Birationally rigid varieties with a pencil of Fano double covers. III

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We complete the study of birational geometry of Fano fiber spaces $\pi: V \to \mathbb{P}^1$, the fiber of which is a Fano double hypersurface of index 1. For each family of these varieties we either prove birational rigidity or produce explicitly non-trivial structures of Fano fiber spaces. A new linear method of studying movable systems on Fano fiber spaces $V/\mathbb{P}^1$ is developed.

Bibliography: 17 titles.

CONTENTS

Introduction

The present paper is the concluding one in the series of papers investigating birational geometry of fiber spaces $\pi: V \to \mathbb{P}^1$, the fiber of which is a double Fano hypersurface of index 1. For almost all families (except for a finite set) birational rigidity was proved in [1,2]. The remaining families are considered in this paper.

0.1. The list of varieties under consideration. Recall [2] that the parameters of a family of fiber spaces $V/\mathbb{P}^1$ are written down in the following format:

$$((a_1, \dots, a_{M+1}), (a_Q, a_W))$$

(respectively, $((a_1, \dots, a_M), a_W)$, if the pencils of Fano double spaces are considered), where

$$\sigma: V \to Q \subset X = \mathbb{P}(\mathcal{E})$$

is a double cover, $\mathcal{E} = \oplus \mathcal{O}_{\mathbb{P}^1}(a_i)$, $a_0 = 0 \leq a_1 \leq \dots \leq a_{M+1}$, the morphism σ is ramified over the smooth divisor $W_Q = W \cap Q$,

$$Q \sim mL_X + a_Q R, \ W \sim 2lL_X + 2a_W R,$$

$a_Q, a_W \in \mathbb{Z}_+$, R is the class of a fiber of the projection $\pi_X: X \to \mathbb{P}^1$, whereas $L_X \in \operatorname{Pic} X$ is the class of the tautological sheaf of the fibration $X/\mathbb{P}^1$. For compactness of notations in the set $(a_1, \dots, a_{M+1})$ we write down only non-zero entries, if there are any, otherwise we write (0). Here is the list of families that were excluded from consideration in [1,2]:

1. $((0), (1, 0))$;
2. $((0), (0, 1))$;
3. $((0), (0, 0))$;
4. $((1), (1, 0))$;
5. $((1), (0, 0))$;
6. $((1, 1), (1, 0))$;
7. $((1, 1), (0, 0))$.

Besides, in [1,2] the following families of varieties with a pencil of double spaces were excluded from consideration:

1^*. $((0), 1)$;
2^*. $((0), 0)$;
3^*. $((1), 0)$;
4^*. $((1, 1), 0)$.

In Sec. 1 of this paper we prove birational superrigidity of general varieties in the families $6, 7$ and 4^*. For varieties in the families $1 - 3, 5$ and $1^* - 3^*$ we construct infinitely many non-trivial structures of Fano fiber spaces, which excludes birational rigidity. This solves the problem of birational rigidity for all families except for the family 4. Those varieties will be considered in a separate paper.

0.2. Further results. To prove certain crucial facts in [1,2] we used the modern techniques to the very limit. This is true, in the first place, for the proof of Proposition 3.1 in [1]. Thus it is natural to try to improve certain technical parts of the arguments. In Sec. 2 we discuss how the local inequality, on which the proof of Proposition 3.1 in [1] is based, could be improved. A stronger version of that inequality is formulated as a conjecture and completely proved in the case when the graph of the corresponding sequence of blow ups is a chain.

In Sec. 3 we develop a new method of proving birational rigidity of Fano fiber spaces $V/\mathbb{P}^1$ (more exactly, a new method of studying movable linear systems on these fiber spaces), that is, the *linear method.* The name emphasizes its being different from the traditional *quadratic* method, based on the operation of taking the self-intersection of a movable linear system Σ, that is, the operation of making the effective algebraic cycle $Z = (D_1 \circ D_2)$ of codimension 2, where $D_1, D_2 \in \Sigma$ are general divisors. Naturally, all constructions involved in the quadratic method are quadratic in the parameters of the system Σ (e.g. the $4n^2$-inequality and other crucial facts). On the contrary, the linear method is based on the operation of restricting the movable linear system Σ on a certain fiber $F = F_t = \pi^{-1}(t)$ of the fiber space $V/\mathbb{P}^1$. It is clear that the principal computations of the linear method are linear in the parameters of the system Σ (degrees, multiplicities, etc.). The linear method was for the first time successfully applied in [3]. Its idea could be illustrated in the following way.

Definition 0.1. (see [3]). We say that a primitive Fano variety F is *divisorially canonical*, or satisfies the condition (C) (respectively, is *divisorially log canonical*, or satisfies the condition (L)), if for any effective divisor $D \in |-nK_F|$, $n \geq 1$, the pair

$$(F, \frac{1}{n} D) \tag{1}$$

has canonical (respectively, log canonical) singularities. If the pair (1) has canonical singularities for a general divisor $D \in \Sigma \subset |-nK_F|$ of any *movable* linear system Σ, then we say that F satisfies the condition of *movable canonicity*, or the condition (M).

Let $V/\mathbb{P}^1$ be a Fano fiber space, $\Sigma \subset |-nK_V + lF|$ a movable linear system on V, which is not made from the pencil of fibers of the projection π, that is, $n \geq 1$. Assume that the log pair $(V, \frac{1}{n}\Sigma)$ is not canonical, that is, for some exceptional divisor $E \subset \widetilde{V}$, where $\varphi \colon \widetilde{V} \to V$ is a sequence of blow ups, the Noether-Fano inequality

$$\nu_E(\Sigma) > na(E)$$

is satisfied. Assume that the centre $B = \varphi(E) \subset V$ lies in some fiber $F = F_t$ and is of codimension ≥ 3 on V (this case is the hardest one in all particular problems). Since the linear system Σ is movable, its restriction $\Sigma_F = \Sigma\,|\,_F$ is a non-empty linear system (which may have fixed components). According to the inversion of adjunction [4], the log pair

$$(F, \frac{1}{n}\Sigma_F)$$

is not log canonical. Suppose that we knew from the start that the fibers F satisfy the condition (L) (the more so, the condition (C)). Now we get a contradiction excluding the maximal singularity E.

This is the general outline of the scheme of the linear method. In order to realize it, one needs to be able to prove the condition (L), which in most cases is very hard. The only known way to do it was shown in [3]. It requires essentially stronger conditions of general position (regularity) than those used by the quadratic method [5]. However, the linear method has certain advantages compared to the quadratic one, namely, it absolutely does not use the condition of "twistedness" of the fiber space $V/\mathbb{P}^1$ over the base. The only information which is used is the properties of fibers.

In Sec. 3 by means of the linear method we prove birational superrigidity of the fiber spaces $V/\mathbb{P}^1$ into double spaces, double quadrics and Fano hypersurfaces of index 1.

0.3. Acknowledgements. A considerable part of results of the present paper (in particular, the computations in Sec. 2 and the key parts of the linear method) was obtained by the author during his stay at Max-Planck-Institut für Mathematik in Bonn in the summer and autumn of 2003. The author is grateful to the institute for the hospitality and the excellent conditions of work.

1 Birationally rigid and non-rigid Fano fiber spaces

1.1. Varieties of the type $((1,1),(1,0))$. Here $X=\mathbb{P}(\mathcal{E})$, where the sheaf $\mathcal{E}$ is of the form $\mathcal{E}=\mathcal{O}_{\mathbb{P}^1}^{\oplus M}\oplus\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$. The space $H^0(X,\mathcal{L}_X\otimes\pi^*\mathcal{O}_{\mathbb{P}^1}(-1))$ is two-dimensional and defines a pencil of divisors $|L_X-R|$. Its base set $\Delta_X=\mathrm{Bs}\,|L_X-R|$ is of codimension 2: it is easy to see that

$$\Delta_X=\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus M})\cong\mathbb{P}^{M-1}\times\mathbb{P}^1.$$

Furthermore, $Q\sim mL_X+R$ and $W\sim 2lL_X$. Set

$$\Delta_Q=\Delta_X\cap Q,\quad \Delta=\sigma^{-1}(\Delta_Q)\subset V.$$

Obviously, Δ_Q is a smooth divisor of bidegree $(m,1)$ on $\Delta_x=\mathbb{P}^{M-1}\times\mathbb{P}^1$, $\Delta\subset V$ is a smooth irreducible subvariety of codimension 2.

Lemma 1.1. *The anticanonical linear system $|-K_V|$ is movable, and moreover* $\mathrm{Bs}\,|-K_V|=\Delta$. *Furthermore,*

$$-K_V\in\partial A^1_{\mathrm{mov}}V.$$

More precisely, every linear system $|-nK_V+lF|$ is empty for $l<0$.

Proof. The first claim of the lemma follows from the fact that the anticanonical linear system $|-K_V|=|L_V-F|$.

Assume that the linear system $\Sigma=|-nK_V+lF|$ is non-empty for some $l<0$. We must show that this assumption leads to a contradiction. In order to do it, we construct a special family of surfaces sweeping out V. Restricting the linear system Σ onto a general surface of this family, we must get a non-empty linear system of curves. As we will see, the latter is impossible.

Let

$$\alpha\colon \mathcal{O}_{\mathbb{P}^1}^{\oplus M} \to \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \quad \text{and} \quad \beta\colon \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^1}(1)$$

be surjective morphisms of sheaves. Their direct sum $\alpha \oplus \beta$ determines an inclusion of projective bundles

$$S = S(\alpha, \beta) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \hookrightarrow X = \mathbb{P}(\mathcal{E}).$$

The subvariety S is a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$, where the intersection $S \cap \Delta_X \cong \mathbb{P}^1 \times \mathbb{P}^1$ is a *divisor* on S.

Set

$$V_S = \sigma^{-1}(Q \cap S), \quad \Delta_S = \Delta \cap V_S.$$

For sufficiently general α, β the surface V_S is smooth, $\Delta_S \subset V_S$ is a smooth curve. Moreover, the surfaces V_S sweep out the variety V.

Obviously, $\sigma(\Delta_S) = \Delta_Q \cap S \subset \mathbb{P}^1 \times \mathbb{P}^1$ is a curve of bidegree $(m, 1)$. Denote by the symbols L_S and F_S the restrictions of the classes L_V and F onto the surface V_S, respectively.

Lemma 1.2. (i) *The following equivalences hold:*

$$\Delta_S \sim L_S - F_S \sim (-K_V)|_{V_S}.$$

(ii) *The following equality holds:* $(\Delta_S \cdot \Delta_S) = 2(1 - m)$.

Proof. (i) By the construction of the variety S we get:

$$S \cap \Delta_X \sim L_X - R,$$

the rest is obvious.

(ii) By the part (i) we get

$$(\Delta_S \cdot \Delta_S) = (\Delta_S \cdot L_S) - (\Delta_S \cdot F_S).$$

As we noted above, $\sigma(\Delta_S)$ is a curve of bidegree $(m, 1)$ on $S \cap \Delta_X = \mathbb{P}^1 \times \mathbb{P}^1$. It is easy to see that the restriction of the tautological sheaf $L_X|_{S \cap \Delta_X}$ is of bidegree $(1, 0)$. Therefore $(\Delta_S \cdot L_S) = 2$ and $(\Delta_S \cdot F_S) = 2m$. (The intersection indices are doubled because of the double cover σ.) Q.E.D. for the lemma.

Let us complete the proof of Lemma 1.1. Let Σ_S be the restriction of the linear system Σ onto the surface V_S. This is a non-empty linear system of curves. By the previous lemma,

$$\Sigma_S \subset |n\Delta_S + lF_S|,$$

where $n \geq 0$, $l < 0$. Take a general curve $C \in \Sigma_S$ and write down

$$C = m^+ \Delta_S + C^+,$$

where $n^+ \in \mathbb{Z}_+$ and all irreducible components of the curve C^+ are different from Δ_S. Of course, $m^+ \leq n$ (otherwise the class $(-F_S)$ would have been effective). Consequently,

$$C^+ \sim (n - m^+)\Delta_S + lF_S,$$

so that

$$0 \leq (C^+ \cdot \Delta_S) = 2(n - n^+)(1 - m) + 2ml < 0$$

(recall that $l < 0$). This contradiction completes the proof of Lemma 1.1.

Lemma 1.3. *The Fano fiber space $V/\mathbb{P}^1$ satisfies the generalized K^2-condition of depth $1/m$, that is,*

$$K_V^2 - \frac{1}{m} H_F \notin \operatorname{Int} A_+^2 V,$$

where $H_F \in A_{\mathbb{R}}^2 V$ is the class of a hyperplane section of a fiber, that is, $H_F = (-K_V \cdot F)$.

Proof. By the formula (1) of the paper [2], $(K_V^2 \cdot L_V^{M-1}) = 2$. Since for the degree of the fiber we have obviously $(F \cdot L_V^M) = (H_F \cdot L_V^{M-1}) = 2m$, we get

$$((K_V^2 - \frac{1}{m} H_F) \cdot L_V^{M-1}) = 0$$

which immediately implies the claim of the lemma.

Corollary 1.1. *A general Fano fiber space $V/\mathbb{P}^1$ of type $((1,1),(1,0))$ is birationally superrigid. The projection $\pi: V \to \mathbb{P}^1$ gives the only non-trivial structure of a rationally connected fiber space on V. The groups of birational and biregular self-maps of the variety V coincide:*

$$\operatorname{Bir} V = \operatorname{Aut} V = \mathbb{Z}/2\mathbb{Z}$$

Proof: this follows immediately from Theorem 2 of the paper [2] ($1/m < 2$, so that the fiber space $V/\mathbb{P}^1$ satisfies the K^2-condition of depth 2, whereas any movable linear system Σ is a subsystem of the complete linear system $|-nK_V + lF|$ with $n, l \in \mathbb{Z}_+$).

1.2. Varieties of type ((1,1),(0,0)). Birational geometry of varieties of this type is somewhat more complicated than birational geometry of the varieties of type $((1,1),(1,0))$. The projective bundle X and the locally free sheaf $\mathcal{E}$ are the same as in Sec. 1.1. However, in the case under consideration $Q \sim mL_X$, so that $-K_V = L_V$ and thus the linear system

$$|-K_V - F| = \sigma^*(|L_X - R|\Big|_Q)$$

is movable. Consequently, the variety V does not satisfy the K-condition. Let

$$\varphi: V \dashrightarrow \mathbb{P}^1$$

be the rational map, given by the pencil $|-K_V - F|$. Birational geometry of the variety V is completely described by

Proposition 1.1. (i) *The variety V is birationally superrigid: for any movable linear system Σ on V its virtual and actual thresholds of canonical adjunction coincide,*

$$c_{\mathrm{virt}}(\Sigma)=c(\Sigma).$$

(ii) *On the variety V there are exactly two non-trivial structures of a rationally connected fiber space, namely $\pi\colon V\to\mathbb{P}^1$ and $\varphi\colon V\dashrightarrow\mathbb{P}^1$. These structures are birationally distinct, that is, there is no birational self-map $\chi\in\mathrm{Bir}\,V$, transforming the fibers of π into the fibers of φ. The groups of birational and biregular self-maps of the variety V coincide:* $\mathrm{Bir}\,V=\mathrm{Aut}\,V$.

(iii) *There is a unique, up to a fiber-wise isomorphism, Fano fiber space $\pi^+\colon V^+\to\mathbb{P}^1$ of the same type $((1,1),(0,0))$, such that the following diagram commutes:*

$$\begin{array}{ccccc} & V & \stackrel{\chi}{\dashrightarrow} & V^+ & \\ \varphi & \downarrow & & \downarrow & \pi^+\\ & \mathbb{P}^1 & = & \mathbb{P}^1, & \end{array}$$

where χ is a birational map. The correspondence $V\to V^+$ is involutive, that is, $(V^+)^+=V$.

Proof. The symbols Δ_Q, Δ_X and Δ mean the same as above (Sec. 1.1). However, in the case under consideration Δ_Q is a smooth divisor of bidegree $(m,0)$ on $\Delta_X=\mathbb{P}^{M-1}\times\mathbb{P}^1$, that is, $\Delta_Q=\Delta_{\mathbb{P}}\times\mathbb{P}^1$, where $\Delta_{\mathbb{P}}\subset\mathbb{P}^{M-1}$ is a smooth hypersurface of degree m. Similarly, $\Delta=\Delta_F\times\mathbb{P}^1$, where $\Delta_F=\sigma^{-1}(\Delta_{\mathbb{P}})$ is the double cover of the hypersurface $\Delta_{\mathbb{P}}$, branched over $W\cap\Delta_{\mathbb{P}}$.

Lemma 1.4. *The base set of the movable linear system $|-K_V-F|$ is* $\mathrm{Bs}\,|-K_V-F|=\Delta$. *Furthermore,*

$$-K_V-F\in\partial A^1_{\mathrm{mov}}V.$$

More precisely, $|-nK_V+lF|=\emptyset$ for $l<-n$.

Proof is completely similar to the proof of Lemma 1.1. The only difference is that this time the curve $\Delta_S\sim L_S-F_S$ (the surface S is constructed in absolutely the same way as in the proof of Lemma 1.1) is not irreducible. The curve Δ_S is a union of $2m$ disjoint (-1)-curves,

$$\Delta_S=\sum_{i=1}^{m}(C_i^++C_i^-),\quad \Delta_Q=\sum_{i=1}^{m}C_i,\quad \sigma(C_i^{\pm})=C_i.$$

The lines $C_i\subset S\cap\Delta_X=\mathbb{P}^1\times\mathbb{P}^1$ are of bidegree $(1,0)$, they correspond to the m points of intersection $\Delta_{\mathbb{P}}\cap S$. Note that the branch divisor $W\cap\Delta_X\subset\mathbb{P}^{M-1}\times\mathbb{P}^1$ is of bidegree $(2l,0)$, that is,

$$W\cap\Delta_X=W_{\mathbb{P}}\times\mathbb{P}^1,$$

where $W_{\mathbb{P}}\subset\mathbb{P}^{M-1}$ is a general hypersurface of degree $2l$. Thus for a general surface S the lines $C_1,\dots,C_m$ lie outside the branch divisor, so that we get $C_i^+\cap C_i^-=\emptyset$.

Now we prove that the linear system $|aL_V - bF|$ is empty for $b > a$ in exactly the same way as in the proof of Lemma 1.1 (where the irreducible curve Δ_S is replaced by $2m$ disjoint (-1)-curves $C_i^{\pm}$).

Proof of Lemma 1.4 is complete.

Now let us study the rational map $\varphi: V \dashrightarrow \mathbb{P}^1$. In order to do that, we need an explicit coordinate presentation of the varieties X, Q and W, participating in the construction of the Fano fiber space $V/\mathbb{P}^1$.

Consider the locally free subsheaves

$$\mathcal{E}_0 = \mathcal{O}_{\mathbb{P}^1}^{\oplus M} \hookrightarrow \mathcal{E} \quad \text{and} \quad \mathcal{E}_1 = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \hookrightarrow \mathcal{E}.$$

Obviously, $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$. Let $\Pi_0 \subset H^0(X, \mathcal{L}_X)$ be the subspace, corresponding to the space of sections of the sheaf $H^0(\mathbb{P}^1, \mathcal{E}_0) \hookrightarrow H^0(\mathbb{P}^1, \mathcal{E})$. Set also

$$\Pi_1 = H^0(X, \mathcal{L}_X \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1)) = H^0(\mathbb{P}^1, \mathcal{E}_1(-1)).$$

Let $x_0, \dots, x_{M-1}$ be a basis of the space Π_0, y_0, y_1 a basis of the space Π_1. Then the sections

$$x_0, \dots, x_{M-1}, y_0t_0, y_0t_1, y_1t_0, y_1t_1, \qquad (2)$$

where t_0, t_1 is a system of homogeneous coordinates on $\mathbb{P}^1$, make a basis of the space $H^0(X, \mathcal{L}_X)$. It is easy to see that the complete linear system (2) defines a morphism

$$\xi: X \to \bar{X} \subset \mathbb{P}^{M+3},$$

the image X of which is a quadratic cone with the vertex space $\mathbb{P}^{M-1} = \xi(\Delta_X)$ and a smooth quadric in $\mathbb{P}^3$, isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, as a base. The morphism ξ is birational, more precisely,

$$\xi: X \setminus \Delta_X \to \bar{X} \setminus \xi(\Delta_X)$$

is an isomorphism and ξ contracts $\Delta_X = \mathbb{P}^{M-1} \times \mathbb{P}^1$ onto the vertex space of the cone. Let

$$u_0, \dots, u_{M-1}, u_{00}, u_{01}, u_{10}, u_{11}$$

be the homogeneous coordinates on $\mathbb{P}^{M+3}$, corresponding to the ordered set of sections (2). The cone $\bar{X}$ is given by the equation

$$u_{00}u_{11} = u_{01}u_{10}.$$

On the cone $\bar{X}$ there are two pencils of $(M+1)$-planes, corresponding to the two pencils of lines on a smooth quadric in $\mathbb{P}^3$. Let $\tau \in \operatorname{Aut} \mathbb{P}^{M+3}$ be the automorphism permuting the coordinates u_{01} and u_{10} and not changing the other coordinates. Obviously, $\tau \in \operatorname{Aut} \bar{X}$ is an automorphism of the cone $\bar{X}$, permuting the above-mentioned pencils of $(M+1)$-planes. One of these pencils is the image of the pencil of fibers of the projection π, that is, the pencil $\xi(|R|)$. For the other pencil we get the equality

$$\tau\xi(|R|) = \xi(|L_X - R|).$$

The automorphism τ induces an involutive birational self-map

$$\tau^+ \in \operatorname{Bir} X.$$

More precisely, τ^+ is a biregular automorphism outside a closed subset Δ_X of codimension 2. Let

$$\varepsilon\colon \tilde{X} \to X$$

be the blow up of the smooth subvariety Δ_X. Obviously, the variety $\tilde{X}$ is isomorphic to the blow up of the cone $\bar{X}$ at its vertex space $\xi(\Delta_X)$. It is easy to check that τ^+ extends to a biregular automorphism of the smooth variety $\tilde{X}$.

Set $Q^+ = \tau^+(Q) \subset X$, $W^+ = \tau^+(W) \subset X$. The divisors Q^+ and W^+ are well defined because τ^+ is an isomorphism in codimension 1.

Lemma 1.5. *The divisors Q^+ and W^+ are divisors of general position in the linear systems $|mL_X|$ and $|2lL_X|$, respectively. In particular, Q^+, W^+ and $Q^+\cap W^+$ are smooth varieties.*

Proof. The claim follows immediately from the fact that the linear systems $|kL_X|$, $k \in \mathbb{Z}_+$, are invariant under τ^+, whereas Q and W are sufficiently general divisors of the corresponding linear systems. Note that if a divisor $D \in |kL_X|$ is given by a polynomial

$$h(u_0, \dots, u_{M-1}, u_{00}, u_{01}, u_{10}, u_{11}),$$

of degree k, then its image $\tau^+(D)$ is given by the polynomial

$$h^+(u_*) = h(u_0, \dots, u_{M-1}, u_{00}, u_{10}, u_{01}, u_{11})$$

with permuted coordinates u_{01} and u_{10}. Q.E.D. for the lemma.

Let $\sigma^+\colon V^+ \to Q^+$ be the double cover, branched over a smooth divisor $Q^+\cap W^+$. Obviously, $V^+/\mathbb{P}^1$ is a general Fano fiber space of type $((1,1),(0,0))$.

Lemma 1.6. (i) *The map τ^+ lifts to a birational map $\chi\colon V \dashrightarrow V^+$, biregular in codimension 1.*

(ii) *The action of χ on the Picard group is given by the formulas*

$$\chi^* K_{V^+} = K_V, \quad \chi^* F^+ = -K_V - F,$$

where F^+ is the class of the fiber of the projection $V^+ \to \mathbb{P}^1$, so that $\operatorname{Pic} V^+ = \mathbb{Z}K_{V^+} \oplus \mathbb{Z}F^+$.

(iii) *The construction of the variety V^+ is involutive: $(V^+)^+ \cong V$.*

Proof: the claims (i)-(iii) are obvious. Just note that the following presentation holds: $\chi = q^+ \circ q^{-1}$, where $q\colon \tilde{V} \to V$ and $q^+\colon \tilde{V} \to V^+$ are blow ups of the smooth subvarieties of codimension two $\Delta \subset V$ and $\Delta^+ \subset V^+$, respectively. Furthermore, $E = q^{-1}(\Delta)$ is the exceptional divisor of both blow ups, $E = \Delta \times \mathbb{P}^1 = \Delta_F \times \mathbb{P}^1 \times \mathbb{P}^1$, whereas the projections $q\,|_E$ and $q^+\,|_E$ are projections with respect to the second and third direct factors, respectively.

Lemma 1.7. *The Fano fiber space $V/\mathbb{P}^1$ satisfies the generalized K^2-condition of depth 2.*

Proof. This immediately follows from the equality

$$((K_V^2 - 2H_F) \cdot L_V^{M-1}) = 0.$$

Q.E.D. for the lemma.

Finally, let us prove Proposition 1.1. Let $\Sigma \subset |-nK_{V^+} + lF|$ be a movable linear system. If $l \in \mathbb{Z}_+$, then by Theorem 2 of the paper [2] we get the desired coincidence of the thresholds: $c_{\rm virt}(\Sigma) = c(\Sigma)$. Assume that $l < 0$. Consider the linear system $\Sigma^+ = \tau^+(\Sigma)$ on V^+. By Lemma 1.6, $\Sigma^+ \subset | - n_+ K_{V^+} + l_+ F^+|$, where

$$l_+ = -l \geq 1.$$

Since τ^+ is an isomorphism in codimension 1, we get $c(\Sigma) = c(\Sigma^+)$. Again applying Theorem 2 of the paper [2], we obtain the desired coincidence of thresholds

$$c_{\rm virt}(\Sigma^+) = c_{\rm virt}(\Sigma) = c(\Sigma^+) = c(\Sigma) = n_+ = n + l.$$

This proves birational superrigidity.

Let us prove the claim (ii). The standard argument (see [2, Sec. 1.1]) shows that on V there are exactly two non-trivial structures of a rationally connected fiber space (the arguments above imply that if a movable linear system Σ satisfies the equality $c_{\rm virt}(\Sigma) = 0$, then either Σ is made from the pencil $|F|$, or Σ is made from the pencil $| - K_V - F|$, which gives a description of the existing structures). For a general variety V these structures cannot be birationally equivalent. Indeed, by birational superrigidity of Fano double hypersurfaces of index 1, any birational map $\chi^+ \in \operatorname{Bir} V$, which transforms the pencil $|F|$ into the pencil $| - K_V - F|$, induces a biregular isomorphism of the fibers of general position in the pencils $|F|$ and $|F^+|$ (the latter is taken on the variety V^+). Therefore, χ^+ induces a biregular isomorphism of the fibers of general position of the fiber spaces $Q/\mathbb{P}^1$ and $Q^+/\mathbb{P}^1$. Now from [6] for $m \geq 3$ we get that these fiber spaces are globally fiber-wise isomorphic. It checks easily that for a sufficiently general divisor $Q \subset X$ this is impossible. For $m = 2$ we argue in a similar way, using the branch divisor W.

Finally, the claim (iii) follows from the arguments above.

Q.E.D. for Proposition 1.1.

1.3. Varieties of type ((0),(1,0)). Here $X = \mathbb{P} \times \mathbb{P}^1$, the hypersurface $Q \subset X$ is of bidegree $(m, 1)$, the hypersurface $W \subset X$ is of bidegree $(2l, 0)$, that is, $W = W_{\mathbb{P}} \times \mathbb{P}^1$, where $W_{\mathbb{P}} \subset P$ is a hypersurface of degree $2l$. Let $(u : v)$ be homogeneous coordinates on $\mathbb{P}^1$, $(x_0 : \ldots : x_{M+1})$ be homogeneous coordinates on $\mathbb{P}$. The hypersurface Q is given by the equation

$$uf_+ + vf_- = 0,$$

where $f_\pm(x_0, \ldots, x_{M+1})$ are homogeneous polynomials of degree m, and the hypersurface W is given by the equation $h(x_*) = 0$, $\deg h = 2l$.

Let $\beta_{\mathbb{P}}: Y_{\mathbb{P}} \to \mathbb{P}$ (respectively, $\beta_X: Y_X \to X$) be the double space, branched over $W_{\mathbb{P}}$ (respectively, the double cover, branched over W). Obviously, the direct product

$X = \mathbb{P} \times \mathbb{P}^1$ generates the direct product $Y_X = Y_{\mathbb{P}} \times \mathbb{P}^1$, compatible with the double covers $\beta_{\mathbb{P}}$, β_X.

Proposition 1.2. *The projection $q\colon Y_X \to Y_{\mathbb{P}}$ onto the first factor determines the birational morphism*

$$q_V = q|_V\colon V \to Y_{\mathbb{P}},$$

contracting the exceptional divisor $E \subset V$:

$$E = q_V^{-1}(\{\beta_{\mathbb{P}}^* f_+ = \beta_{\mathbb{P}}^* f_- = 0\}).$$

The birational morphism q_V transforms the pencil $|F|$ of fibers of the fiber space $V/\mathbb{P}^1$ into the pencil of divisors

$$\{\lambda_+ \beta_{\mathbb{P}}^* f_+ + \lambda_- \beta_{\mathbb{P}}^* f_- = 0 \,|\, \lambda_\pm \in \mathbb{C}\}$$

on $Y_{\mathbb{P}}$. The inverse birational map q_V^{-1} is the blow up of the base set of this pencil.

Proof: this is obvious.

Remark 1.2. The variety $Y_{\mathbb{P}}$ is a Fano variety of index $m+1$. Any structure of a fiber space $\mathbb{P} \dashrightarrow S$ into Fano complete intersections of type $(b_1, \dots, b_e)$ with $b_1 + \ldots + b_e \leq m$ generates a structure of a fiber space $Y_{\mathbb{P}} \dashrightarrow S$ into Fano varieties of index $m - b_1 - \ldots - b_e + 1 \geq 1$. An example of such structure is given by the pencil $(q_V)_*|F|$ described in Proposition 1.2. For this method of constructing non-trivial structures of a fiber space on $Y_{\mathbb{P}}$ the highest dimension of the base corresponds to the case $b_1 = \ldots = b_m = 1$, that is, if we fiber $\mathbb{P}$ into linear subspaces of codimension m. In this case $\dim S = m$. The variety $Y_{\mathbb{P}}$ is certainly not birationally rigid.

Conjecture 1.1. *If the hypersurface $W_{\mathbb{P}} \subset \mathbb{P}$ is sufficiently general, then for any structure of a rationally connected fiber space $Y_{\mathbb{P}} \dashrightarrow S$ the inequality $\dim S \leq m$ holds. If, moreover, $\dim S = m$, then there is a linear projection $\mathbb{P} \dashrightarrow \mathbb{P}^m$ and a birational map $S \dashrightarrow \mathbb{P}^m$ making the following diagram commutative:*

$$\begin{array}{ccc} Y_{\mathbb{P}} & \stackrel{2:1}{\longrightarrow} & \mathbb{P} \\ \downarrow & & \downarrow \\ S & \dashrightarrow & \mathbb{P}^m. \end{array}$$

1.4. Varieties of type ((0),(0,1)). Here $X = \mathbb{P} \times \mathbb{P}^1$ and for the variety $Q \subset X$ we also get the direct decomposition $Q = G \times \mathbb{P}^1 \subset \mathbb{P} \times \mathbb{P}^1$, where $G \subset \mathbb{P}$ is a hypersurface of degree m. Let $q\colon Q \to G$ be the projection onto the second factor, $L_x = q^{-1}(x) \cong \mathbb{P}^1$ the fiber over an arbitrary point $x \in G$. Set $C_x = \sigma^{-1}(L_x) \subset V$. Thus C_x is the fiber of the projection

$$q_V = q \circ \sigma\colon V \to G.$$

The double cover $\sigma\colon C_x \to L_x$ is branched over two points (because $a_W = 1$): the equation of the hypersurface $W \subset X$ is of the form

$$f_{uu} u^2 + 2 f_{uv} uv + f_{vv} v^2 = 0,$$

where $(u:v)$ are homogeneous coordinates on $\mathbb{P}^1$, f_* are homogeneous polynomials of degree $2l$ on $\mathbb{P}$. If the curve C_x is irreducible, then it is a smooth conic; otherwise, C_x is a pair of smooth rational curves, provided that all three polynomials f_{uu}, f_{uv}, f_{vv} do not vanish identically at the point x. What has been just said implies

Proposition 1.3. *The projection q_V realizes V as a conic bundle over the base $G \subset \mathbb{P}$. The discriminant divisor is given by the equation $f_{uv}^2 = f_{uu}f_{vv}$.*

Corollary 1.2. *The Fano fiber space $V/\mathbb{P}^1$ is not birationally rigid. The group of birational self-maps $\mathrm{Bir}\, V \neq \mathrm{Bir}(V/\mathbb{P}^1)$ is infinite.*

Remark 1.3. Since the structure of a Fano fiber space $\pi\colon V \to \mathbb{P}^1$ is not compatible (that is, fiber-wise) with respect to the structure q_V of the conic bundle V/G, the latter also cannot be birationally rigid. This agrees with the fact that V/G does not satisfy the Sarkisov condition:

$$4K_G + D = -4H_G,$$

where D is the discriminant divisor of the fiber space V/G, H_G is the class of a hyperplane section of the hypersurface $G \subset \mathbb{P}$, $K_G = (-M-2+m)H_G$, $D = 4lH_G$.

The group $\mathrm{Bir}(V/G)$ of birational self-maps, preserving the structure of a conic bundle V/G, is very large. Accordingly, on the variety V there are a lot of pencils of rationally connected varieties: it is sufficient to consider all pencils of the form

$$\chi_*|F|, \quad \chi \in \mathrm{Bir}(V/G).$$

However, it is unclear, whether there are birational self-maps on V that are not compatible with the conic bundle V/G.

Conjecture 1.2. *For general G, W on the variety V there is only one structure of a conic bundle, that is, $q_V\colon V \to G$, whereas the group of birational self-maps of the variety V and the group of fiber-wise birational self-maps of the fiber space V/G coincide:*

$$\mathrm{Bir}\, V = \mathrm{Bir}(V/G).$$

1.5. Varieties of type ((0),(0,0)). Obviously, any variety of this type is isomorphic to a direct product $F \times \mathbb{P}^1$, where $\sigma\colon F \to G$ is a Fano double cover of index 1, branched over the divisor $W_G = G \cap W^*$, $W^* \subset \mathbb{P}$ is a hypersurface of degree $2l$. Thus the variety V is not birationally rigid. The morphism of projection onto the direct factor $q\colon V \to F$ defines on V a conic bundle structure. Acting, as above, by fiber-wise birational self-maps $\chi \in \mathrm{Bir}(V/F)$ on the pencil $|F|$, we obtain infinitely many pencils of rationally connected varieties on V. Let $\tau \in \mathrm{Aut}\, V$ be the Galois involution of the double cover $\sigma\colon V \to Q$, so that for general Q, W we have $\mathrm{Aut}\, F = \mathbb{Z}/2\mathbb{Z} = \{\mathrm{id}, \tau\}$.

Conjecture 1.3. *For general G, W_* there is a unique conic bundle structure on V, that is, $q\colon V \to F$. For the group of birational self-maps of the variety V the following presentation holds:*

$$\mathrm{Bir}\, V = \mathrm{Aut}\, F \times \mathrm{Bir}(V/F) = \mathbb{Z}/2\mathbb{Z} \times \mathrm{Bir}(V/F).$$

1.6. Varieties of type ((1),(0,0)). First of all, recall the following well known fact: for $a_1 = \ldots = a_M = 0$, $a_{M+1} = 1$ the variety $X = \mathbb{P}(\mathcal{E})$ is isomorphic to the blow up of $\mathbb{P}^{M+2}$ at a subspace of codimension 2, where the pencil of fibers of the morphism π_X corresponds with respect to this isomorphism to the pencil of hyperplanes in $\mathbb{P}^{M+2}$ containing the centre of the blow up. More precisely, let $P \subset \mathbb{P}^{M+2}$ be a linear subspace of codimension two, $\varphi\colon \widetilde{\mathbb{P}} \to \mathbb{P}^{M+2}$ its blow up, $E \subset \widetilde{\mathbb{P}}$ the exceptional divisor, $E \cong \mathbb{P}^M \times \mathbb{P}^1$, whereas $\varphi\colon E \to P = \mathbb{P}^M$ is just the projection of E onto the first direct factor. Set $\mathcal{L}_P = \varphi^*\mathcal{O}_{\mathbb{P}}(1)$. In accordance with our notations,

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}^{\oplus(M+1)} \oplus \mathcal{O}_{\mathbb{P}^1}(1),$$

where $\mathbb{P}(\mathcal{E})$ is the projectivization of this sheaf, $\mathcal{L}$ is the corresponding tautological sheaf, $E^+ \subset \mathbb{P}(\mathcal{E})$ is the divisor of common zeros of all sections

$$s \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \hookrightarrow H^0(\mathbb{P}(\mathcal{E}), \mathcal{L})$$

(in the sense of the inclusion $\mathcal{O}_{\mathbb{P}^1}(1) \hookrightarrow \mathcal{E}$). Obviously, $E^+ = \mathbb{P}(\mathcal{E}^+)$, where $\mathcal{E}^+ = \mathcal{O}_{\mathbb{P}^1}^{\oplus(M+1)}$, that is, $E^+ \cong \mathbb{P}^M \times \mathbb{P}^1$.

Lemma 1.8. *The varieties $\widetilde{\mathbb{P}}$ and $\mathbb{P}(\mathcal{E})$ are isomorphic. Moreover, the following diagram commutes:*

$$\begin{array}{ccccc} & \widetilde{\mathbb{P}} & \leftrightarrow & \mathbb{P}(\mathcal{E}) & \\ \varphi & \downarrow & & \downarrow & \varphi_{\mathcal{L}} \\ & \mathbb{P}^{M+2} & = & \mathbb{P}^{M+2}, & \end{array}$$

where $\varphi_{\mathcal{L}}$ is the morphism determined by the space of global sections of the tautological sheaf $\mathcal{L}$.

Proof. This is obvious.

Lemma 1.9. *For any integer $e \geq 1$ and any irreducible divisor $R \subset \mathbb{P}(\mathcal{E})$, $R \neq E^+$, $R = (s)$, where $s \in H^0(\mathbb{P}(\mathcal{E}), \mathcal{L}^{\otimes e})$, the image $\varphi_{\mathcal{L}}(R) \subset \mathbb{P}$ is an irreducible hypersurface of degree e.*

Proof. By the arguments above,

$$\mathbb{P}(\mathcal{E}) \setminus E^+ \cong \mathbb{P}^{M+2} \setminus P,$$

where the tautological divisors on $\mathbb{P}(\mathcal{E})$ correspond to hyperplanes on $\mathbb{P}^{M+2}$. A general line $L \subset \mathbb{P}^{M+2}$ does not meet P and thus

$$\deg(\mathcal{L}\,|_{\varphi_{\mathcal{L}}^{-1}(L)}) = 1.$$

In other words, $(R \cdot \varphi_{\mathcal{L}}^{-1}(L)) = (\varphi_{\mathcal{L}}(R) \cdot L) = e$, which is what we need.

By Lemma 1.9, the variety V is birational to the variety $V^\sharp$ which is realized as the double cover $\sigma^\sharp\colon V^\sharp \to Q^\sharp$, $Q^\sharp = \varphi_{\mathcal{L}}(Q) \subset \mathbb{P}^{M+2}$, and $\sigma^\sharp$ is branched (in codimension 1) over the divisor $Q^\sharp \cap W^\sharp$, where $W^\sharp = \varphi_{\mathcal{L}}(W) \subset \mathbb{P}^{M+2}$. By Lemma 1.9, $\deg Q^\sharp = m$, $\deg W^\sharp = 2l$, so that $V^\sharp$ is a Fano double hypersurface of index 2.

Any pencil of hyperplanes in $\mathbb{P}^{M+2}$ determines a pencil of Fano varieties on $V^\sharp$ and thus a pencil of rationally connected divisors on V.

Conjecture 1.4. *Apart from the pencils of rationally connected divisors described above, there are no other structures of a rationally connected fiber space on V. The groups of birational and biregular automorphisms of the variety V coincide:*

$$\operatorname{Bir} V = \operatorname{Aut} V = \mathbb{Z}/2\mathbb{Z}.$$

1.7. Varieties with a pencil of double spaces. Recall that when Fano double spaces of index 1 are considered, the symbol $\mathbb{P}$ denotes the projective space $\mathbb{P}^M$ of one dimension less.

Varieties of type ((0),1). Here $X = \mathbb{P} \times \mathbb{P}^1$ and the branch divisor $W \subset X$ is of bidegree $(2M, 2)$, that is, it is given by the equation

$$f_{uu}u^2 + 2f_{uv}uv + f_{vv}v^2 = 0,$$

where $(u : v)$ are homogeneous coordinates on $\mathbb{P}^1$, $f_\sharp$ are homogeneous polynomials of degree $2M$ on $\mathbb{P}$. This case is completely similar to the one studied above in Sec. 1.4. The projection $q\colon V \to \mathbb{P}$ realizes V as a conic bundle. Its discriminant divisor $D \subset \mathbb{P}$ is a hypersurface of degree $4M$, that is, the Sarkisov condition is not satisfied.

Conjecture 1.5. *On the variety V there are no other structures of a conic bundle, apart from $V/\mathbb{P}$. The groups of birational and fiber-wise birational self-maps coincide:*

$$\operatorname{Bir} V = \operatorname{Bir}(V/\mathbb{P}).$$

Varieties of type ((0),0). Here $X = \mathbb{P} \times \mathbb{P}^1$, $W = W_{\mathbb{P}} \times \mathbb{P}^1$, where $W_{\mathbb{P}} \subset \mathbb{P}$ is a hypersurface of degree $2M$. Let $\sigma\colon F \to \mathbb{P}$ be the double space branched over the hypersurface $W_{\mathbb{P}}$. This case is completely similar to the one considered above in Sec. 1.5: $V \cong F \times \mathbb{P}^1$.

Conjecture 1.6. *Apart from V/F, there are no other structures of a conic bundle on V. For the group of birational self-maps the following presentation holds:*

$$\operatorname{Bir} V = \operatorname{Aut} F \times \operatorname{Bir}(V/F) = \mathbb{Z}/2\mathbb{Z} \times \operatorname{Bir}(V/F).$$

Varieties of type ((1),0). This case is completely similar to the case considered above in Sec. 1.6. Here the variety $X = \mathbb{P}(\mathcal{E})$ is isomorphic to the blow up of $\mathbb{P}^{M+1}$ at a linear space $P \subset \mathbb{P}^{M+1}$ of codimension two. Elementary computations (see Sec. 1.6) show that with respect to this isomorphism the variety V is birationally equivalent to a double space $\mathbb{P}^{M+1}$ of index 2 (with the branch divisor $W_{\mathbb{P}} \subset \mathbb{P}^{M+1}$ of degree $2M$). Moreover, it is easy to see that all varieties of type ((1),0) are realized in this way: take a double space of index 2 and any pencil of hyperplanes on $\mathbb{P}^{M+1}$. In particular, to any plane of codimension two $P \subset \mathbb{P}^{M+1}$ corresponds a pencil Λ_P of Fano varieties on V.

Conjecture 1.7. *Every structure of a rationally connected Fano fiber space on V is a pencil Λ_P for some subspace $P \subset \mathbb{P}^{M+1}$ of codimension two. The groups of birational and biregular automorphisms of the variety V coincide:*

$$\operatorname{Bir} V = \operatorname{Aut} V.$$

Varieties of type ((1,1),0). This case is completely similar to the case considered above in Sec. 1.2. For these varieties the claim of Proposition 1.1 holds. The proof given in Sec. 1.2 works with some simplifications (there is no divisor Q).

2 Infinitely near singularities of vertical subvarieties

2.1. Set up of the problem. Recall that in [1, Sec. 2.2] the condition (vs) for the case of a singular point of a fiber $o \in F$ lying outside the ramification divisor of the double cover $\sigma\colon F \to G$ was proved in the following way. Assume that there exists a prime divisor $Y \subset F$, satisfying the estimate

$$\frac{\operatorname{mult}_x \widetilde{Y}}{\deg Y} > \frac{1}{m}, \tag{3}$$

where x is an infinitely near point of the first order over the point o, that is, $x \in E$, where $E \subset \widetilde{F}$ is the exceptional divisor of the blow up $\varphi\colon \widetilde{F} \to F$ of the point o. Here $p = \sigma(o) \in G$ is a non-degenerate double point of the hypersurface G, $p \notin W$.

Now the crucial fact is Proposition 2.2 in [1, Sec. 2.2]:

There exists a hyperplane $P \subset \mathbb{P}$, $P \ni p$, such that $\sigma(Y) \not\subset P$ and the effective algebraic cycle $Y_P = (Y \circ_F P_F)$, where $P_F = \sigma^{-1}(P_G)$, $P_G = P \cap G$ is the corresponding hyperplane section, satisfies the estimate

$$\frac{\operatorname{mult}_o}{\deg} Y_P > \frac{3}{2m}. \tag{4}$$

Proof repeats the arguments in the similar case in [7] word for word and for this reason was omitted in [1]. It is based on the following local fact. Set $2\nu = \operatorname{mult}_o Y$, $\mu = \operatorname{mult}_x \widetilde{Y}$, $B = T_x E \cap E$, where E is considered as a quadratic hypersurface in $\mathbb{P}^M$.

Lemma 2.1. *The following estimate holds*

$$\operatorname{mult}_B \widetilde{Y} \geq \frac{1}{2}(\mu - \nu). \tag{5}$$

Proposition 2.2 of the paper [1] cited above follows from Lemma 2.1 almost immediately. Indeed, the exceptional divisor E is embedded in the exceptional divisor $\mathbb{T}$

of the blow up of the point p on the projective space $\mathbb{P}$ as a quadric hypersurface. Thus one may consider the tangent hyperplane $T_x E$ as a hyperplane in $\mathbb{T}$, that is, as the projectivized tangent cone to a uniquely determined hyperplane $P \subset \mathbb{P}$, $p \in P$. Let $P^\sharp$ be the strict transform of P on $\widetilde{\mathbb{P}}$, then

$$P^\sharp \cap \mathbb{T} = T_x E.$$

Set $P_G = P \cap G$ and $P_F = \sigma^{-1}(P_G)$. These are irreducible varieties and moreover

$$\deg P_F = 2m, \quad \mathrm{mult}_x \widetilde{P}_F \leq 2,$$

so that $Y \neq P_F$. Furthermore,

$$B = T_x E \cap E \subset \widetilde{P}_F,$$

so that by the standard formulas of intersection theory [8, Chapter 2] we get for the effective cycle $Y_P = (Y \circ_F P_F)$:

$$\mathrm{mult}_o Y_P \geq 2\nu + (\mathrm{mult}_B \widetilde{Y}) \deg B = \mu + \nu,$$

whence, taking into account that $\deg Y_P = \deg Y$ (since Y_P is a hyperplane section of Y) and by assumption (see (3))

$$2\nu \geq \mu = \mathrm{mult}_x \widetilde{Y} > \frac{1}{m} \deg Y,$$

we obtain the desired inequality (4).

Unfortunately, this simple argument, by means of which Proposition 2.2 of the paper [1] is derived from Lemma 2.1, does not work in the case when the singular point $o \in F$ lies on the ramification divisor: if, in the notations of Sec. 3.1 of [1], the point $x \in E$ lies outside the ramification divisor of the double cover $\tilde{\sigma}_E$, that is, $\tilde{\sigma}(x) \notin W_E$, then the hyperplane $P \subset \mathbb{P}$, described above, does not exist. The best we could take instead of the subvariety P in this case is a quadric hypersurface, the tangent cone to which at the point p contains the quadric $\tilde{\sigma}_E(B)$. However, the ratio of the multiplicity to the degree for the cycle Y_P turns out to be not big enough. That is why in [1] an alternative method of proving the condition (vs) was developed for this case. Let us consider in a more detailed way, how the lower estimate for the number $(\mathrm{mult}_o / \deg) Y_P$ is obtained. Let $P \subset \mathbb{P}$ be a quadric hypersurface, containing the point $p \in W \cap G$, where $p \in P$ is a singular point and $P^\sharp \supset \tilde{\sigma}_E(B)$. Now repeating the arguments above word for word, we get the estimate

$$\mathrm{mult}_o Y_P \geq 4\nu + 2\, \mathrm{mult}_B \widetilde{Y}, \tag{6}$$

whence, taking into account Lemma 2.1 we get

$$\frac{\mathrm{mult}_o}{\deg} Y_P > \frac{5}{4m},$$

but this is not good enough to get a contradiction. However, it is easy to see from the inequality (6) that what we need to obtain an estimate for the multiplicity of the cycle Y_P at the point o, is not so much an estimate of the number $\mathrm{mult}_B \widetilde{Y}$, but rather a combined estimate for ν and $\mathrm{mult}_B \widetilde{Y}$. For instance, if we knew that

$$\nu + \mathrm{mult}_B \widetilde{Y} > \frac{1}{m} \deg Y, \tag{7}$$

then, taking into account that $\nu \geq \mathrm{mult}_B \widetilde{Y}$, we could get

$$\frac{\mathrm{mult}_o}{\deg} Y_P > \frac{3}{2m},$$

after which we could have argued as in the case of a singular point outside the ramification divisor. The author believes that if (in the notations of [2, Sec. 1.4]) the inequality $k_v > 2$ holds, then the estimate (7) is true. The aim of this section is to formulate the corresponding claim precisely and to prove it in the particular case when the graph of the sequence of blow ups is a chain. Proof of this conjecture in full would have essentially simplified our work with Fano fiber spaces $V/\mathbb{P}^1$ in the case of vertical subvarieties.

The structure of this section is as follows. To begin with, we recall the proof of Lemma 2.1 (following [7], but with much more details). We need it to develop new arguments generalizing the method of proof of Lemma 2.1. After that we consider the general case and formulate the above-mentioned conjecture. In the remaining part of the section we prove this conjecture for the case when the graph of the sequence of blow ups is a chain.

2.2. Proof of Lemma 2.1. The claim of the lemma is local. Let $\Pi \ni p$ be a germ of a general section of the fiber G by a smooth four-dimensional subvariety of the ambient space (analytically, G in a neighborhood of the point p is a hypersurface in $\mathbb{C}^{M+1}$), such that its strict transform on $\widetilde{G}$ contains the point x: $\widetilde{\Pi} \ni x$. Set

$$E \cap \widetilde{\Pi} = E_\Pi \cong \mathbb{P}^1 \times \mathbb{P}^1.$$

Obviously,

$$B_\Pi = B \cap E_\Pi = L_1 + L_2$$

is the union of the two lines on the quadric E, that pass through the point x. Since $\Pi \ni p$ is a germ of three-dimensional non-degenerate quadratic singularity, its strict transform $\widetilde{\Pi}$ is smooth. Set $L = L_1$.

Let

$$Y_\Pi = Y \cap \Pi, \quad \widetilde{Y}_\Pi = \widetilde{Y} \cap \widetilde{\Pi}$$

be the restrictions of Y onto Π, $\widetilde{\Pi}$ respectively. Let us prove the inequality

$$\gamma = \mathrm{mult}_L \widetilde{Y}_\Pi \geq \frac{1}{2}(\mu - \nu).$$

Since the germ Π is a general one, this implies the inequality (5).

Let

$$\varphi_L\colon \Pi_L \to \widetilde{\Pi}$$

be the blow up of the line L on the smooth three-dimensional variety $\widetilde{\Pi}$, $E_L \subset \Pi_L$ the exceptional divisor. Since

$$(L \cdot L)_{E_\Pi} = 1, \quad (L \cdot E_\Pi)_{\widetilde{\Pi}} = -1,$$

we obtain the exact sequence

$$\begin{array}{ccccccc} 0 \to & \mathcal{N}_{L/E_\Pi} & \to \mathcal{N}_{L/\widetilde{\Pi}} \to & \mathcal{N}_{E_\Pi/\widetilde{\Pi}}|_L & \to 0, \\ & \| & & \| & \\ & \mathcal{O}_L & & \mathcal{O}_L(-1) & \end{array}$$

so that

$$\mathcal{N}_{L/\widetilde{\Pi}} \cong \mathcal{O}_L \oplus \mathcal{O}_L(-1) \tag{8}$$

and thus E_L is a ruled surface of type $\mathbb{F}_1$ with the exceptional section $\widetilde{E}_\Pi \cap E_L$, the class of which is denoted by s. We have

$$\operatorname{Pic} E_L = \mathbb{Z}s \oplus \mathbb{Z}f,$$

where f is the class of a fiber of the ruled surface. Since

$$(s \cdot E_L|_{E_L})_{E_L} = ((\widetilde{E}_\Pi \cap E_L) \cdot E_L) = (L \cdot L)_{L_\Pi} = 0,$$

we get $(E_L \cdot E_L) = -s - f$ (which can also be seen directly from (8)). Let Y_L be the strict transform of the divisor $\widetilde{Y}_\Pi$ on Π_L. Obviously,

$$Y_L \sim -\nu E_\Pi - \gamma E_L$$

(recall that Π is a germ of a three-dimensional section). Therefore,

$$Y_L|_{E_L} \sim \gamma s + (\gamma + \nu) f.$$

On the other hand, the following fact is well known.

Lemma 2.2. *Let $Z \subset R \subset X$ be a flag of strictly embedded smooth varieties, D an effective divisor on X,*

$$\varphi_R\colon \widetilde{X} \to X$$

the blow up of the subvariety R with the exceptional divisor E_R and $E_Z = \varphi_R^{-1}(Z)$. Let $\widetilde{D} \subset \widetilde{X}$ be the strict transform of the divisor D on $\widetilde{X}$. The following estimate holds:

$$\operatorname{mult}_{E_Z} \widetilde{D} = \operatorname{mult}_Z D - \operatorname{mult}_R D.$$

Proof. Restricting Z, R, D onto a general smooth subvariety in X of a suitable dimension, we assume that $Z = \{z\}$ is a point, R is a curve. Let $S \supset R$ be a general surface, $\widetilde{S} \cong S$ its strict transform on $\widetilde{X}$. We get

$$D|_S = (\operatorname{mult}_R D) R + D_S^\sharp,$$

$$\widetilde{D}|_{\widetilde{S}} = D^{\sharp}_{S}, \quad \mathrm{mult}_{E_Z} \widetilde{D} = \mathrm{mult}_z D^{\sharp}_{S},$$

but since $\mathrm{mult}_Z D = \mathrm{mult}_z D|_S$ and $\mathrm{mult}_z R = 1$, we obtain the claim of the lemma.

Let $\Lambda \subset E_L$ be the fiber over the point x. By the lemma just proved, we get the estimate

$$\mathrm{mult}_{\Lambda}(Y_L|_{E_L}) \geq \mu - \gamma.$$

Therefore,

$$\gamma + \nu \geq \mu - \gamma,$$

so that $2\gamma \geq \mu - \nu$, which is what we need.

2.3. A conjecture on multiplicities. Let $\Delta \ni o$ be a germ of a non-degenerate quadratic singularity, $\dim \Delta = N + 1$, $N \geq 3$. Let us blow up the point o:

$$\varphi \colon \widetilde{\Delta} \to \Delta,$$

$E \subset \widetilde{\Delta}$ is the exceptional divisor, that is, an N-dimensional quadric. Let

$$\varphi_{i,i-1} \colon \Delta_i \to \Delta_{i-1}$$

be a sequence of blow ups of irreducible varieties $B_{i-1} \subset \Delta_{i-1}$, $i = 1, \ldots, k$, $\Delta_0 = \widetilde{\Delta}$, with the exceptional divisors

$$E_i = \varphi^{-1}_{i,i-1}(B_{i-1}) \subset \Delta_i,$$

where, moreover, $B_i \subset E_i$ for $i = 0, \ldots, k-1$, and $E_0 = E$, $B_0 = x \in E$ is a point. Consider the graph Γ, associated with the sequence of blow ups $\varphi_{i,i-1}$, and set

$$p_i = \sharp\{\text{the paths from } E_k \text{ to } E_i\}.$$

Let $D \subset \Delta$ be a prime divisor,

$$D^0 = \widetilde{D} \subset \widetilde{\Delta} = \Delta_0, \quad D^1 \subset \Delta_1, \ldots, D^k \subset \Delta_k$$

its strict transforms on $\Delta_0 = \widetilde{\Delta}$, $\Delta_1, \ldots, \Delta_k$, respectively. We get

$$D^k \sim D - \nu E - \mu_1 E_1 - \ldots - \mu_k E_k,$$

for some $\nu, \mu_1, \ldots, \mu_k \in \mathbb{Z}_+$. More precisely,

$$2\nu = \mathrm{mult}_o D, \quad \mu_i = \mathrm{mult}_{B_{i-1}} D^{i-1}.$$

Set

$$B = T_x E \cap E,$$

where the quadric E is assumed to be embedded in $\mathbb{P}^{N+1}$ in the standard way.

Conjecture 2.1. *Assume that the following estimate holds:*

$$2\nu p_0 + \sum_{i=1}^{k} p_i \mu_i \geq \lambda \left(2p_0 + \sum_{i=1}^{k} p_i \right).$$

Then the following inequality is satisfied:

$$\nu + \mathrm{mult}_B \widetilde{D} \geq \lambda. \qquad (9)$$

We prove this conjecture for the particular case when the graph Γ is a chain, that is,

$$B_i \not\subset E^i_{i-1}$$

for every $i = 2, \ldots, k-1$. In this case all the integers p_i are equal to 1.

2.4. Proof in the case when the graph is a chain.

Proposition 2.1. *Assume that the graph* Γ *is a chain and the inequality*

$$2\nu + \mu_1 + \ldots + \mu_k \geq (k+2)\lambda \qquad (10)$$

holds. Then the estimate (9) is true.

Proof is given in the assumption that all the centres of blow ups B_i are points. The general case reduces to this one in a trivial way. We get

$$E_1 \cong \ldots \cong E_k \cong \mathbb{P}^N.$$

Let us construct by induction a sequence of irreducible non-singular subvarieties of codimension two

$$Y_j \subset \Delta_{j+2},$$

$j = 0, \ldots, k-1$, in the following way. Set $Y_0 = B^1$ to be the strict transform of the quadric cone B on Δ_1. Obviously, Y_0 is the cone B with the blown up vertex; in particular, $Y_0 \cap E_1$ is a non-singular quadratic hypersurface in the hyperplane $E^1 \cap E_1 \subset E_1 \cong \mathbb{P}^N$. Now

$$Y_1, \ldots, Y_{k-1}$$

are uniquely determined by the following condition: the subvariety

$$Y_j^+ = \varphi_{j+1,j}(Y_j) \subset E_j$$

is a non-degenerate quadratic cone in $E_j \cong \mathbb{P}^N$ with the vertex at the point $x_j \in E_j$ and the base $Y_{j-1} \cap E_j$, and moreover Y_j is the strict transform of Y_j^+ on Δ_{j+1}.

This construction is justified exactly by the fact that the non-singular quadric $Y_{j-1} \cap E_j$ is contained in the hyperplane $E^j_{j-1} \cap E_j$, whereas the vertex x_j of the cone is not contained in this hyperplane by assumption: $x_j \notin E^j_{j-1}$, since the graph Γ is a chain.

Set

$$\gamma_i = \mathrm{mult}_{Y_i} D^i = \mathrm{mult}_{Y_i^+} D^{i-1}.$$

For convenience set also $\gamma_{-1} = 0$, $\mu_0 = \nu$, $\gamma_k = 0$.

Lemma 2.3. *For any* $i \in \{0, \ldots, k-1\}$ *the following estimate holds:*

$$2\gamma_i \geq \mu_{i+1} - \mu_i + \gamma_{i-1} + \gamma_{i+1}. \qquad (11)$$

Proof of the lemma is given below. Now let us complete the proof of Proposition 2.1. The estimates (11), put together with the coefficients $(k-i)$, give the inequality

$$\sum_{i=0}^{k-1}(k-i)(2\gamma_i-\mu_{i+1}+\mu_i-\gamma_{i-1}-\gamma_{i+1})\geq 0. \tag{12}$$

For $i\neq 0$ the multiplicity γ_i comes into the left-hand side of (12) with the coefficient

$$2(k-i)-(k-i-1)-(k-i+1)=0;$$

the multiplicity γ_0 comes into the left-hand side of (12) with the coefficient

$$-(k-1)+2k=k+1;$$

for $i\neq 0$ the multiplicity μ_i comes with the coefficient

$$(k-i)-(k-i+1)=-1;$$

finally, the multiplicity μ_0 comes into (12) with the coefficient k. As a result, we get the inequality

$$(k+1)\gamma_0+k\nu\geq\mu_1+\ldots+\mu_k,$$

whence, taking into account the inequality (10), we get

$$(k+1)\gamma_0+(k+2)\nu\geq(k+2)\lambda,$$

so that the more so,

$$\gamma_0+\nu\geq\lambda.$$

Now recall that $\gamma_0=\operatorname{mult}_B\widetilde{D}$, which completes the proof of Proposition 2.1.

2.5. Proof of Lemma 2.3. Let us consider the following general situation. Let $\Pi\ni o$ be a smooth three-dimensional germ,

$$\psi\colon\widetilde{\Pi}\to\Pi$$

the blow up of the point o, $E\subset\widetilde{\Pi}$ the exceptional divisor, $E\cong\mathbb{P}^2$. Let $a\in E$ be an arbitrary point,

$$\psi^+\colon\Pi^+\to\widetilde{\Pi}$$

its blow up, $E^+\subset\Pi^+$ the exceptional plane, $\widetilde{E}$ the strict transform of the plane E on Π^+. Obviously, $\widetilde{E}$ is a ruled surface of type $\mathbb{F}_1$. Finally, let $L^*\subset E$ be a line, passing through the point a, $L\subset\widetilde{E}$ its strict transform on Π^+, and let $R\subset E^+$ be a line, passing through the point $L\cap E^+$, but different from the line $\widetilde{E}\cap E^+$. On the line L we take a point $b\neq a$, and let $C\subset\Pi$ be a smooth curve, intersecting transversally the plane E at the point b.

Let $D\subset\Pi$ be a germ of a prime divisor, $o\in D$. Set $\widetilde{D}\subset\widetilde{\Pi}$ and $D^+\subset\Pi^+$ to be the strict transforms of D on $\widetilde{\Pi}$ and Π^+, respectively. Finally, set

$$\mu_o=\operatorname{mult}_o D,\quad \mu_a=\operatorname{mult}_a\widetilde{D},$$

$$\mu_C = \mathrm{mult}_C\, \widetilde{D}, \quad \mu_R = \mathrm{mult}_R\, D^+,$$
$$\mu_L = \mathrm{mult}_L\, \widetilde{D}.$$

The following claim holds.

Lemma 2.4. *We have the estimate*

$$2\mu_L \geq \mu_a + \mu_R + \mu_C - \mu_o.$$

Proof. Blow up the curve L:

$$\psi_L \colon \Pi_L \to \Pi^+,$$

and let $E_L \subset \Pi_L$ be the exceptional divisor. It is well known that the normal sheaf of the line $L^* \subset \widetilde{\Pi}$ is of the form

$$\mathcal{N}_{L^*/\widetilde{\Pi}} \cong \mathcal{O}_{L^*}(-1) \oplus \mathcal{O}_{L^*}(1),$$

so that the normal sheaf of the curve L is

$$\mathcal{N}_{L/\Pi^+} \cong \mathcal{O}_L(-2) \oplus \mathcal{O}_L. \tag{13}$$

In particular, E_L is a surface of type $\mathbb{F}_2$. Set E^L to be the strict transform of the surface $\widetilde{E}$ on Π_L. It is easy to see that the exceptional section of the ruled surface E_L is the curve $E_L \cap E^L$. Write down

$$\mathrm{Pic}\, E_L = \mathbb{Z}s \oplus \mathbb{Z}f,$$

where $(s^2) = -2$, f is the class of a fiber. Denote by the symbols C_L and R_L the strict transforms of the curves C and R on Π_L. Since C and R intersect the surface $\widetilde{E}$ transversally at the points $b \in L$ and $L \cap E^+$, the curves C_L and R_L intersect (transversally) the ruled surface E_L at the points $x_C = C_L \cap E_L$ and $x_R = R_L \cap E_L$, which do not lie on the exceptional section $E_L \cap E^L$, respectively. Finally, let $D_L \subset \Pi_L$ be the strict transform of the divisor D. We get

$$D_E = (D_L \circ E_L) \sim \mu_L(-E_L|_{E_L}) + (\mu_o - \mu_a)f,$$

since, obviously,

$$(\psi_L \circ \psi^+)^* E|_{E_L} \sim -f, \quad \psi_L^* E^+|_{E_L} \sim f.$$

From the form of the normal sheaf (13) or from the relation

$$(E - E^+ - E_L)|_{E_L} \sim s$$

(since $E^L \sim E - E^+ - E_L$) we obtain that

$$-E_L|_{E_L} \sim s + 2f,$$

so that we get finally

$$D_E \sim \mu_L s + (2\mu_L + \mu_o - \mu_a)f. \tag{14}$$

However, the curve $D_E \subset E_L$ is effective, and moreover,

$$\mathrm{mult}_{x_C} D_E \geq \mu_C, \quad \mathrm{mult}_{x_R} D_E \geq \mu_R. \tag{15}$$

Write down

$$D_E = A^* + \alpha_C A_C + \alpha_R A_R, \tag{16}$$

where A_C (respectively, A_R) is the fiber of the ruled surface E_L, containing the point x_C (respectively, x_R), and the curve A^* is effective and does not contain A_C or A_R as a component. Obviously,

$$A^* \sim \mu_L s + \alpha^* f,$$

where $\alpha^* \in \mathbb{Z}_+$.

Lemma 2.5. *The following estimate holds*

$$\mathrm{mult}_{x_C} A^* + \mathrm{mult}_{x_R} A^* \leq \min(\alpha^*, 2\mu_L). \tag{17}$$

Proof. Since A_C is not a component of the curve A^*, we get

$$\mathrm{mult}_{x_C} A^* \leq (A^* \cdot A_C)_{x_C} \leq (A^* \cdot A_C) = \mu_L, \tag{18}$$

and similarly for x_R. Furthermore, if S is an irreducible component of the curve A^*, containing at least one of the points x_C, x_R, then S is not the exceptional section of the surface E_L (since $x_C, x_R \notin E_L \cap E^L$). Therefore,

$$S \sim \beta s + \delta f,$$

where $\delta \geq 2\beta$, whence as above in (18) we get

$$\mathrm{mult}_{x_C} S \leq \beta \leq \frac{\delta}{2}$$

and similarly for x_R, so that

$$\mathrm{mult}_{x_C} S + \mathrm{mult}_{x_R} S \leq \delta.$$

This proves Lemma 2.5.

Taking into account all the estimates (14 - 17) and the fact that $x_C \notin A_R$ and $x_R \notin A_C$ by construction, we get

$$2\mu_L + \mu_o - \mu_a \geq \mu_C + \mu_R,$$

as we claimed. Proof of Lemma 2.4 is complete.

Let us come back to the proof of Lemma 2.3. Let us prove first the inequality (11) for $i \geq 1$. Let

$$\Gamma_i^- \subset \Delta_{i-1}$$

be a germ of a smooth curve, satisfying the following conditions:

(i) $x_{i-1} \in \Gamma_i^-$;

(ii) the strict transform $\Gamma_i \subset \Delta_i$ of the curve Γ_i^- contains the point x_i;

(iii) the strict transform $\Gamma_i^+ \subset \Delta_{i+1}$ of the curve Γ_i contains the point x_{i+1}.

Such a germ exists exactly because the graph

$$E_i \leftarrow E_{i+1} \leftarrow E_{i+2}$$

is a chain. Now let $\Pi_i^- \supset \Gamma_i^-$ be a general three-dimensional germ at the point x_{i-1}. Obviously, the strict transform $\Pi_i \subset \Delta_i$ of the germ Π_i^- contains the point x_i, whereas, in its turn, its strict transform Π_i^+ on Δ_{i+1} contains the point x_{i+1}. We denote the plane

$$\Pi_i \cap E_i \subset \Delta_i$$

by the symbol E_Π. Since E_Π contains the vertex of the quadratic cone Y_i^+, the intersection $E_\Pi \cap Y_i^+$ is a pair of distinct lines passing through the point x_i. Set $L_i^* \subset E_\Pi \cap Y_i^+$ to be one of these two lines, $L_i \subset \Delta_{i+1}$ its strict transform on Δ_{i+1}. Let $R_i \subset \Pi_i^+$ be the line in $E_{i+1} \cong \mathbb{P}^N$, joining the points x_{i+1} and $L_i \cap E_{i+1}$. Finally, if the germ Π_i^- is sufficiently general, then the curve $Y_{i-1}^+ \cap \Pi_i^-$ at the point x_{i-1} has a pair of branches with distinct tangent directions, so that the curve

$$C_i = \Pi_i \cap Y_{i-1} \subset E_{i-1}^i$$

intersects E_i transversally at two distinct points. Let $a_i \in C_i \cap E_i$ be that one which lies on the line L_i^* (recall that $Y_{i-1} \cap E_i$ is the base of the quadratic cone Y_i^+, and $\Pi_i \cap E_i$ is a plane, containing its vertex). Now we apply the lemma with x_{i-1}, x_i, L_i, R_i, C_i instead of o, a, L, R, C, respectively. This immediately implies the estimate (11) for $i \geq 1$.

For $i = 0$ the situation is completely similar with the only exception: there is no curve C_i, so that the inequality of Lemma 2.4 is used in the truncated form:

$$2\mu_L \geq \mu_a + \mu_R - \mu_o,$$

where $o \in \Pi$ is a germ of a non-degenerate quadratic singularity,

$$\mu_o = \frac{1}{2} \operatorname{mult}_o D,$$

as in the proof of Lemma 2.1. This completes the proof of Lemma 2.3.

3 The linear method

3.1. The linear method of proving birational rigidity. Let $V/\mathbb{P}^1$ be a Fano fiber space, satisfying the following assumptions:

(i) V is a smooth projective variety with the Picard group $\operatorname{Pic} V = \mathbb{Z}K_V \oplus \mathbb{Z}F$, where F is the class of a fiber of the projection $\pi\colon V \to \mathbb{P}^1$,

(ii) every fiber $F_t = \pi^{-1}(t)$, $t \in \mathbb{P}^1$, is a (factorial) variety of dimension $\dim F_t \geq 4$, with, at most, non-degenerate quadratic singularities, and moreover,

$$A^1 F_t = \mathbb{Z}H_t, \; A^2 F_t = \mathbb{Z}H_t^2,$$

where $H_t = -K_{F_t} = (-K_V \cdot F_t)$ is the ample anticanonical divisor,

(iii) at each point $x \in V$ the following local condition (LR) holds: if $x \in F = F_t$ is a smooth point of the fiber, then for every effective divisor $D \in |-nK_F|$ and every hyperplane $B \subset E$ in the exceptional divisor $E \subset \widetilde{F}$ of the blow up of the point x on F the following inequality is satisfied:

$$\operatorname{mult}_x D + \operatorname{mult}_B \widetilde{D} \leq 2n, \tag{19}$$

where $\widetilde{D} \subset \widetilde{F}$ is the strict transform of the divisor D. If $x \in F$ is a quadratic singularity, then for every effective divisor $D \in |-nK_F|$ and every hyperplane section B of the non-singular quadric $E \subset \widetilde{F}$, the exceptional divisor of the blow up of the point x on F, the following inequality is satisfied:

$$\operatorname{mult}_x D + 2\operatorname{mult}_B \widetilde{D} \leq 4n, \tag{20}$$

where $\widetilde{D} \subset \widetilde{F}$ is the strict transform of the divisor D.

Proposition 3.1. *For any movable linear system $\Sigma \subset |-nK_V + lF|$ with $n \geq 1$ the log pair $(V, \frac{1}{n}\Sigma)$ is canonical. In particular, if $l \in \mathbb{Z}_+$, then the virtual and actual thresholds of canonical adjunction coincide:*

$$c_{\text{virt}}(\Sigma) = c(\Sigma) = n.$$

Proof. Assume the converse: the pair $(V, \frac{1}{n}\Sigma)$ is not canonical. In other words, the linear system Σ has a maximal singularity $E \subset \widetilde{V}$, where $\varphi\colon \widetilde{V} \to V$ is a sequence of blow ups. Let $Z = \varphi(E) \subset V$ be the centre of the maximal singularity.

Lemma 3.1. *The following estimate holds:* $\operatorname{codim}_V Z \geq 3$.

Proof: the claim follows immediately from the condition (ii).

Let $x \in Z$ be a point of general position, $D \in \Sigma$ be a general divisor. Furthermore, let

$$\lambda\colon V^+ \to V,$$

be the blow up of the point x on V. Denote the exceptional divisor $\lambda^{-1}(x) \subset V^+$ by E^+.

Lemma 3.2. *There exists a hyperplane $B^+ \subset E^+$ satisfying the estimate*

$$\operatorname{mult}_x D + \operatorname{mult}_{B^+} D^+ > 2n, \tag{21}$$

where $D^+ \subset V^+$ is the strict transform of the divisor D.

Proof: this follows easily from the connectedness principle of Shokurov and Kollár [4] and the properties of a non-log-canonical isolated singularity on a surface. For a detailed proof, see [3].

Let $F = F_t$ be the fiber of the projection π, containing the point x. Assume first that $x \in F$ is a non-singular point of the fiber. There are two possibilities:

1) $B^+ \not\subset F^+$,

2) $B^+ \subset F^+$,

where $F^+ \subset V^+$ is the strict transform of the fiber F^+. Obviously, $\lambda_F\colon F^+ \to F$ is the blow up of the point $x \in F$, whereas $E_F = E^+ \cap F^+$ is its exceptional divisor. Consider these two possible cases separately.

1) Here $B = B^+ \cap F^+$ is a hyperplane in the exceptional divisor E_F. Set $D_F = D\,|\,_F$ to be the restriction of the divisor D on the fiber F. The effective divisor D_F is well defined since $F \not\subset \operatorname{Supp} D$. The inequality (21) can be rewritten in the following form:

$$\operatorname{mult}_{B^+}(\lambda^* D) > 2n$$

(since, obviously, $\operatorname{mult}_{B^+} E^+ = 1$). Now we get

$$\operatorname{mult}_B(\lambda_F^* D_F) = \operatorname{mult}_B(\lambda^* D)\,|\,_{F^+} \geq \operatorname{mult}_{B^+}(\lambda^* D).$$

Taking into account that $\operatorname{mult}_B(\lambda_F^* D_F) = \operatorname{mult}_x D_F + \operatorname{mult}_B D_F^+$, where $D_F^+ \subset F^+$ is the strict transform of the divisor D_F, we get finally

$$\operatorname{mult}_x D_F + \operatorname{mult}_B D_F^+ > 2n.$$

Note that $D_F \in |-nK_F|$. We get a contradiction with the condition (LR).

2) Here $B^+ = E_F$ is the exceptional divisor of the blow up λ_F. Setting $D_F = D\,|\,_F$, we obtain the estimate $\operatorname{mult}_x D_F = \operatorname{mult}_x D + \operatorname{mult}_{B^+} D^+ > 2n$, which contradicts the condition (LR) again.

Therefore the point $x \in F$ cannot be smooth. The essence of the arguments above is that the inequality (21) can be "restricted onto the fiber F".

However, the point $x \in F$ cannot be a singularity of the fiber, either. Indeed, in the notations above, in this case $E_F \subset E^+$ is a non-singular quadric, so that $B^+ \not\subset F^+$. The argument 1) again yields the inequality

$$\operatorname{mult}_B(\lambda_F^* D_F) > 2n,$$

where $B = B^+ \cap F^+$ is a hyperplane section of the quadric E_F. Taking into account that

$$\operatorname{mult}_B(\lambda_F^* D_F) = \frac{1}{2}\operatorname{mult}_x D_F + \operatorname{mult}_B D_F^+,$$

we get a contradiction with the condition (LR) once again.

Therefore, the log pair $(V, \frac{1}{n}\Sigma)$ is canonical.

If $l \in \mathbb{Z}_+$, then $c(\Sigma) = n$. The standard arguments of [7] show that the strict inequality

$$c_{\text{virt}}(\Sigma) < n$$

implies non-canonicity of the pair $(V, \frac{1}{n}\Sigma)$. Therefore, we get the equality $c_{\text{virt}}(\Sigma) = n$.

Q.E.D. for Proposition 3.1.

Corollary 3.1. *Assume that the Fano fiber space $V/\mathbb{P}^1$ satisfies the conditions* (i)-(iii) *and the K-condition, that is,*

$$-K \notin \operatorname{Int} A^1_{\text{mov}} V.$$

Then the fiber space $V/\mathbb{P}^1$ is birationally superrigid.

3.2. Varieties with a pencil of double spaces. In this section of the paper the symbol $\mathbb{P}$ stands for the complex projective space $\mathbb{P}^M$ of dimension $M \geq 5$. Let $\mathcal{W} = \mathbb{P}(H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(2M)))$ be the space of hypersurfaces of degree $2M$. In [3] the following fact was proved:

Proposition 3.2. *There exists a non-empty Zariski open subset $\mathcal{W}_{\rm reg} \subset \mathcal{W}$ such that:*

(i) $\operatorname{codim}(\mathcal{W} \setminus \mathcal{W}_{\rm reg}) \geq 2$,

(ii) *the Fano double space $\sigma\colon F \to \mathbb{P}$, branched over a hypersurface $W \in \mathcal{W}_{\rm reg}$, satisfies the condition (LR) at every smooth point $x \in F$ and has at most non-denerate quadratic singularities.*

Assume that $x \in F$ is a singular point.

Lemma 3.3. *The variety F satisfies the condition (LR) at the point x.*

Proof. This is almost obvious: if for a divisor $D \in |-nK_F|$ and a hyperplane section B of the quadric $E = \lambda^{-1}(x)$, where $\lambda\colon \widetilde{F} \to F$ is the blow up of the point x, the inequality

$$\operatorname{mult}_x D + 2 \operatorname{mult}_B \widetilde{D} > 4n, \tag{22}$$

holds, then $\operatorname{mult}_x D > 2n$, since $\operatorname{mult}_x D \geq 2 \operatorname{mult}_B \widetilde{D}$. However, the opposite estimate holds: $\operatorname{mult}_x D \leq 2n$. This contradiction completes the proof of the lemma.

Since $\operatorname{codim}(\mathcal{W} \setminus \mathcal{W}_{\rm reg}) \geq 2$, for a general variety $V/\mathbb{P}^1$ with a pencil of Fano double spaces of dimension $M \geq 5$ every fiber satisfies the condition (LR) at every point.

Recall that the variety V is the double cover of the projective bundle $X/\mathbb{P}^1$, $X = \mathbb{P}(\mathcal{E})$, branched over a smooth hypersurface $W^* \sim 2ML_X + 2a_W R$, where L_X, R are the classes of the tautological sheaf and the fiber in $\operatorname{Pic} X$, respectively. By construction, the locally free sheaf $\mathcal{E}$ is generated by its sections, so that the linear system $|L_V|$ is free, where $L_V = \sigma^* L_X$.

Lemma 3.4. *The following equality holds: $(-K_V \cdot L_V^{M-1}) = 4 - 2a_W$.*

Proof: direct computations.

Corollary 3.2. *For $a_W \geq 2$ the Fano fiber space $V/\mathbb{P}^1$ satisfies the K-condition: $-K_V \notin \operatorname{Int} A^1_{\rm mov} V$.*

In fact, for $a_W \geq 2$ a stronger condition holds:

$$-K_V \notin \operatorname{Int} A^1_+ V,$$

but we do not need that.

Corollary 3.3. *For $a_W \geq 2$ a general Fano fiber space $V/\mathbb{P}^1$ is birationally superrigid.*

3.3. Varieties with a pencil of double quadrics. Now set $\mathbb{P} = \mathbb{P}^{M+1}$, $M \geq 6$. Set also $\mathcal{W} = \mathbb{P}(H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(2M-2)))$. For a hypersurface $W \in \mathcal{W}$ consider the double cover of a quadric $Q \subset \mathbb{P}$:

$$\sigma = \sigma_W\colon F \to Q,$$

branched over the intersection $W_Q = W \cap Q$. About the quadric Q, the hypersurface W and their position with respect to each other we assume the following.

1) The quadric Q is either smooth or a cone over a smooth quadric in $\mathbb{P}^M$ with the vertex at a point.

2) The variety F has at most non-degenerate quadratic singularities. If Q is a cone, then the branch divisor W_Q is smooth.

3) Let $p \in W_Q$ be a smooth point of the branch divisor, $(z_1, \dots, z_{M+1})$ a system of affine coordinates on $\mathbb{P}^{M+1}$ with the origin at the point p. With respect to this coordinate system the hypersurfaces Q and W are given by the equations

$$q_1 + q_2 = 0 \text{ and } w_1 + w_2 + \dots + w_{2M-2} = 0, \tag{23}$$

respectively. Without loss of generality assume that $q_1 \equiv z_1$, $w_1 \equiv z_2$. Then for any linear form $\lambda(z_3, \dots, z_{M+1})$ and any constant $c \in \mathbb{C}$

$$c\, q_2(0, 0, z_3, \dots, z_{M+1}) \neq w_2(0, 0, z_3, \dots, z_{M+1}) - \lambda^2(z_3, \dots, z_{M+1}).$$

Similarly, let $p \in W_Q$ be a singularity of the branch divisor, (z_*) a system of affine coordinates, (23) the equations of the hypersurfaces Q and W, where without loss of generality we assume that $q_1 \equiv w_1 \equiv z_1$. Then for any linear form $\lambda(z_2, \dots, z_{M+1})$ and any constant $c \in \mathbb{C}$

$$c\, q_2(0, z_2, \dots, z_{M+1}) \neq w_2(0, z_2, \dots, z_{M+1}) - \lambda^2(z_2, \dots, z_{M+1}).$$

Let $\mathcal{Q} = \mathbb{P}(H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(2)))$ be the space of quadrics.

Proposition 3.3. *There exists a Zariski open subset $\mathcal{F}_{\rm reg} \subset \mathcal{Q} \times \mathcal{W}$, such that*

(i) $\operatorname{codim}(\mathcal{Q} \times \mathcal{W} \setminus \mathcal{F}_{\rm reg}) \geq 2$,

(ii) *for any pair $(Q, W) \in \mathcal{F}_{\rm reg}$ the corresponding Fano double cover F satisfies the conditions* 1)-3).

Proof: an obvious dimension count.

Proposition 3.4. *For any pair $(Q, W) \in \mathcal{F}_{\rm reg}$ the corresponding Fano double cover F satisfies the condition (LR) at every point $x \in F$.*

Proof. Set $p = \sigma(x) \in Q$. Consider first the case when the morphism σ is non-ramified at the point x, that is, $p \notin W$. Let

$$\varphi\colon \widetilde{F} \to F \quad \text{and} \quad \bar{\varphi}\colon \widetilde{Q} \to Q$$

be the blow ups of the points x and p, respectively, $E = \varphi^{-1}(x)$ and $\bar{E} = \bar{\varphi}^{-1}(p)$ the exceptional divisors. The morphism σ extends to a rational map $\tilde{\sigma}\colon \widetilde{F} \dashrightarrow \widetilde{Q}$, regular in a neighborhood of E and identifying the exceptional divisors E and $\bar{E}$.

Assume that the point $p \in Q$ is smooth. Assume, moreover, that the divisor $D \in |-nK_F|$ satisfies the inequality $\operatorname{mult}_x D + \operatorname{mult}_B \widetilde{D} > 2n$, where $B \subset E$ is a hyperplane. Denote by the symbol $\bar{B} = \tilde{\sigma}(B)$ the corresponding hyperplane in $\bar{E} = \mathbb{P}(T_p Q)$.

Now our arguments are similar to the proof of Theorem 2 in [3] (with simplifications). Obviously, there is a pencil Λ_B of hyperplane sections $R \ni x$ of the quadric Q, such that

$$\widetilde{R} \cap \bar{E} = \bar{B},$$

where $\widetilde{R} \subset \widetilde{Q}$ is the strict transform of the divisor R. A general quadric $R \in \Lambda_B$ is non-singular at the point p, so that $\sigma^{-1}(R)$ is an irreducible variety, non-singular at the point x. The class of a hyperplane section of the variety $\sigma^{-1}(R)$ denote by the symbol H_R. Set also $D_R = D\,|_{\sigma^{-1}(R)}$. This is an effective divisor on the irreducible variety $\sigma^{-1}(R)$, satisfying the inequality

$$\mathrm{mult}_x\, D_R = \mathrm{mult}_x\, D + \mathrm{mult}_B\, \widetilde{D} > 2n.$$

However, the divisor $T_R = \sigma^{-1}(R \cap T_p R)$ is irreducible and its multiplicity at the point x is exactly 2. Therefore, one can write down

$$D_R = aT_R + D_R^\sharp,$$

where $a \in \mathbb{Z}_+$ and the effective divisor $D_R^\sharp \in |n^\sharp H_R|$ does not contain the divisor T_R as a component, and moreover $n^\sharp \geq 1$. We get the inequality

$$\mathrm{mult}_x\, D_R^\sharp > 2n^\sharp.$$

Consider the effective cycle $\Delta = (D_R^\sharp \circ T_R)$. Its degree with respect to the ample class H_R is equal to $4n^\sharp$, whereas the following inequality holds:

$$\mathrm{mult}_x\, \Delta > 4n^\sharp.$$

This is impossible. This contradiction excludes the case under consideration.

Now assume that $p \in Q$ is the vertex of the cone. Here $E \cong \bar{E}$ is a non-singular quadric, $B \cong \bar{B}$ is its hyperplane section. Assume that some divisor $D \in |-nK_F|$ satisfies the inequality (22). There is a unique hyperplane $H \subset \mathbb{P}$ such that the section $R = H \cap Q$ satisfies the equality

$$\widetilde{R} \cap \bar{E} = \bar{B},$$

$\widetilde{R} \subset \widetilde{Q}$ is the strict transform of the section R. Since

$$\mathrm{mult}_x\, R = 2,\ \mathrm{mult}_{\bar{B}}\, R = 1 \text{ and } \sigma^{-1}(R) \in |-K_F|,$$

the divisor $\sigma^{-1}(R) \subset F$ satisfies the inequality (20). Therefore, $\sigma^{-1}(R) \neq D$. Let us form the effective cycle $\Delta = (D \circ \sigma^{-1}(R))$. Its $(-K_F)$-degree is $4n$. Furthermore,

$$\mathrm{mult}_x\, \Delta = \mathrm{mult}_x\, D + 2\,\mathrm{mult}_B\, \widetilde{D} > 4n.$$

This is impossible. Thus the case $p \notin W$ is completely excluded.

Now assume that the morphism σ is ramified at the point x, that is, $p \in W$. The notations $\varphi\colon \widetilde{F} \to F$, $\tilde{\varphi}\colon \widetilde{Q} \to Q$, $E \subset \widetilde{F}$ and $\bar{E} \subset \widetilde{Q}$ mean the same as above, but now the morphism σ does *not* identify E and $\bar{E}$. Let $p \in Q$ be a smooth point, $D \in |-nK_F|$ an effective divisor, satisfying the inequality

$$\mathrm{mult}_x\, D + \mathrm{mult}_B\, \widetilde{D} > 2n. \tag{24}$$

Let $\bar{\Lambda}$ be the pencil of sections of the quadric Q by hyperplanes in $\mathbb{P}$, containing the linear space $T_p(W \cap Q) = T_pW \cap T_pQ$. Take a general divisor $\bar{R} \in \bar{\Lambda}$ and set $R = \sigma^{-1}(\bar{R})$. Obviously, $R \in |-K_F|$ is an irreducible divisor, $E_R = \widetilde{R} \cap E$ is an irreducible quadric in E. By the symbol H_R we denote, as above, the class of a hyperplane section of the variety R. We get an effective divisor $D_R = (D \circ R) \in |nH_R|$ on R, satisfying the equalities

$$\operatorname{mult}_x D_R = 2 \operatorname{mult}_x D, \quad \operatorname{mult}_{B\cap E_R} \widetilde{D}_R = \operatorname{mult}_B \widetilde{D}.$$

Since $\operatorname{mult}_x D_R \leq 4n$, the multiplicity $\operatorname{mult}_B \widetilde{D}$ is strictly positive: the support $\operatorname{Supp} \widetilde{D}$ contains the set B. By linearity of the inequality (24) in the divisor D we can assume D_R to be irreducible. As we have just explained,

$$B \cap E_R \subset \operatorname{Supp} D_R.$$

Lemma 3.5. *The strict transform of the divisor $T = \sigma^{-1}(\bar{R} \cap T_p\bar{R})$ on $\widetilde{F}$ does not contain the entire set $B \cap E_R$.*

Proof. The claim follows from the conditions of general position for the variety F. Indeed, set $\mathbb{T} = \mathbb{P}(T_p\bar{R})$. It is easy to see that $\sigma\,|\,_R$ realizes E_R as the double cover of the space $\mathbb{T}$, branched over the quadric

$$W_T = \mathbb{P}T_p(W \cap T_p(W \cap Q))$$

(one should take into account that $T_p\bar{R} = T_p(W \cap Q)$). In the coordinate form (with respect to the coordinate system in the condition 3)) we get: the quadric E_R is given by the equation

$$y^2 = w_2(0, 0, z_3, \dots, z_{M+1})$$

in the projective space with the homogeneous coordinates

$$(y : z_3 : \dots : z_{M+1}),$$

$z_3 : \dots : z_{M+1}$ gives a system of homogeneous coordinates on $\mathbb{T}$, the covering morphism $\sigma_E \colon E_R \to \mathbb{T}$ is the restriction onto E_R of the linear projection from the point $(1 : 0 : \dots : 0)$. The set $\widetilde{T} \cap E_R$ (where $\widetilde{T}$, as usual, is the strict transform of the divisor T on $\widetilde{R}$) is given by the equation $q_2(0, 0, z_3, \dots, z_{M+1}) = 0$.

Let $\alpha y + \lambda(z_3, \dots, z_{M+1}) = 0$ be the equation of the hyperplane section $B \cap E_R$, $\alpha = 0$ or 1, $\lambda(\cdot)$ a linear form. Assume that

$$B \cap E_R \subset \widetilde{T}.$$

Then, obviously, $\sigma_E(B \cap E_R)$ is contained in the quadric $q_2\,|\,_{\mathbb{T}} = 0$. If $\alpha = 0$, then it means that the quadratic polynomial $q_2(0, 0, z_*)$ is divisible by the linear form $\lambda(z_*)$, which is impossible. If $\alpha = 1$, then it means that the quadratic polynomial $q_2(0, 0, z_*)$ up to a scalar factor is $w_2(0, 0, z_*) - \lambda^2(z_*)$, which is again impossible by the conditions of general position. Q.E.D. for the lemma.

Since, as we have just shown, $D_R \neq T$ (and both divisors are irreducible), the effective cycle $(D_R \circ T) = \Delta$ of codimension 2 on R is well defined. It is easy to see that its degree with respect to H_R is $4n$, whereas its multiplicity at the point x satisfies the inequality

$$\operatorname{mult}_x \Delta \geq 2 \operatorname{mult}_x D_R = 4 \operatorname{mult}_x D > 4n,$$

which is impossible. Thus the case of a smooth point $x \in F$, $p \in W$ is excluded.

It remains to consider the only case of the singular point $x \in F$, where $p = \sigma(x) \in W$. By the conditions of general position we get: $p \in Q$ is a smooth point on the quadric. Assume that a divisor $D \in |-nK_F|$ satisfies the inequality (22). By linearity of this inequality in the divisor D we may assume that D is a prime divisor. In the anticanonical system $|-K_F|$ consider the divisor $T = \sigma^{-1}(Q \cap T_p Q)$. Obviously, $\operatorname{mult}_x T = 4$ and by the conditions of general position $\widetilde{T} \not\supset B$. Therefore, $D \neq T$ and the effective cycle $\Delta = (D \circ T)$ is well defined. Since

$$\operatorname{mult}_x \Delta \geq 2 \operatorname{mult}_x D > 4n$$

($x \in F$ is a double point), we get a contradiction again. This completes the proof of Proposition 3.4. Propositions 3.3, 3.4 imply that for a general Fano fiber space $V/\mathbb{P}^1$ into double quadrics of index 1 and dimension $M \geq 6$ every fiber at every point satisfies the condition (LR).

Recall [1,2], that the variety V is realized as the double cover $\sigma: V \to Q^*$ of the hypersurface $Q^* \subset X = \mathbb{P}(\mathcal{E})$, branched over the divisor $W^* \cap Q^*$. The fiber space V is characterized by the triple of non-negative integral parameters (a_X, a_Q, a_W). It follows from the formula

$$(-K_V \cdot L_V^M) = 2(8 - 2a_X - 3a_Q - 4a_W)$$

that if $2a_X + 3a_Q + 4a_W \geq 8$, then the Fano fiber space $V/\mathbb{P}^1$ satisfies the K-condition: $-K \notin \operatorname{Int} A^1_{\mathrm{mov}} V$ (and even $-K \notin \operatorname{Int} A^1_+ V$).

Corollary 3.4. *For $2a_X + 3a_Q + 4a_W \geq 8$ a general Fano fiber space $V/\mathbb{P}^1$ is birationally superrigid.*

3.4. Fano double hypersurfaces. We give here one intermediate result for this class of varieties. Let $\sigma: F \to Q \subset \mathbb{P} = \mathbb{P}^{M+1}$ be a Fano double cover, where Q is a hypersurface of degree $m \geq 3$, $M \geq 6$. The morphism σ is branched over the divisor $W = W^* \cap Q$, where $W^* \subset \mathbb{P}$ is a hypersurface of degree $2l$, $m + l = M + 1$. Assume that F has at most non-degenerate quadratic singularities. Assume also that at any smooth point $x \in F$ outside the ramification divisor the variety F is regular in the sense of Definition 1 of the paper [9]. Besides, assume that the following conditions of general position are satisfied. Let $(z_1, \ldots, z_{M+1})$ be a system of affine coordinates with the origin at the point $p = \sigma(x)$,

$$f = q_1 + q_2 + \ldots + q_m$$

an equation of the hypersurface Q (so that by Definition 1 of the paper [9] the sequence $q_1,\dots,q_m$ is regular). We require that the linear span of any irreducible component of the closed set

$$q_1=q_2=q_3=0$$

in $\mathbb{C}^{M+1}$ coincides with the hyperplane $q_1=0$ (that is, the tangent hyperplane T_pQ). Furthermore, we require that the closed algebraic set

$$\sigma^{-1}(\overline{\{q_1=q_2=0\}\cap Q})\subset F$$

should be irreducible and any section of this set by an anticanonical divisor $P\in|-K_F|$, $P\ni x$, should be also irreducible and reduced.

These conditions are completely similar to the regularity conditions (R1.1)-(R1.3) of the paper [3].

Proposition 3.5. *In these assumptions the variety F satisfies the condition (LR) at any smooth point $x\in F$ outside the ramification divisor.*

To obtain the **proof**, we repeat the arguments in [3, Sec. 2.1] word for word, replacing (in the concluding part of the proof) the hypertangent technique of the paper [5] (which is used in [3]) by the hypertangent technique of the paper [9]. No other changes are necessary. Q.E.D. for the proposition.

Corollary 3.5. *Assume that the Fano fiber space $V/\mathbb{P}^1$ into double hypersurfaces of index 1 satisfies the conditions of general position, formulated above, at every smooth point of every fiber, lying outside the ramification divisor. Then the centre $B=\mathrm{centre}(E)$ of any maximal singularity E of a movable linear system $\Sigma\subset|-nK_V+lF|$ with $l\in\mathbb{Z}_+$ either is contained in the ramification divisor, or coincides with a singular point of a fiber.*

3.5. Varieties with a pencil of Fano hypersurfaces. To conclude this section, let us consider the class of Fano fiber spaces, that were studied in [7]. Let us show how, at the expense of making the conditions of general position stronger, we can use the linear method to simplify the proof of birational rigidity.

Let $\mathcal{F}=\mathbb{P}(H^0(\mathbb{P},\mathcal{O}_{\mathbb{P}}(M)))$, $\mathbb{P}=\mathbb{P}^M$, be the set of Fano hypersurfaces of index 1. In [3, Sec. 2] the following fact was obtained.

Proposition 3.6. *There exists an open subset (in the sense of Zariski topology) $\mathcal{F}_{\mathrm{reg}}\subset\mathcal{F}$ such that any hypersurface $F\in\mathcal{F}_{\mathrm{reg}}$ is smooth and satisfies the condition (LR) at every point.*

Singular hypersurfaces form a closed subset in $\mathcal{F}$ of codimension 1, so that the complement $\mathcal{F}\setminus\mathcal{F}_{\mathrm{reg}}$ is of codimension 1, too. However, from the computations of [3, Sec. 2.3] one can see that violation of the regularity conditions $(R1.1)-(R1.3)$ (which define the set $\mathcal{F}_{\mathrm{reg}}$) at at least one smooth point $x\in F$ defines a closed subset of $\mathcal{F}$ of codimension at least 2. Therefore, the following fact takes place.

Proposition 3.7. *There exists an open (in the sense of Zariski topology) subset $\mathcal{F}^+_{\mathrm{reg}}\subset\mathcal{F}$ such that*

(i) codim$(\mathcal{F}\setminus\mathcal{F}^+_{\mathrm{reg}})\geq 2$,

(ii) *any hypersurface $\mathcal{F}\in\mathcal{F}^+_{\mathrm{reg}}$ has at most non-degenerate quadratic singularities, and moreover, if $x\in F$ is such a point, then the sequence of homogeneous* *polynomials*

$$q_2, \dots, q_M$$

is regular in $\mathcal{O}_{x,\mathbb{P}}$*, where* $f = q_2 + \ldots + q_M$ *is the polynomial, defining the hypersurface* F *in a system of affine coordinates with the origin at the point* x*,*

(iii) *any hypersurface* $\mathcal{F} \in \mathcal{F}^+_{\mathrm{reg}}$ *satisfies the condition (LR) at every smooth point* $x \in F$.

It turns out that non-degenerate quadratic singularities, satisfying the regularity condition (ii), generate no serious problems.

Proposition 3.8. *Any hypersurface* $\mathcal{F} \in \mathcal{F}^+_{\mathrm{reg}}$ *satisfies the condition (LR) at every point* $x \in F$.

Proof. We need to check only the case of a quadratic singularity $x \in F$. Let

$$\varphi\colon \widetilde{\mathbb{P}} \to \mathbb{P} \text{ and } \varphi_F\colon \widetilde{F} \to F$$

be the blow up of the point $x \in \mathbb{P}$ and its restriction onto F, respectively,

$$E = \varphi^{-1}(x) \subset \widetilde{\mathbb{P}} \text{ and } E_F = \varphi_F^{-1}(x) = \widetilde{F} \cap E$$

the exceptional divisors, $E_F \subset E$ a smooth quadric. Assume that the divisor $D \in |-nK_F|$ satisfies the inequality (22). By linearity of this inequality in D we may assume that the divisor D is irreducible. We get

$$B = E_F \cap \Pi,$$

where $\Pi \subset E$ is a hyperplane. Let $P \subset \mathbb{P}$, $P \ni x$ be the only hyperplane, for which the equality $\widetilde{P} \cap E = \Pi$ holds, $\widetilde{P} \subset \widetilde{\mathbb{P}}$ is the strict transform. Set $R = P \cap F$. This is a divisor on F.

Obviously, R is irreducible and satisfies the inequality (20) with $n = 1$ and R instead of D. Therefore, $R \neq D$, so that the effective cycle $Y = (R \circ D)$ of codimension 2 is well defined. This cycle (we may assume it to be irreducible) satisfies the inequality

$$\frac{\mathrm{mult}_x}{\deg} Y > \frac{4}{M}.$$

Now the proof is completed in the standard way: let $\Lambda_2, \dots, \Lambda_{M-1}$ be the hypertangent linear systems. By the regularity condition we get

$$\mathrm{codim}_F \operatorname{Bs} \Lambda_k = k - 1.$$

Let $(D_4, \dots, D_{M-1}) \in \Lambda_4 \times \ldots \times \Lambda_{M-1}$ be a generic set of hypertangent divisors. Obviously, $Y \not\subset \operatorname{Supp} \Lambda_4$. Let us construct by induction a sequence of irreducible subvarieties Y_i, $i = 2, \dots, M-2$, where $\mathrm{codim}_F Y_i = i$, $Y_2 = Y$, satisfying the inequalities

$$\frac{\mathrm{mult}_x}{\deg} Y_{i+1} \geq \frac{\mathrm{mult}_x}{\deg} Y_i \cdot \frac{i+3}{i+2},$$

Y_{i+1} is an irreducible component of the effective cycle $(Y_i \circ D_{i+2})$. This is possible, because

$$\operatorname{codim}_F Y_i = i < \operatorname{codim}_F \operatorname{Bs} \Lambda_{i+2} = i + 1,$$

so that $Y_i \not\subset \operatorname{Supp} D_{i+2}$ for a general divisor $D_{i+2} \in \Lambda_{i+2}$. The curve $C = Y_{M-2}$ satisfies the inequality

$$\frac{\operatorname{mult}_x}{\deg} C > \frac{4}{M} \cdot \frac{5}{4} \cdot \ldots \cdot \frac{M}{M-1} = 1,$$

which is impossible. The contradiction obtained above completes the proof of Proposition 3.8.

Corollary 3.6. *Assume that for every fiber $F_t = \pi^{-1}(t)$ of the Fano fiber space $\pi: V \to \mathbb{P}^1$ we have $F_t \in \mathcal{F}^+_{\text{reg}}$. Then:*

(i) *for any movable linear system $\Sigma \subset |-nK_V + lF|$ with $l \in \mathbb{Z}_+$ the pair $(V, \frac{1}{n}\Sigma)$ is canonical,*

(ii) *if the variety V satisfies the K-condition, then it is birationally superrigid.*

Remark 3.1. The linear method makes it possible to simplify the proofs of birational rigidity in [7,10]. However, the technique used for proving the conditions of the type (L), works for varieties of sufficiently high (at least 3) dimension. For this reason, the linear method in its present form gives not much for fibrations into curves [11,12] and surfaces [13-15]. For instance, in the case of one-dimensional fibers, restricting the linear system onto a conic, intersecting the centre of a maximal singularity, we get no new information. In the case of two-dimensional fibers (del Pezzo surfaces) the situation is better, however, it is easy to show that under any conditions of general position there are a fiber F, a point $x \in F$ and a divisor $D \in |-nK_F|$ such that the pair $(F, \frac{1}{n}D)$ is not log canonical at the point x. Thus it is impossible to exclude a maximal singularity with the centre at this point by the linear method. That the linear method works in smaller dimensions with difficulties, can be seen already in the absolute case (that is, for Fano varieties): the condition (L) remains unproved for three-dimensional quartics and four-dimensional quintics, whereas the quadratic method works quite successfully for these varieties [16,17].

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