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$\mathbb{G}_a$-ACTIONS ON AFFINE CONES

TAKASHI KISHIMOTO, YURI PROKHOROV, AND MIKHAIL ZAIDENBERG

Abstract. An affine algebraic variety X is called *cylindrical* if it contains a principal Zariski dense open cylinder $U \simeq Z \times \mathbb{A}^1$. A polarized projective variety (Y, H) is called *cylindrical* if it contains a cylinder $U = Y \setminus \operatorname{supp} D$, where D is an effective $\mathbb{Q}$-divisor on Y such that $[D] \in \mathbb{Q}_+[H]$ in $\operatorname{Pic}_{\mathbb{Q}}(Y)$. We show that cylindricity of a polarized projective variety is equivalent to that of a certain Veronese affine cone over this variety. This gives a criterion of existence of a unipotent group action on an affine cone. In [KPZ$_1$]-[KPZ$_3$] this criterion is applied to the question of existence of additive group actions on certain affine cones over del Pezzo surfaces and Fano threefolds.

Contents

Introduction

We fix an algebraically closed field $\Bbbk$ of characteristic zero. We let $\mathbb{G}_a = \mathbb{G}_a(\Bbbk)$. We wonder as to when the affine cone over an irreducible, normal projective variety over $\Bbbk$ admits a non-trivial action of a unipotent group. Since any unipotent group contains a one parameter unipotent subgroup, instead of considering general unipotent group actions we stick to the $\mathbb{G}_a$-actions. Our main purpose in this paper is to provide a geometric criterion for existence of such an action (see Theorem 2.2 and Corollary 2.12). The former version of such a criterion in [KPZ$_1$] involved some unnecessary assumptions. In Theorem 2.2 we remove these assumptions. What is more important, we extend our criterion so that it can be applied more generally to affine quasicones. An affine quasicone is an affine variety V equipped with a $\mathbb{G}_m$-action such that the fixed point set $V^{\mathbb{G}_m}$ attracts the whole V. Thus the variety $Y = (V \setminus V^{\mathbb{G}_m})/\mathbb{G}_m$ is projective over the affine variety $S = V^{\mathbb{G}_m}$. We assume in this paper that Y is normal. Our criterion is formulated in terms of a geometric property called *cylindricity*, which merits to be studied on its own sake.

0.1. Cylindricity. Let us fix the notation. For two $\mathbb{Q}$-divisors H and H' on a quasiprojective variety Y we write $H \sim H'$ if H and H' are linearly equivalent that is,

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$H - H' = \operatorname{div}(f)$ for a rational function f on Y. We write $[H'] \in \mathbb{Q}_+[H]$ in $\operatorname{Pic}_{\mathbb{Q}}(Y)$ meaning that $H' \sim \frac{p}{q}H$ for some coprime positive integers p and q.

Definition 0.2 (cf. [KPZ$_1$, 3.1.4])**.** Let Y be a quasiprojective variety over $\Bbbk$ polarized by an ample $\mathbb{Q}$-divisor $H \in \operatorname{Div}_{\mathbb{Q}}(Y)$. We say that the pair (Y, H) is *cylindrical* if there exists an effective $\mathbb{Q}$-divisor D on Y such that $[D] \in \mathbb{Q}_+[H]$ in $\operatorname{Pic}_{\mathbb{Q}}(Y)$ and $U = Y \setminus \operatorname{supp} D$ is a *cylinder* i.e.

$$U \simeq Z \times \mathbb{A}^1$$

for some variety Z. Here U and Z are quasiaffine varieties. Such a cylinder U is called *H-polar* in [KPZ$_1$, 3.1.7]. Notice that the cylindricity of (Y, H) depends only on the ray $\mathbb{Q}_+[H]$ generated by H in $\operatorname{Pic}_{\mathbb{Q}} Y$.

Remark 0.3. The pair (Y, H) can admit several essentially different cylinders. For instance, let $Y = \mathbb{P}^1$ and H is a $\mathbb{Q}$-divisor on Y of positive degree. Then any divisor $D = rP$, where $P \in \mathbb{P}^1$ and $r \in \mathbb{Q}_+$, defines an H-polar cylinder on Y.

Definition 0.4. An affine variety X is called *cylindrical* if it contains a principal cylinder

$$\mathbb{D}(h) := X \setminus \mathbb{V}(h) \simeq Z \times \mathbb{A}^1, \quad \text{where} \quad \mathbb{V}(h) = h^{-1}(0),$$

for some variety Z and some regular function $h \in \mathcal{O}(X)$. Hence U and Z are affine varieties.

The cylindricity of affine varieties is important due to the following well known fact (see e.g. [KPZ$_1$, Proposition 3.1.5]).

Proposition 0.5. *An affine variety $X = \operatorname{Spec} A$ over $\Bbbk$ is cylindrical if and only if it admits an effective $\mathbb{G}_a$-action, if and only if $\operatorname{LND}(A) \neq \emptyset$, where $\operatorname{LND}(A)$ stands for the set of all nonzero locally nilpotent derivations of A.*

The proof is based upon the slice construction, which we recall in subsection 1.1. In Section 1 we gather necessary preliminaries on positively graded rings. In particular, we give a graded version of the slice construction, and recall the DPD (Dolgachev-Pinkham-Demazure) presentation of a positively graded affine domain A over $\Bbbk$ [1] in terms of an ample $\mathbb{Q}$-divisor H on the variety $Y = \operatorname{Proj} A$. In Section 2 we prove the main theorem (see Theorem 2.2).

Theorem 0.6. *Let $A = \bigoplus_{\nu \geq 0} A_\nu$ be a positively graded affine domain over $\Bbbk$. If the affine quasicone $V = \operatorname{Spec} A$ over the variety $Y = \operatorname{Proj} A$ is cylindrical then the associated pair (Y, H) is. Vice versa, if the pair (Y, H) is cylindrical then for some $d \in \mathbb{N}$ the Veronese cone $V^{(d)} = \operatorname{Spec} A^{(d)}$ is cylindrical, where $A^{(d)} = \bigoplus_{\nu \geq 0} A_{d\nu}$.*

In Lemma 2.10 we precise the range of values of d satisfying the second assertion. In particular, it holds with $d = 1$ provided that $H \in \operatorname{Div}(Y)$ (see Corollary 2.12).

In Section 3 we provide several examples that illustrate our criterion. Besides, we discuss a possibility to lift a $\mathbb{G}_a$-action on a Veronese cone $V^{(d)}$ over Y to the affine cone V over Y.

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[1] I.e. A is a finitely generated $\mathbb{N}$-graded $\Bbbk$-algebra with a unit element and without zero divisors. In particular, A is Noetherian.

Liendo and Alexandr Perepechko were especially pertinent and allowed us to improve significantly the presentation.

1. Preliminaries

1.1. Slice construction. Let A be an affine domain over $\Bbbk$, and let $\partial \in \mathrm{LND}(A)$. The filtration

$$A^\partial = \ker \partial \subsetneq \ker \partial^2 \subsetneq \ker \partial^3 \subsetneq \ldots \tag{1}$$

being strictly increasing one can find an element $g \in \ker \partial^2 \setminus \ker \partial$. Letting $h = \partial g \in \ker \partial \cap \operatorname{im} \partial$, where $h \neq 0$, one considers the localization $A_h = A[h^{-1}]$ and the principal Zariski dense open subset

$$\mathbb{D}(h) = X \setminus \mathbb{V}(h) \simeq \operatorname{Spec} A_h\,, \quad \text{where} \quad \mathbb{V}(h) = h^{-1}(0)\,.$$

The derivation ∂ extends to a locally nilpotent derivation on A_h denoted by the same letter. The element $s = g/h \in A_h$ is a slice of ∂ that is, $\partial(s) = 1$. Hence

$$A_h = A_h^\partial[s], \qquad \text{where} \quad \partial = d/ds \quad \text{and} \quad A_h^\partial \simeq A_h/(s)$$

('Slice Theorem', [Fr, Corollary 1.22]). Thus $\mathbb{D}(h) \simeq Z \times \mathbb{A}^1$ is a principal cylinder in X over $Z = \operatorname{Spec} A_h^\partial$. The $\mathbb{G}_a$-action on $\mathbb{D}(h)$ associated with ∂ is defined by the translations along the second factor. The natural projection $p_1 : \mathbb{D}(h) \to Z$ identifies $\mathbb{V}(g) \setminus \mathbb{V}(h) \subseteq \mathbb{D}(h)$ with Z. Choosing $f \in A_h^\partial = \mathcal{O}(Z)$ such that $\operatorname{Sing}(Z) \subseteq \mathbb{V}(f)$ we can replace g and h by fg and fh, respectively, so that the slice s remains the same, but the new cylinder $\mathbb{D}(fh)$ over an affine variety $Z' = \mathbb{D}(fh)/\mathbb{G}_a$ is smooth.

1.2. Graded slice construction. Suppose that the ring A is graded, and consider $\eta \in \mathrm{LND}(A)$. Decomposing η into a sum of homogeneous components

$$\eta = \sum_{i=1}^n \eta_i, \quad \text{where} \quad \eta_i \in \mathrm{Der}(A), \quad \deg \eta_i < \deg \eta_{i+1} \quad \forall i, \quad \text{and} \quad \eta_n \neq 0\,,$$

we let $\partial = \eta_n$ be the principal homogeneous component of η. Then ∂ is again locally nilpotent and homogeneous (see [Re]). Hence all kernels in (1) are graded. So one can choose homogeneous elements g, h, and s as above. With this choice we call the construction of a cylinder a *graded slice construction*.

1.3. Graded rings and associated schemes. We recall some well known facts on positively graded rings and associated schemes. The presentation below is borrowed from [De], [Fl, sect. 2], [FZ$_1$, §2.1], and [Do, Lecture 3].

Notation 1.4. Given a graded affine domain $A = \bigoplus_{\nu \in \mathbb{Z}} A_\nu$ over $\Bbbk$ the group $\mathbb{G}_m$ acts on A via $t.a = t^\nu a$ for $a \in A_\nu$. This action is effective if and only if the *saturation index* $e(A)$ equals 1, where

$$e(A) = \gcd\{\nu \mid A_\nu \neq (0)\}\,.$$

If A is positively graded i.e. $A_{\le 0} = (0)$ then the associated scheme $Y = \operatorname{Proj} A$ is projective over[2] the affine scheme $S = \operatorname{Spec} A_0$ [EGA, II]. Furthermore, Y is covered by the affine open subsets

$$\mathbb{D}_+(f) = \mathbb{D}_+(fA) := \{\mathfrak{p} \in \operatorname{Proj} A : f \notin \mathfrak{p}\} \cong \operatorname{Spec} A_{(f)}\,,$$

[2]Notice that A_0 can be here an arbitrary affine domain.

where $f \in A_{>0}$ is a homogeneous element and $A_{(f)} = (A_f)_0$ stands for the degree zero part of the localization A_f. The affine variety $V = \operatorname{Spec} A$ is called a *quasicone* over Y with *vertex* $\mathbb{V}(A_{>0})$ and with *punctured quasicone* $V^* = V \backslash \mathbb{V}(A_{>0})$, where $\mathbb{V}(I)$ stands for the zero set of an ideal $I \subseteq A$. For a homogeneous ideal $I \subseteq A$, $\mathbb{V}_+(I)$ stands for its zero set in $Y = \operatorname{Proj} A$. There is a natural surjective morphism $\pi : V^* \to Y$. If $A = A_0[A_1]$ i.e., A is generated as an A_0-algebra by the elements of degree 1 then $V^* \to Y = \operatorname{Proj} A$ is a locally trivial $\mathbb{G}_{\mathrm{m}}$-bundle. In the general case the following holds.

Lemma 1.5. $\operatorname{Proj} A \cong V^*/\mathbb{G}_{\mathrm{m}}$.

Proof. Indeed, V^* is covered by the $\mathbb{G}_{\mathrm{m}}$-invariant affine open subsets $\mathbb{D}(f) = \operatorname{Spec} A_f$, where $f \in A_d$ with $d > 0$. Since $(A_f)^{\mathbb{G}_{\mathrm{m}}} = A_{(f)} = (A_f)_0$ we have $\mathbb{D}_+(f) = \mathbb{D}(f)/\mathbb{G}_{\mathrm{m}}$ and the lemma follows. □

Remark 1.6. Assuming that A is a domain over $\Bbbk$ and $e(A) = 1$ one can find a pair of nonzero homogeneous elements $a \in A_\nu$ and $b \in A_\mu$ of coprime degrees. Let $p, q \in \mathbb{Z}$ be such that $p\nu + q\mu = 1$. Then the localization A_{ab} is graded, the element $u = a^p b^q \in (A_{ab})_1$ is invertible, and $A_{ab} = A_{(ab)}[u, u^{-1}]$. This gives a trivialization of the orbit map $\pi : V^* \to Y = \operatorname{Proj} A$ over the principal open set $\mathbb{D}_+(ab) \subseteq Y$:

$$\mathbb{D}(ab) = \pi^{-1}(\mathbb{D}_+(ab)) \simeq \mathbb{D}_+(ab) \times \mathbb{A}^1_*, \quad \text{where} \quad \mathbb{A}^1_* = \mathbb{A}^1 \setminus \{0\}\,.$$

1.7. Cyclic quotient construction. Let $h \in A_m$ be a homogeneous element of degree $m > 0$, and let $F = A/(h-1)$. For $a \in A$ we let $\bar{a}$ denote the class of a in F. The projection $\rho : A \to F$, $a \mapsto \bar{a}$, extends to the localization A_h via $\rho(a/h^l) = \rho(a) = \bar{a}$. The cyclic group $\boldsymbol{\mu}_m \subseteq \mathbb{G}_{\mathrm{m}}$ of the mth roots of unity acts on F effectively and so defines a $\mathbb{Z}_m$-grading

$$F = \bigoplus_{[i] \in \mathbb{Z}_m} F_{[i]}\,,$$

where $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ and $[i] \in \mathbb{Z}_m$ stands for the residue class of $i \in \mathbb{Z}$ modulo m. It is easily seen that the morphism $\rho : A_h \to F$ restricts to an isomorphism $\rho : A_{(h)} \xrightarrow{\simeq} F_{[0]}$. This yields a cyclic quotient

$$Y_h \to Y_h/\boldsymbol{\mu}_m = \mathbb{D}_+(h) \subseteq Y\,, \quad \text{where} \quad Y_h = h^{-1}(1) \subseteq V\,,$$
$$V = \operatorname{Spec} A, \quad \text{and} \quad \mathbb{D}_+(h) = \operatorname{Spec} A_{(h)} \simeq \operatorname{Spec} F_{[0]}\,.$$

Let ∂ be a homogeneous locally nilpotent derivation of A. If $h \in A^\partial_m$ then the principal ideal $(h-1)$ of A is ∂-stable. Hence the hypersurface $Y_h = \mathbb{V}(h-1)$ is stable under the $\mathbb{G}_{\mathrm{a}}$-action on V generated by ∂, and ∂ induces a homogeneous locally nilpotent derivation $\bar{\partial}$ of the $\mathbb{Z}_m$-graded ring F. The kernel $F^{\bar{\partial}} = \ker \bar{\partial}$ is a $\mathbb{Z}_m$-graded subring of F:

$$F^{\bar{\partial}} = \bigoplus_{[i] \in \mathbb{Z}_m} F^{\bar{\partial}}_{[i]}\,, \quad \text{where} \quad F^{\bar{\partial}}_{[i]} = F_{[i]} \cap F^{\bar{\partial}}\,.$$

Assume further that F is a domain. Then the set $\{[i] \in \mathbb{Z}_m \mid F_{[i]} \neq (0)\}$ is a cyclic subgroup, say, $\boldsymbol{\mu}_n \subseteq \boldsymbol{\mu}_m$. Letting $k = m/n$ we can write

$$F^{\bar{\partial}} = \bigoplus_{i=0}^{n-1} F^{\bar{\partial}}_{[ki]}\,.$$

Lemma 1.8. *We have* $k = e(A^\partial)(= e(A_h^\partial))$ *and* $\gcd(k, d) = 1$*, where* $d = -\deg \partial$.

Proof. The second assertion follows from the first since $\gcd(d,\, e(A^\partial)) = 1$. Indeed, notice that for any nonzero homogeneous element $a \in A_j$ $(j > 0)$ there is $r \in \mathbb{N}$ such that $\partial^{(r)} g \in A^\partial_{j-rd} \setminus \{0\}$. Hence $j = rd + se(A^\partial)$ for some $s \in \mathbb{Z}$. Since by our assumption $e(A) = 1$ it follows that $\mathbb{Z} = \langle d,\, e(A^\partial)\rangle$ and so $\gcd(d,\, e(A^\partial)) = 1$.

To prove the first equality we let $g \in A_j$ be such that $\bar{g} \in F^{\bar\partial}$, where $j > 0$,. The restriction $g|_{Y_h}$ being invariant under the induced $\mathbb{G}_a$-action on Y_h, this restriction is constant on any $\mathbb{G}_a$-orbit in Y_h. For a general point $x \in D(h) \subseteq V$ there exists $\lambda \in \mathbb{G}_m$ such that $h(\lambda.x) = 1$. Since ∂ is homogeneous the $\mathbb{G}_m$-action on V induced by the grading normalizes the $\mathbb{G}_a$-action. Therefore $\lambda.(\mathbb{G}_a.x) = \mathbb{G}_a.(\lambda.x) \subseteq Y_h$ and so $g|_{G_a.(\lambda.x)}$ is constant. Hence also $g|_{\lambda.(\mathbb{G}_a.x)} = \lambda^j g|_{\mathbb{G}_a.x}$ is. It follows that $g \in A^\partial_j$. Clearly $\rho(A^\partial) \subseteq F^{\bar\partial}$, so finally $\rho(A^\partial) = F^{\bar\partial}$. Thus $k = e(A^\partial)$. □

1.9. Quasicones and ample $\mathbb{Q}$-divisors. To any pair (Y, H), where $Y \to S$ is a proper normal integral S-scheme, $S = \operatorname{Spec} A_0$ is a normal affine variety over $\mathbb{k}$, and H is an ample $\mathbb{Q}$-divisor on Y, one can associate a positively graded integral domain over $\mathbb{k}$,

$$A = A(Y, H) = \bigoplus_{\nu \geq 0} A_\nu, \quad \text{where} \quad A_\nu = H^0(Y, \mathcal{O}_Y(\lfloor \nu H \rfloor))\,. \tag{2}$$

The algebra A has saturation index $e(A) = 1$, is finitely generated and normal[3]. So the associated affine quasicone $V = \operatorname{Spec} A$ over $Y = \operatorname{Proj} A$ is normal.

Vice versa, every affine quasicone $V = \operatorname{Spec} A$, where A is a normal affine positively graded $\mathbb{k}$-domain of dimension at least 2 and with saturation index 1 arises in this way ([De, 3.5]). The corresponding ample $\mathbb{Q}$-divisor H on Y is defined uniquely by the quasicone V up to the linear equivalence[4]. In particular, its fractional part $\{H\} = H - \lfloor H \rfloor$ is uniquely determined by V; it is called the *Seifert divisor* of the quasicone V, see [Do, 3.3.2].

1.10. Let again $A = A(Y, H)$ be as in (2). By virtue of Remark 1.6 there exists on V a homogeneous rational function $u \in (\operatorname{Frac} A)_1$ of degree 1. Notice that the divisor $\operatorname{div} u$ on V is $\mathbb{G}_m$-invariant. Choosing this function suitably one can achieve that $\operatorname{div}(u|_{V^*}) = \pi^*(H)$, where $\pi : V^* \to Y$ is the quotient by the $\mathbb{G}_m$-action (see Lemma 1.5).

Furthermore, $\operatorname{Frac} A = (\operatorname{Frac} A)_0(u)$. So any homogeneous rational function $f \in (\operatorname{Frac} A)_d$ of degree d on V can be written as ψu^d for some $\psi \in (\operatorname{Frac} A)_0$. For any $d > 0$ the $\mathbb{Q}$-divisor class $[dH]$ is ample. It is invertible and trivial on any open set $\mathbb{D}_+(ab) \subseteq Y$ as in Remark 1.6.

A rational function on Y can be lifted to a $\mathbb{G}_m$-invariant rational function on V. Thus the field $(\operatorname{Frac} A)_0$ can be naturally identified with the function field of Y. Under this identification we get the equalities

$$A_\nu = H^0(Y, \mathcal{O}_Y(\lfloor \nu H \rfloor)) u^\nu \quad \forall \nu \geq 0\,.$$

1.11. Given a normal positively graded $\mathbb{k}$-algebra $A = \bigoplus_{\nu \geq 0} A_\nu$ a divisor H' on Y satisfying $A_\nu \simeq H^0(Y, \mathcal{O}_Y(\lfloor \nu H' \rfloor))$ $\forall \nu \geq 0$ can be defined as follows (see [De, 3.5]).

[3]See [Ha, Ch. II, Exercice 5.14(a)], [De, 3.1], [Do, 3.3.5], and [AH, Theorem 3.1].

[4]See [Do, Theorem 3.3.4]; cf. also [FZ$_3$] and [AH, Theorem 3.4].

Choose a homogeneous rational function $u' \in (\operatorname{Frac} A)_1$ on V. Write $\operatorname{div}(u'|_{V^*}) = \sum_i p_i \Delta_i$, where the components Δ_i are prime $\mathbb{G}_{\mathrm{m}}$-stable Weil divisors on V^* and $p_i \in \mathbb{Z} \setminus \{0\}$. For every irreducible component Δ_i of $\operatorname{div}(u'|_{V^*})$ we have $\bar{\Delta}_i = \operatorname{Spec}(A/I_{\bar{\Delta}_i})$, where $\bar{\Delta}_i$ is the closure of Δ_i in V and $I_{\bar{\Delta}_i}$ is the graded prime ideal of $\bar{\Delta}_i$ in A. Thus the affine domain $A/I_{\bar{\Delta}_i}$ is graded. Let $q_i = e(A/I_{\bar{\Delta}_i})$. Then $q_i > 0\ \forall i$, the Weil $\mathbb{Q}$-divisor $H' = \sum_i \frac{p_i}{q_i} \pi_* \Delta_i$ satisfies $\pi^* H' = \operatorname{div}(u'|_{V^*})$, and $A \simeq A(Y, H')$. Furthermore, for every component Δ_i the divisor $p_i \pi_* \Delta_i$ is Cartier (see [De, Proposition 2.8]).

If $A = A(Y, H)$ for a $\mathbb{Q}$-divisor H on Y then $H' \sim H$ and $\pi^*(H' - H) = \operatorname{div}(\varphi|_{V^*})$, where $\varphi = u'/u \in (\operatorname{Frac} A)_0$. [5]

1.12. We keep the notation of 1.9. For some $d > 0$ the dth *Veronese subring* of A,

$$A^{(d)} = A(Y, dH) = \bigoplus_{\nu \geq 0} A_{\nu d}\,,$$

is generated over A_0 by its first graded piece A_d:

$$A^{(d)} = A_0[A_d]$$

(see Proposition 3.3 in [Bou, Ch. III, §1] or [Do, Lemma 3.1.3]). This leads to an embedding of $Y \simeq \operatorname{Proj} A^{(d)}$ in the projective space $\mathbb{P}^N_S \simeq \mathbb{P}_{A_0}(A_d)$ and so H is ample over S. Since S is Noetherian, H is ample over $\operatorname{Spec} \Bbbk$.

We call $V^{(d)} = \operatorname{Spec} A^{(d)}$ the dth *Veronese quasicone* associated to the pair (Y, H).

1.13. The discussion in 1.10 and 1.11 leads to the following presentations:

$$A = A(Y, H) = \bigoplus_{\nu \geq 0} H^0(Y, \mathcal{O}_Y(\lfloor \nu H \rfloor)) u^\nu = \bigoplus_{\nu \geq 0} H^0(Y, \mathcal{O}_Y(\lfloor \nu H' \rfloor)) u'^\nu = A(Y, H')\,,$$

where $H' \sim H$ and $u'/u \in (\operatorname{Frac} A)_0$ (see 1.10) is such that $\pi^*(H' - H) = \operatorname{div}(u'/u)$.

Remark 1.14 (*Polar cylinders).* We keep the notation as in 1.13. Assume that for some nonzero homogeneous element $f \in A_\nu$, where $\nu > 0$, the open set $\mathbb{D}_+(f) \subseteq Y$ is a cylinder i.e., $\mathbb{D}_+(f) \simeq Z \times \mathbb{A}^1$ for some variety Z. Then this cylinder is H-polar. Indeed, let $n \in \mathbb{N}$ be such that $n\nu H$ is a Cartier divisor on Y. We have $f^n \in A_{n\nu} = H^0(Y, \mathcal{O}_Y(n\nu H)) u^{n\nu}$. The rational function $f^n u^{-n\nu} \in (\operatorname{Frac} A)_0$ being $\mathbb{G}_{\mathrm{m}}$-invariant it descends to a rational function, say, ψ on Y such that

$$D := \operatorname{div} \psi + n\nu H = \pi_* \operatorname{div}(f^n) \geq 0\,.$$

Hence $D \in n\nu[H]$ is an effective Cartier divisor on Y with $\operatorname{supp} D = \mathbb{V}_+(f^n) = \mathbb{V}_+(f)$. Therefore the cylinder $\mathbb{D}_+(f) = \operatorname{Spec} A_{(f)}$ is H-polar.

1.15. Generalized cones. A quasicone $V = \operatorname{Spec} A$ is called a *generalized cone* if $A_0 = \Bbbk$ so that $\operatorname{Spec} A_0$ is reduced to a point. Let us give the following example.

Example 1.16 (see e.g. [KPZ$_1$])**.** Let (Y, H) be a polarized projective variety over $\Bbbk$, where $H \in \operatorname{Div}(Y)$ is ample. Consider the total space $\tilde{V}$ of the line bundle $\mathcal{O}_Y(-H)$ with zero section $Y_0 \subseteq \tilde{V}$. We have $\mathcal{O}_{Y_0}(Y_0) \simeq \mathcal{O}_Y(-H)$ upon the natural identification of Y_0 with Y. Hence there is a birational morphism $\varphi : \tilde{V} \to V$ contracting Y_0. The resulting affine variety $V = \operatorname{cone}_H(Y)$ is called the *generalized affine cone over* (Y, H) *with vertex* $\bar{0} = \varphi(Y_0) \in V$. It comes equipped with an effective $\mathbb{G}_{\mathrm{m}}$-action induced by the standard $\mathbb{G}_{\mathrm{m}}$-action on the total space $\tilde{V}$ of the line bundle $\mathcal{O}_Y(-H)$. The

[5] Notice that $(\operatorname{Frac} A)_0 = \operatorname{Frac} A_0$ only in the case where $\dim_S Y = 0$.

coordinate ring $A = \mathcal{O}(V)$ is positively graded: $A = \bigoplus_{\nu\geq 0} A_\nu$, and the saturation index $e(A)$ equals to 1. So the graded pieces A_ν with $\nu \gg 0$ are all nonzero and the induced representation of $\mathbb{G}_m$ on A is faithful. The quotient

$$Y = \operatorname{Proj} A = V^*/\mathbb{G}_m\,, \quad \text{where} \quad V^* = V \setminus \{\bar{0}\}\,,$$

can be embedded into a weighted projective space $\mathbb{P}^n(k_0, \ldots, k_n)$ by means of a system of homogeneous generators $(a_0, \ldots, a_n)$ of A, where $a_i \in A_{k_i}$, $i = 0, \ldots, n$.

Remarks 1.17. 1. Assume that $A_0 = \mathbb{k}$ and V is normal. According to 1.9-1.13,

$$(3) \qquad A = \bigoplus_{\nu\geq 0} H^0(Y, \mathcal{O}_Y(\nu H))u^\nu \quad \text{i.e.} \quad A_\nu = H^0(Y, \mathcal{O}_Y(\nu H))u^\nu \quad \forall \nu \geq 0\,,$$

where $u \in (\operatorname{Frac} A)_1$ is such that $\operatorname{div}(u|_{V^*}) = \pi^* H$. Since H is ample this ring is finitely generated (see e.g. Propositions 3.1 and 3.2 in [Pr]).

2. If the polarization H is very ample then $A = A_0[A_1]$ and the affine variety V coincides with the usual affine cone over Y embedded in $\mathbb{P}^n$ by the linear system $|H|$. In this case the $\mathbb{G}_m$-action on V^* is free. However, (3) holds if and only if V is normal that is $Y \subseteq \mathbb{P}^n$ is projectively normal.

2. The criterion

2.1. In this section we fix the following setup. Letting $A = \bigoplus_{\nu\geq 0} A_\nu$ be a positively graded normal affine domain over $\mathbb{k}$ with $e(A) = 1$ we consider the affine quasicone $V = \operatorname{Spec} A$ and the variety $Y = \operatorname{Proj} A$ projective over the affine scheme $S = \operatorname{Spec} A_0$. We let $\pi : V^* \to Y$ be the projection to the geometric quotient of V^* by the natural $\mathbb{G}_m$-action. We can write

$$A = A(Y, H) = \bigoplus_{\nu\geq 0} A_\nu, \quad \text{where} \quad A_\nu = H^0(Y, \mathcal{O}_Y(\lfloor \nu H \rfloor))u^\nu$$

with an ample $\mathbb{Q}$-divisor H on Y such that $\pi^* H = \operatorname{div}(u|_{V^*})$ for some homogeneous rational function $u \in (\operatorname{Frac} A)_1$ (see 1.13).

Notice that in the 'parabolic case' where $\dim_S Y = 0$ there exists a homogeneous locally nilpotent derivation on A 'of fiber type' (that is, an A_0-derivation), whatever is the affine variety $S = \operatorname{Spec} A_0$, see [Li$_2$, Corollary 2.8]. In contrast, such a derivation does not exist if $\dim_S Y \geq 1$. We suppose in the sequel that $\dim_S Y \geq 1$.

Given $d > 0$ we consider the associated Veronese cone $V^{(d)} = \operatorname{Spec} A^{(d)}$, where $A^{(d)} = \bigoplus_{\nu\geq 0} A_{\nu d}$.

The following criterion is inspired by Theorem 3.1.9 in [KPZ$_1$].

Theorem 2.2. *Let the notation and assumptions be as in 2.1 above. If the affine quasicone $V = \operatorname{Spec} A$ is cylindrical then the pair (Y, H) is. Vice versa, if the pair (Y, H) is cylindrical then for some $d \in \mathbb{N}$ the Veronese cone $V^{(d)} = \operatorname{Spec} A^{(d)}$ is cylindrical.*

In Lemma 2.10 below we precise a range of values of d where the latter assertion can be applied. In the next example we illustrate our setting without carrying the normality assumption.

Example 2.3. In the affine space $\mathbb{A}^3 = \operatorname{Spec} \mathbb{k}[x, y, z]$ consider the hypersurface

$$V = \mathbb{V}(x^2 - y^3) \simeq \Gamma \times \mathbb{A}^1\,,$$

where Γ is the affine cuspidal cubic given in $\mathbb{A}^2 = \operatorname{Spec} \Bbbk[x,y]$ by the same equation $x^2 - y^3 = 0$. Notice that V is stable under the $\mathbb{G}_{\mathrm{m}}$-action on $\mathbb{A}^3$ given by

$$\lambda.(x,y,z) = (\lambda^3 x, \lambda^2 y, \lambda z)\,.$$

With respect to this $\mathbb{G}_{\mathrm{m}}$-action, $\mathbb{A}^3$ is the generalized affine cone over the weighted projective plane $\mathbb{P}(3,2,1)$ polarized via an anticanonical divisor H. The divisor H is ample, and $\mathbb{P}(3,2,1)$ is a singular del Pezzo surface of degree 6. The quotient $Y = V/\mathbb{G}_{\mathrm{m}}$ is a unicuspidal rational curve in $\mathbb{P}(3,2,1)$ with an ordinary cusp at the point $P = (0:0:1)$. It can be polarized by a divisor $D \in |H|_Y|$ supported at P. The affine surface $V \simeq \Gamma \times \mathbb{A}^1$ is a cylinder, and (Y,H) is cylindrical as well. The cylinder in Y consists of a single affine curve $Y \setminus \operatorname{supp} D = Y \setminus \{P\} \simeq \mathbb{A}^1$. The natural projection $\pi : V^* \to Y$ sends any generator $\{Q\} \times \mathbb{A}^1$, where $Q \in \Gamma \setminus \{\bar{0}\}$, of the cylinder V onto $Y \setminus \{P\}$.

In Corollary 2.9 below we show that *if a normal affine quasicone* $V = \operatorname{Spec} A$ *is cylindrical then* (Y,H) *is.* This proves the first part of Theorem 2.2. Let us start with a particular case, where the proof is rather short.

Lemma 2.4. *Let* $A = \bigoplus_{\nu \ge 0} A_\nu$ *be a positively graded normal affine domain with* $e(A) = 1$, *and let* $\partial \in \operatorname{LND}(A)$ *be a nonzero homogeneous locally nilpotent derivation on* A *of degree* $-d$, *where* $d \in \mathbb{Z}$. *Suppose that* $e(A^\partial) = 1$ *i.e.,*

$$A^\partial_\nu \neq (0) \quad \forall \nu \gg 0\,. \tag{4}$$

Then $f\partial \in \operatorname{LND}(A_{(h)})$ *for some nonzero homogeneous elements* $h \in \mathbb{A}^\partial_m$ *and* $f \in (A^\partial_h)_d$, *where* $m > 0$ *and* $A_{(h)} = (A_h)_0$.

Proof. By (4) for a sufficiently large $m \in \mathbb{N}$ there are nonzero elements, say, $h_1 \in A^\partial_{m+d}$ and $h \in A^\partial_m$. Letting $f = h_1/h \in (A^\partial_h)_d$ we consider a homogeneous locally nilpotent derivation $\delta = f\partial$ of degree zero on the localization A_h. It restricts to a locally nilpotent derivation on $A_{(h)}$. Let us show that this restriction is nonzero, as required. Indeed, we have

$$A_{(h)} = \bigoplus_{j \ge 0} A_{mj} h^{-j}\,.$$

Hence $\partial | A_{(h)} = 0$ if and only if $A^{(m)} \subseteq A^\partial$, where $A^{(m)} = \bigoplus_{j \ge 0} A_{mj}$ is the mth Veronese subring of A. However, the latter is impossible since $\operatorname{tr.deg}(A^{(m)}) = \operatorname{tr.deg}(A) = \operatorname{tr.deg}(A^\partial) + 1$. □

Corollary 2.5. *Under the assumptions of Lemma 2.4 suppose in addition that* A *is normal and so it admits a presentation*[6] $A = A(Y,H)$, *where* $Y = \operatorname{Proj}(A)$ *and* H *is an ample* $\mathbb{Q}$*-divisor on* Y. *Then the pair* (Y,H) *is cylindrical.*

Proof. Let a pair $(A_{(h)}, f\partial | A_{(h)})$ verify the conclusion of Lemma 2.4. Applying to this pair the homogeneous slice construction (see 1.2) we obtain a principal cylinder $\mathbb{D}_+(\tilde{h})$ in $\mathbb{D}_+(h)$, where $\tilde{h} \in \ker(f\partial) \cap \operatorname{im}(f\partial) \subseteq A_{(h)}$ is a nonzero homogeneous element of degree zero. We can write $\tilde{h} = a h^{-\beta}$ for some $\beta \ge 0$ and some $a \in A_\alpha$, where $\alpha = m\beta$. Hence the cylinder $\mathbb{D}_+(\tilde{h}) = \mathbb{D}_+(ah) = Y \setminus \mathbb{V}_+(ah)$ is H-polar, see Remark 1.14. □

[6] See 1.9.

Corollary 2.6. *If $H \in \mathrm{Div}(Y)$ is very ample then $e(A^\partial) = 1$ and, moreover, $h^d\partial \in \mathrm{LND}(A_{(h)})$ for a nonzero element $h \in A_1^\partial$.*

In contrast, in case where the assumption $e(A^\partial) = 1$ of Lemma 2.4 does not hold it is not so evident how can one produce a locally nilpotent derivation on A stabilizing $A_{(h)}$ starting with a given one. Let us provide a simple example.

Example 2.7. Consider the affine plane $X = \mathbb{A}^2 = \mathrm{Spec}\,\mathbb{k}[x, y]$ equipped with the $\mathbb{G}_{\mathrm{m}}$-action $\lambda.(x, y) = (\lambda^2 x, \lambda y)$. The homogeneous locally nilpotent derivation $\partial = \frac{\partial}{\partial y}$ on the algebra $A = \mathbb{k}[x, y]$ graded via $\deg x = 2$, $\deg y = 1$ defines a principal cylinder on X with projection $x : X \to \mathbb{A}^1 = Z$. Note that $e(A^\partial) = 2$. The derivation ∂ extends to a locally nilpotent derivation of the algebra

$$\tilde{A} = A[z]/(z^2 - x) = \mathbb{k}[z, y] \supseteq A$$

such that $e(\tilde{A}^\partial) = 1$. The localization $A_x = \mathbb{k}[x, x^{-1}, y]$ extends to $\tilde{A}_z = \mathbb{k}[z, z^{-1}, y] = \mathbb{k}[z, z^{-1}, s]$, where

$$s = y/z \in \tilde{A}_{(z)} = (\tilde{A}_z)_0 = \mathbb{k}[s]$$

is a slice of the homogeneous derivation $\partial_0 = z\partial \in \mathrm{LND}(\tilde{A}_{(z)})$ of degree zero. Thus $\mathrm{Spec}\,\tilde{A}_{(z)} = \mathrm{Spec}\,\mathbb{k}[s] \simeq \mathbb{A}^1$ is a polar cylinder in $\tilde{Y} = \mathrm{Proj}\,\tilde{A}$.

The subrings $A \subseteq \tilde{A}$ and $A_x \subseteq \tilde{A}_z$ are the rings of invariants of the involution $\tau : (z, y) \mapsto (-z, y)$ resp. $(z, s) \mapsto (-z, -s)$. This defines the Galois $\mathbb{Z}/2\mathbb{Z}$-covers $\mathrm{Spec}\,\tilde{A}_z \to \mathrm{Spec}\,A_x$ and $\mathrm{Spec}\,\tilde{A}_{(z)} \to \mathrm{Spec}\,A_{(x)}$. Hence $\mathrm{Spec}\,A_{(x)} = \mathrm{Spec}\,\mathbb{k}[s^2] \simeq \mathbb{A}^1$, where $s^2 = y^2/x \in A_{(x)}$, is a polar cylinder in $Y = \mathrm{Proj}\,A$ with a locally nilpotent derivation d/ds^2.

So in order to construct a polar cylinder for (Y, H) in the general case one needs to apply a different strategy. We use below the cyclic quotient construction (see 1.7)

$$Y_f = \mathrm{Spec}\,F \xrightarrow{/\mathbb{Z}_m} \mathbb{D}_+(h) = \mathrm{Spec}\,A_{(h)}\,, \tag{5}$$

where $h \in A_m^\partial$ is nonzero and $F = A/(h-1)$. The key point is the following proposition.

Proposition 2.8. *Let $A = \bigoplus_{\nu \geq 0} A_\nu$ be a positively graded affine domain over $\mathbb{k}$, and let $\partial \in \mathrm{LND}(A)$ be a nonzero homogeneous locally nilpotent derivation. If $\dim_S Y \geq 1$, where $Y = \mathrm{Proj}\,A$ and $S = \mathrm{Spec}\,A_0$, then there exists a homogeneous element $f \in A^\partial$ such that $\mathbb{D}_+(f) = \mathrm{Spec}\,A_{(f)}$ is an H-polar cylinder*[7] *in Y, where Y is polarized via an ample $\mathbb{Q}$-divisor H such that $A = A(Y, H)$.*

Proof. Let $d = -\deg\partial$. We apply the homogeneous slice construction 1.2. One can find a homogeneous element $g \in (\ker\partial^2 \setminus \ker\partial) \cap A_{d+m}$ such that $h = \partial g \in A_m^\partial$, where $m > 0$ (in particular h is non-constant). Indeed, assuming to the contrary that $A^\partial \subseteq A_0$ we obtain $\mathrm{tr.deg}(A_0) \geq \mathrm{tr.deg}(A) - 1$. It follows that the morphism $Y \to S$ is finite, contrary to our assumption that $\dim_S Y \geq 1$. Thus there exists $a \in A_\alpha^\partial$, where $\alpha > 0$ and $a \neq 0$. Replacing (g, h) by (ag, ah), if necessary, we can consider that $\deg h > 0$. In this case the fibers $h^*(c)$ with $c \neq 0$ are all isomorphic under the $\mathbb{G}_{\mathrm{m}}$-action on $V = \mathrm{Spec}\,A$ induced by the grading of A.

[7] In particular $\mathrm{LND}(A_{(f)}) \neq \emptyset$.

We use further the cyclic quotient construction, see 1.7. In particular, we consider the quotient

$$F = A/(h-1)A = A_h/(h-1)A_h = F^{\bar\partial}[\bar s], \quad \text{where} \quad \bar s = g + (h-1)A \in F \tag{6}$$

is a slice of the induced locally nilpotent derivation $\bar\partial$ on F. We have

$$\operatorname{Spec} F^{\bar\partial} \simeq \mathbb{V}(g) \cap \mathbb{V}(h-1)\,,$$

where both schemes are regarded with their reduced structure. Choosing g appropriately we may suppose that the variety $\operatorname{Spec} F \simeq h^*(1)$ (of positive dimension) is reduced and irreducible. Then also the variety $\operatorname{Spec} F^{\bar\partial}$ is since $\operatorname{Spec} F \simeq \operatorname{Spec} F^{\bar\partial} \times \mathbb{A}^1$ by (6). Indeed, the Stein factorization applied to h gives $h = h_1^l$, where $m = kl$, $l \geq 1$, and $h_1 \in A_k^\partial$ is such that the fibers $h_1^*(c)$, $c \neq 0$, are all reduced and irreducible. Now we replace (g,h) by the new pair (g_1, h_1), where $g_1 = g/h_1^{l-1} \in A_h = A_{h_1}$ and $h_1 = \partial g_1$. Since the variety $\operatorname{Spec} F^{\bar\partial}$ is reduced and irreducible $F^{\bar\partial}$ is a domain. Thus Lemma 1.8 can be applied.

The subgroup $\boldsymbol{\mu}_m \subseteq \mathbb{G}_{\mathrm{m}}$ of mth roots of unity acts effectively on F stabilizing the kernel $F^{\bar\partial}$. This action provides the $\mathbb{Z}_m$-gradings

$$F = \bigoplus_{\sigma \in \mathbb{Z}_m} F_\sigma \quad \text{and} \quad F^{\bar\partial} = \bigoplus_{\nu=0}^{n-1} F^{\bar\partial}_{[k\nu]}\,,$$

where $m = kn$ and $k = e(A^\partial)$ is such that $F^{\bar\partial}_{[k\nu]} \neq 0\ \forall \nu$ (see 1.7). The $\boldsymbol{\mu}_m$-action on F yields an effective $\boldsymbol{\mu}_n$-action on $F^{\bar\partial}$. We have $\bar\partial : F_\sigma \to F_{\sigma - r}$, where $r = [d] \in \mathbb{Z}_m$.

According to (5) one can write

$$A_{(h)} \simeq F^{\boldsymbol{\mu}_m} = F_{[0]} = (F^{\bar\partial}[\bar s])_{[0]} = \bigoplus_{j \geq 0} F^{\bar\partial}_{[-rj]} \bar s^j\,.$$

For $\alpha \in \mathbb{N}$, $\alpha \gg 1$, one can find a nonzero element $t \in A^\partial_{e(A^\partial)+\alpha m} = A^\partial_{k+\alpha m}$ (see Lemma 1.8). Then also $\bar t = t + (h-1)A \in F^{\bar\partial}_{[k]}$ is nonzero. The subgroup $\langle \bar t\rangle \subseteq (F^{\bar\partial}_{\bar t})^\times$ acts on $F^{\bar\partial}_{\bar t}$ via multiplication permuting cyclically the graded pieces $(F^{\bar\partial}_{\bar t})_{[ik]}$, $i = 0, \ldots, n-1$. Thus $(F_{\bar t})_{[k\nu]} = (F_{\bar t})_{[0]} \bar t^\nu\ \forall \nu$. It follows that

$$\begin{aligned} A_{(ht)} \simeq F_{\bar t}^{\boldsymbol{\mu}_m} = (F_{\bar t})_{[0]} = (F^{\bar\partial}_{\bar t}[\bar s])_{[0]} &= \bigoplus_{j \geq 0} (F^{\bar\partial}_{\bar t})_{[-krj]} \bar s^{kj} \\ &= \bigoplus_{j \geq 0} (F^{\bar\partial}_{\bar t})_{[0]} \left(\bar s^k \bar t^{-r}\right)^j = \bigoplus_{j \geq 0} (F^{\bar\partial}_{\bar t})_{[0]} \bar s_1^j\,, \end{aligned}$$

where $s_1 = s^k t^{-r} \in A_{(ht)}$ and $\bar s_1 \in F_{[0]}$. Letting $f = ht \in A^\partial_{k+(\alpha+1)m}$ we obtain that $A_{(f)} \simeq F^{\bar\partial}_{(\bar t)}[\bar s_1]$ is a polynomial ring. Thus $\mathbb{D}_+(f) = Y \setminus \mathbb{V}_+(f)$ is a cylinder. According to Remark 1.14 this cylinder is H-polar. Now the proof is completed. □

The proof of the next corollary is similar to that of Corollary 2.5.

Corollary 2.9. *If under the assumptions of Theorem 2.2 the affine quasicone V over Y is cylindrical then the pair (Y, H) is.*

This finishes the proof of the first assertion of Theorem 2.2. The second follows from the next lemma.

Lemma 2.10. *Assume that the pair (Y, H) as in 2.1 is cylindrical with a cylinder*

$$Y \setminus \operatorname{supp} D \simeq Z \times \mathbb{A}^1 ,$$

where D is an effective $\mathbb{Q}$-divisor on Y such that $D \sim \frac{p}{q} H$ in $\operatorname{Pic}_{\mathbb{Q}}(Y)$ for some coprime integers $p, q > 0$. Then the Veronese quasicone $V^{(p)}$ over Y is cylindrical and possesses a principal cylinder $\mathbb{D}(h) \simeq Z' \times \mathbb{A}^1$, where $Z' \simeq Z \times \mathbb{A}^1_*$ and $h^q \in A_p$.

Proof. We have $D = \frac{p}{q} H + \operatorname{div}(\varphi)$ for a rational function φ on Y. Hence $\operatorname{div}(\varphi^q) + pH = qD \geq 0$ and so in the notation as in 1.13

$$h := \varphi^q u^p \in A_p = H^0(Y, \mathcal{O}_Y(\lfloor pH \rfloor)) u^p \subseteq A ,$$

where $u \in (\operatorname{Frac} A)_1$ satisfies $\operatorname{div}(u|_{V^*}) = \pi^* H$. So $\operatorname{div}(h|_{V^*}) = q\pi^* D$. Since

$$\operatorname{Spec}(A^{(p)})_{(h)} = \mathbb{D}_+(h) = Y \setminus \operatorname{supp} D \simeq Z \times \mathbb{A}^1$$

is a cylinder we have

$$(A^{(p)})_{(h)} \simeq \mathcal{O}(Z)[s], \quad \text{where} \quad s \in (A^{(p)})_{(h)} \quad \text{and} \quad \mathcal{O}(Z) \simeq (A^{(p)})_{(h)}/(s) .$$

Similarly as in Remark 1.6 we obtain

$$(A^{(p)})_h = (A^{(p)})_{(h)}[h, h^{-1}] \simeq \mathcal{O}(Z)[s, h, h^{-1}] = \mathcal{O}(Z')[s] , \tag{7}$$

where $Z' = \operatorname{Spec} \mathcal{O}(Z)[h, h^{-1}] = Z \times \mathbb{A}^1_*$. Letting $\mathbb{A}^1 = \operatorname{Spec} \Bbbk[s]$ we see that

$$\mathbb{D}(h) = \operatorname{Spec} (A^{(p)})_h \simeq Z' \times \mathbb{A}^1$$

is a principal cylinder in $V^{(p)}$, as required. Now the proof of Theorem 2.2 is completed. □

Remarks 2.11. 1. The assumption $D \sim \frac{p}{q} H$ of Lemma 2.10 implies the equality $\{pH\} = \{qD\}$. Hence the irreducible components Δ_i of the fractional part $\{pH\}$ of the $\mathbb{Q}$-divisor pH on Y (cf. 1.11) are contained in $\operatorname{supp}\{qD\}$ and do not meet the cylinder $Y \setminus \operatorname{supp} D$.

2. Suppose that $H \in \operatorname{Div} Y$ is an ample Cartier divisor. According to Lemma 2.10 with $p = 1$, the existence of an effective divisor $D \in |H|$ such that $Y \setminus \operatorname{supp} D$ is a cylinder guarantees the cylindricity of the quasicone $V = \operatorname{Spec} A(Y, H)$. On the other hand, the cylindricity of V does not guarantee the existence of such a divisor D in the linear system $|H|$, but only in the linear system $|nH|$ for some $n \in \mathbb{N}$ (see Theorem 2.2). We wonder whether there exists an upper bound for such n in terms of the numerical invariants of the pair (Y, H). This important question is non-trivial already in the case of del Pezzo surfaces Y and pluri-anticanonical divisors $H = -mK_Y$, see Example 3.1 below.

The latter remark together with Proposition 0.5, Theorem 2.2, and Lemma 2.10 lead to the following corollary.

Corollary 2.12. *Let Y be a normal algebraic variety over $\Bbbk$ projective over an affine variety S with $\dim_S Y \geq 1$. Let $H \in \operatorname{Div}(Y)$ be an ample divisor on Y, and let $V = \operatorname{Spec} A(Y, H)$ be the associated affine quasicone over Y. Then V admits an effective $\mathbb{G}_a$-action if and only if Y contains an H-polar cylinder.*

3. Final remarks and examples

For the details of the following examples we send the reader to [KPZ_1, KPZ_3]. The latter paper inspired the present work.

Example 3.1. The generalized cone over a smooth del Pezzo surface Y_d of degree d (proper over $S = \operatorname{Spec} \Bbbk$) polarized by the (integral) pluri-anticanonical divisor $-rK_{Y_d}$ admits an additive group action if $d \geq 4$ and does not admit such an action for $d = 1$ and $d = 2$, whatever is $r \geq 1$. The latter follows from the criterion of Theorem 2.2. Indeed, in the case $d \leq 2$ the pair $(Y_d, -rK_{Y_d})$ is not cylindrical ([KPZ_3]). The case $d = 3$ remains open.

Remark 2.11.2 initiates the following definitions.

Definition 3.2. The *cylindricity spectrum* of a pair (Y, H) is

$$\mathrm{Sp}_{\mathrm{cyl}}(Y, H) = \{r \in \mathbb{Q}_+ \mid \exists D \in [rH] \text{ such that } D \geq 0 \text{ and } Y \setminus \operatorname{supp} D \simeq Z \times \mathbb{A}^1\}\,.$$

Clearly, $\mathrm{Sp}_{\mathrm{cyl}}(Y, H) \subseteq \mathbb{Q}_+$ is stable under multiplication by positive integers. An element $r \in \mathrm{Sp}_{\mathrm{cyl}}(Y, H)$ is called *primitive* if it is not divisible in $\mathrm{Sp}_{\mathrm{cyl}}(Y, H)$. The set of primitive elements will be called a *primitive spectrum* of (Y, H). We conjecture that the primitive spectrum is finite.

Examples 3.3. 1. It may happen that the pair (Y, H) as in Theorem 2.2 is cylindrical while the quasicone V is not. Consider, for instance, a normal generalized cone V over $Y = \mathbb{P}^1$, that is, a normal affine surface with a good $\mathbb{G}_{\mathrm{m}}$-action and a quasirational singularity.[8] Notice that (Y, H) is cylindrical for any $\mathbb{Q}$-divisor H on Y of positive degree (see Remark 0.3). However, it was shown in [FZ_4, Theorem 3.3] that V admits a $\mathbb{G}_{\mathrm{a}}$-action (that is, is cylindrical) if and only if $V \simeq \mathbb{A}^2/\mathbb{Z}_m$ is a toric surface, if and only if it has at most cyclic quotient singularity. The singularities of the generalized cones

$$x^2 + y^3 + z^7 = 0 \qquad \text{and} \quad x^2 + y^3 + z^3 = 0$$

in $\mathbb{A}^3$ being non-cyclic quotient, these cones over $\mathbb{P}^1$ are not cylindrical (see [FZ_2]), whereas suitable associated Veronese cones are. In terms of the polarizing $\mathbb{Q}$-divisor H on Y, a criterion of [Li_1, Corollary 3.30] says that V is cylindrical if and only if the fractional part of H is supported on at most two points of $Y = \mathbb{P}^1$. In the above examples it is supported on three points.

2. Similarly, let a, b, c be a triple of positive integers coprime in pairs, and consider the normal affine surface $x^a + y^b + z^c = 0$ in $\mathbb{A}^3$ with a good $\mathbb{G}_{\mathrm{m}}$-action. According to [De, Example 3.6] an associated $\mathbb{Q}$-divisor H on $Y = \mathbb{P}^1$ can be given as $H = \frac{\alpha}{a}[0] + \frac{\beta}{b}[1] + \frac{\gamma}{c}[\infty]$, where α, β, γ are integers satisfying $\alpha bc + \beta ac + \gamma ab = 1$. This divisor is ample since $\deg H = \frac{1}{abc} > 0$. For $a, b, c > 1$ the fractional part of H is again supported on three points. Hence this cone, say, $V = V_{a,b,c}$ is not cylindrical and does not admit any $\mathbb{G}_{\mathrm{a}}$-action. At the same time the Veronese cone $V^{(d)}$ does if and only if at least one of the integers a, b, c divides d. Indeed in the latter case the fractional part of the associated divisor dH of the Veronese cone $V^{(d)}$ is supported on at most two points. It is easily seen that the primitive spectrum of $(\mathbb{P}^1, H)$ has cardinality 3.

[8] An isolated surface singularity is called *quasirational* if the components of the exceptional divisor of its minimal resolution are all rational.

Remarks 3.4. 1. Given a homogeneous derivation $\partial \in \mathrm{LND}(A)$ of degree d there exists a replica $a\partial \in \mathrm{LND}(A^{(m)})$ of ∂ stabilizing the mth Veronese subring $A^{(m)} = \bigoplus_{k\geq 0} A_{km}$ of A, where $a \in A_j^{\partial}$ for some $j \gg 0$ such that $j + d \equiv 0 \mod m$. In this way a $\mathbb{G}_a$-action on a generalized cone $V = \mathrm{cone}_H(Y)$ induces such an action on the associated Veronese cone $V^{(m)}$. Notice that the locally nilpotent derivation on the localization A_h constructed in the proof of Lemma 2.10 has degree zero. Hence it preserves any Veronese subring $A_h^{(m)}$. It follows that if V is cylindrical then the associated Veronese cone $V^{(m)}$ is for any positive $m \equiv 0 \mod e(A^{\partial})$.

2. The question arises as to when a $\mathbb{G}_a$-action on a Veronese power $V^{(m)}$ of a generalized cone $V = \mathrm{cone}_H(Y)$ (normalized by the standard $\mathbb{G}_m$-action) is induced by such an action on V. The natural embedding $A^{(m)} \hookrightarrow A$ yields an m-sheeted cyclic Galois cover $V \to V^{(m)}$ with the Galois group being a subgroup of the 1-torus $\mathbb{G}_m$ acting on V. This cover can be ramified in codimension 1. For instance, this is the case if Y is smooth and the ample $\mathbb{Q}$-divisor H is not integral, while mH is.

In case that this cover is non-ramified in codimension 1 the $\mathbb{G}_a$-action on $V^{(m)}$ can be lifted to V commuting with the Galois group action (see Theorem 1.3 in [MaMi] [9]).

The following simple example[10] shows that without the normality assumption for the quasicone V, it is impossible in general to lift to V a given $\mathbb{G}_a$-action on a Veronese cone $V^{(m)}$.

Example 3.5. Consider the polynomial algebra $\tilde{A} = \Bbbk[x, y]$ with the standard grading and a homogeneous locally nilpotent derivation $\partial = y\frac{\partial}{\partial x}$ of degree 0. Consider also a non-normal subring

$$\tilde{B} = \Bbbk[x^2, xy, y^2, x^3, y^3] \subseteq \tilde{A}$$

with normalization $\tilde{A}$. Note that ∂ does not stabilize $\tilde{B}$. On the other hand, the involution $\tau : (x, y) \mapsto (-x, -y)$ acts on $\tilde{A}$ leaving $\tilde{B}$ invariant. Furthermore, letting $G = \langle \tau \rangle \simeq \mathbb{Z}/2\mathbb{Z}$ we obtain

$$\tilde{A}^G = \tilde{B}^G = \bigoplus_{\nu=0}^{\infty} \tilde{A}_{2\nu} =: A\,.$$

Let $\tilde{V} = \mathrm{Spec}\,\tilde{B}$ and $V = \mathrm{Spec}\,A$. Then $\tilde{V} = \mathrm{cone}(\tilde{\Gamma})$ is a generalized affine cone over the smooth projective rational curve $\tilde{\Gamma} \subseteq \mathbb{P}^4(2, 2, 2, 3, 3)$ given by $(x : y) \mapsto (x^2 : xy : y^2 : x^3 : y^3)$, while $V = \mathrm{cone}(\Gamma)$ is the usual quadric cone over a smooth conic $\Gamma \subseteq \mathbb{P}^2$. The embedding $A \hookrightarrow \tilde{B}$ induces a 2-sheeted Galois cover $\tilde{V} \to V$ ramified only over the vertex of V. The derivation ∂ stabilizes A, and the induced $\mathbb{G}_a$-action on V lifts to the normalization $\mathbb{A}^2$ of $\tilde{V}$, and also to $\tilde{V}^* = \tilde{V} \setminus \{\bar{0}\} \simeq \mathbb{A}^2 \setminus \{\bar{0}\}$. However, since ∂ does not stabilize $\tilde{B}$ this action cannot be lifted to the cone $\tilde{V}$.

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[9]See also [Ka], [FZ$_4$, 1.7], and the proof of Lemma 2.16 in [FZ$_2$], where the argument must be completed. Cf. Proposition 2.4 in [BJ] for a more general fact on lifting algebraic group actions to an étale cover over a complete base in arbitrary characteristic.

[10]Cf. [FZ$_2$, Example 2.17]. The double definition of A in [FZ$_2$, Example 2.17] is not correct; the correct definition is given by the second equality $A = \bigoplus_{\nu\neq 1} A_\nu$, while for our purposes the first equality is more suitable.

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