Pseudo-differential operators, ellipticity and asymptotics on manifolds with edges

B.-W. Schulze

Max-Planck-Arbeitsgruppe
Partielle Differentialgleichungen
und Komplexe Analysis
FB Mathematik
Universität Potsdam
Am Neuen Palais 10
14415 Potsdam
Germany

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
53225 Bonn
Germany

MPI / 94-40

Contents

1. Introduction

The analysis of differential equations on spaces with singularities (e.g., with piece-wise smooth geometry) is necessary in various applications in physics and engineering. Also pure mathematics (such as geometry, Lie Group theory, index theory, topology) require the analysis near singularities. We will simply speak of manifolds with singularities, although we actually allow stratified spaces that are in general no C manifolds. It is a natural problem to extend the classical concept of pseudo-differential operators from the C^∞ case to manifolds with singularities, in particular, ellipticity and parametrix constructions within corresponding operator algebras. We shall see (here in the case of edges) that the construction of a parametrix of a given single operator in a singular situation does employ the insight from rather general operator algebras associated with corresponding lower singularities (here conical ones).
The local model of a space with conical singularities is the geometric cone

$$X^\Delta = (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$$

where the base X is a closed compact C^∞ manifold. We may always think of an embedded cone in $\mathbb{R}^N$ for sufficiently large N, where $X \subset S^{N-1}$ is a submanifold of the unit sphere and

$$X^\Delta = \{\lambda x : \ x \in X, \lambda \geq 0\} \ .$$

The analysis will always be performed on the open stretched cone

$$X^\wedge = \mathbb{R}_+ \times X \ni (t, x) \ . \tag{1.1}$$

The natural differential operators A of order μ on $X^\wedge$ are those of Fuchs type, namely

$$A = t^{-\mu} \sum_{j=0}^{\mu} a_j(t) \left(-t\frac{\partial}{\partial t}\right)^j \tag{1.2}$$

with operator-valued coefficients $a_j(t) \in C^\infty\big(\overline{\mathbb{R}}_+, \mathit{Diff}^{\mu-j}(X)\big)$. Here $\mathit{Diff}^\nu(X)$ is the (Fréchet) space of all differential operators of order ν on X with smooth coefficients with respect to every chart. Note that when $g(t)$ is a t-dependent Riemannian metric on X, C^∞ in t up to $t = 0$, then the Laplace-Beltrami operator to the metric $dt^2 + t^2 g(t)$ on $X^\wedge$ is just of Fuchs type, of order $\mu = 2$. Another easy observation is that for every differential operator $\tilde{A}$ in $\mathbb{R}^{n+1}$ of order μ

$$\tilde{A} = \sum_{|\alpha| \leq \mu} a_\alpha(\tilde{x}) D_{\tilde{x}}^\alpha$$

with $a_\alpha(\tilde{x}) \in C^\infty(\mathbb{R}^{n+1})$, the restriction $A = \tilde{A}|_{\mathbb{R}^{n+1}\setminus\{0\}}$ takes the form (1.2) under the substitution of polar coordinates, with $t = |\tilde{x}|$ and $x \in S^n = X$. Differential operators of Fuchs type in adequate weighted Sobolev spaces have been studied by many authors.

Let us mention, in particular, Kondrat'ev's technique for characterizing the Fredholm property under an ellipticity condition and the asymptotics of solutions in terms of meromorphic families of operators on X, cf. [K1]. Algebras of pseudo-differential operators of Fuchs type containing the parametrices of elliptic elements have been constructed by Schulze [S7]. Note that the Fredholm property in this context may be studied both for compact manifolds with conical singularities as well as for the infinite (open stretched) cone $X^\wedge$. In the latter case there have to be imposed so-called exit conditions for $t \to \infty$, where $t \to \infty$ is regarded as an exit of the manifold to infinity, cf. Cordes [C1], Schrohe [S1].
The case of operators on manifolds with conical singularities is of independent interest. The structure of the corresponding pseudo-differential algebras (with various sorts of asymptotics, discrete and continuous ones) is rather complex compared with the C^∞ situation. The role of the conical singularities in the present discussion is that manifolds with edges are locally close to an edge of dimension q of the form of a wedge

$$X^\Delta \times \Omega \quad \text{with open} \quad \Omega \subseteq \mathbb{R}^q \, .$$

The analysis will refer to the open stretched wedge

$$X^\wedge \times \Omega = \mathbb{R}_+ \times X \times \Omega \ni (t, x, y) \, . \tag{1.3}$$

The method of treating operators on $X^\wedge \times \Omega$ will be to perform a calculus of pseudo-differential operators on Ω with operator-valued symbols, taking values in the pseudo-differential algebra of Fuchs operators on $X^\wedge$. In this application of the analysis for conical singularities it is actually necessary to have an algebra, because of the point-wise compositions between symbols.
Similarly to the differential operators of Fuchs type on $X^\wedge$ there is a class of natural (so-called edge-degenerate) differential operators on the wedge $X^\wedge \times \Omega$, cf. Section 2 below. Our calculus gives the answer to the problem of characterizing a pseudo-differential algebra on the wedge that contains these operators and the parametrices of elliptic elements. This theory covers many special cases, in particular

(i) pseudo-differential boundary value problems without (and with) the transmission property, cf. Višik, Eskin [V1], Eskin [E1], Boutet de Monvel [B1], Rempel, Schulze [R1],
(ii) mixed and transmission problems, cf. Schulze [S8], Rempel, Schulze [R2], [R3],
(iii) problems of Sobolev type, cf. Sternin [S10], Rempel, Schulze [R1],
(iv) operators on "branched" spaces when the singular sets are edges in our sense.

Note that (i) corresponds to the case when the base X of the model cone $X^\wedge$ of the wedge is of dimension 0. Then Ω is a neighbourhood on the boundary and $\mathbb{R}_+ = X^\wedge$ is

the inner normal. Similarly to boundary value problems, where the Fredholm property depends on elliptic boundary conditions (in the pseudo-differential case of trace and potential type) the theory on manifolds with edges requires elliptic conditions along the edges, also being of trace and potential type in general, both occurring even for differential operators when $\dim X > 0$. In Schulze [S4] as well as in the book [S7] there has been described an algebra of pseudo-differential edge problems for the case $X = S^n$, with trace and potential conditions. The present paper will allow general cone bases X. A brief survey on results of this type for arbitrary X was also given in [S6]. It was applied in the latter paper to operators on manifolds with corners, locally being cones with bases X that have conical points, again. It is obvious for geometric reasons in that case that the edge theory is necessary along the one-dimensional edges $\cong \mathbb{R}_+$, emanated from the corners. Close to such corners there was also established an edge theory with the Mellin transform on $\mathbb{R}_+$. This program, called the Mellin-edge approach, was continued and deepend in Dorschfeldt, Schulze [D1].
The present Fourier-edge approach to pseudo-differential operators (i.e. with the Fourier transform along the edges) will point out the aspect of edge-degenerate operators that are completed to an algebra. More details may also be found in the second part of the book Egorov, Schulze [E2].

2. The typical differential operators on a manifold with edges

A differential operator on $X^\wedge \times \Omega \ni (t, x, y)$ of order $\mu \in \mathbb{N}(= \{0, 1, 2, \ldots\})$ is called edge-degenerate if it has the form

$$A = t^{-\mu} \sum_{j+|\alpha| \leq \mu} a_{j\alpha}(t, y) \left(-t\frac{\partial}{\partial t}\right)^j (tD_y)^\alpha \tag{2.1}$$

with operator-valued coefficients $a_{j\alpha}(t, y) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \mathit{Diff}^{\,\mu-(j+|\alpha|)}(X))$. The edge-degenerate differential operators will be regarded as the typical ones on the (open stretched) wedge $X^\wedge \times \Omega$. If $g(t, y)$ is a (t, y)-dependent Riemannian metric on X, C^∞ in (t, y) up to $t = 0$, then the Laplace-Beltrami operator for the metric of the "geometric wedge" $dt^2 + t^2 g(t, y) + t^2 dy^2$ on $X^\wedge \times \Omega$ is just edge-degenerate, of order $\mu = 2$. Let us also note that for every differential operator $\tilde{A}$ in $\mathbb{R}^{n+1} \times \Omega \ni (\tilde{x}, y)$ of order μ

$$\tilde{A} = \sum_{|\beta| \leq \mu} a_\beta(\tilde{x}, y) D^\beta_{\tilde{x}, y}$$

with $a_\beta(\tilde{x}, y) \in C^\infty(\mathbb{R}^{n+1} \times \Omega)$, the restriction A of $\tilde{A}$ to $\left(\mathbb{R}^{n+1} \setminus \{0\}\right) \times \Omega$ takes the form (2.1) under the substitution of polar coordinates in $\mathbb{R}^{n+1} \setminus \{0\}$, with $t = |\tilde{x}|$, $X \in S^n = X$. In this case (2.1) may be regarded as an anisotropic reformulation of the operator $\tilde{A}$ with respect to the fictitious edge Ω. This shows that the edge-degenerate symbols, i.e. those of the form

$$t^{-\mu}p(t,x,y,t\tau,\xi,t\eta)\ , \tag{2.2}$$

are much more general than the "usual ones". (τ,ξ,η) are the covariables to (t,x,y). For convenience, x will indicate points of X as well as local coordinates on X under corresponding charts.
Let W be a manifold with edge Y, i.e. $W\setminus Y$ and Y are C^∞ manifolds of dimensions $n+1+q$ and q, respectively, and W is locally close to every $y\in Y$ of the form $X^\Delta\times\Omega$ with some closed compact C^∞ manifold X and an open neighbourhood Ω of y in Y. Then we can analogously talk about edge-degenerate differential operators A on W, by demanding that A takes the form (2.1) in local coordinates near y. The analysis of such operators will refer to the stretched manifold $\mathbb{W}$ "with edge" Y, that is a C^∞ manifold with boundary, described near the boundary by $\overline{\mathbb{R}}_+\times X\times\Omega$. The transition diffeomorphisms are assumed to be smooth up to $t=0$. The edge-degenerate behaviour of operators remains invariant under such diffeomorphisms. If U is a coordinate neighbourhood on X with local coordinates x then the restriction of A to $\overline{\mathbb{R}}_+\times U\times\Omega\ni(t,x,y)$ will have a complete symbol of the form (2.2), where

$$p(t,x,y,\tilde\tau,\xi,\tilde\eta) \tag{2.3}$$

is C^∞ in t up to $t=0$. If $p_{(\mu)}(t,x,y,\tilde\tau,\xi,\tilde\eta)$ is the homogeneous principal part of (2.3) in $(\tilde\tau,\xi,\tilde\eta)$ of order μ, then

$$\sigma^\mu_\psi(A)(t,x,y,\tilde\tau,\xi,\tilde\eta)=t^{-\mu}p_{(\mu)}(t,x,y,\tilde\tau,\xi,\tilde\eta)|_{\tilde\tau=t\tau,\tilde\eta=t\eta} \tag{2.4}$$

is the homogeneous principal symbol of A of order μ. Let us also set

$$\sigma^\mu_{\psi,b}(A)(t,x,y,\tilde\tau,\xi,\tilde\eta)=p_{(\mu)}(t,x,y,\tilde\tau,\xi,\tilde\eta)\ . \tag{2.5}$$

For the ellipticity we shall need a further (operator-valued) symbol, dependent on $(y,\eta)\in T^*Y\setminus 0$, acting as a family of cone operators on $X^\wedge$. This is

$$\sigma^\mu_\wedge(A)(y,\eta)=t^{-\mu}\sum_{j+|\alpha|\le\mu}a_{j\alpha}(0,y)\left(-t\frac{\partial}{\partial t}\right)^j(t\eta)^\alpha\ , \tag{2.6}$$

called the homogeneous principal edge symbol of A of order μ. (2.6) will be an operator family

$$\sigma^\mu_\wedge(A)(y,\eta):\ \mathcal{K}^{s,\gamma}(X^\wedge)\longrightarrow\mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \tag{2.7}$$

between the weighted Sobolev spaces $\mathcal{K}^{s,\gamma}(X^\wedge)$ of smoothness $s\in\mathbb{R}$ and weight $\gamma\in\mathbb{R}$. Here, by definition,

$$\mathcal{K}^{s,\gamma}(X^\wedge)=\omega(t)t^\gamma\mathcal{H}^s(X^\wedge)+\big(1-\omega(t)\big)H^s(X^\wedge) \tag{2.8}$$

for a cut-off function $\omega(t)$ $\big($i.e., $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$, $\omega(t) = 0$ for $0 \le t < \varepsilon$ with some $\varepsilon > 0\big)$, further $\mathcal{H}^s(X^\wedge)$ is the Sobolev space of smoothness s, based on the Mellin transform in t and (locally) on the Fourier transform in x. Details may be found in Schulze [S7]. For $s \in \mathbb{N}$ we have $u \in \mathcal{H}^s(X^\wedge)$ for an u supported by $\overline{\mathbb{R}}_+ \times U$ iff

$$(t\partial_t)^{\alpha_0}\partial_{x_1}^{\alpha_1} \cdot \ldots \cdot \partial_{x_n}^{\alpha_n} u(t,x) \in t^{-\frac{n}{2}} L^2(\mathbb{R}_+ \times \mathbb{R}^n)$$

for all $\alpha = (\alpha_0, \ldots, \alpha_n)$ with $|\alpha| \le s$. Finally $H^s(X^\wedge)$ is the standard Sobolev space on the infinite cone that corresponds to $H^s(\mathbb{R}^{n+1})$ outside the origin in the special case $X = S^n$, cf. [S7]. In view of $\mathcal{H}^s(X^\wedge) \subset H^s_{loc}(X^\wedge)$, the space (2.8) is independent of the concrete choice of ω.

Remark 2.1 *Set* $(\kappa_\lambda u)(t,x) = \lambda^{\frac{n+1}{2}} u(\lambda t, x)$ *for* $\lambda \in \mathbb{R}_+$. *Then* $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ *is a group of isomorphisms on* $\mathcal{K}^{s,\gamma}(X^\wedge)$ *belonging to* $C\big(\mathbb{R}_+, \mathcal{L}_\sigma\big(\mathcal{K}^{s,\gamma}(X^\wedge)\big)\big)$ *for all* $s, \gamma \in \mathbb{R}$. *Here* σ *indicates the strong operator topology.*

Remark 2.2 *The operator family* (2.6) *is homogeneous of order* μ *in the sense*

$$\sigma_\wedge^\mu(A)(y, \lambda\eta) = \lambda^\mu \kappa_\lambda \sigma_\wedge^\mu(A)(y,\eta)\kappa_\lambda^{-1} \tag{2.9}$$

for all $\lambda \in \mathbb{R}_+$, $\eta \neq 0$.

The operators (2.7) belong to the cone algebra on $X^\wedge$ for every fixed (y, η), cf. [S7], [S8]. The so-called conormal symbol of order μ follows by inserting $z \in \mathbb{C}$ for $-t\partial_t$ and putting $t = 0$. It will be denoted by $\sigma_M^\mu(\cdot)$. Then

$$\sigma_M^\mu \sigma_\wedge^\mu(A)(y,z) = \sum_{j=0}^{\mu} a_{j0}(0,y) z^j : \; H^s(X) \longrightarrow H^{s-\mu}(X)$$

is an operator family between the standard Sobolev spaces $H^s(X)$ on X, that is independent of η.

Remark 2.3 *Let* A *be edge-degenerate, of order* μ, *and*

$$\sigma_{\psi,b}^\mu(A)(0,x,y,\tilde{\tau},\xi,\tilde{\eta}) \neq 0 \quad \textit{for all} \quad (x,y) \in X \times \Omega, (\tilde{\tau},\xi,\tilde{\eta}) \in \mathbb{R}^{n+1+q} \setminus \{0\} \; . \tag{2.10}$$

Then, for every $y_0 \in \Omega$ *there exists a countable discrete subset* $D = D(y_0) \subset \mathbb{C}$ *with* $K \cap D$ *finite for every compact* $K \subset \mathbb{C}$ *such that*

$$\sigma_M^\mu \sigma_\wedge^\mu(A)(y_0, z) : \; H^s(X) \longrightarrow H^{s-\mu}(X)$$

is an isomorphism for all $z \in \mathbb{C} \setminus D(y_0)$ *and all* $s \in \mathbb{R}$.

Let $\beta \in \mathbb{C}$ and

$$\Gamma_\beta = \{z \in \mathbb{C} : \; \mathrm{Re}\, z = \mathrm{Re}\, \beta\} \; . \tag{2.11}$$

Theorem 2.4 *Let A be edge-degenerate, of order μ, and assume* (2.10). *Then*

$$\sigma_\wedge^\mu(A)(y,\eta): \ \mathcal{K}^{s,\gamma}(X^\wedge) \longrightarrow \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \tag{2.12}$$

is a Fredholm operator for all $s \in \mathbb{R}$, $\eta \in \mathbb{R}^q \setminus \{0\}$, and all $\gamma = \gamma(y) \in \mathbb{R}$ for which $D(y) \cap \Gamma_{\frac{n+1}{2}-\gamma(y)} = \emptyset$.

Remark 2.5 *The index of* (2.12) *for an allowed weight γ is independent of s. Further we have*

$$\operatorname{ind} \sigma_\wedge^\mu(A)(y,\eta) = \operatorname{ind} \sigma_\wedge^\mu(A)\left(y, \frac{\eta}{|\eta|}\right) . \tag{2.13}$$

The latter relation follows from (2.9). *Finally the index may depend on the choice of γ. In all cases when* (2.12) *is Fredholm there are finite-dimensional subspaces $L_\gamma^-(y,\eta)$, $L_\gamma^+(y,\eta)$ of $\mathcal{S}(\overline{\mathbb{R}}_+, C^\infty(X))\left(= \mathcal{S}(\mathbb{R}, C^\infty(X)|_{\overline{\mathbb{R}}_+})\right)$ such that*

$$L_\gamma^+(y,\eta) = \ker \sigma_\wedge^\mu(A)(y,\eta), \quad L_\gamma^-(y,\eta) \oplus \operatorname{im} \sigma_\wedge^\mu(A)(y,\eta) = \mathcal{K}^{s,\gamma}(X^\wedge)$$

for every $y \in Y$, $\eta \neq 0$.

If we set $N_\pm = \dim L_\gamma^\pm(y,\eta)$, we find a matrix of operators

$$\begin{pmatrix} \sigma_\wedge^\mu(A) & c \\ b & r \end{pmatrix} : \begin{array}{c} \mathcal{K}^{s,\gamma}(X^\wedge) \\ \oplus \\ \mathbb{C}^{N_-} \end{array} \longrightarrow \begin{array}{c} \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \\ \oplus \\ \mathbb{C}^{N_+} \end{array} \tag{2.14}$$

that is an isomorphism for the given fixed (y,η). For obtaining c and b it suffices to choose arbitrary isomorphisms $c: \ \mathbb{C}^{N_-} \to L_\gamma^+(y,\eta)$ and $b: \ L_\gamma^-(y,\eta) \to \mathbb{C}^{N_+}$, respectively. Further we may set $r = 0$. The edge calculus below will require such a choice of c, b, r, such that (2.14) smoothly depends on $y \in \Omega$ and $\eta \neq 0$ and that

$$\begin{pmatrix} a & c \\ b & r \end{pmatrix}(y,\lambda\eta) = \lambda^\mu \begin{pmatrix} \kappa_\lambda & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & c \\ b & r \end{pmatrix}(y,\eta) \begin{pmatrix} \kappa_\lambda & 0 \\ 0 & 1 \end{pmatrix}^{-1} \tag{2.15}$$

for all $\lambda > 0$, with $a := \sigma_\wedge^\mu(A)$. This homogeneity relation is satisfied for the left upper corner anyway, cf. Remark 2.2. For the remaining entries it suffices to have an isomorphism (2.14) for arbitrary y and $|\eta| = 1$, and then to define the values of c, b, r at $\eta \neq 0$ by (2.15) by putting $\lambda = |\eta|$, and replacing η by $\frac{\eta}{|\eta|}$. It remains to choose c, b, r smoothly in y and η with $|\eta| = 1$. The existence of such a choice for y varying over a compact $K \Subset \Omega$ is actually an easy consequence of generalities on families of Fredholm operators, parametrized by a compact parameter set, which is here $K \times S^{q-1} \ni (y,\eta)$, (cf. Schulze [S7], [S8]). The restriction to compact K will be sufficient for our purposes, since Ω below plays the role of a piece from a compact C^∞

manifold Y (the edge). Since the dimensions N_- and N_+ will jump in general under varying y, we finally get operator families

$$\begin{pmatrix} \sigma_\psi^\mu(A) & c \\ b & r \end{pmatrix} : \begin{matrix} \mathcal{K}^{s,\gamma}(X^\wedge) \\ \oplus \\ J^-_{(y,\eta)} \end{matrix} \longrightarrow \begin{matrix} \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \\ \oplus \\ J^+_{(y,\eta)} \end{matrix} \tag{2.16}$$

where $J^\pm$ are vector bundles over $K \times S^{q-1}$ (sub (y,η) indicates the fiber over (y,η)) and $\dim J^+_{(y,\eta)} - \dim J^-_{(y,\eta)} = N^+ - N^-$. Then, in general, we have to allow $r \neq 0$.
It is now a topological condition to the original operator A that J^-, J^+ may be chosen as the local representatives of vector bundles on Y, i.e. the dependence on η in (2.16) disappears. Under that condition we obtain

$$\begin{pmatrix} \sigma_\psi^\mu(A) & c \\ b & r \end{pmatrix}(y,\eta) : \begin{matrix} \mathcal{K}^{s,\gamma}(X^\wedge) \\ \oplus \\ J^-_y \end{matrix} \longrightarrow \begin{matrix} \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \\ \oplus \\ J^+_y \end{matrix} \tag{2.17}$$

which is then to be used below as the homogeneous principal edge symbol of some "edge problem" associated with A.
For purposes below we shall introduce here operator-valued meromorphic functions

$$h(z) \in \mathcal{A}(\mathbb{C} \setminus D, L^\mu_{cl}(X)) \ . \tag{2.18}$$

Here D is a countable set in $\mathbb{C}$ with finite $K \cap D$ for every compact K. The space $L^\mu_{cl}(X)$ of classical pseudo-differential operators on X of order μ is endowed with its natural Fréchet topology, cf. [S8], and $\mathcal{A}(U)$ for an open $U \subseteq \mathbb{C}$ is the space of holomorphic functions in U. Further $\mathcal{A}(U,E)$ for any, say Fréchet, space E is the space of holomorphic E-valued functions on U. Clearly $\mathcal{A}(U,E)$ has a natural Fréchet structure, again. Denote the points of D by $\{p_j\}_{j\in\mathbb{Z}}$. We will assume that $D \cap \{z : \ c \leq \mathrm{Re}\, z \leq c'\}$ is finite for every $c < c'$. Furthermore let us fix a sequence $\{m_j\}_{j\in\mathbb{Z}}$, $m_j \in \mathbb{N}$, and a sequence $\{N_j\}_{j\in\mathbb{Z}}$ of finite-dimensional subspaces N_j of finite-dimensional operators in $L^{-\infty}(X)$. Write

$$P = \{(p_j, m_j, N_j)\}_{j\in\mathbb{Z}} \ , \quad D = \pi_{\mathbb{C}} P \ .$$

Every such P will also be called a (discrete) asymptotic type for Mellin symbols. We will also need $L^\mu_{cl}(X;\Lambda)$ for $\Lambda := \mathbb{R}^l$ with some $l \in \mathbb{N}$ which is the space of all λ-dependent classical pseudo-differential operators on X, i.e. the amplitude functions, given locally, are classical of order μ in (ξ,λ), and $L^{-\infty}(X;\Lambda) = \mathcal{S}(\Lambda, L^{-\infty}(X))$. The space $L^\mu_{cl}(X;\Lambda)$ is also Fréchet in a natural way. Now

$$M^\mu_P(X) \tag{2.19}$$

will denote the subspace of all $h(z)$ like (2.18) such that for every D-excision function $\chi(z)$ (i.e. $\chi(z) \in C^\infty(\mathbb{C})$, $\chi(z) = 0$ for dist $(z, D) < \varepsilon_0$, $\chi(z) = 1$ for dist $(z, D) > \varepsilon_1$, with certain $0 < \varepsilon_0 < \varepsilon_1$)

$$\chi(z)h(z)|_{\Gamma_\beta} \in L^\mu_{cl}(X; \mathbb{R}_\tau) \quad \text{for} \quad \tau = \operatorname{Im} z \ ,$$

for all $\beta \in \mathbb{R}$, uniformly in $c \leq \beta \leq c'$ for every $c < c'$; furthermore it is required that $h(z)$ is meromorphic with poles at p_j of multiplicities $m_j + 1$, and the Laurent expansion at p_j is

$$h(z) = \sum_{k=0}^{m_j} \kappa_j (z - p_j)^{-(k+1)} + h_j(z)$$

with $\kappa_j \in N_j$, $h_j(z) \in \mathcal{A}(U, L^\mu_{cl}(X))$ for some neighbourhood U of p_j. The space $M^\mu_P(X)$ has a natural Fréchet topology. We will write

$$M^\mu_O(X) \quad \text{when} \quad D = \emptyset \ . \tag{2.20}$$

By replacing $L^\mu_{cl}(X)$ by $L^\mu_{cl}(X; \Lambda)$ we get analogously the space

$$M^\mu_O(X; \Lambda) \ , \tag{2.21}$$

i.e. $h(z, \lambda) \in M^\mu_O(X; \Lambda)$ means $h(z, \lambda) \in \mathcal{A}(\mathbb{C}, L^\mu(X; \Lambda))$ and

$$h(z, \lambda)|_{\Gamma_\beta} \in L^\mu_{cl}(X; \mathbb{R}_\tau \times \Lambda)$$

for all $\beta \in \mathbb{R}$, uniformly in β for $c \leq \beta \leq c'$ for all $c < c'$.

3. The wedge Sobolev spaces

We now briefly remind of the material on the wedge Sobolev spaces from [S4], [S7]. Let first E be a Banach space and

$$\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+} \subset C(\mathbb{R}_+, \mathcal{L}_\sigma(E))$$

be a group of isomorphisms in E with $\kappa_\lambda \kappa_\rho = \kappa_{\lambda\rho}$ for all $\lambda, \rho \in \mathbb{R}$ (σ indicates the strong operator topology). Let us fix a strictly positive function $\eta \to [\eta]$ in $C^\infty(\mathbb{R}^q_\eta)$ with

$$[\eta] = |\eta| \quad \text{for} \quad |\eta| \geq c$$

with some $c > 0$. We will set

$$\kappa(\eta) := \kappa_{[\eta]} \ . \tag{3.1}$$

An example for a choice of

$$\{E, \{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}\} \tag{3.2}$$

was given in Remark 2.1.

If E is any Fréchet space we will denote by $\mathcal{S}(\mathbb{R}^q, E)$ the Schwartz space of E-valued functions on $\mathbb{R}^q$. Then $\mathcal{S}(\mathbb{R}^q, E) = \mathcal{S}(\mathbb{R}^q) \otimes_\pi E$, where $\otimes_\pi$ is the completed projective tensor product. In an analogous sense we will form other vector-valued spaces that extend scalar ones, for instance

$$\mathcal{A}(U, E) = \mathcal{A}(U) \otimes_\pi E$$

when $U \subseteq \mathbb{C}$ is open and $\mathcal{A}(U)$ the space of all holomorphic functions in U in the standard Fréchet topology. We can also form the Sobolev space $H^s(\mathbb{R}^q, E)$ of smoothness s, defined as the subspace of all $u \in \mathcal{S}'(\mathbb{R}^q, E) := \mathcal{L}(\mathcal{S}(\mathbb{R}^q), E)$ for which

$$\int [\eta]^{2s} \big(r_j(F_{y\to\eta}u)(\eta)\big)^2 d\eta < \infty$$

for all $j \in \mathbb{Z}$. Here $\{r_j\}_{j\in\mathbb{Z}}$ is a semi-norm system for the topology of E, and $F_{y\to\eta}$ is the Fourier transform in $\mathbb{R}^q$.
Let us now return to (3.2) for a Banach space E.

Definition 3.1 $\mathcal{W}^s(\mathbb{R}^q, E)$, $s \in \mathbb{R}$, *is the subspace of all* $u \in \mathcal{S}'(\mathbb{R}^q, E)$ *with*

$$\left\{ \int [\eta]^{2s} \|\kappa^{-1}(\eta)(F_{y\to\eta}u)(\eta)\|_E^2 d\eta \right\}^{\frac{1}{2}} < \infty .$$

The basic properties of spaces like $\mathcal{W}^s(\mathbb{R}^q, E)$ may be found in Schulze [S7]. In the present calculus it is sufficient to deal with Hilbert spaces E. Such a case is

$$E = \mathcal{K}^{s,\gamma}(X^\wedge)$$

with $\{\kappa_\lambda\}$ from Remark 2.1. We obtain by definition the weighted wedge Sobolev space of smoothness $s \in \mathbb{R}$ and weight $\gamma \in \mathbb{R}$ on $X^\wedge \times \mathbb{R}^q$

$$\mathcal{W}^{s,\gamma}(X^\wedge \times \mathbb{R}^q) := \mathcal{W}^s\big(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge)\big) . \tag{3.3}$$

Another example is

$$E = H^s(\mathbb{R}^{n+1}_{\tilde{x}}) \quad \text{with} \quad (\kappa_\lambda u)(\tilde{x}) = \lambda^{\frac{n+1}{2}} u(\lambda \tilde{x}) , \quad \lambda > 0 .$$

Then, an easy calculation shows

$$\mathcal{W}^s\big(\mathbb{R}^q, H^s(\mathbb{R}^{n+1})\big) = H^s(\mathbb{R}^{n+1+q}) .$$

If E is Fréchet, with the system $\{r_j\}_{j\in\mathbb{Z}}$ of semi-norms with $r_j(\kappa_\lambda e) \in C(\mathbb{R}_+)$ for all $e \in E$, $j \in \mathbb{Z}$, then $\mathcal{W}^s(\mathbb{R}^q, E)$ will denote the space of all $u \in \mathcal{S}'(\mathbb{R}^q, E)$ such that

$$\int [\eta]^{2s} \big(r_j(\kappa^{-1}(\eta)(F_{y\to\eta}u)(\eta))\big)^2 d\eta < \infty$$

for all $j \in \mathbb{Z}$. In all concrete applications here E will be either a Hilbert space or a projective limit of Hilbert spaces. From now on we will assume that E is of that type, though many assertions also hold in more general cases. Note that there is a canonical isomorphism

$$T := F^{-1}_{\eta \to y} \kappa(\eta) F_{y' \to \eta} : \ \mathcal{W}^s(\mathbb{R}^q, E) \longrightarrow H^s(\mathbb{R}^q, E) \ .$$

This allows one to form $\mathcal{W}^s(\mathbb{R}^q, V) \subset \mathcal{W}^s(\mathbb{R}^q, E)$ to (closed) subspaces $V \subset E$ also in cases when V is not invariant under $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$. It suffices to set

$$\mathcal{W}^s(\mathbb{R}^q, V) = T^{-1} H^s(\mathbb{R}^q, V) \ .$$

In the applications it will not be a problem that $\mathcal{W}^s(\mathbb{R}^q, V)$ may depend on the concrete choice of the function $\eta \to [\eta]$; the error terms can be characterized and fit to the elements of the calculus. A typical example is $E = \mathcal{K}^{s,\gamma}(X^\wedge)$ with the above $\{\kappa_\lambda\}$ and

$$V_P = \text{ linear span of } \Big\{ \omega(t) c_{jk}(x) t^{-p_j} \log^k t : \ c_{jk} \in L_j, 0 \le k \le m_j, \\ \frac{n+1}{2} - \gamma + \vartheta < \operatorname{Re} p_j < \frac{n+1}{2} - \gamma, j = 0, \dots, N \Big\} \ .$$

Here

$$P = \big\{ (p_j, m_j, L_j) \big\}_{j=0,\dots,N} \tag{3.4}$$

is a so-called discrete asymptotic type, i.e. a tuple of data with $p_j \in \mathbb{C}$ in the strip $\frac{n+1}{2} - \gamma + \vartheta < \operatorname{Re} p_j < \frac{n+1}{2} - \gamma$, with fixed $\vartheta < 0$, and L_j are finite-dimensional subspaces of $C^\infty(X)$. The cut-off function $\omega(t)$ is fixed. V_P is a finite-dimensional subspace of $\mathcal{K}^{s,\gamma}(X^\wedge)$ for every $s \in \mathbb{R}$. Then the space $\mathcal{W}^{s,\gamma}(\mathbb{R}^q, V_P)$ can be characterized as the linear span of all distributions of the form

$$F^{-1}_{\eta \to y} \Big\{ \omega(t[\eta]) [\eta]^{\frac{n+1}{2}} c_{jk}(x) \big(t[\eta]\big)^{-p_j} \log^k \big(t[\eta]\big) \hat{v}_{jk}(\eta) \Big\} \tag{3.5}$$

with arbitrary $v_{jk} \in H^s(\mathbb{R}^q)$ and $\hat{v}_{jk} = F_{y' \to \eta} v_{jk}$. Let us set

$$\mathcal{K}^{s,\gamma}_P(X^\wedge) = V_P + \mathcal{K}^{s,\gamma}_\Theta(X^\wedge)$$

with

$$\mathcal{K}^{s,\gamma}_\Theta(X^\wedge) = \bigcap_{\varepsilon > 0} \mathcal{K}^{s,\gamma - \vartheta - \varepsilon}(X^\wedge) \ ,$$

and $\Theta = (\vartheta, 0]$ indicating a weight strip of length $-\vartheta$. Then, both $\mathcal{K}^{s,\gamma}_\Theta(X^\wedge)$ and $\mathcal{K}^{s,\gamma}_P(X^\wedge)$ are projective limits of Hilbert spaces that are $\{\kappa_\lambda\}$-invariant. We then obtain the spaces

$$\mathcal{W}^{s,\gamma}_\Theta(X^\wedge \times \mathbb{R}^q) := \mathcal{W}^s\big(\mathbb{R}^q, \mathcal{K}^{s,\gamma}_\Theta(X^\wedge)\big) \ , \tag{3.6}$$

$$\mathcal{W}^{s,\gamma}_P(X^\wedge \times \mathbb{R}^q) := \mathcal{W}^s\big(\mathbb{R}^q, \mathcal{K}^{s,\gamma}_P(X^\wedge)\big) \ , \tag{3.7}$$

by the above scheme. The elements of (3.6) may be interpreted as distributions of edge-flatness $-\vartheta - 0$ relative to the weight γ, whereas (3.7) is the subspace of $\mathcal{W}^{s,\gamma}(X^\wedge \times \mathbb{R}^q)$ of distributions with (discrete) edge asymptotics of type P. It can be proved that

$$\mathcal{W}_P^{s,\gamma}(X^\wedge \times \mathbb{R}^q) = \mathcal{W}_\Theta^{s,\gamma}(X^\wedge \times \mathbb{R}^q) + \mathcal{W}^s(\mathbb{R}^q, V_P) + \mathcal{W}^\infty\left(\mathbb{R}^q, \mathcal{K}_P^{\infty,\gamma}(X^\wedge)\right) .$$

In this sense the "singular functions" of the edge asymptotics have the form (3.5) modulo $\mathcal{W}^\infty\left(\mathbb{R}^q, \mathcal{K}_P^{\infty,\gamma}(X^\wedge)\right)$, whereas $\mathcal{W}^\infty\left(\mathbb{R}^q, \mathcal{K}_P^{\infty,\gamma}(X^\wedge)\right)$ consists $\big($modulo $\mathcal{W}^\infty\left(\mathbb{R}^q, \mathcal{K}_\Theta^{\infty,\gamma}(X^\wedge)\right)\big)$ of all elements of the form

$$\omega(t) c_{jk}(x) t^{-p_j} \log^k t \; v_{jk}(y)$$

with arbitrary $v_{jk} \in H^\infty(\mathbb{R}^q)$ and $c_{jk} \in L_j$, $0 \le k \le m_j$, $j = 0, \ldots, N$. The edge asymptotics in the form (3.5) have been first obtained in the book Rempel, Schulze [R3], cf. also Schulze [S7].
If M is a paracompact C^∞ manifold we form $H^s_{loc}(M)$ in the usual manner as the space of all distributions on M of local Sobolev smoothness s. Then $H^s_{comp}(M)$ is the subspace of elements with compact support. Analogously it is possible to form

$$\mathcal{W}^s_{loc}(M, E) \,, \qquad \mathcal{W}^s_{comp}(M, E) \,.$$

Here it is used the coordinate invariance of the spaces.

Proposition 3.2 *For every* $s, \gamma \in \mathbb{R}$ *we have*

$$\mathcal{W}^{s,\gamma}(X^\wedge \times \mathbb{R}^q) \subset H^s_{loc}(X^\wedge \times \mathbb{R}^q) \,.$$

Now let W be a compact manifold with edge Y in the sense of Section 2. Then we form the weighted Sobolev space on the associated stretched manifold $\mathbb{W}$

$$\mathcal{W}^{s,\gamma}(\mathbb{W}) \;\text{ for }\; s, \gamma \in \mathbb{R} \tag{3.8}$$

as the subspace of all $u \in H^s_{loc}(\text{int}\, \mathbb{W})$ with $\varphi u \in \mathcal{W}^s\left(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge)\right)$ close to $\partial\mathbb{W}$ for every $\varphi \in C_0^\infty(\mathbb{W})$, in the corresponding local coordinates. Analogously, to every asymptotic type (3.4) satisfying $(p, m, L) \in P \Rightarrow (p - j, m, L) \in P$ for all $j \in \mathbb{N}$ such that $\text{Re}\, p - j > \frac{n+1}{2} - \gamma + \vartheta$, we can form the global spaces with discrete edge asymptotics

$$\mathcal{W}_P^{s,\gamma}(\mathbb{W}) \,, \quad s \in \mathbb{R} \,, \tag{3.9}$$

defined by the condition $u \in (3.9) \Leftrightarrow u \in H^s_{loc}(\text{int}\, \mathbb{W})$, $\varphi u \in \mathcal{W}^s\left(\mathbb{R}^q, \mathcal{K}_P^{s,\gamma}(X^\wedge)\right)$ for all φ as mentioned. Note that the transition diffeomorphisms for $\mathbb{W}$ close to $\partial\mathbb{W}$ are assumed to be (t, x)-independent for small t. Otherwise the global notion of asymptotic types needs a little further discussion on the coefficient spaces L_j. This will be dropped here.

Proposition 3.3 *Let A be an edge-degenerate differential operator on W of order μ. Then A induces continuous operators*

$$A: \mathcal{W}^{s,\gamma}(\mathbb{W}) \longrightarrow \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W}) \tag{3.10}$$

for all $s, \gamma \in \mathbb{R}$. Furthermore, to every asymptotic type P (associated with the data (γ, Θ), $\Theta = (\vartheta, 0]$) there exists an asymptotic type Q (analogously associated with $(\gamma - \mu, \Theta)$) such that A induces continuous operators

$$A: \mathcal{W}_P^{s,\gamma}(\mathbb{W}) \longrightarrow \mathcal{W}_Q^{s-\mu,\gamma-\mu}(\mathbb{W})$$

for all $s \in \mathbb{R}$.

The operator A is called elliptic with respect to σ_ψ^μ and $\sigma_{\psi,b}^\mu$ if $\sigma_\psi^\mu(A)$ (the homogeneous principal symbol of A of order μ as a function on $T^*(\operatorname{int} \mathbb{W}) \setminus 0$) is non-vanishing, and if $\sigma_{\psi,b}^\mu(A)(t, x, y, \tilde{\tau}, \xi, \tilde{\eta}) \neq 0$ for all $(t, x, y) \in \overline{\mathbb{R}}_+ \times X \times \Omega$, $(\tilde{\tau}, \xi, \tilde{\eta}) \neq 0$ with respect to every chart of $\mathbb{W}$ near $\partial\mathbb{W}$ (cf. (2.5)).
It is now a natural question whether (3.10) is a Fredholm operator for arbitrary $s, \gamma \in \mathbb{R}$, once A is elliptic with respect to σ_ψ^μ and $\sigma_{\psi,b}^\mu$. The answer is negative in general. A result of our theory will be that the Fredholm property requires the bijectivity of the following operator-valued edge symbol, namely of

$$\sigma_\wedge^\mu(A)(y, \eta): \mathcal{K}^{s,\gamma}(X^\wedge) \longrightarrow \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \tag{3.11}$$

for all $y \in Y$ and $\eta \neq 0$. However from Theorem 2.4 we only know that (3.11) is a family of Fredholm operators for those weights $\gamma \in \mathbb{R}$ such that $D(y) \cap \Gamma_{\frac{n+1}{2}-\gamma} = \emptyset$ for all $y \in Y$. This shows that we may expect exceptional weights where the Fredholm property of (3.10) will be violated. On the other hand the Fredholm property (3.11) is not sufficient. The idea from Rempel, Schulze [R3] and Schulze [S4], [S7] is now to enlarge the class of operators by allowing matrices

$$\mathcal{A} = \begin{pmatrix} A & C \\ B & R \end{pmatrix} : \begin{matrix} \mathcal{W}^{s,\gamma}(\mathbb{W}) \\ \oplus \\ H^s(Y, J^-) \end{matrix} \longrightarrow \begin{matrix} \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W}) \\ \oplus \\ H^{s-\mu}(Y, J^+) \end{matrix} \tag{3.12}$$

where $J^\pm$ are finite-dimensional complex vector bundles on Y. The meaning of the additional operators B (trace with respect to Y), C (potential with respect to Y), R (pseudo-differential on Y) is analogous to that from pseudo-differential boundary problems, cf. Višik, Eskin [V1], Boutet de Monvel [B1], Rempel, Schulze [R1]. The operators B, C, R will also be called edge conditions. They can be generated in local terms over $\Omega \subseteq \mathbb{R}^q$ as pseudo-differential operators in y with the operator-valued symbols

$$\chi(\eta) b(y, \eta), \quad \chi(\eta) c(y, \eta), \quad \chi(\eta) r(y, \eta),$$

with $b(y, \eta)$, $c(y, \eta)$, $r(y, \eta)$ from (2.17) and an excision function $\chi(\eta)$, cf. the notions from Proposition 4.3 below. The problem of constructing a parametrix of (3.12) will motivate considering analogous operator matrices with edge-degenerate pseudo-differential operators in the left upper corners.

4. Pseudo-differential operators with operator-valued symbols in the Fourier-edge approach

We now pass to the elements of the calculus of pseudo-differential operators with symbols taking values in $\mathcal{L}(E,\widetilde{E})$ for Banach spaces E, $\widetilde{E}$. The notation "Fourier-edge approach" indicates that the operators are based on the Fourier transform, and the symbol estimates contain groups of isomorphisms

$$\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+} \in C(\mathbb{R}_+, \mathcal{L}_\sigma(E)) \ , \quad \{\tilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+} \in C(\mathbb{R}_+, \mathcal{L}_\sigma(\widetilde{E})) \ . \tag{4.1}$$

We then set $\kappa(\eta) = \kappa_{[\eta]}$, $\tilde{\kappa}(\eta) = \tilde{\kappa}_{[\eta]}$. Similarly to Definition 3.1 the objects will depend on the concrete choice of (4.1), but the groups are fixed once and for all in any concrete case. So we will omit indicating the role of (4.1) in the notations, except when the operators are the identical ones for all λ.
The spaces E (or $\widetilde{E}$) are allowed to be of finite dimension. In this case κ_λ (or $\tilde{\kappa}_\lambda$) are always assumed to be the identities for all λ.

Definition 4.1 *$S^\mu(\Omega \times \mathbb{R}^q; E, \widetilde{E})$ for open $\Omega \subseteq \mathbb{R}^p$ and $\mu \in \mathbb{R}$ is the space of all $a(y,\eta) \in C^\infty(\Omega \times \mathbb{R}^q; \mathcal{L}(E,\widetilde{E}))$ such that*

$$\left\| \tilde{\kappa}^{-1}(\eta)\{ D_y^\alpha D_\eta^\beta a(y,\eta)\}\kappa(\eta)\right\|_{\mathcal{L}(E,\widetilde{E})} \le c[\eta]^{\mu-|\beta|} \tag{4.2}$$

for all multi-indices $\alpha \in \mathbb{N}^p$, $\beta \in \mathbb{N}^q$ and all $y \in K$ for arbitrary $K \Subset \Omega$, $\eta \in \mathbb{R}^q$, with constants $c = c(\alpha,\beta,K) > 0$. The elements in $S^\mu(\Omega \times \mathbb{R}^q; E, \widetilde{E})$ are called operator-valued symbols (or amplitude functions) of order μ.

The best constants in the symbol estimates (4.2) for given a form a semi-norm system on $S^\mu(\Omega \times \mathbb{R}^q; E, \widetilde{E})$ under which this space is Fréchet. The space $S^\mu(\mathbb{R}^q; E, \widetilde{E})$ of elements that are y-independent is closed in the induced topology. Then

$$\begin{aligned} S^\mu(\Omega \times \mathbb{R}^q; E, \widetilde{E}) &= C^\infty(\Omega, S^\mu(\mathbb{R}^q; E, \widetilde{E})) \\ &= C^\infty(\Omega) \otimes_\pi S^\mu(\mathbb{R}^q; E, \widetilde{E}) \end{aligned}$$

(with $\otimes_\pi$ as completed projective tensor product). Many elements of the theory of analogous scalar spaces (i.e. where $E = \widetilde{E} = \mathbb{C}$) may be obtained analogously also in the operator-valued case, cf. [S7]. We will not repeat here all those things. Let us only mention that asymptotic sums of symbols a_j of orders μ_j, $j \in \mathbb{N}$ tending to $-\infty$ as $j \to \infty$ can be carried out within the symbol classes. This means that there is a symbol a of order $\mu = \max\{\mu_j\}$ such that for every N there is an M with $a - \sum_{j=0}^{M} a_j$ being of order $\mu - N$. Then a is unique modulo

$$S^{-\infty}(\Omega \times \mathbb{R}^q; E, \widetilde{E}) = \bigcap_{\mu\in\mathbb{R}} S^\mu(\Omega \times \mathbb{R}^q; E, \widetilde{E}) \ ,$$

and we write $a \sim \sum_{j=0}^{\infty} a_j$. Note that

$$S^{-\infty}(\Omega \times \mathbb{R}^q; E, \widetilde{E}) = C^\infty\Big(\Omega, \mathcal{S}(\mathbb{R}^q, \mathcal{L}(E, \widetilde{E}))\Big) ,$$

which is independent of the concrete choice of $\{\kappa_\lambda\}$.

Example 4.2 Let

$$a(y,\eta) = t^{-\mu} \sum_{j+|\alpha| \le \mu} a_{j\alpha}(t,y) \left(-t\frac{\partial}{\partial t}\right)^j (t\eta)^\alpha$$

with coefficients $a_{j\alpha}(t,y) \in C^\infty\big(\overline{\mathbb{R}}_+ \times \Omega, \mathit{Diff}^{\mu-(j+|\alpha|)}(X)\big)$ that are independent of t for $t \ge c$ with some $c > 0$. Then

$$a(y,\eta) \in S^\mu(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))$$

for every $s, \gamma \in \mathbb{R}$, with κ_λ, $\tilde{\kappa}_\lambda$ from Remark 2.1.

Proposition 4.3 *Let*

$$a_{(\mu)}(y,\eta) \in C^\infty(\Omega \times (\mathbb{R}^q \setminus \{0\}, \mathcal{L}(E, \widetilde{E}))$$

be homogeneous of order μ in $\eta \neq 0$, i.e.

$$a_{(\mu)}(y, \lambda\eta) = \lambda^\mu \tilde{\kappa}_\lambda a_{(\mu)}(y,\eta)\kappa_\lambda^{-1}$$

for all $\lambda > 0$, $y \in \Omega$, $\eta \neq 0$. Further let $\chi(\eta)$ be an excision function $\big($i.e. $\chi(\eta) \in C^\infty(\mathbb{R}^q)$, $\chi(\eta) = 0$ for $|\eta| \le c_0$, $\chi(\eta) = 1$ for $|\eta| \ge c_1$, with constants $0 < c_0 < c_1 < \infty\big)$. Then

$$\chi(\eta) a_{(\mu)}(y,\eta) \in S^\mu(\Omega \times \mathbb{R}^q; E, \widetilde{E}) .$$

Definition 4.4 *A symbol $a(y,\eta) \in S^\mu(\Omega \times \mathbb{R}^q; E, \widetilde{E})$ is called classical if there are functions $a_{(\mu-j)}(y,\eta) \in C^\infty\big(\Omega \times (\mathbb{R}^q \setminus \{0\}), \mathcal{L}(E, \widetilde{E})\big)$ that are homogeneous of order $\mu - j$, $j \in \mathbb{N}$, with*

$$a(y,\eta) \sim \sum_j \chi(\eta) a_{(\mu-j)}(y,\eta)$$

for any excision function χ. We denote by $S^\mu_{cl}(\Omega \times \mathbb{R}^q; E, \widetilde{E})$ the space of all classical symbols.

Set

$$\sigma^\mu_\wedge(a)(y,\eta) = a_{(\mu)}(y,\eta) , \tag{4.3}$$

called the homogeneous principal symbol of $a \in S^\mu_{cl}(\Omega \times \mathbb{R}^q; E, \widetilde{E})$ of order μ.

Remark 4.5 *The symbol* $a(y,\eta)$ *of Example* 4.2 *belongs to* $S^{\mu}_{cl}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))$ *once the coefficients* $a_{j\alpha}$ *are independent of* t *and* $\sigma^{\mu}_{\wedge}(a)(y,\eta) =$(2.6).

Let $F = F_{y\to\eta}$ be as above the Fourier transform in $\mathbb{R}^q$. Then, similarly to the scalar theory of pseudo-differential operators, we can form

$$Op(a)u(y) = \iint e^{i(y-y')\eta} a(y,y',\eta)u(y')dy'\, đ\eta$$

with $đ\eta = (2\pi)^{-q}d\eta$, for every $a(y,y',\eta) \in S^{\mu}(\Omega \times \Omega \times \mathbb{R}^q; E, \widetilde{E})$ with open $\Omega \subseteq \mathbb{R}^q$. Here we first assume $u \in C_0^\infty(\Omega, E)$. Then $Op(a)$ induces a continuous operator

$$Op(a): \ C_0^\infty(\Omega, E) \longrightarrow C^\infty(\Omega, \widetilde{E}) \ .$$

Definition 4.6 *Let* $\Omega \subseteq \mathbb{R}^q$ *be open,* $\mu \in \mathbb{R}$. *Then*

$$L^{\mu}(\Omega; E, \widetilde{E}) = \left\{Op(a): \ a(y,y',\eta) \in S^{\mu}(\Omega \times \Omega \times \mathbb{R}^q; E, \widetilde{E})\right\}$$

is called the space of pseudo-differential operators (with operator-valued symbols, in the edge approach). The subset $L^{\mu}_{cl}(\Omega; E, \widetilde{E})$*, defined by classical* $a(y,y',\eta)$*, consists by definition of the classical pseudo-differential operators.*

Remember that

$$L^{-\infty}(\Omega; E, \widetilde{E}) = \bigcap_{\mu\in\mathbb{R}} L^{\mu}(\Omega; E, \widetilde{E})$$

consists of all $Op(a)$ with $a(y,y',\eta) \in S^{-\infty}(\Omega \times \Omega \times \mathbb{R}^q; E, \widetilde{E})$, which is the same as the space of all integral operators with kernel in

$$C^\infty(\Omega \times \Omega; \mathcal{L}(E, \widetilde{E})) \ .$$

Theorem 4.7 *Every* $A \in L^{\mu}(\Omega; E, \widetilde{E})$ *extends to a continuous operator*

$$A: \ \mathcal{W}^{s}_{comp}(\Omega, E) \longrightarrow \mathcal{W}^{s-\mu}_{loc}(\Omega, \widetilde{E})$$

for every $s \in \mathbb{R}$.

Remark 4.8 *The basic elements of the scalar pseudo-differential calculus have corresponding analogues in the operator-valued case. This concerns, in particular, compositions with the Leibniz product on symbolic level, and the result how to pass from* $Op(a)$ *with* $a(y,y',\eta)$ *to an* $Op(\underline{a})$ *with* $\underline{a}(y,\eta)$ *being independent of* y' *(everything mod* $L^{-\infty}(\Omega; E, \widetilde{E})$*). Any such* $\underline{a}(y,\eta)$ *is called a complete symbol of the operator.*

Remark 4.9 *It is not hard to generalize the pseudo-differential calculus with operator-valued symbols to the case of Fréchet spaces E and $\widetilde{E}$. In our applications E and $\widetilde{E}$ can be written as countable projective limits of Hilbert spaces E^j and $\widetilde{E}^k$, that are invariant under $\{\kappa_\lambda\}$ and $\{\tilde{\kappa}_\lambda\}$, respectively, for all j, k. The details may be found in* [S7], [S8].

Let us finish this section by some examples. To this end we remind of the fact that for every (discrete) asymptotic type P associated with the weight data (ρ, Θ) the space $\mathcal{K}_P^{r,\rho}(X^\wedge)$ can be written as projective limit of Hilbert spaces that are $\{\kappa_\lambda\}$ invariant. This is also the case for $\mathcal{K}_P^{\infty,\rho}(X^\wedge)$ which gives us the symbol spaces

$$S_{cl}^{\mu}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}_P^{\infty,\rho}(X^\wedge)) \tag{4.4}$$

for every $s, \mu, \gamma, \rho \in \mathbb{R}$. For every element in $\mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{r,\rho}(X^\wedge))$ we can define the formal adjoint as an element of $\mathcal{L}(\mathcal{K}^{-r,-\rho}(X^\wedge), \mathcal{K}^{-s,-\gamma}(X^\wedge))$, via the non-degenerate sesqui-linear pairings

$$\mathcal{K}^{s,\gamma}(X^\wedge) \times \mathcal{K}^{-s,-\gamma}(X^\wedge) \longrightarrow \mathbb{C} ,$$

induced by the $\mathcal{K}^{0,0}(X^\wedge)$-scalar product, for every $s, \gamma \in \mathbb{R}$. To every $g(y, \eta)$ in (4.4) we can point-wise pass to the formal adjoint $g^*(y, \eta)$ and demand that it belongs to

$$S_{cl}^{\mu}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,-\rho}(X^\wedge), \mathcal{K}_Q^{\infty,-\gamma}(X^\wedge))$$

for some other (discrete) asymptotic type Q, associated with $(-\gamma, \Theta)$.
We shall actually replace $\mathcal{K}_P^{\infty,\rho}(X^\wedge)$, $\mathcal{K}_Q^{\infty,-\gamma}(X^\wedge)$ by subspaces $\mathcal{S}_P^{\rho}(X^\wedge)$ and $\mathcal{S}_Q^{-\gamma}(X^\wedge)$, respectively, where

$$\mathcal{S}_P^{\rho}(X^\wedge) := \omega \mathcal{K}_P^{\infty,\rho}(X^\wedge) + (1-\omega)\mathcal{S}(\overline{X}^\wedge) . \tag{4.5}$$

Here $\omega(t)$ is any fixed cut-off function and $\mathcal{S}(\overline{X}^\wedge) = \mathcal{S}(\overline{\mathbb{R}}_+, C^\infty(X))$. Then, the space (4.5) does not depend on the concrete choice of ω.

Definition 4.10 *$R_G^{\mu}(\Omega \times \mathbb{R}^q, \underline{g})_{P,Q}$ with $\underline{g} = (\gamma, \rho, \Theta)$ is defined as the space of all*

$$g(y, \eta) \in \bigcap_{s \in \mathbb{R}} S_{cl}^{\mu}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{S}_P^{\rho}(X^\wedge))$$

for which

$$g^*(y, \eta) \in \bigcap_{s \in \mathbb{R}} S_{cl}^{\mu}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,-\rho}(X^\wedge), \mathcal{S}_Q^{-\gamma}(X^\wedge)) .$$

Here Ω is an open set in $\mathbb{R}^q$ or in $\mathbb{R}^q \times \mathbb{R}^q$. In the latter case y will also be written as (y, y'). The elements of $R_G^{\mu}(\Omega \times \mathbb{R}^q, \underline{g})_{P,Q}$ will be called Green symbols, to the weight data $\underline{g} = (\gamma, \rho, \Theta)$, with the asymptotic types P, Q. For the edge pseudo-differential calculus it is necessary also to consider the space

$$\mathcal{R}_G^{\mu}(\Omega \times \mathbb{R}^q, \underline{g}; \mathbb{C}^{N_-}, \mathbb{C}^{N_+})_{P,Q} \tag{4.6}$$

consisting of all

$$g(y,\eta) \in \bigcap_{s\in\mathbb{R}} S^{\mu}_{cl}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^{\wedge}) \oplus \mathbb{C}^{N_-}, \mathcal{S}^{\rho}_{P}(X^{\wedge}) \oplus \mathbb{C}^{N_+})$$

with

$$g^*(y,\eta) \in \bigcap_{s\in\mathbb{R}} S^{\mu}_{cl}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,-\rho}(X^{\wedge}) \oplus \mathbb{C}^{N_+}, \mathcal{S}^{-\gamma}_{Q}(X^{\wedge}) \oplus \mathbb{C}^{N_-}) \ .$$

We will also speak of Green symbols in the corresponding generalized sense, where for $g = (g_{ij})_{i,j=1,2}$ *the element* g_{21} *has the meaning of a trace,* g_{12} *of a potential symbol with respect to the edge.* g_{22} *is nothing else than an* $N_- \times N_+$ *matrix of classical scalar symbols. Remember that the involved groups act on finite-dimensional spaces as the identities, i.e. the groups are of the form* $\{\kappa_\lambda \oplus 1\}$ *with the original* κ_λ *in the first component and the identity 1 in the second finite-dimensional component.*

$$\mathcal{R}^{\mu}_{G}(\Omega \times \mathbb{R}^q, \underline{g}; \mathbb{C}^{N_-}, \mathbb{C}^{N_+}) \tag{4.7}$$

will denote the union of the classes (4.6) over all P, Q. An analogous notation makes sense with $R^{\mu}_{G}(\Omega \times \mathbb{R}^q, \underline{g})$.

5. The algebra of edge problems

Let us now return to the program to obtain a class of "edge-degenerate" pseudo-differential operators on a (stretched) manifold $\mathbb{W}$ with edges Y that completes the edge-degenerate differential operators (cf. Proposition 3.3) to an algebra, containing the parametrices of elliptic elements. Many results allow formulations in local terms, in particular, with respect to the "interior symbol classes". The operators we are talking about will belong to $L^{\mu}(\operatorname{int}\mathbb{W})$ and $L^{\mu}_{cl}(\operatorname{int}\mathbb{W})$, respectively. Thus we concentrate on a collar neighbourhood V of $\partial\mathbb{W}$ in $\mathbb{W}$, $V \cong \overline{\mathbb{R}}_+ \times \partial\mathbb{W}$, written as a union of stretched wedges

$$\mathbb{W}_j \cong \overline{\mathbb{R}}_+ \times X \times Y_j \ , \quad j = 1, \ldots, N$$

with subsets $Y_j \subset Y$ forming an open covering of Y. Let

$$\kappa_j : \ Y_j \longrightarrow \Omega_j \ , \quad j = 1, \ldots, N$$

be charts with open $\Omega_j \subseteq \mathbb{R}^q$ and denote the points in $\overline{\mathbb{R}}_+ \times X \times \Omega_j$ by (t, x, y) (j will often be fixed in concrete considerations). Further choose a covering of X by coordinate neighbourhoods X_k, $k = 1, \ldots, M$. The local coordinates under charts $\chi_k : X_k \to U_k$ with open $U_k \subseteq \mathbb{R}^n$ will also be denoted by x if no confusion is possible. In other words the interior symbols are locally defined on

$$\mathbb{R}_+ \times U \times \Omega \ni (t, x, y) \ \text{ with open } \ U \subseteq \mathbb{R}^n \ , \quad \Omega \subseteq \mathbb{R}^q \ .$$

They are assumed to be of the form

$$t^{-\mu}p(t,x,y,t\tau,\xi,t\eta) \tag{5.1}$$

with

$$p(t,x,y,\tilde{\tau},\xi,\tilde{\eta}) \in S^{\mu}_{cl}(\overline{\mathbb{R}}_+ \times U \times \Omega \times \mathbb{R}^{1+n+q}_{\tilde{\tau},\xi,\tilde{\eta}}) \ . \tag{5.2}$$

We might allow as well non-classical symbols, but the question here is to obtain a "minimal" algebra with the mentioned properties. The choice of classical symbols also leads to a stronger locally convex topology of the operator spaces. Note that (5.2) implies

$$p(t,x,y,t\tau,\xi,t\eta) \in S^{\mu}_{cl}(\overline{\mathbb{R}}_+ \times U \times \Omega \times \mathbb{R}^{1+n+q}_{\tau,\xi,\eta}) \ .$$

Since $t > 0$ corresponds to int $\mathbb{W}$ and because of the cut-off factors in a partition of unity on $\mathbb{W}$ we may assume that $p(t,x,y,\tilde{\tau},\xi,\tilde{\eta})$ is independent of t for $t > c$ with some $c > 0$. For analogous reasons it is allowed to assume for a moment that p has its support with respect to x in a compact subset of U. It is now an important technical point to find an operator $A \in L^{\mu}_{cl}(X^\wedge \times \Omega)$ with (5.1) as a complete symbol such that

$$\varphi A \psi : \ \mathcal{W}^s_{comp}(\Omega, \mathcal{K}^{s,\gamma}(X^\wedge)) \longrightarrow \mathcal{W}^{s-\mu}_{loc}(\Omega, \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)) \tag{5.3}$$

is continuous (for all $s \in \mathbb{R}$) for arbitrary $\varphi, \psi \in C^\infty(X)$, supported by a compact subset of the coordinate neighbourhood on X, corresponding to U. Since A is uniquely determind by (5.1) mod $L^{-\infty}(X^\wedge \times \Omega)$, the problem consists of finding a suitable representative in the class of operators modulo smoothing ones. A choice of A for which (5.3) actually holds is by no means evident (unless we have an edge-degenerate differential operator) and requires a careful discussion of a corresponding operator convention that assigns an operator to such a symbol. The crucial observation is that there are Mellin operator conventions. Let us recall the standard form of the Mellin transform M, namely

$$Mu(z) = \int_0^\infty t^{z-1}u(t)dt \ , \quad z \in \mathbb{C} \ ,$$

first defined on $C_0^\infty(\mathbb{R}_+)$ and then extended to various distribution spaces on $\mathbb{R}_+$. In an analogous manner we will understand M when u depends on further variables to which there may be applied the Fourier transform. The inverse of M has the form

$$Mu(z) = \int_{\Gamma_{\frac{1}{2}}} t^{-z}g(z)dz \ ,$$

$\Gamma_\beta = \{z : \ \mathrm{Re}\, z = \mathrm{Re}\, \beta\}$. If $\Omega \subseteq \mathbb{R}^p$ is some open set then

$$S^\mu(\Omega \times \Gamma_\beta \times \mathbb{R}^N)$$

will denote the space of all $a(x,z,\xi)$, where (x,ξ) is for a moment the variable in $\Omega\times\mathbb{R}^N$, such that $a(x,\beta+i\tau,\xi)$ belongs to $S^\mu(\Omega\times\mathbb{R}_\tau\times\mathbb{R}^N_\xi)$. In an analogous manner we obtain symbol spaces with subscript cl, or spaces like

$$S^\mu_{cl}(\overline{\mathbb{R}}_+\times U\times\Omega\times\Gamma_\beta\times\mathbb{R}^n\times\mathbb{R}^q)\ . \tag{5.4}$$

Remember that (5.4) has a natural Fréchet topology. Now let

$$S^\mu_{cl}(\overline{\mathbb{R}}_+\times U\times\Omega\times\mathbb{C}\times\mathbb{R}^n\times\mathbb{R}^q)_{hol}$$

be the subspace of all $a(t,x,y,z,\xi,\eta)$ in $C^\infty(\overline{\mathbb{R}}_+\times U\times\Omega\times\mathbb{C}\times\mathbb{R}^n\times\mathbb{R}^q)$ that are holomorphic in $z\in\mathbb{C}$ and such that $a(t,x,y,\beta+i\tau,\xi,\eta)$ belongs to (5.4) uniformly in $c\le\beta\le c'$ for arbitrary $c<c'$. Analogously we have such symbol classes with $\mathbb{R}_+$ instead of $\overline{\mathbb{R}}_+$. Let us set

$$op^\delta_M(h)u(t)=t^\delta op_M(T^{-\delta}h)t^{-\delta}u(t) \tag{5.5}$$

with $h(t,\dots,z,\dots)$ being a symbol in the covariable $\operatorname{Im} z$, varying over $\Gamma_{\frac{1}{2}-\delta}$, $(T^{-\delta}h)(\dots,z,\dots)=h(\dots,z-\delta,\dots)$ and

$$\begin{aligned} op_M(f)v(t) &= \frac{1}{2\pi i}\int\limits_{\Gamma_{\frac{1}{2}}} t^{-z}f(t,z)\left\{\int\limits_0^\infty t'^{z-1}u(t')dt'\right\}dz \\ &= M^{-1}_{z\to t}fM_{t'\to z}v\ . \end{aligned}$$

The latter expression is a Mellin pseudo-differential operator, relative to the weight line $\Gamma_{\frac{1}{2}}$, whereas (5.5) corresponds to the weight line $\Gamma_{\frac{1}{2}-\delta}$. If h in (5.5) depends on (t,x,y,z,ξ,η), then we can form the pseudo-differential action with respect to the other variables with the Fourier transform. This will be indicated by $op_{\psi,x}$, $op_{\psi,(x,y)}$, ... In particular, we have

$$op_{\psi,x}(p)w(x)=(2\pi)^{-n}\iint e^{i(x-x')\xi}p(\dots,x,\dots,\xi,\dots)w(x')dx'\,d\xi$$

with p being a symbol in (x,ξ). To any given

$$h(t,x,y,z,\xi,\eta)\in S^\mu_{cl}(\mathbb{R}_+\times U\times\Omega\times\mathbb{C}\times\mathbb{R}^n\times\mathbb{R}^q)_{hol}$$

we obtain the (y,η)-dependent operator family

$$op^\delta_M op_{\psi,x}(h)(y,\eta):\ C^\infty_0(\mathbb{R}_+\times U)\longrightarrow C^\infty(\mathbb{R}_+\times U)$$

for every $\delta\in\mathbb{R}$. We then have

$$op^\delta_M op_{\psi,x}(h)(y,\eta)\in C^\infty\big(\Omega,L^\mu_{cl}(\mathbb{R}_+\times U;\mathbb{R}^q_\eta)\big)\ .$$

The Mellin representation in the t-variable is always possible for such operators because of the equivalence of the phase functions $(t-t')\tau$ and $(\log t' - \log t)\tau$. That does not mean at once that a pseudo-differential operator family, written in the Mellin convention for suitable δ, will lead to (5.3) after applying $op_{\psi,y}$. This will only be ensured by a corresponding control of the amplitude function close to $t=0$. The following theorem on the existence of such Mellin operator convention will show that the smoothness in t up to $t=0$ remains preserved.

Theorem 5.1 *To every* $p(t,x,y,t\tau,\xi,t\eta) \in S^{\mu}_{cl}(\overline{\mathbb{R}}_+ \times U \times \Omega \times \mathbb{R}^{1+n+q}_{\tau,\xi,\eta})$ *with* (5.2) *there exists an*

$$f(t,x,y,z,\xi,\tilde{\eta}) \in S^{\mu}_{cl}(\overline{\mathbb{R}}_+ \times U \times \Omega \times C^{\infty} \times \mathbb{R}^n_{\xi} \times \mathbb{R}^q_{\tilde{\eta}})_{hol}$$

such that for $h(t,x,y,z,\xi,\eta) = f(t,x,y,z,\xi,t\eta)$

$$op_{\psi,(t,x)}(p)(y,\eta) - op^{\delta}_{M} op_{\psi,x}(h)(y,\eta) \in C^{\infty}(\Omega, L^{-\infty}(\mathbb{R}_+ \times U; \mathbb{R}^q_{\eta}))$$

for all $\delta \in \mathbb{R}$.

Remark 5.2 *The choice of* f *as a holomorphic function in* z *follows by a cut-off argument on the level of distributional kernels. This technique as well as other details of the proof of Theorem* 5.1 *may be found in Schulze* [S5].

From Theorem 5.1 it is now easy (and essential from the point of view of operator-valued symbolic structure) to pass to a corresponding global Mellin convention with respect to X. To this end we choose symbols

$$p_k(t,x,y,t\tau,\xi,t\eta) \in S^{\mu}_{cl}(\mathbb{R}_+ \times U_k \times \Omega \times \mathbb{R}^{1+n+q})$$

for $k=1,\ldots,M$, that satisfy the conditions of Theorem 5.1 for $U=U_k$. Further let $\{\varphi_k\}_{k=1,\ldots,M}$ be a partition of unity on X, belonging to the open covering $\{U_k\}_{k=1,\ldots,M}$, and let $\{\psi_k\}_{k=1,\ldots,M}$ be functions in $C^{\infty}_0(U_k)$ with $\varphi_k\psi_k = \varphi_k$ for all k. Then

$$P(y,\eta) := \sum_{k=1}^{M} \varphi_k op_{\psi,(t,x)}(p_k)(y,\eta)\psi_k \tag{5.6}$$

is an element of $C^{\infty}(\Omega, L^{\mu}_{cl}(X^{\wedge}; \mathbb{R}^q_{\eta}))$ (for simplicity the pull-backs of local operators over U_k to X_k were suppressed). Analogously we can form the f_k in the sense of Theorem 5.1 for U_k, and we set

$$\begin{aligned} f(t,y,z,\tilde{\eta}) &= \sum_{k=1}^{N} \varphi_k op_{\psi,x}(f_k)(t,y,z,\tilde{\eta})\psi_k \,, \\ h(t,y,z,\eta) &:= f(t,y,z,t\eta)\,. \end{aligned} \tag{5.7}$$

Then $f(t,y,z,\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, M_O^\mu(X;\mathbb{R}^q_{\tilde{\eta}}))$, cf. (2.21). It follows for the operator family (5.6)

$$P(y,\eta) - op_M^\delta(h)(y,\eta) \in C^\infty(\Omega, L^{-\infty}(X^\wedge;\mathbb{R}^q)) \ . \tag{5.8}$$

From now on we shall tacitly assume that all p_k are independent of t in the first t-argument for large t.
Now let us fix arbitrary cut-off functions $\omega(t)$, $\omega_0(t)$, $\omega_1(t)$ satisfying

$$\omega\omega_0 = \omega \ , \qquad \omega\omega_1 = \omega_1 \ . \tag{5.9}$$

Indicate by $\sim$ the equivalence mod $C^\infty(\Omega, L^{-\infty}(X^\wedge;\mathbb{R}^q))$. Then (5.8) implies

$$\begin{aligned} P(y,\eta) &= \omega(t[\eta])P(y,\eta) + \left(1-\omega(t[\eta])\right)P(y,\eta) \\ &\sim \omega(t[\eta])op_M^\delta(h)(y,\eta) + \left(1-\omega(t[\eta])\right)P(y,\eta) \ . \end{aligned}$$

In view of

$$\begin{aligned} \omega(t[\eta])op_M^\delta(h)(y,\eta)\left(1-\omega_0(t[\eta])\right) &\sim 0 \ , \\ \left(1-\omega(t[\eta])\right)P(y,\eta)\omega_1(t[\eta]) &\sim 0 \ , \end{aligned}$$

which is a consequence of (5.9), it follows

$$P(y,\eta) \sim a_0(y,\eta)$$

with

$$a_0(y,\eta) = \omega(t[\eta])op_M^\delta(h)(y,\eta)\omega_0(t[\eta]) + \left(1-\omega(t[\eta])\right)P(y,\eta)\left(1-\omega_1(t[\eta])\right) \ . \tag{5.10}$$

Proposition 5.3 *For every operator family of the form* (5.6) *with associated* (5.7) *we have for*

$$a(y,\eta) = \omega(t[\eta])t^{-\mu}op_M^{\gamma-\frac{n}{2}}(h)(y,\eta)\omega_0(t[\eta]) + \left(1-\omega(t[\eta])\right)t^{-\mu}P(y,\eta)\left(1-\omega_1(t[\eta])\right) \ . \tag{5.11}$$

with fixed $\gamma \in \mathbb{R}$

$$a(y,\eta) \in S^\mu(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))$$

for all $s \in \mathbb{R}$. *Here, as above, the operator-valued symbol classes rely on the families of isomorphisms* $u(\lambda t, x) \to \lambda^{\frac{n+1}{2}}u(\lambda t,x)$, $\lambda > 0$.

Corollary 5.4 *Let* $A = op_{\psi,y}(a)$. *Then* $A \in L^\mu_{cl}(X^\wedge \times \Omega)$ *induces continuous operators*

$$A: \ \mathcal{W}^s_{comp}(\Omega, \mathcal{K}^{s,\gamma}(X^\wedge)) \longrightarrow \mathcal{W}^{s-\mu}_{loc}(\Omega, \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))$$

for all $s \in \mathbb{R}$.

The operator families of the form (5.11) will also be called (complete) edge symbols. Our next objective is to add the so-called smoothing Mellin operators with discrete asymptotics. Let

$$P = \{(p_j, m_j, N_j)\}_{j\in\mathbb{Z}}$$

be a sequence with $p_j \in \mathbb{C}$, $|\mathrm{Re}\, p_j| \to \infty$ as $|j| \to \infty$, $m_j \in \mathbb{N}$, and N_j being a finite-dimensional subspace of finite-dimensional operators in $L^{-\infty}(X)$, $j \in \mathbb{Z}$. Set $\pi_{\mathbb{C}} P = \{p_j\}_{j\in\mathbb{Z}}$. We then have the space

$$M_P^{-\infty}(X) = \bigcap_{\mu\in\mathbb{R}} M_P^{\mu}(X) \ ,$$

cf. (2.19), as the set of all operator-valued functions $h(z)$ in the complex plane with the following properties

(i) $h(z)$ is an $L^{-\infty}(X)$-valued meromorphic function with poles at the points p_j of multiplicities $m_j + 1$, $j \in \mathbb{Z}$,

(ii) the Laurent coefficients of $h(z)$ at $(z - p_j)^{-(k+1)}$ belong to N_j for $0 \leq k \leq m_j$, $j \in \mathbb{Z}$,

(iii) if $\chi(z)$ is any $\pi_{\mathbb{C}} P$-excision function (i.e. $\chi(z) = 0$ in some neighbourhood of $\pi_{\mathbb{C}} P$, $\chi(z) = 1$ for $\mathrm{dist}\,(z, \pi_{\mathbb{C}} P) > \varepsilon$ with an $\varepsilon > 0$) then

$$\chi(\beta + i\tau) h(\beta + i\tau) \in \mathcal{S}(\mathbb{R}_\tau, L^{-\infty}(X))$$

for all $\beta \in \mathbb{R}$ and uniformly in $c \leq \beta \leq c'$ for every $c < c'$.

The space $M_P^{-\infty}(X)$ has a canonical Fréchet topology (which is nuclear). Thus it makes sense to talk about $C^\infty(\Omega, M_P^{-\infty}(X))$ for every fixed P. For every $\delta \in \mathbb{R}$ with $\pi_{\mathbb{C}} P \cap \Gamma_{\frac{1}{2}-\delta} = \emptyset$ we can form the Mellin pseudo-differential operators

$$op_M^{\delta}(h) \ , \qquad h(z) \in M_P^{-\infty}(X) \ .$$

They are elements of $L^{-\infty}(X^\wedge)$.

Proposition 5.5 *Let $\omega(t)$, $\omega_0(t)$ be arbitrary cut-off functions, $h(y, z) \in C^\infty(\Omega, M_P^{-\infty}(X))$, and $\pi_{\mathbb{C}} P \cap \Gamma_{\frac{n+1}{2}-\gamma} = \emptyset$. Then, the operator family*

$$t^{-\nu} \omega(t[\eta]) op_M^{\gamma-\frac{n}{2}}(h)(y) \omega_0(t[\eta]) \eta^\alpha$$

for $\alpha \in \mathbb{N}^q$, $\nu \in \mathbb{R}$, is an element of

$$S^{\nu+|\alpha|}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{\infty,\gamma-\nu}(X^\wedge))$$

for every $s \in \mathbb{R}$.

Remark 5.6 *Let us fix $\mu, \gamma \in \mathbb{R}$ and $j \in \mathbb{N}$. Choose any $\delta \in \mathbb{R}$ with*

$$\gamma \geq \delta \geq \gamma - j \ , \quad \pi_{\mathbb{C}} P \cap \Gamma_{\frac{n+1}{2}-\delta} = \emptyset \ . \tag{5.12}$$

Then for arbitrary $h(y,z) \in C^\infty(\Omega, M_P^{-\infty}(X))$ we have

$$\begin{aligned} m(y,\eta) \ &:= \ t^{-\mu+j}\omega(t[\eta])op_M^{\delta-\frac{n}{2}}(h)(y)\omega_0(t[\eta])\eta^\alpha \\ &\in S_{cl}^{\mu-(j-|\alpha|)}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{\infty,\gamma-\mu}(X^\wedge)) \end{aligned} \tag{5.13}$$

for every $s \in \mathbb{R}$.

Note that the homogeneous principal symbol of (5.13) of order $\mu - (j - |\alpha|)$ in the sense of (4.3) equals

$$\sigma_\wedge^{\mu-(j-|\alpha|)}(m)(y,\eta) = t^{-\mu+j}\omega(t|\eta|)op_M^{\delta-\frac{n}{2}}(h)(y)\omega_0(t|\eta|)\eta^\alpha \ . \tag{5.14}$$

Proposition 5.7 *Let $\tilde{m}(y,\eta)$ be of analogous form as (5.13) with the same $h(y,z)$ but for another choice of the cut-off functions ω, ω_0, of the function $\eta \to [\eta]$, and of δ, satisfying (5.12). Then*

$$m(y,\eta) - \tilde{m}(y,\eta) \in R_G^{\mu-(j-|\alpha|)}(\Omega \times \mathbb{R}^q, \underline{g})$$

with $\underline{g} = (\gamma, \gamma - \mu, \Theta)$.

Definition 5.8 *Let $\gamma, \mu \in \mathbb{R}$, $\underline{g} = (\gamma, \gamma - \mu, \Theta)$ with $\Theta = (\vartheta, 0]$, $-\infty < \vartheta < 0$. Then*

$$R^\mu(\Omega \times \mathbb{R}^q, \underline{g})$$

denotes the set of all

$$a(y,\eta) + m(y,\eta) + g(y,\eta)$$

with $a(y,\eta)$ being of the form (5.11), $m(y,\eta)$ a finite sum of operator families like (5.13) over j, α with (j,α)-dependent $h = h_{j\alpha}$, $\delta = \delta_{j\alpha}$, and further $g(y,\eta) \in R_G^\mu(\Omega \times \mathbb{R}^q, \underline{g})$.

For technical reasons we will also talk about

$$R^\nu(\Omega \times \mathbb{R}^q, \underline{g}) \ , \qquad \underline{g} = (\gamma, \gamma - \mu, \Theta)$$

for any $\nu \in \mathbb{R}$ with $\mu - \nu \in \mathbb{N}$, defined in an analogous manner, where now $a(y,\eta)$ refers to ν instead of μ and for the summands of $m(y,\eta)$ it is required $j \geq \mu - \nu$. Further define

$$\mathcal{R}^\nu(\Omega \times \mathbb{R}^q, \underline{g}; \mathbb{C}^{N_-}, \mathbb{C}^{N_+}) \tag{5.15}$$

as the set of all operator families

$$r(y,\eta) = \begin{pmatrix} r_{11}(y,\eta) & g_{12}(y,\eta) \\ g_{21}(y,\eta) & g_{22}(y,\eta) \end{pmatrix} \tag{5.16}$$

where $r_{11}(y,\eta) = (a + m + g_{11})(y,\eta)$ belongs to $R^\nu(\Omega \times \mathbb{R}^q, \underline{g})$ and the matrix (g_{ij}) to $\mathcal{R}^\nu_G(\Omega \times \mathbb{R}^q, \underline{g};\ \mathbb{C}^{N_-}, \mathbb{C}^{N_+})$ in the sense of (4.7). In these notations Ω is an open set in $\mathbb{R}^q$. We may also allow all objects to depend on $(y,y') \in \Omega \times \Omega$. Then we shall write

$$\mathcal{R}^\nu(\Omega \times \Omega \times \mathbb{R}^q, \underline{g}; \mathbb{C}^{N_-}, \mathbb{C}^{N_+}) \ni r(y,y',\eta) .$$

Proposition 5.9 *We have for* $\underline{g} = (\gamma, \gamma - \mu, \Theta)$

$$\mathcal{R}^\nu(\Omega \times \Omega \times \mathbb{R}^q, \underline{g}; \mathbb{C}^{N_-}, \mathbb{C}^{N_+}) \subset S^\nu(\Omega \times \Omega \times \mathbb{R}^q; E, \widetilde{E})$$

with $E = \mathcal{K}^{s,\gamma}(X^\wedge) \oplus \mathbb{C}^{N_-}$, $\widetilde{E} = \mathcal{K}^{s-\nu,\gamma-\mu}(X^\wedge) \oplus \mathbb{C}^{N_+}$, *for arbitrary* $s \in \mathbb{R}$.

Next we shall introduce the principal symbols of order ν to $r(y,y',\eta) \in \mathcal{R}^\nu(\Omega \times \Omega \times \mathbb{R}^q, \underline{g}; \mathbb{C}^{N_-}, \mathbb{C}^{N_+})$. They will be defined on $y = y'$, so it suffices to neglect y'. First the complete symbols (5.1) (now for μ replaced by ν) related to coordinate neighbourhoods U on X, Ω on Y, lead to the invariant system of homogeneous components of order ν

$$t^{-\nu} p_{(\nu)}(t, x, y, t\tau, \xi, t\eta) .$$

This extends together with the homogeneous principal symbols globally on int $\mathbb{W}$ to an element

$$\sigma^\nu_\psi(r) \in C^\infty(T^*(\Omega \times X^\wedge) \setminus 0) \tag{5.17}$$

that is homogeneous of order ν in the covariables. In a collar neighbourhood of $\partial\mathbb{W}$ in the (t,x,y)-coordinates we get that

$$t^\nu \sigma^\nu_\psi(r)(t, x, y, t^{-1}\tau, \xi, t^{-1}\eta)$$

is C^∞ up to $t = 0$. As above we set

$$\sigma^\nu_{\psi,b}(r)(t, x, y, \tau, \xi, \eta) = t^\nu \sigma^\nu_\psi(r)(t, x, y, t^{-1}\tau, \xi, t^{-1}\eta) . \tag{5.18}$$

If we write r as a 2×2 matrix (5.16), we get

$$\tilde{g}(y,\eta) := \begin{pmatrix} 0 & g_{12} \\ g_{21} & g_{22} \end{pmatrix} (y,\eta) \in S^\nu_{cl}(\Omega \times \mathbb{R}^q; E, \widetilde{E})$$

in the notations of Proposition 5.9. Thus there is a well-defined homogeneous principal symbol in the sense of (4.3), namely $\sigma^\nu_\wedge(\tilde{g})(y,\eta)$. Furthermore, by construction,

$$r_{11}(y,\eta) = a(y,\eta) + m(y,\eta) + g_{11}(y,\eta)$$

with $a(y,\eta)$ being of the form (5.13) with ν instead of μ, and

$$m(y,\eta) + g_{11}(y,\eta) \in S^\nu_{cl}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\nu,\gamma-\mu}(X^\wedge)) .$$

From the latter relation we get a homogeneous principal symbol $\sigma_\wedge^\nu(m+g_{11})(y,\eta)$. Finally we set

$$\begin{aligned} \sigma_\wedge^\nu(a)(y,\eta) \quad := \quad & \omega(t|\eta|)t^{-\nu}op_M^{\gamma-\frac{n}{2}}(h_0)(y,\eta)\omega_0(t|\eta|) \\ & +\left(1-\omega(t|\eta|)\right)t^{-\nu}P_0(y,\eta)\left(1-\omega_1(t|\eta|)\right) \ . \end{aligned} \tag{5.19}$$

Here, in the notations of Proposition 5.3,

$$h_0(t,y,z,\eta)=f(0,y,z,t\eta) \ ,$$

cf. (5.7), and

$$P_0(y,\eta)=\sum_{k=1}^{M}\varphi_k op_{\psi,(t,x)}(p_{k,(\nu),0})(y,\eta)\psi_k$$

cf. (5.6), with $p_{k,(\nu),0} = p_{k,(\nu)}(0,x,y,t\tau,\xi,t\eta)$, sub (ν) indicating the homogeneous principal symbol of order ν. Note that we always assume here $\eta\neq 0$. We thus obtain

$$\sigma_\wedge^\nu(r_{11})(y,\eta)=\sigma_\wedge^\nu(a)(y,\eta)+\sigma_\wedge^\nu(m+g_{11})(y,\eta) \ .$$

This yields

$$\sigma_\wedge^\nu(r)(y,\eta)=\begin{pmatrix} \sigma_\wedge^\nu(r_{11}) & 0 \\ 0 & 0 \end{pmatrix}(y,\eta)+\sigma_\wedge^\nu(\tilde{g})(y,\eta)$$

as an element

$$\sigma_\wedge^\nu(r)(y,\eta)\in C^\infty\Big(T^*\Omega\setminus 0,\bigcap_s \mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge)\oplus\mathbb{C}^{N_-},\mathcal{K}^{s-\nu,\gamma-\mu}(X^\wedge)\oplus\mathbb{C}^{N_+})\Big) \tag{5.20}$$

satisfying

$$\sigma_\wedge^\nu(r)(y,\lambda\eta)=\lambda^\nu\begin{pmatrix} \kappa_\lambda & 0 \\ 0 & 1 \end{pmatrix}\sigma_\wedge^\nu(r)(y,\eta)\begin{pmatrix} \kappa_\lambda & 0 \\ 0 & 1 \end{pmatrix}^{-1} \tag{5.21}$$

for all $\lambda>0$. (5.21) is the homogeneous principal edge symbol of r. Every $r(y,y',\eta)\in\mathcal{R}^\nu(\Omega\times\Omega\times\mathbb{R}^q,\underline{g};\mathbb{C}^{N_-},\mathbb{C}^{N_+})$ gives rise to a pseudo-differential operator with respect to the y-variables

$$Op(r)u(y)=\iint e^{i(y-y')\eta}r(y,y',\eta)u(y')dy'\,đ\eta \ ,$$

$đ\eta=(2\pi)^{-q}d\eta$. From Theorem 4.7 we get continuous operators

$$Op(r):\ \mathcal{W}^s_{comp}(\Omega,\mathcal{K}^{s,\gamma}(X^\wedge))\longrightarrow\mathcal{W}^{s-\mu}_{loc}(\Omega,\mathcal{K}^{s-\nu,\gamma-\mu}(X^\wedge))$$

for every $s\in\mathbb{R}$. We now look at a compact manifold W with edges Y, cf. notations in the beginning of this section. Choose a collar neighbourhood V of $\partial\mathbb{W}$, written as a finite union of stretched wedges $\mathbb{W}_j$, and consider $\kappa_j:\ \mathbb{W}_j\to\overline{\mathbb{R}}_+\times X\times\Omega_j$, according to our geometric assumptions, with open $\Omega_j \subseteq \mathbb{R}^q$. Fix a system of functions $\psi_j \in C_0^\infty(\mathbb{W}_j)$, $\sum_j \psi_j = 1$ on $\bigcup_j \kappa_j^{-1}\{0 \le t \le \delta\}$ with some $\delta > 0$, $\sum_j \psi_j = 0$ on $\bigcup_j \kappa_j^{-1}\{t \ge 2\delta\}$. Further choose $\tilde{\psi} \in C_0^\infty(\mathbb{W}_j)$ with $\tilde{\psi}_j \psi_j = \psi_j$ for all j and $\sum_j \tilde{\psi}_j = 0$ on $\bigcup_j \kappa_j^{-1}\{t \ge 3\delta\}$. Then, to every system of symbols

$$r_j(y, y', \eta) \in \mathcal{R}^\nu(\Omega_j \times \Omega_j \times \mathbb{R}^q, \underline{g}; \mathbb{C}^{N_-}, \mathbb{C}^{N_+})$$

we can form a global operator

$$\mathcal{A}_V = \sum_j \psi_j \kappa_j^* Op(r_j) \tilde{\psi}_j$$

with the operator pull-back κ_j^*.

Definition 5.10 *$\mathcal{Y}^\nu(\mathbb{W}, \underline{g}; \mathbb{C}^{N_-}, \mathbb{C}^{N_+})$ is the space of all operators of the form*

$$\mathcal{A} = \begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix} \mathcal{A}_V \begin{pmatrix} \omega_0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1-\omega & 0 \\ 0 & 0 \end{pmatrix} \mathcal{P} \begin{pmatrix} 1-\omega_1 & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{G}\,, \tag{5.22}$$

where $\mathcal{A}_V$ is as mentioned whereas

$$\mathcal{P} = \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}$$

are operators with $P \in L_{cl}^\nu(\mathrm{int}\,\mathbb{W})$; ω, ω_0, ω_1 are in $C^\infty(\mathbb{W})$, supported by a collar neighbourhood of $\partial\mathbb{W}$ and satisfy $\omega\omega_0 = \omega$, $\omega\omega_1 = \omega_1$; finally $\mathcal{G}$ is a so-called smoothing Green operator, i.e. it induces continuous maps

$$\mathcal{G} \; : \; \begin{array}{c} \mathcal{W}^{s,\gamma}(\mathbb{W}) \\ \oplus \\ H^s(Y, \mathbb{C}^{N_-}) \end{array} \longrightarrow \begin{array}{c} \mathcal{W}_P^{\infty,\gamma-\mu}(\mathbb{W}) \\ \oplus \\ H^\infty(Y, \mathbb{C}^{N_+}) \end{array},$$

$$\mathcal{G}^* \; : \; \begin{array}{c} \mathcal{W}^{s,-\gamma+\mu}(\mathbb{W}) \\ \oplus \\ H^s(Y, \mathbb{C}^{N_+}) \end{array} \longrightarrow \begin{array}{c} \mathcal{W}_Q^{\infty,-\gamma}(\mathbb{W}) \\ \oplus \\ H^\infty(Y, \mathbb{C}^{N_-}) \end{array}$$

for all $s \in \mathbb{R}$, with discrete ($\mathcal{G}$-dependent) asymptotic types P and Q, associated with the corresponding weight data. $*$ indicates the formal adjoint in the sense

$$(\mathcal{G}u, v)_{\mathcal{W}^{0,0}(\mathbb{W}) \oplus H^0(Y, \mathbb{C}^{N_+})} = (u, \mathcal{G}^* v)_{\mathcal{W}^{0,0}(\mathbb{W}) \oplus H^0(Y, \mathbb{C}^{N_-})}$$

for all $u \in C_0^\infty(\mathrm{int}\,\mathbb{W}) \oplus C^\infty(Y, \mathbb{C}^{N_-})$, $v \in C_0^\infty(\mathrm{int}\,\mathbb{W}) \oplus C^\infty(Y, \mathbb{C}^{N_+})$.

The elements of $\mathcal{Y}^\nu(\mathbb{W}, \underline{g}; \mathbb{C}^{N_-}, \mathbb{C}^{N_+})$ are called edge pseudo-differential operators of order ν with respect to the weight data $\underline{g} = (\gamma, \gamma - \mu, \Theta)$. An analogous definition makes sense for (complex) vector bundles J^-, J^+ over Y instead of the trivial ones $Y \times \mathbb{C}^{N_-}$ and $Y \times \mathbb{C}^{N_+}$, respectively. This yields the operator classes

$$\mathcal{Y}^\nu(\mathbb{W}, \underline{g}; J^-, J^+) . \tag{5.23}$$

Here we have fixed once and for all Hermitean structures in the occurring bundles J over Y such that non-degenerate pairings such as

$$H^s(Y, J) \times H^{-s}(Y, J) \longrightarrow \mathbb{C} , \quad s \in \mathbb{R} ,$$

make sense $\big(H^s(Y, J)$ being the Sobolev space of distributional sections in J of smoothness $s\big)$. Every such $\mathcal{A}$ in (5.23) induces continuous operators

$$\mathcal{A} : \begin{array}{c} \mathcal{W}^{s,\gamma}(\mathbb{W}) \\ \oplus \\ H^s(Y, J^-) \end{array} \longrightarrow \begin{array}{c} \mathcal{W}^{s-\nu,\gamma-\mu}(\mathbb{W}) \\ \oplus \\ H^{s-\nu}(Y, J^+) \end{array} \tag{5.24}$$

for all $s \in \mathbb{R}$. Furthermore for every (discrete) asymptotic type P to (γ, Θ) there is a resulting (discrete) asymptotic type R to $(\gamma - \mu, \Theta)$, also dependent on $\mathcal{A}$, such that $\mathcal{A}$ induces continuous operators

$$\mathcal{A} : \begin{array}{c} \mathcal{W}_P^{s,\gamma}(\mathbb{W}) \\ \oplus \\ H^s(Y, J^-) \end{array} \longrightarrow \begin{array}{c} \mathcal{W}_R^{s-\nu,\gamma-\mu}(\mathbb{W}) \\ \oplus \\ H^{s-\nu}(Y, J^+) \end{array} \tag{5.25}$$

for all $s \in \mathbb{R}$.

Another obvious generalization of (5.23) concerns the case of operators, also acting between distributional sections of vector bundles E, F over $\mathbb{W}$. In other words we would have operators like

$$\begin{array}{c} \mathcal{W}^{s,\gamma}(\mathbb{W}, E) \\ \oplus \\ H^s(Y, J^-) \end{array} \longrightarrow \begin{array}{c} \mathcal{W}^{s-\nu,\gamma-\mu}(\mathbb{W}, F) \\ \oplus \\ H^{s-\nu}(Y, J^+) \end{array} .$$

All results here have immediate generalizations to this situation. Details will be dropped for brevity. However non-trivial bundles J^+, J^- are interesting even when the left upper corners are scalar elliptic operators. This is known already from the more special elliptic pseudo-differential boundary value problems. Therefore we shall formulate things from now on for non-trivial J^+, J^-. Every $\mathcal{A} \in \mathcal{Y}^\nu(\mathbb{W}, \underline{g}; J^-, J^+)$ allows two leading symbols, namely first

$$\sigma_\psi^\nu(\mathcal{A}) \in C^\infty\big(T^*(\mathrm{int}\, \mathbb{W}) \setminus 0\big) , \tag{5.26}$$

which is homogeneous of order ν in the covariables. It coincides by definition with $\sigma_\psi^\nu(\text{l.u.c. } \mathcal{A})$ from the left upper corner (abbreviated by l.u.c.). The second one is the homogeneous edge symbol of order ν

$$\sigma_\wedge^\nu(\mathcal{A}) : \pi_Y^* \begin{pmatrix} \mathcal{K}^{s,\gamma}(X^\wedge) \\ \oplus \\ J^- \end{pmatrix} \longrightarrow \pi_Y^* \begin{pmatrix} \mathcal{K}^{s-\nu,\gamma-\mu}(X^\wedge) \\ \oplus \\ J^+ \end{pmatrix} , \tag{5.27}$$

$s \in \mathbb{R}$. Here $\pi_Y : T^*Y \setminus 0 \to Y$ is the canonical projection, and π_Y^* indicates the pull-back under π_Y. The homogeneity means

$$\sigma_\wedge^\nu(\mathcal{A})(y, \lambda\eta) = \lambda^\nu \begin{pmatrix} \kappa_\lambda & 0 \\ 0 & 1 \end{pmatrix} \sigma_\wedge^\nu(\mathcal{A})(y,\eta) \begin{pmatrix} \kappa_\lambda & 0 \\ 0 & 1 \end{pmatrix}^{-1} , \tag{5.28}$$

where the identities in the right lower corners refer to the fibers in the bundles J^- und J^+, respectively. If $J^\mp$ are trivial, of fibre dimension $N_\mp$ then $\sigma_\wedge^\nu(\mathcal{A})$ can also be interpreted as an operator-valued function

$$\sigma_\wedge^\nu(\mathcal{A}) \in C^\infty(T^*Y \setminus 0, \bigcap_s \mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge) \oplus \mathbb{C}^{N_-}, \mathcal{K}^{s-\nu,\gamma-\mu}(X^\wedge) \oplus \mathbb{C}^{N_+})) .$$

satisfying (5.28), cf. also (5.21).

Theorem 5.11 *$\mathcal{A} \in \mathcal{Y}^\nu(\mathbb{W}, \underline{g}; J^-, J^+)$ for $\underline{g} = (\gamma, \gamma - \mu, \Theta)$, $\Theta = (-k, 0]$, $\mu - \nu < k$, and $\sigma_\psi^\nu(\mathcal{A}) = 0$, $\sigma_\wedge^\nu(\mathcal{A}) = 0$ imply that (5.24) is a compact operator for all $s \in \mathbb{R}$.*

Theorem 5.12 *Let $\mu, \rho, \nu, \kappa \in \mathbb{R}$, $\Theta = (-k, 0]$, $k \in \mathbb{N} \setminus \{0\}$, and*

$$\begin{aligned} \mathcal{A} &\in \mathcal{Y}^\nu(\mathbb{W}, \underline{g}; G, J^+) , & \underline{g} &= (\gamma - \rho, \gamma - (\mu + \rho), \Theta) , \\ \mathcal{B} &\in \mathcal{Y}^\kappa(\mathbb{W}, \underline{h}; J^-, G) , & \underline{h} &= (\gamma, \gamma - \rho, \Theta) , \end{aligned}$$

for vector bundles G, J^-, J^+ over Y. Then $\mathcal{AB} \in \mathcal{Y}^{\nu+\kappa}(\mathbb{W}, \underline{m}; J^-, J^+)$ with $\underline{m} = (\gamma, \gamma - (\mu + \rho), \Theta)$, and

$$\sigma_\psi^{\nu+\kappa}(\mathcal{AB}) = \sigma_\psi^\nu(\mathcal{A})\sigma_\psi^\kappa(\mathcal{B}) , \tag{5.29}$$

$$\sigma_\wedge^{\nu+\kappa}(\mathcal{AB}) = \sigma_\wedge^\nu(\mathcal{A})\sigma_\wedge^\kappa(\mathcal{B}) . \tag{5.30}$$

Let

$$\mathcal{Y}_{M+G}^\nu(\mathbb{W}, \underline{g}; J^-, J^+) \tag{5.31}$$

be the subspace of all $\mathcal{A} \in \mathcal{Y}^\nu(\mathbb{W}, \underline{g}; J^-, J^+)$ for which the operator $\mathcal{P}$ in the representation (5.22) vanishes as well as $a(y,\eta)$ in the local description of the symbols $r_{11}(y,\eta)$. If in addition all $m(y,\eta)$ vanish we get by definition the subspace

$$\mathcal{Y}_G^\nu(\mathbb{W}, \underline{g}; J^-, J^+) . \tag{5.32}$$

The operators in (5.31) are called smoothing Mellin+Green, those in (5.32) Green operators.

Remark 5.13 *If $\mathcal{A}$ or $\mathcal{B}$ in Theorem* 5.12 *belongs to the operator space with subscript $M+G$ (G) then also the composition $\mathcal{AB}$.*

Remark 5.14 *The edge pseudo-differential problems $\mathcal{A} \in \mathcal{Y}^{\nu}(\mathbb{W}, \underline{g}; J^-, J^+)$ allow also a formal adjoint $\mathcal{A}^* \in \mathcal{Y}^{\nu}(\mathbb{W}, \underline{g}^*; J^-, J^+)$ in an obvious manner, with $\underline{g}^* = (-\gamma + \mu, -\gamma, \Theta)$ for $\underline{g} = (\gamma, \gamma - \mu, \Theta)$, and there is a natural symbolic rule under passing to the formal adjoint. This will not be used in the sequel, so the details are dropped. We shall denote by*

$$Y^{\nu}(\mathbb{W}, \underline{g}) \ , \quad Y^{\nu}_{M+G}(\mathbb{W}, \underline{g}) \ , \quad Y^{\nu}_{G}(\mathbb{W}, \underline{g}) \tag{5.33}$$

the subclasses of elements of (5.23), (5.31) *and* (5.32), *respectively, for which the fibre dimension of J^- and J^+ are zero. In other words* (5.32) *just consist of the left upper corner of the corresponding spaces of operator matrices.*

6. Ellipticity and parametrices

Definition 6.1 *An operator $\mathcal{A} \in \mathcal{Y}^{\mu}(\mathbb{W}, \underline{g}; J^-, J^+)$ for $\underline{g}(\gamma, \gamma - \mu, \Theta)$, $\Theta = (-k, 0]$, is called elliptic if*

(i) *$\sigma^{\mu}_{\psi}(\mathcal{A}) \neq 0$ on $T^*(\operatorname{int} \mathbb{W}) \setminus 0$ and $t^{\mu}\sigma^{\mu}_{\psi}(\mathcal{A})(t, x, y, t^{-1}\tau, \xi, t^{-1}\eta) \neq 0$ in a collar neighbourhood of $\partial\mathbb{W}$ in the coordinates (t, x, y), including $t = 0$, and for all $(\tau, \xi, \eta) \neq 0$,*

(ii)

$$\sigma^{\mu}_{\wedge}(\mathcal{A})(y, \eta) : \begin{array}{c} \mathcal{K}^{s,\gamma}(X^{\wedge}) \\ \oplus \\ J^-_y \end{array} \longrightarrow \begin{array}{c} \mathcal{K}^{s-\mu,\gamma-\mu}(X^{\wedge}) \\ \oplus \\ J^+_y \end{array} \tag{6.1}$$

is an isomorphism for all $y \in Y$, $\eta \neq 0$, for some fixed $s = s_0 \in \mathbb{R}$ (sub y indicates the fibre of the corresponding bundle over y).

Remark 6.2 *From the theory of pseudo-differential operators on the infinite open stretched cone $X^{\wedge} = \mathbb{R}_+ \times X$ it is known that the condition* (ii) *of Definition* 6.1 *is satisfied for all $s \in \mathbb{R}$ as soon as it holds for a particular $s_0 \in \mathbb{R}$.*

Remark 6.3 *The condition* (i) *of Definition* 6.1 *may be regarded as the interior ellipticity of $\mathcal{A}$. It is the ellipticity of the left upper corner in* int $\mathbb{W}$ *in the sense of edge-degenerate symbols and independent of the weight γ. The condition* (ii) *is an analogue of the Shapiro-Lopatinski condition in boundary value problems. In the edge case it depends on γ. The index of $\sigma^{\mu}_{\wedge}(A)(y, \eta)$ for $A =$* l.u.c. *$\mathcal{A}$ depends on γ, and it may happen that for a particular γ_0 the operator A cannot be completed to a matrix $\mathcal{A}$ for which* (ii) *is satisfied.*

If we write

$$\mathcal{A} = \begin{pmatrix} A & C \\ B & R \end{pmatrix} , \tag{6.2}$$

then $A = \text{l.u.c.}\ \mathcal{A}$ is the "interior" pseudo-differential operator of the "edge problem" $\mathcal{A}$, up to some smoothing Mellin and Green operator. B may be regarded as a trace, C as a potential operator with respect to the edge Y. The operator R is pseudo-differential on Y. It may happen that C, R or B, R vanish in an elliptic edge problem to A.
Note that in contrast to the case of boundary value problems (i.e. when the model cone of the wedge equals $\mathbb{R}_+$) the ellipticity of $\mathcal{A}$ for an edge-degenerate differential operator A does require in general both trace and potential conditions. In other words

$$\sigma_\wedge^\mu(A)(y,\eta) : \ \mathcal{K}^{s,\gamma}(X^\wedge) \longrightarrow \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \tag{6.3}$$

will be a family of Fredholm operators with non-trivial kernels and cokernels (of γ-dependent dimension). For every fixed y, η the operator (6.3) belongs to the cone algebra, i.e. to the algebra of pseudo-differential operators on the (open infinite stretched) cone $X^\wedge$. Hence it has a leading conormal symbol which is an operator family (η-independent)

$$\sigma_M^\mu \sigma_\wedge^\mu(A)(y,z) : \ H^s(X) \longrightarrow H^{s-\mu}(X) \tag{6.4}$$

for $y \in Y$, $z \in \Gamma_{\frac{n+1}{2}-\gamma}$ ($n = \dim X$). The condition (ii) of Definition 6.1 implies that (6.4) is a family of isomorphisms for $|\mathrm{Im}\, z| > c$ with $c > 0$ sufficiently large, for all $y \in Y$. This is true for all $\mathcal{A} \in \mathcal{Y}^\mu(\mathbb{W}, \underline{g}; J^-, J^+)$, $A = \text{l.u.c.}\ \mathcal{A}$. In addition (6.4) is a meromorphic family of Fredholm operators. Then there is only a discrete set $D(y) \subset \mathbb{C}$ of exeptional values of z, where (6.4) is no isomorphism.

Definition 6.4 *Let* $\mathcal{A} \in \mathcal{Y}^\mu(\mathbb{W}, \underline{g}; J^-, J^+)$ *with* $\underline{g} = (\gamma, \gamma - \mu, \Theta)$ *and* $\mathcal{P} \in \mathcal{Y}^{-\mu}(\mathbb{W}, \underline{g}^{-1}; J^+, J^-)$ *with* $\underline{g}^{-1} = (\gamma - \mu, \gamma, \Theta)$. *Then* $\mathcal{P}$ *is called a parametrix of* $\mathcal{A}$ *if*

$$\mathcal{AP} - 1 \ \in \ \mathcal{Y}_G^{-\infty}(\mathbb{W}, \underline{h}_r; J^+, J^+) , \tag{6.5}$$

$$\mathcal{PA} - 1 \ \in \ \mathcal{Y}_G^{-\infty}(\mathbb{W}, \underline{h}_l; J^-, J^-) , \tag{6.6}$$

for $\underline{h}_r = (\gamma - \mu, \gamma - \mu, \Theta)$, $\underline{h}_l = (\gamma, \gamma, \Theta)$.

The definition of a parametrix can be weakend in many ways. The present one is reasonable for the purpose here. In the elliptic case we shall impose a technical assumption which is not necessary for the Fredholm property of the operator (6.8) below, but simplifies things considerably, namely that

$$\left(\sigma_M^\mu \sigma_\wedge^\mu(A)(y,z)\right)^{-1} \in C^\infty\left(Y, M_R^{-\mu}(X)\right) \tag{6.7}$$

for an y-independent asymptotic type R. Then, in particular, the set $D(y)$ does not depend on y.

Theorem 6.5 *Let* $\mathcal{A} \in \mathcal{Y}^{\mu}(\mathbb{W}, \underline{g}; J^-, J^+)$ *for* $\underline{g} = (\gamma, \gamma - \mu, \Theta)$, $\gamma, \mu \in \mathbb{R}$, $\Theta = (-k, 0]$, $k \in \mathbb{N} \setminus \{0\}$. *Then, the following conditions are equivalent*

(i) $\mathcal{A}$ *is elliptic,*

(ii) *the operator*

$$\mathcal{A}: \ \mathcal{W}^{s,\gamma}(\mathbb{W}) \oplus H^s(Y, J^-) \longrightarrow \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W}) \oplus H^{s-\mu}(Y, J^+) \qquad (6.8)$$

is Fredholm for an $s = s_0 \in \mathbb{R}$.

In that case (6.8) *is a Fredholm operator for all* $s \in \mathbb{R}$. *Under the condition* (6.7) *there is a parametrix* $\mathcal{P} \in \mathcal{Y}^{-\mu}(\mathbb{W}, \underline{g}^{-1}; J^+, J^-)$.

Remark 6.6 *If* $A \in Y^{\mu}(\mathbb{W}, \underline{g})$ (*cf.* (5.32)) *satisfies the condition* (i) *of Definition* 6.1 *and if* (6.7) *holds, then there exists a* $P \in Y^{-\mu}(\mathbb{W}, \underline{g}^{-1})$ *with*

$$AP - 1 \in Y^0_{M+G}(\mathbb{W}, \underline{h}_r) \ , \qquad PA - 1 \in Y^0_{M+G}(\mathbb{W}, \underline{h}_l) \ ,$$

cf. the notations of Definition 6.4.

Theorem 6.7 *Let* $\mathcal{A} \in \mathcal{Y}^{\mu}(\mathbb{W}, \underline{g}; J^-, J^+)$ *with* $\underline{g} = (\gamma, \gamma - \mu, \Theta)$ *be elliptic. Then,*

$$\mathcal{A}u \in \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W}) \oplus H^{s-\mu}(Y, J^+)$$

for a fixed $s \in \mathbb{R}$ *and*

$$u \in \mathcal{W}^{-\infty,\gamma}(\mathbb{W}) \oplus H^{-\infty}(Y, J^-) \qquad (6.9)$$

implies $u \in \mathcal{W}^{s,\gamma}(\mathbb{W}) \oplus H^s(Y, J^-)$. *Moreover*

$$\mathcal{A}u \in \mathcal{W}_P^{s-\mu,\gamma-\mu}(\mathbb{W}) \oplus H^{s-\mu}(Y, J^+)$$

for an asymptotic type P *to* $(\gamma - \mu, \Theta)$ *and* (6.7) *imply*

$$u \in \mathcal{W}_Q^{s,\gamma}(\mathbb{W}) \oplus H^s(Y, J^-)$$

with an asymptotic type Q *to* (γ, Θ).

Remark 6.8 *There can be defined more general classes of pseudo-differential edge problems that allow* y*-dependent discrete asymptotic types in the Green and Mellin operators. In the case of boundary value problems this was done in Schulze* [S9]. *The non-trivial edge case is completely analogous. In particular, the weighted Sobolev spaces do allow* y*-dependent discrete (in general branching) asymptotic types. Then the parametrix construction is possible in the larger class of edge problems and the elliptic regularity with asymptotics holds without the condition* (6.7). *Details will be published in forthcoming papers.*

Remark 6.9 *There is another generalization of* $\mathcal{Y}^{\nu}(\mathbb{W},\ldots)$ *in the sense of so-called continuous asymptotics, cf. analogously Schulze* [S4], [S7]. *The notion of ellipticity does not refer to that nature of asymptotics. However an elliptic operator* $\mathcal{A} \in \mathcal{Y}^{\mu}(\mathbb{W},\ldots)$ *in the present sense has always a parametrix in the sense of the class* $\mathcal{Y}^{-\mu}(\mathbb{W},\ldots)$ *with continuous asymptotics, without the condition* (6.7).

Theorem 6.10 *Let A be an edge-degenerate pseudo-differential operator on* $\mathbb{W}$ *of order* μ, *A being of the type of a left upper corner in* $\mathcal{Y}^{\mu}(\ldots)$. *Let A be elliptic in the sense of condition* (i) *of Definition* 6.1. *Further assume that there is an elliptic operator* $\mathcal{A} \in \mathcal{Y}^{\mu}(\mathbb{W},\underline{g};J^-,J^+)$ *with* $A = \text{l.u.c.}\ \mathcal{A}$. *Then*

$$A:\ \mathcal{W}^{s,\gamma}(\mathbb{W}) \longrightarrow \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W})$$

has closed image in the space $\mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W})$ *for every* $s \in \mathbb{R}$.

7. Remarks

Our operator algebra of Definition 5.10 concerns the case of a manifold with edges without boundary. It is natural to ask the same things in the case of boundary value problems, where the base X of the model cone is a smooth compact manifold with boundary. There is no canonical choice for an analogue of the pseudo-differential operators in that case. However for constructing a "minimal algebra" containing the parametrices of elliptic differential boundary value problems (for elliptic edge-degenerate operators) one may choose Boutet de Monvel's algebra on X. This program is planned to be carried out in a series of papers jointly with Schrohe, starting with [S2]. On the other hand it is also very natural to start with the algebras

$$\mathcal{Y}^{\nu}(X,\underline{g};J^-,J^+)\ , \tag{7.1}$$

here with respect to X, regarded as a manifold with edge ∂X, where the model cone is trivial. Then it would be necessary first to pass to the conification of (7.1), i.e. to the cone algebra on $X^{\wedge} = \mathbb{R}_+ \times X$, relative to (7.1), and then to perform the edgification, again, analogously to the above constructions for the wedge without boundary. This is to be done in future and not so easy as in Boutet de Monvel's case. This theory does contain near the edges three singular directions $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$ and hence three times Mellin operator constructions within the subordinated operator structures. The papers of Schulze [S6] and Dorschfeldt, Schulze [D1] are devoted to the necessary steps for repeated conifications and edgifications of given pseudo-differential operator algebras from lower singularity orders to higher ones. The general program will be to establish an axiomatic approach for reaching the adequate calculus on arbitrary stratified spaces (e.g. polyhedra). This requires always to formulate the corresponding parameter-dependent variants of every algebra that is already constructed. The additional parameters are then used as the further covariables in the next cone axis direction or the next edge. In Behm [B2] there will be obtained the parameter-dependent edge theory as the necessary extension of the present edge algebra. Like in "ordinary" boundary value problems where elliptic trace (and potential) conditions are natural in the concept of ellipticity, also the algebras for higher singularities will contain subalgebras related to the lower-dimensional skeletons of the given piece-wise smooth configuration. Those skeletons are by no means C manifolds but branched spaces. This shows that it was adequate from the very beginning to allow more general cone bases than spheres. It should finally be mentioned that whenever some singularity is of the type of an edge, in general being locally a Cartesian product between an open set $\Omega \subseteq \mathbb{R}^q$ and a model cone of singular behaviour, i.e. with a polyhedron as base, we have to expect a rather complex behaviour of asymptotics of solutions in the sense of a variable discrete behaviour as it was studied in Schulze [S9].

References

[B1] L. Boutet de Monvel: *Boundary problems for pseudo-differential operators.* Acta Math. 126 (1971) 11-51

[B2] S. Behm: *Pseudo-differential operators with parameters on manifolds with edges.* Doctor thesis Potsdam University 1994

[C1] H. O. Cordes: *A global parametrix for pseudo-differential operators over* $\mathbb{R}^n$ *with applications.* SFB 72 Preprints, University Bonn 1976

[D1] Ch. Dorschfeldt, B.-W. Schulze: *Pseudo-differential operators with operator-valued symbols in the Mellin-edge-approach.* Preprint des Sonderforschungsbereichs 288 "Differentialgeometrie und Quantenphysik", No 69, Berlin 1993 (to appear in Ann. Glob. Anal. and Geom. 1994)

[E1] G. I. Eskin: *Boundary problems for elliptic pseudo-differential equations.* Moscow, Nauka 1973 (Transl. of Mathem. Monographs 52, A.M.S., Providence, Rhode Island 1980)

[E2] Ju. V. Egorov, B.-W. Schulze: *Pseudo-differential operators, singularities, applications.* Birkhäuser Verlag (to appear)

[H1] T. Hirschmann: *Functional analysis in cone and edge Sobolev spaces.* Ann. Global Anal. and Geometry 8, 2 (1990), 167-192

[H2] L. Hörmander: *The analysis of linear partial differential operators.* Vols. 1-4 Springer-Verlag New York 1983/85

[K1] V. A. Kondrat'ev: *Boundary value problems for elliptic equations in domains with conical points.* Trudy Mosk. Mat. Obšč. 16 (1967), 209-292 (Translations Moscow Math. Soc. (1967), 227-313)

[R1] S. Rempel, B.-W. Schulze: *Index theory of elliptic boundary problems.* Akademie-Verlag Berlin 1982

[R2] S. Rempel, B.-W. Schulze: *A theory of pseudo-differential boundary value problems with discontinuous conditions I,II.* Ann. Global Anal. and Geometry 2, 2 (1984), 163-251, 2, 3 (1984), 289-384

[R3] S. Rempel, B.-W. Schulze: *Asymptotics for elliptic mixed boundary problems (pseudo-differential and Mellin operators in spaces with conormal singularity).* Math. Research 50, Akademie-Verlag Berlin 1989

[S1] E. Schrohe: *Spaces of weighted symbols and weighted Sobolev spaces on manifolds.* Springer Lecture Notes in Math. 1256, Springer-Verlag 1987, 360-377

[S2] E. Schrohe, B.-W. Schulze: *Boundary value problems in Boutet de Monvel's algebra for manifolds with conical singularities I.* (to appear in Advances in Partial Differential Equations, Akademie-Verlag Berlin 1994)

[S3] B.-W. Schulze: *Regularity with continuous and branching asymptotics for elliptic operators on manifolds with edges.* Integral Equ. and Operator Theory 11 (1988) 557-602

[S4] B.-W. Schulze: *Pseudo-differential operators on manifolds with edges.* Symp. "Partial Diff. Equ." Holzhau 1988, Teubner Texte zur Mathematik 112, BSB B. G. Teubner Verlagsgesellschaft Leipzig 1989, 259-288

[S5] B.-W. Schulze: *Mellin representations of pseudo-differential operators on manifolds with corners.* Ann. Global Anal. and Geometry 8, 3 (1990), 261-297

[S6] B.-W. Schulze: *The Mellin pseudo-differential calculus on manifolds with corners.* Symp. "Analysis on Manifolds with Singularities" Breitenbrunn 1990, Teubner Texte zur Mathematik 131, Teubner-Verlag Stuttgart-Leipzig 1992, 208-289

[S7] B.-W. Schulze: *Pseudo-differential operators on manifolds with singularities.* North-Holland Amsterdam 1991

[S8] B.-W. Schulze: *Pseudo-differential boundary value problems, conical singularities and asymptotics.* Akademie-Verlag Berlin 1994 (to appear as a monograph consisting of the material of the Reports "Pseudo-differential operators and asymptotics on manifolds with corners" of the Karl-Weierstraß-Institut für Mathematik Berlin I, 07 (1989), II, 02 (1990), III, 04 (1990), IV, 06 (1990), VI, 08 (1990), VIII, 01 (1991), VIII, 02 (1991), IX, 03 (1991))

[S9] B.-W. Schulze: *Pseudo-differential operators and asymptotics on manifolds with corners, Parts XII, XIII.* SFB 256 Preprints 214, 220 Bonn 1992 (to appear in Advances in Partial Differential Equations, Akademie-Verlag Berlin 1994, 1995)

[S10] B. Ju. Sternin: *Topological aspects of a problem of S. L. Sobolev.* (Russian), Izd-vo Mosk. instituta elektronogo mašinostrojenija Moscow 1971

[V1] M. I. Višik, G. I. Eskin: *Convolution equations in a bounded region.* Uspechi Mat. Nauk 20, 3 (1965), 89-152