{\underline{O}}_S(E_i+L)) \neq 0$. Then there exists an effective divisor D such that

$$(3.9)\qquad D \sim E_i + L.$$

Since $L \in \operatorname{Pic}^0(S)$, we have $D\cdot E = 0$. Therefore, D consists of components of fibres, hence

$$D \sim nE + \sum_{j=1}^{\lambda} \alpha_j E_j$$

with suitable integers $n, \alpha_1, \ldots, \alpha_\lambda$. By (3.9) we have

$$-L \sim -nE - \alpha_1 E_1 - \cdots - \alpha_{i-1}E_{i-1} + (1-\alpha_i)E_i$$
$$- \alpha_{i+1}E_{i+1} - \cdots\cdots - \alpha_\lambda E_\lambda.$$

Restricting $\underline{\underline{O}}_S(-L)$ on E_i, we have by (3.3)

$$\underline{\underline{O}}_S(E_i)|_{E_i} = \underline{\underline{O}}_S((1-\alpha_i)E_i)|_{E_i}.$$

Therefore we have

$$1 - \alpha_i \equiv 1 \mod \nu_i.$$

Intersecting a hyperplane section of S with L, we obtain

$$0 = -n - \alpha_1/m_1 - \cdots\cdots - (1-\alpha_i)/m_i - \cdots\cdots - \alpha_\lambda/m_\lambda .$$

Hence condition U_i is satisfied.

Finally, suppose $h^2(\underline{O}_S(E_i+L)) \neq 0$. Then $h^0(\underline{O}_S(K_S-E_i-L)) \neq 0$. Then, by the similar argument as above, we see that condition U_i is satisfied. q.e.d.

§4. A necessary condition for algebraicity, II.

The following corollaries are easy consequences of Theorem 3.:

Corollary 4.1. Let $f : S \longrightarrow \mathbb{P}^1$ be an algebraic elliptic surface of type $(m_1, m_2, \ldots\ldots, m_\lambda)$. Let m be the least common multiple of $m_1, m_2, \ldots\ldots, m_\lambda$. For a prime number q we let α be the maximal integer such that q^α divides m. Then there exist at least two indices i and j such that q^α divides both m_i and m_j.

Corollary 4.2. Let $f : S \longrightarrow \mathbb{P}^1$ be an algebraic elliptic surface of type $(m \mid \nu)$. Then, the only one multiple fibre is a wild fibre with $m = p^\delta$ and $\nu = 1$, where δ is a positive interger.

Corollary 4.3. Let $f : S \longrightarrow \mathbb{P}^1$ be an algebraic elliptic surface of type (m, n). Then $m = n$.

Corollary 4.4. Let $f : S \longrightarrow \mathbb{P}^1$ be an algebraic elliptic surface of type (m_1, m_2, m_3) with $\kappa(S) = 1$. Then we have

$$1/m_1 + 1/m_2 + 1/m_3 \leq 5/6.$$

The equality holds if and only if $(m_1, m_2, m_3) = (2, 6, 6)$.

Proof. By the canonical divisor formula (1.3), $\kappa(S) = 1$ if and only if

$$-2 + (m_1 - 1)/m_1 + (m_2 - 1)/m_2 + (m_3 - 1)/m_3 > 0.$$

Hence we have

$$1/m_1 + 1/m_2 + 1/m_3 < 1.$$

Then by Corollary 4.1, it is easy to show the above inequality.

q.e.d.

Remark 4.5 (Iitaka [I2]). There exists an analytic elliptic surf[ace] $g : X \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ which has only three singular fibres $2E_1, 3E_2, 7E_3$ with elliptic curves E_i, $i = 1,2,3$ and $\chi(X, O_X) = 0$, $\kappa(X) = 1$. It is easy to show that if $g : X \longrightarrow \mathbb{P}^1$ is an analytic elliptic surface with type (m_1, m_2, m_3) with $\kappa(X) = 1$, then we have

$$1/m_1 + 1/m_2 + 1/m_3 \leq 41/42.$$

Example 4.6. Here we give an example of algebraic elliptic surface of type (2,6,6), in case char. $k \neq 2,3$.

Let C be a non-singular complete curve of genus two defined by

$$y^2 = x^6 - 1.$$

Let us consider two automorphisms of C defined by

$$\sigma : (x,y) \longmapsto (x, -y),$$
$$\tau : (x,y) \longmapsto (\rho x, y),$$

where ρ is a primitive sixth root of unity. Let G be a group generated by σ and τ. The group G is isomorphic to $\mathbb{Z}/(2) + \mathbb{Z}/(6)$. Fix an elliptic curve E, a torsin point $a \in E$ of order 2 and a point $b \in E$ of order 6. Then the group G operates on $C \times E$ by

$$\sigma : (x,y,\zeta) \longmapsto (x, -y, \zeta + a)$$
$$\tau : (x,y,\zeta) \longmapsto (\rho x, y, \zeta + b).$$

The operation is free and we have an elliptic surface

$$f : S = C \times E/_G \longrightarrow C/_G = \mathbb{P}^1,$$

where f is obtained by the natural projection $C \times E \longrightarrow C$. The elliptic surface thus obtained is of type (2,6,6) and a canonical divisor has a form

$$f^*(-2p) + E_1 + 5E_2 + 5E_3,$$

where $2E_1$, $6E_2$, $6E_3$ are the multiple fibres. It is easy to show that $\dim|13K_S| = 0$ and $\dim|mK_S| \geq 1$ for $m \geq 14$.

Finally we give three typical examples of multiple fibres in characteristic p.

<u>Example</u> 4.7. Let g be the automorhpism of the projective line $\mathbb{P}^1$ defined by

$$g : t \longmapsto t + 1,$$

where t is a coordinate of an affine line $\mathbb{A}^1$ in $\mathbb{P}^1$. Let E be an ordinary elliptic curve and $a \in E$ a point of order p. Then the group $G = \langle g \rangle \simeq \mathbb{Z}/p\mathbb{Z}$ acts on $\mathbb{P}^1 \times E$ by

$$g : (t, \xi) \longmapsto (t + 1, \xi + a).$$

Then we have an elliptic surface $f : S = \mathbb{P}^1 \times E/G \longrightarrow \mathbb{P}^1/G = \mathbb{P}^1$, where f is the morphism induced by the projection. The elliptic surface S has only one multiple singular fibre pE_∞ over the point at infinity of $\mathbb{P}^1$. Since the canonical morphism π : $\mathbb{P}^1 \times E \longrightarrow S$ is étale, we have $\chi(\underline{O}_S) = \chi(\underline{O}_{\mathbb{P}^1 \times E}) = 0$. Hence, by Corollary 4.2, the multiple fibre pE_∞ is a wild fibre. By (1.6) and (1.8), $\mathrm{ord}[E_\infty]|_{E_\infty} = 1$. Moreover, we have $\kappa(S) \leq \kappa(\mathbb{P}^1 \times E) = -\infty$. In particular, we have $p_g(S) = 0$, hence $h^1(O_S) = 1$. Therefore, by Corollary 2.5, the canonical divisor of S is given by

$$K_S = f^*\underline{O}_{\mathbb{P}^1}(-1) + (p-2)E_\infty .$$

Another method to show this fact can be found in § 8.

Example 4.8. As the group scheme $G = \alpha_p$ is a subgroup scheme of $\mathbb{G}_a$, G acts naturally on $\mathbb{A}^1$ and the action can be extended to that on $\mathbb{P}^1$. Let E be a supersingular elliptic curve. Since α_p is also a subgroup scheme of E, it acts on $\mathbb{P}^1 \times E$ naturally. Hence, we have an elliptic surface $f : S = \mathbb{P}^1 \times E/\alpha_p \longrightarrow \mathbb{P}^1/\alpha_p \cong \mathbb{P}^1$, where f is induced from the natural projection. The elliptic surface has only one singular fiber which is a multiple fibre pE_∞ over the point at infinity. Since the quotient morphism $\pi : \mathbb{P}^1 \times E \longrightarrow S$ is a purely inseparable finite flat morphism of degree p, we have $\chi(\underline{O}_S) = \chi(O_{\mathbb{P}^1 \times E}) = 0$ and $\kappa(S) \leq \kappa(\mathbb{P}^1 \times E) = -\infty$ (see [RS], for instance). Hence, by Corollary 4.2, pE_∞ is a wild fibre and we have $p_g(S) = 0$. Hence, $h^1(\underline{O}_S) = 1$, By the same method as above, we obtain

$$K_S = f^*\underline{O}_{\mathbb{P}^1}(-1) + (p-2)E_\infty .$$

Example 4.9. Since μ_p is a subgroup scheme of $\mathbb{G}_m$, it acts naturally on $\mathbb{A}^1 - \{0\}$. This action can be extended to that on $\mathbb{P}^1$. Since μ_p is also a subgroup scheme of an ordinary elliptic curve E, μ_p acts naturally on $\mathbb{P}^1 \times E$. Hence, we obtain an elliptic surface $f : S = \mathbb{P}^1 \times E/\mu_p \longrightarrow \mathbb{P}^1/\mu_p$, where f is induced from the natural projection. The elliptic surface has two multiple singular fibres pE_0 over 0 and pE_∞ over the point at infinity of $\mathbb{P}^1$. Since the quotient morphism is purely inseparable, finite and flat, we have $\chi(\underline{O}_S) = 0$, $p_g(S) = 0$, and $h^1(\underline{O}_S) = 1$. Since neighbourhoods of E_0 and E_∞ are isomorphic, if one of the multiple fibres is wild, the other is also wild. Then, by the Leray spectral sequence associated with the morphism f, we have $h^1(\mathcal{O}_S) = h^0(R^1f_*\mathcal{O}_S) \geq h^0((R^1f_*\mathcal{O}_S)_{tor}) \geq 2$. This is a contradiction. Therefore, both multiple fibres are ordinary and we have

$$K_S = f^*\mathcal{O}_{\mathbb{P}^1}(-2) + (p-1)E_0 + (p-1)E_\infty.$$

Remark 4.10. In the above three examples, let $\tilde{\pi}: \mathbb{P}^1 \times E \longrightarrow S$ be the quotient morphism and $\tilde{f}: \mathbb{P}^1 \times E \longrightarrow \mathbb{P}^1$ the natural projection. For general points $q \in \mathbb{P}^1 = \mathbb{P}^1/G$ and $u \in \mathbb{P}^1$, we have

$$\begin{array}{ccc} \mathbb{P}^1 \times E & \xrightarrow{\tilde{\pi}} & S \\ {\scriptstyle \tilde{f}} \downarrow & & \downarrow {\scriptstyle f} \\ \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 \end{array}$$

$p\tilde{\pi}^*(E_\infty) = \tilde{\pi}^*(pE_\infty) \sim \tilde{\pi}^*(f^{-1}(q)) \sim$ $\sim p(\tilde{f}^{-1}(u))$. Therefore, we have

(4.1) $\tilde{\pi}^*(E_\infty) = \infty \times E.$

The restriction of $\tilde{\pi}$, $\tilde{\pi}_\infty = \tilde{\pi}\,|_{\infty \times E} : \infty \times E \longrightarrow E_\infty$ is nothing but the quotient morphism $E \longrightarrow E/G$. Taking the dual, we have a homomorphism $\tilde{\pi}_\infty^* : \mathrm{Pic}^0(E_\infty) \longrightarrow \mathrm{Pic}^0(E)$. Then, the kernel of $\tilde{\pi}_\infty^*$ is the dual group scheme $\hat{G}$ of G. On the other hand $[E_\infty]|_{E_\infty} \in \mathrm{Pic}^0(E_\infty)$ and by (4.1) $[E_\infty]|_{E_\infty}$ is in $\mathrm{Ker}\,\tilde{\pi}_\infty^*$. If $G = \mathbb{Z}/p\mathbb{Z}$ (resp. α_p, resp. μ_p), then $\hat{G}$ is μ_p (resp. α_p, resp. $\mathbb{Z}/p\mathbb{Z}$). Therefore, $\mathrm{ord}[E_\infty]|_{E_\infty}$ is one (resp. p), if G is $\mathbb{Z}/p\mathbb{Z}$ or α_p (resp. μ_p). Thus, the wildness of a multiple fibre of an elliptic surface obtained by a quotient by a finite group shceme does depend on the dual group scheme.

§5. Pluricanonical mappings of elliptic surfaces.

In this section we study pluricanonical mappings of elliptic surfaces with $\kappa = 1$. The following lemma is well-known and easy to prove (see [I2], Proposition 7, for instance).

Lemma. 5.1. Let $f : S \longrightarrow C$ be an elliptic surface with $\kappa = 1$. Then, S carries the unique structure of the elliptic surface.

Theorem 5.2. Let $f : S \longrightarrow C$ be an algebraic elliptic surface with $\kappa(S) = 1$. Then, the complete linear system $|mK_S|$ gives the unique structure of the elliptic surface if $m \geq 14$.

Remark 5.3. 1). By Example 4.6, the number 14 is the best possible, if $\mathrm{char}.k \neq 2,3$.

2). If we consider an analytic elliptic surface with $\kappa = 1$, then 86 is the best possible number ([I2] and see also Remark 4.5).

Proof of Theorem 5.2.

The idea of the proof is due to Iitaka [I2]. The uniqueness is clear from Lemma 5.1. Using the notation in §1, we have

$$|mK_S| = f^*\left(|mK_C - mf| + \sum_{i=1}^{\lambda}\left[\frac{ma_i}{m_i}\right]p_i\right),$$

where by $[\mu]$ we denotes the largest integer less than or equal to μ. Put

$$\Delta = mK_C - mf + \sum_{i=1}^{\lambda}\left[\frac{ma_i}{m_i}\right]p_i,$$

$$g = g(C), \quad t = \text{length } T.$$

If $\deg \Delta \geq 2g + 1$, then Δ is very ample, hence $f^*|\Delta|$ gives the structure of the elliptic surface. On the other hand the condition $\kappa(S) = 1$ is equivalent to

$$(5.1) \qquad 2g - 2 + \chi(\underline{\underline{O}}_S) + t + \sum_{i=1}^{\lambda}\frac{a_i}{m_i} > 0.$$

Therefore it is enough to show that if $m \geq 14$, we have

$$(*) \qquad \deg \Delta = m(2g - 2 + \chi(\underline{\underline{O}}_S) + t) + \sum_{i=1}^{\lambda}\left[\frac{ma_i}{m_i}\right] \geq 2g + 1.$$

Since $\chi(\underline{\underline{O}}_S) \geq 0$, we have the following six cases.

Case (I) $\chi(\underline{\underline{O}}_S) + t \geq 3$,

Case (II) $0 \leq \chi(\underline{\underline{O}}_S) + t \leq 2$ and $g \geq 1$,

Case (III) $\chi(\underline{\underline{O}}_S) + t = 2$ and $g = 0$,

Case (IV-1) $\chi(\underline{\underline{O}}_S) = 1$, $t = 0$ and $g = 0$,

Case (IV-2) $\chi(\underline{\underline{O}}_S) = 0$, $t = 1$ and $g = 0$,

Case (V) $\chi(\underline{\underline{O}}_S) = t = 0$ and $g = 0$.

In case (I), (*) is satisfied for $m \geq 1$. In case (II), (*) is satisfied for $m \geq 6$. In case (IV-1), since all multiple fibres are tame, it is easy to show that (*) is satisfied for $m \geq 6$. In case (V), using Corollary 4.4, we can prove the theorem by the same method as in [I2]. For the reader's convenience, we give the proof. Put

$$A = -2 + \sum_{i=1}^{\lambda} (m_i - 1)/m_i .$$

By (5.1), we have $A > 0$.

Therefore, we have $\lambda \geq 3$. If $\lambda \geq 4$, we have

$$A \geq -2 + 1/2 + 1/2 + 1/2 + 2/3 = 1/6.$$

But (2,2,2,3) does not satisfied condition U_4. Hence we have $A > 1/6$ if $\lambda \geq 4$. On the other hand, if $\lambda = 3$, by Corollary 4.4, we have $A \geq 1/6$. Since we have

$$\left[m(1 - 1/m_i)\right] - m(1 - 1/m_i) \geq -(1 - 1/m_i),$$

and

$$\sum_{i=1}^{\lambda}\left[m(1 - 1/m_i)\right] = \sum_{i=1}^{\lambda} m(1 - 1/m_i) + \sum_{i=1}^{\lambda}\left\{\left[m(1 - 1/m_i)\right] - m(1 - 1/m_i)\right\}$$

$$\geq (m-1)\left\{\sum_{i=1}^{\lambda}(1 - 1/m_i)\right\} = (m-1)(2+A),$$

to prove (*), it is enough to show that $(m-1)(2+A) > 2m$ if $m \geq 14$.

But this is clear by $A \geq 1/6$.

Next consider Case (III). By (5.1) we have

(5.2) $$\sum_{i=1}^{\lambda} a_i/m_i > 0.$$

If there exists at least one tame multiple fibre, say m_1D_1, then we have

$$\deg \Delta \geq [m(1-1/m_1)] \geq [m/2] \geqslant 1, \text{ if } m \geq 2.$$

Therefore, assume that there are no tame fibres. Since $t \leq 2$, there exist either only one wild fibre or two wild fibres, and $\text{char}.k = p > 0$. First consider the case in which there is only one wild fibre. Then (5.2) is equivalent to $a_1 > 0$. As m_1D_1 is a wild fibre, by (1.6) we have $m_1 = p^{\gamma}\nu_1$ with an integer $\gamma \geqslant 1$. By (1.8) and (1.9) we have

$$2 \leq h^0(\underline{O}_{m_1D_1}) = h^1(\underline{O}_{m_1D_1}) = h^0((R^1f_*\underline{O}_S)_{P_1}) = 1 + t \leq 3.$$

Therefore, by Lemma 2.4, we have the following three possibilities.

(i) $a_1 + \nu_1 + 1 = m_1$, (ii) $a_1 + 2\nu_1 + 1 = m_1$,

(iii) $a_1 + (p+1)\nu_1 + 1 = m_1$.

Case (i). In this case, we have

$$\deg \Delta = [m(1-(\nu_1+1)/m_1)] = [m(1 - 1/p^{\gamma} - 1/p^{\gamma}\nu_1)].$$

If $p \geq 3$, then $\deg \Delta \geq [m(1-1/3-1/3)] \geqslant 1$ for $m \geq 3$. Next assume $p = 2$. If $m_1 = 2$, then $a_1 = 0$. This contradicts our assumption. Therefore $m_1 \geq 4$, since m_1D_1 is wild. Hence (*) is satisfied if $m \geq 4$.

Case (ii). In this case we have

$$\deg \Delta = [m\{1 - (2\nu_1+1)/m_1\}] = [m(1 - 2/p^{\gamma} - 1/p^{\gamma}\nu_1).$$

As we have $a_1 = p^{\gamma}\nu_1 - 2\nu_1 - 1 > 0$, we have the following four cases:

(1) $p \geq 5$, (2) $p = 3$, $\gamma \geq 2$, (3) $p = 3$, $\gamma = 1$, $\nu_1 \geqslant 2$,

(4) $p = 2$, $\gamma \geq 2$.

We can check that in each case $\deg \Delta \geq 1$ if $m \geq 6$.

Case (iii). In this case, as we have $a_1 = p^{\gamma}\nu_1 - (p+1)\nu_1 - 1 > 0$, we have the following three cases :

(1) $p \geq 3, \gamma \geq 2$, (2) $p = 2$, $\gamma \geq 3$, (3) $p = 2$, $\gamma = 2$, $\nu_1 \geq 2$.

We can show easily that (*) is satisfied for $m \geq 8$.

Next assume that there are two wild fibres. Then we have $t = 2$ and by (1.8) and (1.9) we have

$$h^0(\underline{O}_{m_iD_i}) = h^1(\underline{O}_{m_iD_i}) = h^0((R^1f_*O_S)_{P_i}) = 2, \ i = 1,2.$$

Therefore by Lemma 2.4, we have $m_i = a_i + \nu_i + 1$, $i=1,2$. By (5.2), we may assume $a_1 > 0$. Then by the same method as in Case (i), condition (*) is satisfied for $m \geq 4$.

Finally consider the case (IV-2). As $t = 1$, we have only one wild fibre, say m_1D_1. Then by (1.8) and Lemma 2.4, we have $a_1 + \nu_1 + 1 = m_1$ and $m_1 = p^{\gamma}\nu_1$ with an integer $\gamma \geq 1$. Condition (5.1) is written as

(5.3) $$\sum_{i=1}^{\lambda} a_i/m_i > 1.$$

Hence, $\lambda \geq 2$. If $\lambda \geq 4$, there are at least three tame fibres. Hence,

$\deg \Delta \geq -m + 3[m/2] \geq 1$ if $m \geq 4$. If $\lambda = 3$, we have two tame fibres m_2D_2 and m_3D_3 with $m_2 \leq m_3$. If $m_2 \geq 2$ and $m_3 \geq 3$, then $\deg \Delta \geq -m + [m/2] + [2m/3] \geq 1$ if $m \geq 8$. If $m_2 = m_3 = 2$, then by (5.3), we have $a_1 > 0$. Hence $m_1 = p^{\gamma}\nu_1 \geq 3$. Hence we have

$\deg \Delta \geq -m + [m(1 - 1/2 - 1/3)] + 2[m/2] \geq 1$ if $m \geq 12$.

If $\lambda = 2$, there are a wild fibre m_1D_1 and a tame fibre m_2D_2. By Theorem 3.3, $(m_1, m_2 \mid \nu_1, m_2)$ satisfies conditions U_i, $i = 1,2$. By condition U_2, we have $m_2 | m_1$. If $p \mid \nu_1$, then $m_1 | m_2$ by condition U_1. If $p \nmid \nu_1$, then we have $m_2 = p^{\beta}\nu_1$ with a non-negative integer $\beta \leq \gamma$ by condition U_1. Therefore, we have the following two cases:

(i) $p \mid \nu_1$, $m_1 = m_2 = p^{\gamma}\nu_1$, $\gamma \geq 1$,

(ii) $p \nmid \nu_1$, $m_1 = p^{\gamma}\nu_1$, $m_2 = p^{\beta}\nu_1$, $\gamma \geq \beta$, $\gamma \geq 1$.

Case (i). In this case, condition (5.3) is written as $a_1 - 1 = p^{\gamma}\nu_1 - \nu_1 - 2 > 0$. Hence the following three cases occur.

(i-1) $p \geq 3$, $\nu_1 \geq 3$, (i-2) $p = 2$, $\nu_1 \geq 2$, $\gamma \geq 2$

(i-3) $p = 2$, $\nu_1 \geq 4$, $\gamma = 1$.

In each case ,(*) is satisfied for $m \geq 14$.

Case (ii). In this case condition (5.3) is written as

$$(p^{\gamma} - 1)\nu_1 - (p^{\gamma-\beta} + 1) = p^{\gamma-\beta}(p^{\beta} - 1) + (p^{\gamma} - 1)(\nu_1 - 1) - 2 > 0.$$

Therefore, by the condition $p \nmid \nu_1$, the following twelve cases occur:

(ii-1) $\beta > 0$, $p \geq 5$, (ii-2) $\beta > 1$, $p = 3$, (ii-3) $\beta = 1$, $p = 3$, $\nu_1 \geq 2$, (ii-4) $\beta = 1$, $p = 3$, $\nu_1 = 1$, $\gamma \geq 2$, (ii-5) $\beta > 1$, $p = 2$, (ii-6) $\beta = 1$, $p = 2$, $\nu_1 \geq 3$, (ii-7) $\beta = 1$, $p = 2$, $\nu_1 = 1$, $\gamma \geq 3$, (ii-8) $\beta = 0$, $\nu_1 \geq 2$, $p \geq 5$, (ii-9) $\beta = 0$, $\nu_1 \geq 2$, $p = 3$, $\gamma \geq 2$, (ii) $\beta = 0$, $\nu_1 \geq 4$, $p = 3$, $\gamma = 1$, (ii-11) $\beta = 0$, $\nu_1 \geq 5$, $p = 2$, (ii-12) $\beta = 0$, $\nu_1 = 3$, $p = 2$, $\gamma \geq 2$.

In each case, it is easy to show that (*) is satisfied if $m \geq 14$.

q.e.d.

§6. Reduction of a wild fibre to a tame fibre.

In this section, we use the notation in §1 and denote m_1D_1 by mD. Put $f(D) = p$ and $\nu = \mathrm{ord}\, [D]|_D$. By (1.6), we have $m = p^{\gamma}\nu$ with a non-negative integer γ. We know that $\gamma \geq 1$ if and only if mD is a wild fibre.

Lemma 6.1.(i) If $f^{-1}(p) = mD$ is a wild fibre, then the natural mapping $\rho : H^1(S, \underline{O}_S) \longrightarrow H^1(D, \underline{O}_D)$ is surjective.

(ii) Assume $\deg f < 0$. (e.g. $f : S \rightarrow C$ has a wild fibre.) Let $f^{-1}(q) = E$, $q \in C$ be not a wild fibre. Then the natural restriction mapping $\rho: H^1(S, \underline{O}_S) \longrightarrow H^1(E, \underline{O}_E)$ is the zero mapping.

Proof. (i) From the exact sequence

$$0 \rightarrow \underline{O}_S(-D) \longrightarrow \underline{O}_S \longrightarrow \underline{O}_D \rightarrow 0,$$

we obtain a long exact sequence

$$H^1(\underline{O}_S) \rightarrow H^1(\underline{O}_D) \longrightarrow H^2(\underline{O}_S(-D)) \longrightarrow H^2(\underline{O}_S) \longrightarrow 0.$$

Since mD is a wild fibre, we have $h^0(\underline{O}_S(K_S+D)) = h^0(O_S(K_S))$. Therefore, in the above exact sequence, the natural restriction mapping $\rho : H^1(O_S) \longrightarrow H^1(\underline{O}_D)$ is surjective.

(ii) From the Leray spectral sequence associated with $f : S \rightarrow C$, we have the edge exact sequence

(6.1) $$0 \longrightarrow H^1(C, \underline{O}_C) \longrightarrow H^1(S, \underline{O}_S) \longrightarrow H^0(C, R^1f_*\underline{O}_S) \longrightarrow 0.$$

By (1.2) and $\deg \underline{f} < 0$, $H^0(C, R^1f_*\underline{O}_S) = T$. The natural restriction mapping $\rho : H^1(S, \underline{O}_S) \longrightarrow H^1(E, \underline{O}_E)$ factors through $H^1(S, \underline{O}_S) \longrightarrow H^0(C, R^1f_*\underline{O}_S) \longrightarrow H^1(E, \underline{O}_E)$. Hence $\rho = 0$. q.e.d.

Let X be a complete algebraic variety defined over an algebraically closed field k with $\mathrm{char}.k = p > 0$. By the Frobenius mapping F_X of X we means that F_X acts on the structure sheaf $\underline{\underline{O}}_X$ by $g \longmapsto g^p$. The Frobenius mapping F_X induces a p-linear mapping of $H^1(X,\underline{\underline{O}}_X)$. We also denote it by F_X. We assume that mD is a wild fibre and we shall reduce the wild fibre into a tame fibre in the following three cases: (I) D is an ordinary elliptic curve, (II) D is of type I_n, (III) D is a supersingular elliptic curve.

Case (I). As D is an ordinary elliptic curve, the Frobenius mapping F_D acts semi-simply on $H^1(D,\underline{\underline{O}}_D)$. Considering the Fitting decomposition of $H^1(S,\underline{\underline{O}}_S)$ with respect to the Frobenius mapping F_S, we can find a non-zero element $\alpha \in H^1(S,\underline{\underline{O}}_S)$ such that

(6.2) $$\rho(\alpha) \neq 0, \quad F_S(\alpha) = \alpha, \quad F_D(\rho(\alpha)) = \rho(\alpha).$$

We make a covering of S using the element α. For this purpose, let $\{U_i\}_{i \in I}$ be an affine open covering of S and let α be represented by a Čech cocycle $\{f_{ij}\}$ with respect to this covering. By (6.2), there exist elements $f_i \in \Gamma(U_i, \underline{\underline{O}}_S)$, $i \in I$ such that

$$f_{ij}^p = f_{ij} + f_i - f_j \quad \text{on} \quad U_i \cap U_j.$$

Let $\pi_1 : S^{(1)} \longrightarrow S$ be the covering defined by

(6.3) $$\begin{cases} z_i^p - z_i = f_i & \text{on } U_i, \quad i \in I, \\ z_i = z_j + f_{ij} & \text{on } U_i \cap U_j. \end{cases}$$

This is an étale covering of degree p.

If we restrict the covering $\pi_1 : S^{(1)} \longrightarrow S$ to D, we obtain a non-trivial étale covering of degree p of D, since $\rho(\alpha) \neq 0$. On the other hand if we restrict the covering to a general fibre E, then by Lemma 6.1 (ii), the covering splits into p copies of E. Hence by the Stein factorization of $f \circ \pi_1$, we obtain a curve $C^{(1)}$ and morphisms g_1, f_1 with a commutative diagram

$$\begin{array}{ccc} S & \xleftarrow{\pi_1} & S^{(1)} \\ f\downarrow & & \downarrow f_1 \\ C & \xleftarrow{g_1} & C^{(1)}, \end{array}$$

where g_1 is totally ramified at p with deg g_1 = p. We denote by $p^{(1)}$ the point on $C^{(1)}$ such that $g_1(p^{(1)}) = p$. Put $f_1^{-1}(p^{(1)}) = m^{(1)}D^{(1)}$. Then we have

$$\pi_1^*[D]|_D = [D^{(1)}]|_{D^{(1)}},$$

and

$$m^{(1)} = m/p.$$

Since D is an ordinary elliptic curve and $\pi_1|_{D^{(1)}} : D^{(1)} \longrightarrow D$ is an étale covering of degree p, $\pi_1|_{D^{(1)}}^* : \mathrm{Pic}^0(D) \longrightarrow \mathrm{Pic}^0(D^{(1)})$ is a purely inseparable homomorphism of degree p. Therefore, we have

$$\nu = \mathrm{ord}[D]|_D = \mathrm{ord}\ \pi_1|_{D^{(1)}}^*[D]|_D = \mathrm{ord}[D^{(1)}]|_{D^{(1)}}.$$

Now continue this procedure γ times. We obtain the following diagram:

$$(6.4)\qquad \begin{array}{ccccccccc} S & \xleftarrow{\pi_1} & S^{(1)} & \xleftarrow{\pi_2} & S^{(2)} & \leftarrow & \cdots\cdots & \xleftarrow{\pi_\gamma} & S^{(\gamma)} \\ f\downarrow & & f_1\downarrow & & f_2\downarrow & & & & f_\gamma\downarrow \\ C & \xleftarrow{g_1} & C^{(1)} & \xleftarrow{g_2} & C^{(2)} & \leftarrow & \cdots\cdots & \xleftarrow{g_\gamma} & C^{(\gamma)} \\ \cup & & \cup & & \cup & & & & \cup \\ p^{(0)} = p & & p^{(1)} & & p^{(2)} & & & & p^{(\gamma)}, \end{array}$$

where π_i's are étale morphisms of degree p as in (6.3),

$f_i : S^{(i)} \longrightarrow C^{(i)}$ is an elliptic surface for each i, and g_i is a morphism of degree p totally ramified at $p^{(i-1)}$ for each i with $g_i(p^{(i)}) = p^{(i-1)}$. Put $f_i^{-1}(p^{(i)}) = m^{(i)}D^{(i)}$. Then we have

$$\operatorname{ord}[D^{(i)}]\big|_{D^{(i)}} = \operatorname{ord}[D^{(i-1)}]\big|_{D^{(i-1)}},$$

$$m^{(i)} = m^{(i-1)}/p.$$

Hence, $\operatorname{ord}[D^{(\gamma)}]\big|_{D^{(\gamma)}} = \gamma = m^{(\gamma)}$. This means that $m^{(\gamma)}D^{(\gamma)}$ is a tame fibre by (1.7).

Case (II). Let $\operatorname{Pic}^0(D)$ be the group of isomorphism classes of invertible sheaves on D which are of degree 0 on each irreducible component. As D is of type I_n, $\operatorname{Pic}^0(D) = \mathbb{G}_m$, hence, the Frobenius mapping F_D acts semi-simply on $H^1(O_D)$. Therefore by the same method as above, we have the same diagram as in (6.4). Put $f_i^{-1}(p^{(i)}) = m^{(i)}D^{(i)}$. Then, the divisor $D^{(i)}$ is of type $I_{p^i n}$ and we have $m^{(i)} = m^{(i-1)}/p$. It is easy to show that the homomorphism $\pi_i\big|^*_{D^{(i)}} : \operatorname{Pic}^0(D^{(i-1)}) \longrightarrow \operatorname{Pic}^0(D^{(i)})$ is the Frobenius morphism $F : \mathbb{G}_m \longrightarrow \mathbb{G}_m$. Therefore, we have

$$\operatorname{ord}[D^{(i-1)}]\big|_{D^{(i-1)}} = \operatorname{ord}\ \pi_i\big|^*_{D^{(i)}}[D^{(i-1)}]\big|_{D^{(i-1)}} = \operatorname{ord}[D^{(i)}]\big|_{D^{(i)}}$$

Hence, $D^{(\gamma)}$ is of type $I_{p^\gamma n}$ and $m^{(\gamma)}D^{(\gamma)}$ is a tame fibre.

Case (III) Since D is a supersingualr elliptic curve, the Frobenius mapping F_D acts nilpotently on $H^1(\underset{\approx}{O}_D)$. Considering the Fitting decomposition of $H^1(S,O_S)$ with respect to the Frobenius mapping F_S, we find a non-zero element $\alpha \in H^1(S,\underset{\approx}{O}_S)$ such that

(6.5) $\rho(\alpha) \neq 0,\ F_S^{n-1}(\alpha) \neq 0,\ F_S^n(\alpha) = 0,\ F_n(\rho(\alpha)) = 0$

with a suitable positive integer n.

We make a covering of S, using this element α. Let $\{U_i\}_{i\in I}$ be an affine open covering of S such that α is represented by a Čech cocycle $\{f_{ij}\}$ with respect to this covering. By (6.5), there are $f_i \in \Gamma(U_i, \underline{\underline{O}}_S)$, $i \in I$, such that

$$f_{ij}^{p^n} = f_i - f_j \quad \text{on} \quad U_i \cap U_j.$$

We define the covering $\pi_1 : S^{(1)} \longrightarrow S$ by

$$(6.6) \qquad \begin{cases} z_i^{p^n} = f_i, & \text{on } U_i,\ i \in I, \\ z_i = z_j + f_{ij} & \text{on } U_i \cap U_j. \end{cases}$$

This is a flat covering of degree p^n.

Let $\mu : \tilde{S}^{(1)} \longrightarrow S^{(1)}$ be the minimal non-singular model (as an elliptic surface) of the normalization of $S^{(1)}$. By the Stein factorization, we have the following diagram

$$\begin{array}{ccccc} S & \xleftarrow{\ \pi_1\ } & S^{(1)} & \xleftarrow{\ \mu\ } & \tilde{S}^{(1)} \\ f \downarrow & & & & \downarrow f_1 \\ C & & \xleftarrow{\qquad g_1 \qquad} & & C^{(1)} \\ \Psi & & & & \Psi \\ p & & & & p^{(1)} \end{array}$$

where $C^{(1)}$ is a non-singular complete curve and $g_1(p^{(1)}) = p$. Since the restriction of π_1 to a general fibre of f is trivial by Lemma 6.1 (ii), the morphism g_1 is purely inseparable of degree p^n. Moreover, since D is an elliptic curve, the normalization of $S^{(1)}$ is already non-singular in a neighbourhood of $f_1^{-1}(p^{(1)})$ by the structure of singular fibres of elliptic surfaces. Put $f_1^{-1}(p^{(1)}) = m^{(1)}D^{(1)}$, $g_{ij} = f_{ij}|_D$, $g_i = f_i|_D$. Since F_D is the zero mapping on $H^1(\underline{\underline{O}}_D)$, there are elements $h_i \in \Gamma(U_i \cap D, \underline{\underline{O}}_D)$ such that

$$g_{ij}^p = h_i - h_j \quad \text{on} \quad U_i \cap U_j \cap D.$$

By a suitable choice of h_i, we may assume

$$(6.7) \qquad g_i = h_i^{p^{n-1}}, \quad i \in I.$$

Let us consider the covering $\pi_1' : D' \longrightarrow D$ defined by

$$\begin{cases} w_i^p = h_i & \text{on } U_i \cap D, \\ w_i = w_j + g_{ij} & \text{on } U_i \cap U_j \cap D. \end{cases}$$

By (6.5), this covering is a non-trivial flat covering of degree p. By (6.6), (6.7) and normalization, we obtain the following commutative diagram

$$\begin{array}{ccccc} S & \xleftarrow{\pi_1} & S^{(1)} & \xleftarrow{\mu} & S^{(1)} \\ \uparrow & & \uparrow & & \uparrow \\ D & \xleftarrow{\pi'} & D' & \xleftarrow{\pi''} & D^{(1)}, \end{array}$$

with $\deg \pi' = p$. Therefore, setting $\tilde{\pi}_1 = \pi_1 \circ \mu$, we have

$$\tilde{\pi}_1^{-1}(D) = p^{n-j} D^{(1)},$$

for a suitable integer j, $1 \leq j \leq n$. This implies

$$(6.8) \qquad m^{(1)} = m / p^j.$$

Since D is a supersingular elliptic curve, $D^{(1)}$ is also a supersingular elliptic curve. As $\tilde{\pi}_1|_{D^{(1)}}$ is a purely inseparable morphism, $\tilde{\pi}_1|_{D^{(1)}}^* : \mathrm{Pic}^0(D) \longrightarrow \mathrm{Pic}^0(D^{(1)})$ is also purely inseparable. Therefore, we have

$$(6.9) \qquad \nu = \mathrm{ord}[D]|_D = \mathrm{ord}\, \tilde{\pi}_1|_{D^{(1)}}^* [D]|_D =$$

$$= \mathrm{ord}\left[p^{n-j} D^{(1)}\right]\Big|_{D^{(1)}} = \mathrm{ord}\left[D^{(1)}\right]\Big|_{D^{(1)}},$$

because the supersingularity of $\mathrm{Pic}^0(D^{(1)})$ implies that $\mathrm{ord}[D^{(1)}]|_{D^{(1)}}$ is prime to p. Continuing this procedure, we obtain a tame fibre by virtue of (6.8) and (6.9).

§7. Reduction of a tame fibre to a non-multiple fibre.

We use the same notation as in the previous section. In this section we assume that mD is a tame multiple fibre. Hence $\mathrm{ord}[D]|_D = m$. We shall reduce the multiple fibre to a non-multiple fibre in the following three cases.

(I) D is an ordinary elliptic curve, (II) D is of type I_n, (III) D is a supersingular elliptic curve.

The missing case is a tame fibre pD with $\mathrm{Pic}^0(D) = \mathbb{G}_a$. For this case, very few examples are known ([KI]).

First consider Case (I). Write

$$m = p^{\delta}m', \qquad (p,m') = 1, \quad \delta \geq 0.$$

<u>First Step</u>. Let t be a local parameter of p on an open affine neighourhood of p in the curve C. Taking U small enough, we may assume that $[p^{\delta}D]$ is of order m' on $f^{-1}(U)$. Let $\{U_i\}$ be an affine open covering of $f^{-1}(U)$ and let $f_i = 0$ be a defining equation of $p^{\delta}D$ in U_i. Then, $[p^{\delta}D]|_{f^{-1}(U)}$ is defined by trasition functions

$$f_{ij} = f_i/f_j \qquad \text{on} \quad U_i \cap U_j.$$

Pulling back t by f, we consider t as a regular function on $f^{-1}(U)$. Then we have

$$t = u_i f_i^{m'} \qquad \text{on } U_i, \quad u_i \in \Gamma(U_i, \underline{O}_S^*).$$

Hence we have

$$u_j = f_{ij}^{m'} u_i \qquad \text{on} \quad U_i \cap U_j.$$

Define an étale covering $\pi_1 : V^{(1)} \longrightarrow f^{-1}(U)$ of degree m' by

$$\begin{cases} z_i^{m'} = u_i & \text{on } U_i, \\ z_j = f_{ij} z_i & \text{on } U_i \cap U_j. \end{cases}$$

For a general point $q \in U$, the restriction of $[p^\delta D]$ to $f^{-1}(q)$ is trivial. Hence the restriction of the covering π_1 to $f^{-1}(q)$ splits into m' copies of $f^{-1}(q)$. As $[p^\delta D]|_D$ is of order m', the restriction of the covering π_1 to D is a non-trivial connected êtale covering of degree m'. Therefore, by the Stein factorization, we have the following commutative diagram.

$$\begin{array}{ccc} f^{-1}(U) & \xleftarrow{\ \pi_1\ } & V^{(1)} \\ f \downarrow & & \downarrow f_1 \\ U & \xleftarrow{\ g_1\ } & U^{(1)} \\ \cup & & \cup \\ p & & p^{(1)} \end{array},$$

where $U^{(1)}$ is a non-singular curve, f_1 is an elliptic fibration and $g_1(p^{(1)}) = p$. The morphism g_1 is totally ramified at p. Put $f_1^{-1}(p^{(1)}) = m^{(1)}D^{(1)}$. Then we have

$$\pi_1|_D^*[D]|_D = [D^{(1)}]|_{D^{(1)}}, \quad m^{(1)} = p^\delta .$$

Moreover, we have

$$\operatorname{ord}[D^{(1)}]|_{D^{(1)}} = p^\delta .$$

<u>Second Step.</u> Let s be a local parameter of $p^{(1)}$ on an affine open neighbourhood V of $p^{(1)}$ in $U^{(1)}$. Taking V small enough, we may assume that $[D^{(1)}]|_{f^{-1}(V)}$ is of order p^δ. Let $\{V_i\}_{i \in I}$ be an open affine covering of $f_1^{-1}(V)$ and $g_i = 0$ a defining equation of $D^{(1)}$ on V_i. Then, $[D^{(1)}]|_{f_1^{-1}(V)}$ is defined by transition functions

$$g_{ij} = g_i / g_j \qquad \text{on} \quad V_i \cap V_j .$$

Pulling back by f_1, we may consider s as a regular function on $f_1^{-1}(V)$. Then, we have

$$s = v_i g_i^{p^\delta} \quad \text{on } V_i, \quad v_i \in \Gamma(V_i, \underline{O}_{V_i}^*).$$

Then, we have

$$(7.1) \qquad v_j = g_{ij}^{p^\delta} v_i \quad \text{on } V_i \cap V_j .$$

Define a flat covering $\pi_2 : V^{(2)} \longrightarrow f^{-1}(V)$ of degree p^δ by

$$(7.2) \qquad \begin{cases} w_i^{p^\delta} = v_i & \text{on } V_i, \\ w_i = g_{ij} w_j & \text{on } V_i \cap V_j . \end{cases}$$

By the same argument as in the first step and the Stein factorization, we have the following diagram.

$$\begin{array}{ccccc} V^{(1)} & \supset & f^{-1}(V) & \xleftarrow{\ \pi_2\ } & V^{(2)} \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_1'} & & \downarrow{\scriptstyle f_2} \\ U^{(1)} & \supset & V & \xleftarrow{\ g_2\ } & U^{(2)} \\ & & \ni p^{(1)} & & \ni p^{(2)} \end{array} ,$$

where $U^{(2)}$ is a non-singular curve, $f_1' = f_1|_{f^{-1}(V)}$, f_2 is an elliptic fibration and $g_2(p^{(2)}) = p^{(1)}$. The morphism g_2 is purely inseparable and of degree p^δ .

The following lemma is well-known.

<u>Lemma</u> 7.1. Let E be an ordinary elliptic curve, $\{U_i\}_{i \in I}$ an affine open covering of E and L a line bundle on E of order p^δ with $\delta \geq 1$ defined by a Čech cocycle $\{f_{ij}\}$ with respect to the covering $\{U_i\}$. Then there exist $f_i \in \Gamma(U_i, \underline{O}_E^*)$, $i \in I$ such that $f_{ij}^{p^\delta} = f_i / f_j$. Moreover, df_i / f_i, $i \in I$ give a non-zero regular 1-form ω on E.

By (7.2) the singular points of $V^{(2)}$ are contained in the zeros of the 1-form

$$dv_i \quad \text{on} \quad \pi_2^{-1}(V_i),\ i \in I.$$

Put $v_i' = v_i\Big|_{V_i \cap D^{(1)}}$. Then, by (7.1) and Lemma 7.1, dv_i'/v_i' on $V_i \cap D^{(1)}$, $i \in I$ define a non-zero regular 1-form on E. Therefore, in particular, dv_i on V_i has no zeros arround $D^{(1)}$. Hence $V^{(2)}$ is non-singular arround $f_2^{-1}(p^{(2)})$. It is easy to see that $f_2^{-1}(p^{(2)})$ is a regular fibre. Thus we obtain a reduction of the tame fibre to a regular fibre.

Cases (II) and (III) are treated as follows. Since D is of type I_n (resp. a supersingular elliptic curve), $\mathrm{Pic}^0(D)$ is $\mathbb{G}_m$ (resp. a supersingular elliptic curve). Therefore, $\mathrm{Pic}^0(D)$ has no points of order p. Hence if mD is a tame fibre, we have $(m, p) = 1$, since $m = \mathrm{ord}[D]\big|_D$, $[D]\big|_D \in \mathrm{Pic}^0(D)$. Thus, by the same method as in the first step in Case (I), we obtain a reduction of the tame fibre to a non-multiple fibre.

§ 8. Examples.

In this section we give examples of elliptic surfaces with wild fibres. We also give examples of a smooth deformation and a lifting of certain elliptic surfaces. In this section, we fix an algebraically closed field k of $\mathrm{char}.k = p > 0$.

First we generalize Example 4.7 in the following way.

I) Etale quotients. Let $\pi : C \longrightarrow \mathbb{P}^1_k$ be a cyclic Galois covering of degree p^n ramified only at the point at infinity of $\mathbb{P}^1_k$. For simplicity, assume that π is totally ramified at ∞. Put $\infty_1 = \pi^{-1}(\infty)$. Let g be an automorphism of C which generates the Galois group G of the covering π. Fix an ordinary elliptic curve E over k and a torsion point $a \in E(k)$ of order p^n. Then the cyclic group G operates on $C \times E$ by

$$\begin{array}{cccc} g : & C \times E & \longrightarrow & C \times E \\ & \mathrel{\rotatebox{90}{$\in$}} & & \mathrel{\rotatebox{90}{$\in$}} \\ & (u, \zeta) & \longmapsto & (g(u), \zeta + a). \end{array}$$

By $\tilde{\pi} : C \times E \longrightarrow S = C \times E/_G$, we denote the quotient morphism. The morphism $f : S \longrightarrow \mathbb{P}^1 = C/_G$ induced from the natural projection $C \times E \rightarrow C$ gives a structure of an elliptic surface. By our construction, the elliptic surface has only one multiple singular fibre $p^n E_\infty$ over the point at infinity of $\mathbb{P}^1$. By a canonical divisor formula, we have

(8.1) $$K_S = f^* \underline{O}_{\mathbb{P}^1}(-2+\ell) + aE_\infty ,$$

(8.2) $$\ell = -\deg \underset{\sim}{f}, \quad 0 \leq a \leq p^n - 1.$$

By a similar argument as in Remark 4.10, we have

$$\tilde{\pi}^*(E_\infty) = \infty_1 \times E.$$

Since $\tilde{\pi}$ is étale, we have $\tilde{\pi}^* K_S = K_{C \times E} = \tilde{f}^* K_C$ where $\tilde{f} : C \times E \longrightarrow C$ is the natural projection.

$$\begin{array}{ccc} C \times E & \xrightarrow{\tilde{\pi}} & S \\ \tilde{f} \downarrow & & \downarrow f \\ C & \xrightarrow{\pi} & \mathbb{P}^1 \end{array}$$

Hence, by (8.1) we have

$$(\ell - 2)p^n + a = 2g(C) - 2.$$

Therefore, by (8.2), we obtain

$$(8.3) \quad \begin{cases} -\deg \underline{f} = \left[\dfrac{2g(C) - 2}{p^n}\right] + 2, \\ a = 2g(C) - 2 - \left[\dfrac{2g(C) - 2}{p^n}\right] \cdot p^n \end{cases}$$

By a similar argument as in Remark 4.10, we have

$$\operatorname{ord} \tilde{\pi}^*_\infty [E_\infty] \big|_{E_\infty} = \operatorname{ord} [\infty_1 \times E] \big|_{\infty_1 \times E} = 1.$$

Moreover, as $\tilde{\pi}$ is étale, $\chi(\underline{\underline{O}}_S) = 0$. Hence, $f : S \longrightarrow \mathbb{P}^1$ is of type $(p^n|1)$ (see Definition 3.1).

As special cases, we obtain the following examples.

<u>Example</u> 8.1. Let C be the complete non-singular model of the curve defined by the equation

$$x^p - x = t^m, \quad (m, p) = 1.$$

The curve C has the automorphism g of order p defined by

$$(8.4) \qquad g : (t, x) \longrightarrow (t, x + 1).$$

The genus of C is given by

$$(8.5) \qquad g(C) = \frac{1}{2}(p-1)(m-1).$$

Therefore, by (8.3), if we write

$$m = dp + b, \qquad 1 \leq b < p, \qquad d \geq 0,$$

then we have

$$\begin{cases} -\deg \underline{f} = m - d, \\ a = p - b - 1. \end{cases}$$

Moreover, we have

$$p_g(S) = m - d - 1,$$

$$\kappa(S) = \begin{cases} -\infty & \text{if } m = 1, \\ 0 & \text{if } m = 2 \text{ and } p = 3, \text{ or } m = 3 \text{ and } p = 2, \\ 1 & \text{otherwise.} \end{cases}$$

Thus, there exist elliptic surfaces of type $(p|1)$ with fixed a, $0 \leq a \leq p-2$ and arbitrary large $-\deg \underline{f}$.

In general, any $\mathbb{Z}/p^n\mathbb{Z}$ covering of $\mathbb{P}^1$ is constructed by means of Witt vectors (see [W]). Here, for simplicity, we only consider certain $\mathbb{Z}/p^2\mathbb{Z}$ coverings.

Example 8.2. Let C be the non-singular complete curve whose function field is the extension of $k(t)$ defined by the equations

$$\begin{cases} x^p = x + t^m, & (m, p) = 1, \\ y^p = y - \sum_{i=1}^{p-1} \frac{(p-1)!}{(p-i)!i!} x^i t^{m(p-i)}. \end{cases}$$

The curve C has the automorphism g defined by

$$g : (t, x, y) \longrightarrow (t, x + 1, y - \sum_{i=1}^{p-1} \frac{(p-1)!}{(p-i)!i!} x^i).$$

The automorphism is of order p^2 and has only one fixed point ∞_2, which lies over the point at infinity ∞ of $\mathbb{P}^1$.

Fix an ordinary elliptic curve E over k . and a torsion point $b \in E(k)$ of order p^2. Define the action of g on $C \times E$ by

$$g : (t, x, y, \zeta) \longmapsto (t, x+1, y - \sum_{i=1}^{p-1} \frac{(p-1)!}{(p-i)!i!} x^i, \quad \zeta + b).$$

Then the quotient variety $S = C \times E/\langle g \rangle$ with the natural morphism $f : S \longrightarrow \mathbb{P}^1 = C/\langle g \rangle$ induced from the natural projection is an elliptic surface with only one multiple fibre $p^2 E$ over the point at infinity. By the same argument as above, the elliptic surface S is of type $(p^2|1)$. The genus of the curve C is given by

$$g(C) = \frac{1}{2}(p-1)(mp^2-p+m-1),$$

Hence, by (8.3), we have

$$\begin{cases} -\deg \underline{f} = mp - m + 1 + \left[\dfrac{mp-(m+1)}{p^2}\right], \\ a = mp-(m+1) - \left[\dfrac{mp-(m+1)}{p^2}\right]p^2. \end{cases}$$

Here, we give some numerical results.

$p = 2$,

m mod 4	1	3
a	0	2

$p = 3$

m mod 9	1	2	4	5	7	8
a	1	3	7	0	4	6

$p = 5$

m mod 25	1	2	3	4	6	7	8	9	11	12	13	14	16	17	18	19	21	22	23	24
a	3	7	11	15	23	2	6	10	18	22	1	5	13	17	21	0	8	12	16	20

It is easy to see that all numbers $a_1p + a_0$ ($a_0 = 0,1,2,\ldots,p-2$, $a_1 = 0,1,2,\ldots,p-1$) appear as the number a, but numbers $a_1p + (p-1)$ ($a_1 = 0,1,2,\ldots,p-1$) never appear as the number a. In general, the following proposition holds.

Proposition 8.3. Let $f : S \longrightarrow C$ be an elliptic surface with only one multiple fibre $f^{-1}(p) = p^{\gamma}\nu E, \gamma \geq 1$. Assume that E is an ordinary elliptic curve with $\mathrm{ord}([E]|_E) = \nu$. Then, for the number a, $[a/\nu]$ is not equal to $-1 \pmod p$.

Proof. We consider the reduction of the wild fibre $p^{\gamma}\nu E$ to a tame fibre which we described in §6,(6.4). We set $g = g_1 \circ g_2 \circ \cdots \circ g_{\gamma}$ and $\pi = \pi_1 \circ \pi_2 \cdots \circ \pi_{\gamma}$. Then, in our case, g is ramified only at p. Let t (resp. s) be a local coordinate at p (resp. $p^{(\gamma)}$). Then, with a suitable unit u at $p^{(\gamma)}$, we have $g^*(t) = us^{p^{\gamma}}$. Considering u in the completion of the local ring $\underline{\underline{O}}_{p^{(\gamma)}}$ at $p^{(\gamma)}$, we have $\mathrm{ord}_s(du) \not\equiv -1 \pmod p$. Therefore, we have $\mathrm{ord}_s(g^*(dt)) \not\equiv -1 \pmod p$. By Hurwitz's formula, we have

$$2g(C^{(\gamma)}) - 2 = p^{\gamma}(2g(C) - 2) + \mathrm{ord}_s(g^*(dt)).$$

Therefore, we have

$$2g(C^{(\gamma)}) - 2 \not\equiv -1 \pmod p.$$

On the other hand, by the same method as in (8.3), we have

$$p^{\gamma}(2g(C) - 2 + \chi) + [a/\nu] = 2g(C^{(\gamma)}) - 2.$$

Hence, we have $[a/\nu] \not\equiv -1 \pmod p$. q.e.d.

Example 8.4. Let C be the complete non-singular model of the curve defined by

$$x^p - x = t^{nm}, \qquad (nm, p) = 1.$$

The curve C has two automorphims

$$g : (t, x) \longmapsto (t, x+1)$$

$$h : (t, x) \longmapsto (\rho t, x),$$

where ρ is a primitive n-th root of unity. The automorphims g, h generate the group $G = \mathbb{Z}/p\mathbb{Z} \times \mu_n$. The automorphism h has $(p+1)$ fixed points : $g^k((0,0))$, $k = 0,1,2,\ldots\ldots,p-1$ and the point at infinity. Fix an ordinary elliptic curve E over k and a torsion point $c \in E(k)$ of order pn. Then, we introduce the action of G on $C \times E$ by

$$g : (t, x, \zeta) \longmapsto (t, x+1, \zeta + nc),$$

$$h : (t, x, \zeta) \longmapsto (\rho t, x, \zeta + pc).$$

Let $f : S = C \times E/_G \longrightarrow \mathbb{P}^1 = C/_G$ be the quotient. The elliptic surface $f : S \longrightarrow \mathbb{P}^1$ has two multiple fibres : nE_0 over the origin and pnE_∞ over the point at infinity. As we assume $(nm, p) = 1$, nE_0 is a tame fibre. By the similar argument as above, we have $\mathrm{ord}[E_\infty]|_{E_\infty} = n$, hence, pnE_∞ is a wild fibre and $f : S \longrightarrow \mathbb{P}^1$ is of type $(pn, n|n,n)$. If we write

$$m = dp + b, \qquad 1 \leq b \leq p-1,$$

then we have

$$K_S = f^* O_{\mathbb{P}^1}(-2+\ell) + (n-1)E_0 + aE_\infty ,$$

where

$$\begin{cases} \ell = -\deg \underline{f} = m - d, \\ a = pn - bn - 1. \end{cases}$$

II. α_p Quotients.

<u>Example</u> 8.5. Let C be a singular curve in $\mathbb{P}^2_k$ defined by the equation

$$\xi_0 \xi_2^{p-1} = \xi_1^p .$$

The singular point of the curve C is $p_0 = (1:0:0)$. The group scheme $\alpha_p = \operatorname{Spec}(k[\varepsilon]/(\varepsilon^p))$ operates on C by

$$(\xi_0 : \xi_1 : \xi_2) \longmapsto (\xi_0 : \xi_1 + \varepsilon\xi_0 : \xi_2).$$

Let E be a supersingular elliptic curve defined over k. Then E contains α_p as a subgroup scheme. Therefore, the group scheme α_p operates on $C \times E$. Put $S = C \times E/_{\alpha_p}$ and

$f : S \longrightarrow C/_{\alpha_p}$ is the induced morphism from the natural projection.

First we show that S is smooth. Since E operates on S, it is enough to show smoothness at the image of a formal neighbourhood F of $C \times 0$ in $C \times E$ into S. Let $\operatorname{Spf}(k[[\eta]])$ be the formal completion of E at the origin. Then the action of α_p on the formal neighbourhood is given by

$$\eta \longmapsto \eta + \varepsilon$$

(see Appendix 2).

The completion of the local ring of F at $(1:0:0)\times 0$ is given by $k[[x,y,\eta]]/(x^p-y^{p-1})$ where $x = \xi_1/\xi_0$, $y = \xi_2/\xi_0$.

Since α_p acts on this local ring by

$$\begin{cases} x \longmapsto x+\varepsilon, \\ y \longmapsto y, \\ \eta \longmapsto \eta+\varepsilon, \end{cases}$$

the invariant ring is isomorphic to $k[[y, x-\eta]]$, hence, regular. For another point $p = (1:x_0:y_0) \in C$, the completion of the local ring of F at $p\times 0$ is given by $k[[u_1,v_1,\eta]]/(f(u_1,v_1))$, where

$$\begin{cases} u_1 = x - x_0, \\ v_1 = y - y_0, \\ f(u_1,v_1) = u_1^p - (v_1 + y_0)^{p-1} + x_0^p. \end{cases}$$

The action of α_p is given by

$$\begin{cases} u_1 \longmapsto u_1+\varepsilon, \\ v_1 \longmapsto v_1, \\ \eta \longmapsto \eta+\varepsilon. \end{cases}$$

Hence the invariant ring is isomorphic to $k[[v_1,u_1-\eta]]$, hence, regular. The completion of the local ring of F at $(0:0:1)\times 0$ is given by $k[[s,w,\eta]]/(s^p-w) \cong k[[s,\eta]]$, where

$$s = x/y, \quad w = 1/y.$$

The action of α_p is given by

$$\begin{cases} s \longmapsto s+\varepsilon w = s+\varepsilon s^p, \\ \eta \longmapsto \eta+\varepsilon. \end{cases}$$

Hence, the invariant ring is isomorphic to $k[[u_2, v_2]]$, where

$$u_2 = \eta^p, \quad v_2 = s - \eta s^p,$$

hence, regular. Therefore, S is smooth. On the other hand, C/α_p is isomorphic to $\mathbb{P}^1$ whose affine coordinates are given by y, w with $yw = 1$. Then the morphism f is given in the above formal coordinates by

$$y = \begin{cases} y \\ v_1 + y_0 \end{cases}$$

$$w = v_2^p(1 - u_2 v_2^{p-1} + \ldots\ldots)^{-1},$$

where the last expression is given by solving the equation

$$s^p = v_2^p(1 - u_2 s^{p(p-1)})^{-1}.$$

Hence, $f : S \longrightarrow \mathbb{P}^1$ has only one multiple singular fibre pE over the point at infinity. The canonical divisor is calcula by using the above formal coordinates, since we know that K_S has the form $f^*O_{\mathbb{P}^1}(-2+\ell) + aE_\infty$. Since we have

$$v_2^p = s^p(1 - \eta^p s^{p(p-1)}) = w(1 - u_2 w^{p-1}),$$

we have

$$f^*(dw) = w^p du_2.$$

Therefore, we have

$$dy \wedge d(x-\eta) = d(\frac{1}{w}) \wedge d(\frac{s-\eta w}{w}) = -w^{p-3} du_2 \wedge dv_2.$$

This means that $K_S|_{\hat{F}}$ is a pull back of a line bundle $O_{\mathbb{P}^1}(p-3)$, where $\hat{F}$ is the image of F into S. Since f has only one multiple fibre, we conclude

$$a = 0, \quad \ell = p - 1.$$

Thus we have

$$\begin{cases} K_S = f^* O_{\mathbb{P}^1}(p-3), \\ -\deg \underline{f} = p - 1. \end{cases}$$

Hence, $\kappa(S) = -\infty$ if $p = 2$ (in this case, S is a ruled surface of genus one), $\kappa(S) = 0$ if $p = 3$ and $\kappa(S) = 1$, if $p \geq 5$. On the other hand, by our construction, the elliptic surface S is uniruled.

Remark 8.6. Using p-closed vector fields on $\mathbb{P}^1 \times E$ with supersingular elliptic curve E, we can construct many examples of elliptic surfaces which are similar to the above example ([RS]).

Next we give an example of a smooth deformation of the elliptic surface in Example 8.5.

Example 8.7. Let us consider a group scheme $G = \mathrm{Spec}\, A$, over $R = k[[\lambda]]$
$A = k[[\lambda]][\varepsilon]/(\varepsilon^p)_\lambda$, where the comultiplication Δ, coinverse S and counit ι are given by

$$\begin{cases} \Delta(\varepsilon) = \varepsilon \otimes 1 + 1 \otimes \varepsilon + \lambda \varepsilon \otimes \varepsilon, \\ S(\varepsilon) = -\dfrac{\varepsilon}{1+\lambda\varepsilon} = -\varepsilon + \lambda\varepsilon^2 - \lambda^2\varepsilon^3 + \cdots\cdots + (-1)^{p-1}\lambda^{p-2}\varepsilon^{p-1}. \\ \iota(\varepsilon) = 0. \end{cases}$$

We define a curve C in $\mathbb{P}^2_R$ by the equation

$$\xi_0 \xi_2^{p-1} - \xi_1^p = 0.$$

The action of the group scheme G over R is given by

$$\begin{cases} \xi_0 \longmapsto \xi_0 \\ \xi_1 \longmapsto (1+\lambda\varepsilon)\xi_1 + \varepsilon\xi_0, \\ \xi_2 \longmapsto \xi_2. \end{cases}$$

Let E be an abelian scheme over R of relative dimension 1. We assume that E contains G as a group subscheme. Let $\hat{E}$ be the formal completion of E along the zero section of E over R. Then, we may assume that the coordinate ring of $\hat{E}$ is written by $R[[\eta]]$ with the action of G

$$\eta \longmapsto (1+\lambda\varepsilon)\eta + \varepsilon .$$

We show that $S = C\times_R E/G \longrightarrow \mathrm{Spec}(R)$ is smooth and factors through $S \xrightarrow{\hat{f}} C/_G = \mathbb{P}^1_R \longrightarrow \mathrm{Spec}(R)$ such that $\hat{f}_0 : S_0 \longrightarrow \mathbb{P}^1_k$ and $\hat{f}_\eta : S_\eta \to \mathbb{P}^1$ are elliptic surfaces. Since E acts on S as translation on each fibres, to show smothness, it is enough to consider the image $\hat{F}$ of a formal neighbourhood F of 0-section in $C\times E$ into S. In $\mathrm{Spec}(R)\times(1:0:0)$, the completion of local ring of F is given by

$$R[[x,y,\eta]]/(x^p - y^{p-1}), \quad x = \xi_1/\xi_0 \ , \ y = \xi_2/\xi_0 ,$$

where the action of G is given by

$$\begin{cases} x \longmapsto (1+\lambda\varepsilon)x + \varepsilon , \\ y \longmapsto y , \\ \eta \longmapsto (1+\lambda\varepsilon)\eta + \varepsilon . \end{cases}$$

Hence the invariant ring is isomorphic to $R[[y, \dfrac{\eta - x}{1+\lambda x}]]$,

hence, smooth over R. In $\mathrm{Spec}(R)\times(0:0:1)$, the completion of the local ring of F is written by

$$R[[s,w,\eta]]/(w - s^p) \cong R[[s,\eta]] ,$$

where

$$s = \xi_1/\xi_2, \quad w = \xi_0/\xi_2,$$

and the action of G is written by

$$\begin{cases} s \longmapsto (1+\lambda\varepsilon)s + \varepsilon s^p, \\ \eta \longmapsto (1+\lambda\varepsilon)\eta + \varepsilon . \end{cases}$$

Hence, the invariant ring is isomorphic to $R[[\eta^p, \dfrac{s - \eta s^p}{1+\lambda\eta}]]$,

hence, smooth over R. In this way, we can show that S is smooth over R. Moreover, it is easy to show that $C/_G \cong \mathbb{P}^1_R$ and $\hat{f}_0 : S_0 \longrightarrow \mathbb{P}^1_k$ is nothing but the elliptic surface in Example 8.5. The generic fibre is an elliptic surface $\hat{f}_\eta : S_\eta \longrightarrow \mathbb{P}^1$ obtained by μ_p quotient, since the group scheme G is μ_p over $L = \overline{k((\lambda))}$. Moreover, fixed points of $G \otimes L$ are the point at infinity and (p-1) points defined by

(8.6) $$p_i = (x, y) = \left(-\frac{1}{\lambda},\ \omega^i \left(-\frac{1}{\lambda^p}\right)^{1/(p-1)} \right),\quad i = 1,2,\ldots,p-1,$$

where ω is a primitive (p-1)-th root of unity. Hence, we have

$$K_{S_\eta} = f_\eta^* O_{\mathbb{P}^1}(-2) + \sum_{i=1}^{p-1} (p-1)E_i + (p-1)E_\infty,$$

where $pE_i = f_\eta^{-1}(p_i)$ and $pE_\infty = f_\eta^{-1}(\infty)$. Note that

$$K_{S_0} = f_0^* O_{\mathbb{P}^1}(p-3),$$

and by (8.6), all points p_i's are specialized to the point at infinity of C_0. Hence the multiple fibre pE_∞ of f_0 is a specialization of p tame multiple fibres.

Finally we give an example of lifting of the elliptic surface in Example 8.1.

<u>Example</u> 8.8. Let ω be a primitive p-th root of unity and $K = \mathbb{Q}(\omega)$ a cyclotomic field. The prime p is totally ramified in $K/_{\mathbb{Q}}$. By Z we denote the ring of algebraic integers of K. Put $\mathfrak{p} = (1-\omega)$, $R = Z_{\mathfrak{p}}$. Then R is a discrete valuation ring with prime element $\pi = 1-\omega$ whose residue field is the prime field of characteristic p. Let us consider an automorphism of $\mathbb{P}^1_K$ defined by

$$g : x \longmapsto \omega x + 1.$$

The order of g is p. Put $P(x) = \prod_{i=0}^{p-1} g^i(x)$. Then $P(x)$ is a polynomial of degree p with coefficients in R and we have

(8.7) $$P(x) \equiv x^p - x \mod (\pi).$$

Moreover, for any field extension L/K, we have

(8.8) $$L[x]^G \simeq L[P(x)],$$

where $G = \langle g \rangle$.

Let C be a curve in $\mathbb{P}^2_R$ defined by

$$\zeta_0^p P(\zeta_1/\zeta_0) - \zeta_0\zeta_2^{p-1} = 0.$$

The curve C is smooth over $\mathrm{Spec}(R)$. By (8.7), the closed fibre C_0 is a curve defined by the equation

$$\zeta_1^p - \zeta_0^{p-1}\zeta_1 - \zeta_0\zeta_2^{p-1} = 0.$$

Fix an elliptic curve E defined over K which has the following properties: 1) there exists a finite extension L/K such that there exists a non-trivial p-torsion point a in $E(L)$, 2) the elliptic curve E can be extended to an abelian scheme $\varphi : \underline{E} \longrightarrow \mathrm{Spec}(\tilde{R})$ over $\mathrm{Spec}(R)$, where $\tilde{R}$ is the integral closure of R in L, the p-torsion point a defines the section $\tilde{a}$ of order p, that is, on the closed fibre $\tilde{a}_0$ is a non-trivial p-torsion point of E_0. Now we define the action of g on $C_{\tilde{R}} \times_{\tilde{R}} E$ which is written symbolically by

$$g : ((\zeta_0 : \zeta_1 : \zeta_2), \eta) \longmapsto ((\zeta_0 : \omega\zeta_1 + \zeta_0 : \zeta_2), \eta + \tilde{a}).$$

Put $S = C_{\tilde{R}} \times_{\tilde{R}} E \,/\, G$ with structure morphism $\psi : S \longrightarrow \mathrm{Spec}(\tilde{R})$, where $G = \langle g \rangle$. Moreover, there is a morphism $\tilde{f} : S \longrightarrow C_{\tilde{R}}/G$.

By (8.8), we have $C_{\tilde{R}/G} \simeq \mathbb{P}^1_{\tilde{R}}$. It is easy to show that γ is smooth. Moreover, $\tilde{f}_0 : S_0 \longrightarrow \mathbb{P}^1$ and $\tilde{f}_\eta : S_\eta \longrightarrow \mathbb{P}^1$ are elliptic surfaces. Since the automorphism g of C_η has p fixed points, namely $(p-1)$ points $p_i = (1 : 1/(1-\omega) : y_i)$, $i = 1,2,\ldots$ $p-1$, where y_i's are all roots of $y^{p-1} = P(1/(1-\omega))$ and the point at infinity. Hence, we have

$$K_{S_\eta} = \tilde{f}_\eta^* O_{\mathbb{P}^1}(-2) + \sum_{i=1}^{p} (p-1)E_i ,$$

where $pE_i = f_\eta^{-1}(p_i)$, $i = 1,2,\ldots,p-1$, and $pE_p = f^{-1}(\infty)$.
On the other hand, $\tilde{f}_0 : S_0 \longrightarrow \mathbb{P}^1$ is one of the elliptic surfaces in Example 8.1 and we have

$$K_{S_0} = \tilde{f}_0^* O_{\mathbb{P}^1}(p-3).$$

Similarly as in Example 8.7, p tame fibres of S are specialized to one wild fibre.

§9. Deformation and lifting invariance of Kodaira dimension for surfaces.

Let R be a discrete valuation ring. By η (resp. o) we denote the generic (resp. closed) point of Spec(R). Set $k_1 = k(\eta)$, the field of fractions of R (resp. $k_o = k(o)$, the residue field of R). In the following we assume that k_o is algebraically closed. Let X be an algebraic space , proper, separated and of finite type over Spec(R) with structure morphism $\varphi : X \longrightarrow \mathrm{Spec}(R)$. By X_1 (resp. X_o) we mean the generic geometric (resp. closed) fibre of φ. Note that a smooth algebraic space of dimension 2 , proper, separated and of finite type over an algebraically closed field is projective.

The main purpose of the present section is to prove the following theorem.

Theorem 9.1. Let $\varphi : X \longrightarrow \mathrm{Spec}(R)$ be the same as above. Assume that φ is smooth and of relative dimension 2, and has connected geometric fibres. Then we have

$$\kappa(X_o) = \kappa(X_1).$$

Corollary 9.2. Under smooth deformation and lifting, the Kodaira dimensions of smooth projective surfaces are invariant:

To prove the theorem, first we give a remark on the intersection theory. For two invertible sheaves L and L′ on a projective surface S, the intersection number $(L \cdot L')_S$ is defined by the coefficient of $n_1 n_2$ of a polynomial $\chi(S, L^{n_1} \otimes L'^{n_2})$ in n_1 and n_2 . For divisors the intersection number is defined by that of the corresponding invertible sheaves. Now let $\varphi : X \to \mathrm{Spec}(R)$ be the same as in Theorem 9.1 and D, D′ divisors on X. Let L, L′ be the invertible sheaves corresponding to D , D′, respectively.

Put $D_1 = D|_{X_1}$, $D_o = D|_{X_o}$ etc. Then as φ is flat, we have $\chi(X_o, L_o^{n_1} \otimes L_o'^{n_2}) = \chi(X_1, L_1^{n_1} \otimes L_1'^{n_2})$. Hence we have $(D_o \cdot D'_o)_{X_o} = (D_1 \cdot D'_1)_{X_1}$. Thus we proved the following (see also [SGA 6], Appendice à Exposé X).

Lemma 9.3. Let $\varphi: X \longrightarrow \mathrm{Spec}(R)$ be the same as in Theorem 9.1 and D, D' divisors on X. Then we have

$$(9.1) \qquad \begin{cases} K_{X_o}^2 = K_{X_1}^2 , \\ (K_{X_o} \cdot D_o)_{X_o} = (K_{X_1} \cdot D_1)_{X_1}, \\ (D_o \cdot D'_o)_{X_o} = (D_1 \, D'_1)_{X_1}. \end{cases}$$

The following lemma plays an important role in our proof.

Lemma 9.4. Let $\varphi : X \longrightarrow \mathrm{Spec}(R)$ be the same as in Theorem 9.1. If X_o contains an exceptional curve of the first kind $\underset{\sim}{e}$, there exist a discrete valuatin ring $\widetilde{R} \supset R$ and a proper smooth morphism $\widetilde{\varphi} : \widetilde{X} \to \mathrm{Spec}(\widetilde{R})$ of algebraic spaces which is separated and of finite type and a proper surjective morphism $\pi : X \otimes \widetilde{R} \longrightarrow \widetilde{X}$ over $\mathrm{Spec}(\widetilde{R})$ such that on the closed fibre, π induces the contraction of the exceptional curve $\underset{\sim}{e}$. Moreover, on the generic fibre, π also induces a contraction of an exceptional curve of the first kind.

Proof. By [A1] I, Corollary 6.2, $\mathrm{Hilb}_{X/\mathrm{Spec}(R)}$ is represented by an algebraic space $\underline{H}$, locally of finite type over $\mathrm{Spec}(R)$. Let Y be the irreducible component containing the point $\{\underset{\sim}{e}\}$ corresponding to $\underset{\sim}{e}$. As we have $\underset{\sim}{e} \simeq \mathbb{P}^1_{k_o}$ and $N_{\underset{\sim}{e}/X} \simeq \underline{O}_{\mathbb{P}^1}(-1) \oplus \underline{O}_{\mathbb{P}^1}$, Y is smooth at $\{\underset{\sim}{e}\}$ and of dimension 1. Since $\underset{\sim}{e}$ cannot move inside X_o, the structure morphism $p: Y \longrightarrow \mathrm{Spec}(R)$ is surjective. Therefore we can find a discrete valuation ring $\widetilde{R} \supset R$ and a morphism $j : \mathrm{Spec}(\widetilde{R}) \longrightarrow Y$ over $\mathrm{Spec}(R)$ with $j(\hat{o}) = \{\underset{\sim}{e}\}$. We let $\hat{p} : \underset{\sim}{E} \longrightarrow \mathrm{Spec}(\widetilde{R})$ be the pull-back of the universal family over Y. As the closed fibre $\underset{\sim}{E}_o$ is a projective line, we may choose the morphism j in such a way that the generic fibre $\underset{\sim}{E}_1$ of $\hat{p}$ is also a projective line. Moreover $\underset{\sim}{E}$ can be considered as a smooth closed algebraic subspace of codimension 1 in $\hat{X} = X \otimes \tilde{R}$. By Lemma 9.2, we have $-1 = E_0^2 = E_1^2$. Hence E_1 is also an exceptional curve of the first kind. Hence by [A1], II, Corollary 6.11, there exists a contraction morphism $\pi : \hat{X} \longrightarrow \tilde{X}$ over $\mathrm{Spec}(\tilde{R})$ which contracts E to a section of $\tilde{\varphi} : \tilde{X} \longrightarrow \mathrm{Spec}(R)$ where $\tilde{\varphi}$ is proper, smooth, separated and of finite type over $\mathrm{Spec}(\tilde{R})$. q.e.d.

Proof of Theorem 9.1.

In the following proof, we freely use the results on the classification theory of algebraic surfaces.

Step I. $\kappa(X_0) = -\infty$ if and only if $\kappa(X_1) = -\infty$.

For a surfacs S, $\kappa(S) = -\infty$ if and only if $P_{12}(S) = 0$. Hence by the upper semi-continuity, $\kappa(X_0) = -\infty$ implies $\kappa(X_1) = -\infty$. Conversely, assume that $\kappa(X_1) = -\infty$. By Lemma 9.4, we may assume that X_0 is relatively minimal. Since $\kappa(X_1) = -\infty$, there exists a curve C_1 in X_1 with $(K_{X_1} \cdot C_1)_{X_1} < 0$. By the flat extension of C_1, there exists a curve C_0 on X_0 with $(K_{X_0} \cdot C_0)_{X_0} = (K_{X_1} \cdot C_1)_{X_1} < 0$. Since X_0 is relatively minimal, this implies that $\kappa(X_0) = -\infty$.

Step II. $\kappa(X_0) = 2$ if and only if $\kappa(X_1) = 2$.

Assume $\kappa(X_0) = 2$. By Lemma 9.4, we may assume that X_0 is minimal. Then we have $0 < K_{X_0}^2 = K_{X_1}^2$. By Step I, this implies $\kappa(X_1) = 2$. Conversely assume $\kappa(X_1) = 2$. Then, by [K4] and [T] we have

$$P_m(X_0) \geq P_m(X_1) = \frac{1}{2} m(m-1) K_{X^*}^2 + \chi(\mathcal{O}_{X^*}) \quad \text{for } m \gg 0,$$

where X^* is the minimal model of X_1. Hence we have $\kappa(X_0) = 2$.

Step III. $\kappa(X_0) = 0$ if and only of $\kappa(X_1) = 0$.

Assume $\kappa(X_0) = 0$. Then $P_m(X_0) \leq 1$ for all $m \leq 1$. Hence $\kappa(X_1) \leq 0$. By Step I, we have $\kappa(X_1) = 0$. Conversely assume $\kappa(X_1) = 0$. By the above steps, we have $\kappa(X_0) = 0$ or 1. By Lemma 9.4 we may assume that X_0 is minimal. Hence we have $0 = K_{X_0}^2 = K_{X_1}^2$. Therefore

X_1 is also minimal. Therefore $12K_{X_1}$ is trivial, hence we have

$1 = h^0(X_1, \underline{\underline{O}}(-12K_{X_1})) \leq h^0(X_o, \underline{\underline{O}}(-12K_{X_o}))$. Hence, $\kappa(X_o) = 0$.

Step IV. $\kappa(X_o) = 1$ if and only if $\kappa(X_1) = 1$.

This is clear from the above steps.

Thus the theorem was proved.

Remark 9.5. By a similar argument as above, it is easy to show that each class of Enriques' classification of surfaces is invariant under smooth deformation and lifting (we consider quasi-elliptic surfaces as in the class of elliptic surfaces).

The following lemma will be used in the next section.

Lemma 9.6. Let $\varphi : X \to \mathrm{Spec}(R)$ be the same as in Theorem 9.1. Assume $\kappa(X_o) \geq 0$. Then X_1 is minimal if and only if so is X_o.

Proof. By Theorem 9.1 we have $0 \leq \kappa(X_o) = \kappa(X_1)$. Hence if X_o is not minimal, then by Lemma 9.4 X_1 is not minimal. Therefore, assume that X_o is minimal but X_1 is not minimal. Then we have $K^2_{X_1} = K^2_{X_o} \geq 0$. Hence, if $\kappa(X_o) = 0$ or 1, then X_1 is minimal. Hence $\kappa(X_o) = \kappa(X_1) = 2$. Then by [K4] and [T] there exist positive intergers m_o, m_1 such that

$$\begin{cases} P_m(X_o) = \frac{1}{2}m(m-1)K^2_{X_o} + \chi(\underline{\underline{O}}_{X_o}), & m \geq m_o, \\ P_m(X_1) = \frac{1}{2}m(m-1)K^2_{X^*} + \chi(\underline{\underline{O}}_{X_1}), & m \geq m_1, \end{cases}$$

where X^* is the minimal model of X_1. As we have $\chi(O_{X_1}) = \chi(O_{X_o})$, $K^2_{X^*} > K^2_{X_1} = K^2_{X_o} > 0$, we have $P_m(X_1) > P_m(X_o)$ for sufficiently large m. This is a contradiction. q.e.d.

§ 10. Invariance of the genus of the base curve under smooth deformation and lifting.

By a smooth family of elliptic surface $\varphi : X \longrightarrow S = \mathrm{Spec}(R)$, we mean that X is an algebraic space of finite type over R, φ is smooth, proper and separated and φ has connected geometric fibres which are elliptic surfaces. We let X_o be the closed fibre and X_1 the generic geometric fibre of φ. In this section, we assume that $\kappa(X_o) = \kappa(X_1) = 1$. Moreover, by Lemma 9.4 and Lemma 9.6, we may assume that X_o and X_1 are minimal.

<u>Theorem</u> 10.1. Let $\varphi : X \longrightarrow \mathrm{Spec}(R)$ be the same as above. We let $f_o : X_o \longrightarrow C_o$ and $f_1 : X_1 \longrightarrow C_1$ be the elliptic fibrations. Then we have

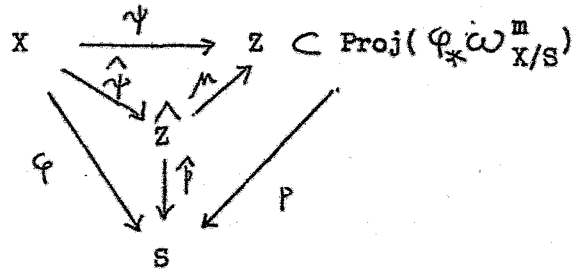
$$g(C_o) = g(C_1).$$

That is, the genus of the base curve of an elliptic surface with $\kappa = 1$ is invariant under smooth deformation and lifting.

Proof. We choose a positive integer $m \geq 14$ in such a way that m is a common multiple of multiplicities of all multiple fibres of f_1. Then $\varphi_*\omega^m_{X/S}$ is locally free and $\varphi_*\omega^m_{X/S} \otimes_R \overline{k(\eta)} \xrightarrow{\sim} H^0(X_1, \underline{O}(mK_{X_1}))$.

We let $\psi : X \longrightarrow \mathrm{Proj}(\varphi_*\omega^m_{X/S})$ be the rational mapping over $S = \mathrm{Spec}(R)$. Then ψ_η is nothing but a rational mapping associated with the complete linear system $|mK_{X_\eta}|$, hence by Theorem 5.2, it is a morphism and $\psi_\eta(X_\eta)$ is a non-singular curve C_η with $C_\eta \otimes \overline{k(\eta)} \xrightarrow{\sim} C_1$. On the other hand, the map ψ_o is a rational map associated with a sublinear system L of $|mK_{X_o}|$ given by $\varphi_*\omega^m_{X/S} \otimes k_o$. Hence, the linear system L has only fixed components consisting of fibres of f_o. Therefore ψ_o is a morphism.

Thus ψ is a morphism. Put $Z = \psi(X)$. Let $\hat{\psi}: X \longrightarrow \hat{Z}$, $\mu: \hat{Z} \longrightarrow Z$ be the Stein factorization of $\psi: X \longrightarrow Z$ and $\hat{p}: \hat{Z} \longrightarrow S$, the structure morphism.



By the above consideration, μ is isomorphic on the generic fibre. Let $\nu: \tilde{Z}_o \longrightarrow \hat{Z}_o$ be the normalization of $\hat{Z}_o$ in X_o and $\tilde{\psi}_o: X_o \longrightarrow \tilde{Z}_o$ the canonical morphism. Then, as ψ_o is defined by a sublinear system L of $|mK_{X_o}|$, $\tilde{\psi}_o: X_o \longrightarrow \tilde{Z}_o$ is isomorphic to the elliptic fibration $f_o: X_o \longrightarrow C_o$ (see Lemma 5.1). Since $\hat{\psi}_o: X_o \longrightarrow \hat{Z}_o$ has connected fibres, $\nu: \tilde{Z}_o \longrightarrow \hat{Z}_o$ is factored through a purely inseparable morphism $\nu_1: \tilde{Z}_o \longrightarrow \hat{Z}_o^*$ and a desingularizaion $\nu_2: \hat{Z}_o^* \longrightarrow \hat{Z}_o$. Hence we have

(10.1) $$g(C_o) = g(\tilde{Z}_o) = g(\hat{Z}_o^*).$$

On the other hand, since $\hat{p}: \hat{Z} \longrightarrow S$ is a degeneration of curves, considering a non-singular model of $\hat{Z}$, we get

(10.2) $$g(\hat{Z}_o^*) \leq g(\hat{Z}_\eta) = g(C_1).$$

Moreover, if the equlity holds in (10.2), then $\hat{Z}$ is non-singular.

By the étale cohomology theory, we have

(10.3) $$\begin{cases} b_1(X_o) = 2 \dim \mathrm{Alb}(X_o), \\ b_1(X_1) = 2 \dim \mathrm{Alb}(X_1). \end{cases}$$

Since φ is smooth, we have

(10.4) $$b_1(X_o) = b_1(X_1).$$

Moreover, by Lemma 3.4, we have

(10.5) $$\begin{cases} \dim \mathrm{Alb}(X_o) = g(C_o) \text{ or } g(C_o) + 1, \\ \dim \mathrm{Alb}(X_1) = g(C_1) \text{ or } g(C_1) + 1. \end{cases}$$

By (10.1) and (10.2), we have $g(C_o) \leq g(C_1)$. Therefore, by (10.3), (10.4) and (10.5), we have $g(C_o) = g(C_1)$ or $g(C_1) = g(C_o) + 1$. Suppose that $g(C_1) = g(C_o) + 1$. Then we have $\dim \mathrm{Alb}(X_1) = g(C_1)$ and $\dim \mathrm{Alb}(X_o) = g(C_o) + 1$. Hence, by Lemma 3.4, by the Albanese mapping $\alpha_1 : X_1 \longrightarrow \mathrm{Alb}(X_1)$ each fibre of f_1 is mapped to a point, but by the Albanese mapping $\alpha_o : X_o \longrightarrow \mathrm{Alb}(X_o)$ each fibre of f_o is mapped to a curve. We show that this gives a contradiction. For that purpose we need the following Lemma.

Lemma 10.2. Let $\varphi : X \longrightarrow \mathrm{Spec}(R)$ be a proper smooth separated morphism of finite type of algebraic spaces with connected geometric fibres. Then there exist a discrete valuation ring $\tilde{R} \supset R$, an abelian scheme $\gamma : A \longrightarrow \mathrm{Spec}(\tilde{R})$ and a morphism $\tilde{\alpha} : \tilde{X} = X \otimes \tilde{R} \longrightarrow A$ over $\mathrm{Spec}(\tilde{R})$ such that on the generic fibre $\tilde{\alpha}_\eta : \tilde{X}_\eta \longrightarrow A_\eta$ is the Albanese mapping and on the closed fibre $\tilde{\alpha}_o : \tilde{X}_o \longrightarrow A_o$ is isogenous to the Albanese mapping, that is, there exists an isogeny $\beta_o : \mathrm{Alb}(X_o) \longrightarrow A_o$ such that $\tilde{\alpha}_o = \beta_o \cdot \alpha_o$ where α_o is the Albanese mapping of X_o.

Now we assume Lemma 10.2 and we apply it to our situation. We may assume $\tilde{R} = R$. Let H be a relative hyperplane section of $\gamma : A \longrightarrow \mathrm{Spec}(R)$. Then we have

$$\begin{cases} (\tilde{\alpha}_\eta^{-1}(H_\eta) \cdot F_\eta)_{X_\eta} = 0, \\ (\tilde{\alpha}_o^{-1}(H_o) \cdot F_o)_{X_o} > 0, \end{cases}$$

where F_η (resp. F_o) is a general fibre of f_1(resp. f_o). On the other hand, as we saw above, we have

$$(\tilde{\alpha}_\eta^{-1}(H_\eta) \cdot F_\eta)_{X_\eta} = (\tilde{\alpha}_o^{-1}(H_o) \cdot p^m F_o)_{X_o}$$

for a suitable non-negative integer m. A contradiction. q.e.d.

Proof of Lemma 10.2. We may assume that R is complete and $\varphi : X \longrightarrow \mathrm{Spec}(R)$ has a section. Hence the Albanese mapping $\alpha_\eta : X_\eta \longrightarrow A_\eta$ is defined over $k(\eta)$. Let $\gamma : A \longrightarrow \mathrm{Spec}(R)$ be the Néron minimal model of A_η in the category of algebraic spaces, separated and of finite type over R. We show that γ is an abelian scheme. As $\varphi : X \longrightarrow \mathrm{Spec}(R)$ is smooth, the inertia group I of the Galois group $G = \mathrm{Gal}(k^s(\eta)/k(\eta))$ operates trivially on $H^1_{\acute{e}t}(X_{\bar{\eta}}, \mathbb{Z}_\ell)$ where $k^s(\eta)$ is the separable closure of $k(\eta)$ and $X_{\bar{\eta}} = X_\eta \otimes k^s(\eta)$ and $\ell \neq \mathrm{char}.k_o$ ([SGA 7-I], 2.4 and 2.5). Since in our case $H^1_{\acute{e}t}(X_{\bar{\eta}}, \mathbb{Z}_\ell) \xrightarrow{\sim} H^1_{\acute{e}t}(A_{\bar{\eta}}, \mathbb{Z}_\ell)$ as G-module, by [ST], Theorem 1, we conclude that A_η has good reduction. Hence, $\gamma : A \longrightarrow \mathrm{Spec}(R)$ is an abelian scheme. By the definition of the Néron model, there exists a morphism $\tilde{\alpha} : X \longrightarrow A$ over $\mathrm{Spec}(R)$ such that on the generic fibre it coincides with the Albanese mapping. Moreover, we have G-module isomorphisms $H^1_{\acute{e}t}(X_{\bar{\eta}}, \mathbb{Z}_\ell) \xrightarrow{\sim} H^1_{\acute{e}t}(X_o, \mathbb{Z}_\ell)$ and $H^1(A_{\bar{\eta}}, \mathbb{Z}_\ell) \xrightarrow{\sim} H^1(A_o, \mathbb{Z}_\ell)$ ([SGA 7-I], 2.4 and 2.5). As $\tilde{\alpha}_\eta$ induces an isomorphism $H^1(X_{\bar{\eta}}, \mathbb{Z}_\ell) \xrightarrow{\sim} H^1(A_{\bar{\eta}}, \mathbb{Z}_\ell)$, $\tilde{\alpha}_o$ induces an isomorphism $H^1(X_o, \mathbb{Z}_\ell) \xrightarrow{\sim} H^1(A_o, \mathbb{Z}_\ell)$. Since the Albanese mapping induc an isomorphism $H^1(X_o, \mathbb{Z}_\ell) \xrightarrow{\sim} H^1(\mathrm{Alb}(X_o), \mathbb{Z}_\ell)$, this implies that $\tilde{\alpha}_o : X_o \longrightarrow A_o$ is isogenous to the Albanese mapping. q.e.d.

Appendix 1. A criterion for algebraicity over $\mathbb{C}$.

In this appendix we give a necessary and sufficient condition for an analytic elliptic surface $f : S \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ with $\chi(\underline{\underline{O}}_S) = 0$ to be algebraic. The condition $\chi(\underline{\underline{O}}_S) = 0$ implies that $f : S \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ has only multiple singular fibres m_iE_i with elliptic curves E_i, $i = 1,2,\ldots\ldots,\lambda$, and that the moduli of general fibres is constant. Let $\{p_1,p_2,\ldots\ldots\ldots,p_\lambda\}$ be the set of all points of $\mathbb{P}^1_{\mathbb{C}}$ over which f has multiple fibres m_iE_i and f has smooth fibers over $\mathbb{P}^1_{\mathbb{C}} - \{p_1,p_2,\ldots\ldots,p_\lambda\}$. By Kodaira [K2] II, such an elliptic surface is constructed as follows.

Let E be the elliptic curve appearing in a general fibre of f. We express E as a quotient manifold $\mathbb{C}/\Lambda$, where $\Lambda = \mathbb{Z}\cdot 1 + \mathbb{Z}\cdot\tau$, $\mathrm{Im}(\tau) > 0$. We let t_i be a local coordinate of $\mathbb{P}^1_{\mathbb{C}}$ with center p_i and consider $D_i = \{t_i \mid |t_i| < \varepsilon\}$ as an open set in $\mathbb{P}^1_{\mathbb{C}}$. Put $\hat{D}_i = \{s_i \in \mathbb{C} \mid |s_i| < \varepsilon^{1/m_i}\}$.

Fix complex numbers

$$\text{(A.1)} \qquad a_i = \frac{\alpha_i}{m_i} + \frac{\beta_i}{m_i}\tau, \qquad \alpha_i, \beta_i \in \mathbb{Z}, \ (\alpha_i, \beta_i, m_i) = 1.$$

Define an anlytic automorphism g_i of $\hat{D}_i \times E$ by

$$g_i : (s_i, [\zeta]) \longrightarrow (e_{m_i}s_i, [\zeta + a_i]),$$

where $e_{m_i} = \exp(2\pi\sqrt{-1}/m_i)$ and ζ is a global coordinate of $\mathbb{C}$, the universal covering of E, and $[\zeta]$ is the corresponding point of E. By $((s_i, [\zeta]))$, we denote the image of a point $(s_i, [\zeta]) \in \hat{D}_i \times E$ in the quotient manifold $\hat{D}_i \times E/\langle g_i \rangle$. Then a holomorphic mapping

$$\hat{D}_i \times E/\langle g_i \rangle \ni ((s_i, [\zeta])) \longmapsto s_i^{m_i} \in D_i$$

defines an elliptic fibration which has a multiple fibre m_iE_i

over the origin, where the elliptic curve E_i is isomorphic to $E/\langle a_i \rangle$.

Let us consider a holomorphic mapping

$$(A.2) \qquad \varphi_i : (\hat{D}_i - \{0\}) \times E/\langle g_i \rangle \longrightarrow (D_i - \{0\}) \times E$$
$$((s_i, [\zeta])) \longmapsto \left(s_i^{m_i}, \left[\zeta - \frac{m_i a_i}{2\pi\sqrt{-1}} \log s_i \right] \right).$$

It is easy to see that φ_i is biholomorphic. Hence, using the isomorphism φ_i, we can patch together $\hat{D}_i \times E/\langle g_i \rangle$, $i = 1, 2, \ldots, \lambda$, and $(\mathbb{P}^1_{\mathbb{C}} - \{p_1, \ldots, p_\lambda\}) \times E$ and obtain a compact analytic surface S with a surjective holomorphic mapping $f : S \longrightarrow \mathbb{P}^1_{\mathbb{C}}$. The elliptic surface thus obtained has multiple fibres $m_i E_i$ over p_i, $i = 1, 2, \ldots, \lambda$, and other fibres are smooth. We denote this elliptic surface by $L_{p_1}(m_1, a_1) L_{p_2}(m_2, a_2) \cdots\cdots L_p(m_\lambda, a_\lambda)(\mathbb{P}^1_{\mathbb{C}} \times E)$ and call it the elliptic surface obtained from $\mathbb{P}^1_{\mathbb{C}} \times E$ by means of logarithmic transformations. By Kodaira [K2], II, every elliptic surface $f : S \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ with $\chi(\mathcal{O}_S) = 0$ is isomorphic to the above elliptic surface with a suitable choice of a_i's. The following theorem is a consequence of the proof of [K2], II, Theorem 27.

<u>Theorem</u>. The elliptic surface $f : S = L_{p_1}(m_1, a_1) L_{p_2}(m_2, a_2) \cdots\cdots L_p(m_\lambda, a_\lambda)(\mathbb{P}^1_{\mathbb{C}} \times E) \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ is algebraic if and only if

$$(A.3) \qquad \sum_{i=1}^{\lambda} a_i = 0.$$

Proof. Since $p_g(S) = 0$, S is algebraic if and only if $b_1(S)$ is even. For a surface, it is known that $b_1(S) = h^0(S, \Omega^1_S) + h^1(S, \mathcal{O}_S)$ and $h^1(S, \mathcal{O}_S) = h^0(S, \Omega^1_S)$ or $h^0(S, \Omega^1_S) + 1$ (see [K2], I). By $p_g(S) = 0$ and $\chi(\mathcal{O}_S) = 0$, we have $h^1(S, \mathcal{O}_S) = 1$. Therefore, S is algebrai if and only if $h^0(S,\Omega^1_S) = 1$, and this is equivalent to the existence of a non-zero holomorphic 1-form on S in our situation.

Suppose that we have a holomorphic 1-form ω. Then, on $(\mathbb{P}^1_{\mathbb{C}} - \{p_1,p_2,\ldots\ldots,p_\lambda\})\times E$, ω is expressed in a form

$$\omega = A(t)dt + B(t)d\zeta,$$

where t is a coordinate of an affine line in $\mathbb{P}^1_{\mathbb{C}}$ (we assume that all points p_i's are in this affine line) and $A(t)dt$ and $B(t)$ are holomorphic on $\mathbb{P}^1_{C} - \{p_1,p_2,\ldots\ldots,p_\lambda\}$. We may assume $t_i = t - \gamma_i$, $i = 1,2,\ldots\ldots\lambda$. Then, by (A.2) $\varphi_i^*\omega$ is holomorphic on $\hat{D}_i\times E/\langle g_i\rangle$, $i = 1,2,\ldots\ldots,\lambda$. Therefore, $B(t)$ is holomorphic on $\mathbb{P}^1_{\mathbb{C}}$, hence, constant. Therefore, we may assume that $B(t) = 1$. Moreover, $A(t)dt$ has simple poles at p_i with residues $a_i/2\pi\sqrt{-1}$, $i = 1,2,\ldots\ldots,\lambda$, and is holomorphic elsewhere. Therefore, the problem is reduced to the existence of a meromorphic 1-form on $\mathbb{P}^1_{\mathbb{C}}$ which has simple poles at p_i with residues $a_i/2\pi\sqrt{-1}$, $i = 1,2,\ldots\ldots,\lambda$, and is holomorphic elsewhere. Hence, the above condition (A.3) is necessary and sufficient. q.e.d.

Remark. Using this theorem, we can construct an example of an elliptic surface $f : S \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ with $\chi(\underline{\underline{O}}_S) = 0$ which satisfies all condition U_i in §3, and is not algebraic.

Appendix 2. An action of α_p .

Let E be a supersingular elliptic curve defined over an algebraically closed field k of characteristic $p>0$. Let $G_{1,1} = \mathrm{Spf}(k[[\eta]])$ be the formal group obtained by the completion of E at the origin (for the notation, see [M1]). We denote by $F_E : E \longrightarrow E^{(p)}$ (resp. F) the Frobenius morphism of E (resp. $G_{1,1}$). We set

$$N_n = \mathrm{Ker}\left\{F_{E^{(p^{n-1})}} \circ \cdots \circ F_{E^{(p)}} \circ F_E\right\}.$$

Then, we have $N_n \cong \mathrm{Ker}(F^n)$. In particular, we have $N_1 = \alpha_p$. In this appendix, we prove the following proposition due to F. Oort.

<u>Theorem</u> 1. The action of $\alpha_p = \mathrm{spec}(k[\varepsilon]/(\varepsilon^p))$ on $G_{1,1}$ $= \mathrm{Spf}(k[[\eta]])$ which is induced by the addition $E \times E \longrightarrow E$ is given by

$$\eta \longmapsto \eta + \varepsilon$$

with suitable elements η and ε .

We denote by W_n the Witt group scheme. The underlying scheme of W_n is given by $W_n = \mathrm{Spec}(k[x_1, x_2, \ldots\ldots, x_n])$. We also denote by F the Frobenius morphism of W_n.

<u>Lemma</u> 2. Let $(x_1, x_2, \ldots\ldots, x_n)$, $(y_1, y_2, \ldots\ldots, y_n)$ and $(z_1, z_2, \ldots\ldots, z_n)$ be three coordinates of W_n. Then, the addition of W_n is given by

$$z_j = x_j + y_j + P_j(x_1, \ldots, x_{j-1}, y_1, \ldots, y_{j-1}) \quad (j=1,2,\ldots,n)$$

with suitable polynomials $P_j(x_1, \ldots, x_{j-1}, y_1, \ldots, y_{j-1})$. Moreover, we have $P_j(x_1, \ldots, x_{j-1}, y_1, \ldots, y_{j-1}) \equiv 0 \mod (x_1, x_2, \ldots, x_{j-1})$, where $(x_1, x_2, \ldots, x_{j-1})$ is an ideal of the polynomial ring $k[x_1, \ldots, x_{j-1}, y_1, \ldots, y_{j-1}]$.

For the proof, see [W]. We set

$$N'_n = \mathrm{Ker}(F - RV : W_n \longrightarrow W_n),$$

where V is the Verschiebung from W_n to W_{n+1}, and where R is the restriction from W_{n+1} to W_n. The following lemma is rather well-known.

Lemma 3. $N_n \cong N'_n$.

Proof. We use the theory of Dieudonné modules. We consider the ring $A = W[F,V]$, where W is the ring of infinite Witt vectors over k, and where F and V satisfy the well-known relations (see [DG]). We denote by $D(G)$ the Dieudonné module of the group scheme G. Then, we have

$$D(N_n) = A/A(F^n, F-V, V^n) = D(N'_n).$$

Therefore, we conclude $N_n \cong N'_n$. q.e.d.

We denote by I the ideal of $k[x_1,x_2,\ldots,x_n]$ which defines N'_n in W_n. Then, by the definition of N'_n, we have $x_1^p \equiv 0$ (mod I), $x_1 \equiv x_2^p$ (mod I), $x_2 \equiv x_3^p$ (mod I),..., $x_{n-1} \equiv x_n^p$ (mod I).

Since we see that $k[x_1,x_2,\ldots,x_n]/I$ is generated by x_n over k, by Lemma 3 we have the homomorphism φ_n from N_n to W_n defined by

(1) $$\begin{array}{ccccc} k[\tau]/(\tau^{p^n}) & \xleftarrow{\sim} & k[x_1,x_2,\ldots,x_n]/I & \longleftarrow & k[x_1,x_2,\ldots,x_n] . \\ \tau & \longleftarrow\!\!\shortmid & x_n & & \end{array}$$

Moreover, we have the following commutative diagram:

(2) $$\begin{array}{ccc} N_n & \xrightarrow{\varphi_n} & W_n \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle V} \\ N_{n+1} & \xrightarrow{\varphi_{n+1}} & W_{n+1} \end{array},$$

where i is the natural immersion. Since $G_{1,1} \cong \varinjlim N_n$, in order to prove Theorem 1, it suffices to prove the following lemma.

<u>Lemma</u> 4. The action of $\alpha_p = \mathrm{Spec}(k[\varepsilon]/(\varepsilon^p)$ on $N_n = \mathrm{Spec}(k[\tau]/(\tau^{p}$ which is induced by the addition $E \times E \longrightarrow E$ is given by

$$\tau \longmapsto \tau + \varepsilon$$

with suitable elements τ and ε .

Proof. We have the commutative diagram :

$$\begin{array}{ccccc}
\alpha_p \times N_n & \hookrightarrow & N_n \times N_n & \xrightarrow{\varphi_n \times \varphi_n} & W_n \times W_n \\
\rho_1 \downarrow & & \rho_2 \downarrow & & \rho_3 \downarrow \\
N_n & \cong & N_n & \xrightarrow{\varphi_n} & W_n ,
\end{array}$$

where ρ_2 and ρ_3 are additions, and where ρ_1 is the morphism induced by ρ_2 . Therefore, by (1) and Lemma 2, we see that ρ_1 is given by

$$\begin{array}{ccc}
k[\varepsilon]/(\varepsilon^p) \otimes k[\tau]/(\tau^{p^n}) & \longleftarrow & k[\tau]/(\tau^{p^n}) \\
\Psi & & \Psi \\
1 \otimes \tau + \varepsilon \otimes 1 & \longleftarrow\!\!\shortmid & \tau .
\end{array}$$

q.e.d.

References

[A1] Artin,M., Algebraization of formal moduli,I,II, in Global Analysis (D.C.Spencer and S.Iyanaga, eds.), University of Tokyo Press and Princeton University Press (1969), 21-71, Ann.of Math.91 (1970), 88-135.

[A2] ________, Théoremes de représentabilité pour les espaces algébriques, Publ. Sém. Math. Sup. Univ. de Montréal, 1973.

[BM] Bombieri,E. and D.Mumford, Enriques' classification of surfaces in char. p, II, III, in Complex Analysis and Algebraic Geometry (W.L.Baily and T.Shioda,eds.), Iwanami Shoten Publisher and Princeton University Press (1977), 23-42, Invent. Math. 35 (1976), 197-232.

[DG] Demazure,M. and P.Gabriel, Groupes algébriques, Masson & Cie, Paris, and North-Holland, Amsterdam, 1970.

[I1] Igusa,J., Betti and Picard numbers of abstract algebraic surfaces, Proc. Nat. Acad. Sci. U.S.A., 46 (1960), 724-726.

[I2] Iitaka,S., Deformations of compact complex surfaces,II, J. Math. Soc. Japan 22 (1970), 247-261.

[K1] Katsura,T., On Kummer surfaces in characteristic 2, in Proc. Intern. Symp. on Algebraic Geometry, Kyoto, 1977, (M.Nagata, ed.), Kinokuniya, Tokyo 1978, 525-542.

[K2] Kodaira,K., On the structure of compact analytic surfaces, I,II, Amer. J. Math. 86 (1964), 751-798, ibid. 88 (1966), 682-721.

[K3] __________, On compact analytic surfaces,II,III, Ann. of Math. 77 (1963),563-626, ibid. 78 (1963), 1-40.

[K4] __________, Pluricanonical systems on algebraic surfaces of general type, J. Math. Soc. Japan 20 (1968), 170-192.

[M1] Manin, Yu.I., The theory of commutative formal groups over fields of finite characteristic, Russian Math. Surveys 18 (1963), 1-83.

[M2] Mumford, D., Enriques' classification of surfaces in char. p, I, in Global Analysis (D.C.Spencer and S.Iyanaga, eds.), University of Tokyo and Princeton University Press (1969), 325-339.

[O] Oort, F., Finite group schemes, local moduli for abelian varieties, and lifting problems, Compositio Math. 23 (1971), 265-296.

[R1] Raynaud, M., Spécialisation du foncteur de Picard, Publ. Math. IHES 38 (1970), 27-76.

[R2] __________, Surfaces elliptiques et quasi-elliptiques, manuscript.

[R3] __________, Modèles de Néron, C.R. Acad. Sci. Paris 262 (1966), 413-416.

[RS] Rudakov, A.N. and I.R. Shafarevich, Inseparable morphisms of algebraic surfaces, Izv. Akad. Nauk SSSR Ser. Mat. 40 (1976), 1269-1307, English translation, Math. USSR Izv. 5 (1976), 1205-1237.

[SGA 6] Lecture Notes in Math. 225, Springer, 1971.

[SGA 7-I] Lecture Notes in Math. 288, Springer, 1972.

[ST] Serre, J.-P. and J. Tate, Good reduction of abelian varieties, Ann. of Math. 88 (1968), 492-517.

[T] Tankeev, S.G., Pluricanonical mappings of algebraic surfaces of general type, (in Russian) Uspehi Mat. Nauk 30 (1975), 184.

[W] Witt, E., Zyklische Körper und Algebren der Characteristik p vom Grade p^n, J. reine ang. Math., 176 (1936), 126-140.

[Y] Yamada, H., Rational sections and Chern classes of vector bundles, J. Math. Kyoto Univ. 6 (1967), 295-312.

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