Quantum Cosmology and Conformal Invariance

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According to Belinsky, Khalatnikov and Lifshitz, gravity near a space-like singularity reduces to a set of decoupled one-dimensional mechanical models at each point in space. We point out that these models fall into a class of conformal mechanical models first introduced by de Alfaro, Fubini and Furlan (DFF). The deformation used by DFF to render the spectrum discrete corresponds to a negative cosmological constant. The wave function of the universe is the zero-energy eigenmode of the Hamiltonian, also known as the spherical vector of the representation of the conformal group $SO(1,2)$. A new class of conformal quantum mechanical models is constructed, based on the quantization of nilpotent coadjoint orbits, where the conformal group is enhanced to an ADE non-compact group for which the spherical vector is known.

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While cosmological singularities have so far eluded any satisfactory treatment in quantum gravity, one of the few insights into their generic properties is afforded by the Belinsky, Khalatnikov and Lifshitz (BKL) classical analysis [1]. Under a self-consistent hypothesis of decoupling of the dynamics at nearby points upon approaching a space-like singularity, it is shown that a generic solution of four-dimensional Einstein gravity exhibits a chaotic oscillatory behavior near the big bang (or big crunch). This subject has attracted much recent attention, with the discovery that the chaotic properties of the dynamics are tied to the hyperbolicity of the Kac Moody algebra underlying the billiard geometry on which the motion takes place [2, 3]. This is in particular the case in all models descending from eleven-dimensional supergravity or $N=1$ supergravity in ten dimensions [4]. The subleading effect of spatial gradients at the singularity have also been studied [5].

Our aim in this letter is twofold. Firstly, we point out an important feature of the one-dimensional model describing the gravity modes at each decoupled point: namely, it exhibits the conformal invariance found in one-dimensional quantum mechanics by de Alfaro, Fubini and Furlan (DFF) [6]. Conformal quantum mechanical models have already made an appearance in black hole physics [7, 8], yet their relevance to cosmology seems to have been hitherto unnoticed. More specifically, the Wheeler-DeWitt equation is simply the Schrödinger equation considered by DFF, restricted to zero energy. The wave function of the universe thus lies at the bottom of a continuum of delta-function normalizable states. Retaining the effect of a negative cosmological constant discretizes the spectrum while preserving conformal invariance. The wave function of the universe is thus the spherical vector of the representation of the conformal group. Secondly, we construct a new class of conformal quantum mechanical models, where the conformal group is enhanced to an ADE non-compact group. In these models, the spherical vector, and hence the wave function of the universe is known exactly.

We start with $n+1$ dimensional Einstein gravity with a cosmological constant and reduce down to a 0+1 dimensional system. Parameterizing the metric as

$$ds^2 = -\frac{\eta(t)}{V(t)}\,dt^2 + V^{2/n}(t)\,\hat g_{ij}(t)\,dx^i dx^j\,, \qquad (1)$$

the Einstein-Hilbert action reduces to

$$\int dt\; d^n x\;\sqrt{-g}\,(R-2\Lambda)\;= \int\; dt\left\{\frac{1}{2\eta}\left[-\frac{2(n-1)}{n}\dot V^2 + V^2\,\dot U^M G_{MN}\dot U^N\right] - 2\Lambda\eta\right\} \qquad (2)$$

where V denotes the volume of the spatial metric and U^M coordinatize the coset $S = Sl(n)/SO(n)$, with homogeneous metric $dU^M G_{MN} dU^N := -\frac{1}{2} d\hat g_{ij} d\hat g^{ij}$. In this form, we recognize the action for a "fictitious" point particle, with mass squared $m^2 = 4\Lambda$, tachyonic for negative (AdS) Λ, propagating on a Lorentzian cone with base S and metric

$$d\sigma^2 = -\tfrac{2(n-1)}{n}\,dV^2 + V^2 dU^M G_{MN} dU^N\,. \qquad (3)$$

The rescaled lapse η plays the role of an einbein gauge field enforcing invariance under general time reparameterizations. The appearance of the volume V with a negative kinetic term suggests its use as a "cosmological time" variable with which to set up measurements [10]. Indeed, it develops with time as $[(V/\eta)d/dt]^2V = [2n\Lambda/(n-1)]V$, while the particle follows geodesics on the cone and hence on its base S. A similar reduction in the presence of extra scalar and gauge fields would yield a cone over an enlarged homogeneous space (e.g. $SO(n,n)/[SO(n)\times SO(n)]$ in the presence of a Kalb-Ramond two-form and dilaton).

Since the conical moduli space (3) admits an homothetic Killing vector $V\partial_V$, the free particle system should exhibit conformal invariance [8]. In order to display it, let us choose the gauge $\eta = V$ and introduce conjugate momenta p and P_M for the canonically normalized volume coordinate $\rho = \sqrt{8(n-1)V/n}$ and the shape moduli U^M. The equation of motion of η enforces the Hamiltonian constraint,

$$H = \frac{1}{2}p^2 - \frac{8(n-1)}{n\rho^2}\Delta - \frac{n\Lambda}{4(n-1)}\rho^2 \equiv 0\,. \quad (4)$$

Here $\Delta = -P_M G^{MN} P_N$ is the quadratic Casimir of the action of $Sl(n)$ on the homogeneous space $Sl(n)/SO(n)$ and has vanishing Poisson bracket withs ρ and p. Therefore it effectively plays the rôle of a coupling constant $g = 8(n-1)\Delta/n$ for the $1/\rho^2$ potential. Indeed at $\Lambda = 0$, we recognize the Hamiltonian in (4) as that of the conformal mechanical system introduced by de Alfaro, Fubini and Furlan [6]. The generators

$$E_+ = \frac{1}{2}\rho^2\ ,\quad H_0 = \frac{1}{2}\rho p\ ,\quad E_- = \frac{1}{2}\left(p^2 + \frac{g}{\rho^2}\right) \quad (5)$$

represent the conformal group $SO(1,2)$ in 0+1 dimensions, $\{E_+, E_-\} = 2H_0$, $\{H_0, E_\pm\} = \pm E_\pm$. For vanishing Λ, the Hamiltonian $H = E_-$ has unit dimension w.r.t. the generator of conformal rescalings H_0, hence the system is conformally invariant. Remarkably, the introduction of a cosmological term preserves the action of the conformal group $SO(2,1)$, as it simply amounts to choosing a different generator $H = E_- - n\Lambda/(2n-2)E_+$ as the Hamiltonian. This deformation was considered in [6] although without a clear physical motivation.

Let us now discuss some aspects of the quantum dynamics in this toy model of a cosmological singularity. While we understand little about quantum gravity at a space-like singularity, we may assume that the decoupling of nearby points still holds and work in a minisuperspace truncation. Replacing canonical momenta by their Schrödinger representation $p \to i\partial/\partial\rho$, $P_M \to i\partial/\partial U^M$, the Hamiltonian constraint (4) becomes the Wheeler-DeWitt equation,

$$H\psi = \left(-\frac{1}{2}\partial^2 + \frac{4(n-1)}{n\ \rho^2}\Delta - \frac{n\Lambda}{4(n-1)}\rho^2\right)\psi = 0\,, \quad (6)$$

acting on wave functions $\psi(\rho, U^M)$, where Δ is the quadratic Casimir of the $Sl(n)$ action on the homogeneous space S. As usual, the vanishing of the Hamiltonian implies that the wave function ψ is independent of the time t, nevertheless correlations between the volume ρ and the other observables U_M may be used to set up measurements. One may now recognize the Wheeler-DeWitt equation (6) as the Schrödinger equation of DFF's conformal quantum mechanics, with coupling $g = 8(n-1)\Delta/n$, restricted to zero-energy states. The invariance under the conformal group $SO(1,2)$ is retained at the quantum level, after resolving the ordering ambiguity $H_0 \to (\rho p + p\rho)/4$. (Indeed, the requirement of conformal symmetry and its extension below, uniquely fix all ordering ambiguities.) In particular, the effect of a negative cosmological constant Λ is to replace the parabolic generator E_- with continuous spectrum $\mathbb{R}^+$, by a compact, discrete spectrum, generator $E_- - n\Lambda/(2n-2)E_+$. It is intriguing that this favorable case corresponds to a tachyonic particle propagating on the cone (3). In contrast, a positive Λ would lead to an hyperbolic Hamiltonian with unbounded spectrum.

Although the WDW and DFF Hamiltonians agree, one should pay attention to boundary conditions. The traditional conformal quantum mechanics analysis requires a positive coupling g and square integrability around $\rho = 0$. When ρ is viewed as a cosmological time, this requirement should be re-examined. The analogy of (6) with a massive Klein–Gordon type equation suggests to consider the Klein-Gordon norm on "space-like slices" of fixed ρ. "Third" quantization may be used to cure the non-positive definiteness [11], but interpretational problems remain. A related issue is the sign of g which depends on the class of functions admitted on S. Similarly, as mentioned earlier, the sign of the mass/cosmological term determines whether the spectrum of H is bounded-discrete, bounded or unbounded. The physical relevance of this requirement is however not clear since the WDW equation selects only zero energy modes. Due to our lack of understanding of space-like singularities we shall refrain from taking a position on these issues.

At this stage, a few comments are in order about the chaotic properties of these cosmological models. Firstly, our discussion was carried out for free Kasner flights, in the absence of the potential terms originating from allowing spatial gradients. As one approaches the space-like singularity, the potential terms behave as infinitely steep reflection walls [12]. It is possible that their effect could be mimicked by modding out by a discrete symmetry group (e.g, the walls exchanging the various radii are included in a $Sl(n,\mathbb{Z})$ subgroup of $Sl(n)$ acting on S). This group is however of a very wild nature, as it should contain the Weyl group of an hyperbolic Kac-Moody algebra [2, 3] and it is rather unclear how to describe the geodesic motion on such an object. One may speculate that the universal $SO(2,1)$ subalgebra uncovered in [16] in general hyperbolic Kac-Moody algebra may play a role in implementing conformal invariance.

We now proceed to present a construction of a class of conformally invariant quantum systems, associated with finite Lie groups. This class was first found as part of the construction of theta series for non-symplectic groups [13] and a particular example was independently found in [15] for the minimal representation of E_8. For simplicity, we shall illustrate it on the simplest non-trivial case, D_4, which will turn out to coincide with the model (6) originating from $2+1$ gravity.

The classical phase spaces of our systems arise from the coadjoint orbit of a nilpotent element of smallest order in a finite simple Lie algebra G. This element can be conjugated into the generator associated to the highest root, E_ω. The $Sl(2)$ subalgebra generated by $\{E_\omega, H_\omega, E_{-\omega}\}$ will be the conformal group $SO(2,1)$ in 0+1 dimensions and we will choose $E_{-\omega}$ as the Hamiltonian. The generator H_ω grades the algebra into 5 subspaces, $G = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_1 \oplus G_2$. The top and bottom subspaces are one-dimensional and consist of E_ω and $E_{-\omega}$, respectively. The level-one space G_1 is a Heisenberg algebra, which can be diagonalized in the form $[E_{\beta_i}, E_{\gamma_j}] = \delta_{ij} E_\omega$. We can thus represent these generators as canonical coordinates and conjugate momenta, $E_{\beta_i} = y p_i$, $E_{\gamma_i} = x_i$, $E_\omega = y$. The maximal subalgebra H in G commuting with $SO(2,1)$ lies in the level-zero space G_0 and acts linearly as canonical transformations on the coordinates and momenta $\{x_i, p_i\}$, leaving y invariant. The choice of a polarization of G_1 into coordinates and momenta further breaks H to a subgroup H_l acting linearly on the coordinates. A standard polarization is to choose β_0 as the simple root to which the affine root ω attaches on the Dynkin diagram of G and $\beta_{i>0}$ the positive roots such that $\langle \beta_0, \beta_i \rangle = 1$ [14]. The canonical momentum associated to y is in turn obtained from the Cartan generator associated to β_0, $H_{\beta_0} := [E_{\beta_0}, E_{-\beta_0}] = py - p_0 x_0$. In order to bring the generator E_ω to the conformal quantum mechanics form (5), we make a canonical transformation

$$y = \rho^2, \quad x_i = \rho q_i, \quad p = \frac{1}{2\rho} p_\rho - \frac{1}{2\rho^2} q_i \pi_i, \quad p_i = \frac{\pi_i}{\rho} \tag{7}$$

It can then be checked that the generators of the $SO(2,1)$ subalgebra take the form

$$E_\omega = \rho^2 \ , \quad H_\omega = \rho p_\rho \ , \quad E_{-\omega} = \frac{1}{4} p^2 + \frac{\Delta}{\rho^2} \ , \tag{8}$$

where Δ is, up to an additive constant, the quadratic Casimir of H, hence a quartic invariant of the coordinates and momenta $\{q_i, \pi_i\}$. Finally, the symmetry is enhanced from $SO(2,1) \times H$ to all of G using two discrete generators: (i) the Fourier transform on all positions q_i at once, corresponding to the longest word in the Weyl group, and (ii) the Weyl reflection with respect to β_0, acting on wave functions as

$$W\psi(\rho, q_0, q_i) = e^{-\frac{I_3(q_i)}{q_0}} \psi \left(\sqrt{-\rho q_0}, \sqrt{-\rho^3/q_0}, \sqrt{\rho q_i / q_0} \right) \tag{9}$$

where I_3 is the cubic invariant of the positions under the linearly realized H_l The compatibility between the two actions, and indeed the whole construction, relies heavily on the invariance of the non-Gaussian character $\exp(i I_3(x_i)/x_0)$ under Fourier transform.

As an example, the algebra D_4 decomposes as $1_{-2} \oplus (2,2,2)_{-1} \oplus [(1,1,1)+(3,1,1)+\text{perm}]_0 \oplus (2,2,2)_1 \oplus 1_2$ under $Sl(2)^3 \times \mathbb{R}$, where $\mathbb{R}$ denotes the Cartan generator H_ω associated to the highest root. The coordinates and momenta correspond to the grade-one space and transform as a $(2,2,2)$ of $H = Sl(2)^3$. They satisfy the Heisenberg algebra

$$[q^{aA\alpha}, q^{bB\beta}] = \epsilon^{ab} \epsilon^{AB} \epsilon^{\alpha\beta} \tag{10}$$

The actions of each $Sl(2)$ factor in H are represented by the angular momentum-like operators

$$\Sigma^\mu = \sigma^\mu_{\alpha\beta} \ \ \epsilon_{ab} \ \epsilon_{AB} \ q^{aA\alpha} \ q^{bB\beta} \tag{11}$$

with similar definitions for s^m and S^M. It is easy to check that these generators satisfy the $Sl(2)$ algebra, $[\Sigma^\mu, \Sigma^\nu] = \epsilon^{\mu\nu\rho} \Sigma^\rho$. The quadratic Casimirs of all three $Sl(2)$'s are identical and equal to the unique quartic invariant of the $(2,2,2)$ representation. The generators of the $SO(2,1)$ subalgebra then read as in (8) with Δ the common quadratic Casimir of the three $Sl(2)$. In order to clarify the meaning of the Hamiltonian in (8), let us choose a polarization such that the $Q^{A\alpha} = q^{1A\alpha}$ are coordinates and $q^{2A\alpha}$ are momenta. The bispinor $Q^{A\alpha}$ can be thought of as a vector Q^I of $SO(2,2)$, parameterized by three "polar angles" $\Omega \in H_3 = SO(2,2)/SO(2,1) = SO(2,1)$ and its length squared $\kappa^2 = Q^I \eta_{IJ} Q^J$, where η_{IJ} is the signature $(2,2)$ metric. The quadratic Casimir then corresponds to the angular momentum squared on the pseudo-sphere H_3, *i.e.* the Laplacian on $Sl(2)$. Out of the four coordinates Q^I, we thus see that only the three hyperbolic polar angles receive a kinetic terms in the Hamiltonian $E_{-\omega}$, while the radius κ decouples.

A natural object in the theory of minimal representations is the spherical vector, *i.e.* the wave function annihilated by all compact generators $E_\alpha + E_{-\alpha}$. For irreducible and so-called spherical representations, this vector exists and is unique. From the expression (8), we see that a spherical vector corresponds to a zero-energy state of the Hamiltonian with a negative "cosmological constant", *i.e.* $H = E_{-\omega} + E_\omega$. In the D_4 case, the spherical vector should also be invariant under the compact $U(1)$ inside each $Sl(2)$ factor, so that the coordinates of the D_4 conformal quantum mechanics are effectively valued in $H_3/U(1) = Sl(2)/U(1)$. This is indeed the conformal model (6) of $2+1$ gravity with $S = SO(2)$! Therefore, supplementing the fields $(\rho; U_1, U_2) \in \mathbb{R} \times Sl(2)/U(1)$ with a decoupled radius κ, the conformal symmetry of $2+1$ reduced gravity is enhanced to an $SO(4,4)$ non-compact spectrum generating symmetry. The spherical vector of the D_4 minimal representation has been obtained in [13], in the polarization used here reads

$$f_{D_4} = \frac{\exp(-2\pi S)}{\rho \ S} \ , \quad S = \sqrt{\rho^4 + \rho^2 \text{tr}(Q^t Q) + \kappa^4} \tag{12}$$

TABLE I: Non-linearly realized symmetry G, canonically realized H, representation of positions and momenta under H, linearly realized H_l, homogeneous space S including decoupled factors $\mathbb{R}_\kappa$, and functional dimension of the Hilbert space for models with enhanced conformal symmetry.

G	H	G_1	H_l	S	dim
A_{n-1}	$Gl(n-2)$	$[n-2]+[n-2]$	$Sl(n-2)$	$\mathbb{R}_\kappa \times Sl(n-2)/[Sl(n-3)\times\mathbb{R}^{n-3}]$	$n-1$
D_n	$Sl(2)\times D_{n-2}$	$(2,2n-4)$	D_{n-2}	$\mathbb{R}_\kappa \times SO(n-2,n-2)/SO(n-3,n-2)$	$2n-3$
E_6	$Sl(6)$	$20=[0,0,1,0,0]$	$SO(3,3)$	$\mathbb{R}_\kappa \times SO(3,3)/[SO(3)\times SO(3)]$	11
E_7	$SO(6,6)$	32	$Sl(6)$	$\mathbb{R}_\kappa \times Gl(6)/Sp(6)$	17
E_8	E_7	56	$Sl(8)$	$\mathbb{R}_\kappa \times Sl(8)/Sp(8)$	29

It would be very interesting if the new coordinate $\kappa^2 = \det(Q^{A\alpha})$, appearing here as a degeneracy label, had a cosmological interpretation.

A similar conformal quantum mechanical system can be constructed for any finite simple Lie algebra G, except for the $SO(2n+1)$ series. The details of the construction of the minimal representation have been spelled out for the simply laced case in [13] and the spherical vector has been obtained there. In order to translate the results of [13] into the presentation (8) suitable for interpreting $E_{-\omega}$ as the Hamiltonian of a quantum mechanical system, one only needs to perform the canonical transformation (7). We thus obtain a class of quantum mechanical systems with an enhanced conformal symmetry G corresponding to any finite simple Lie group in the (non-compact) split real form. The field content and symmetries of these models are summarized in Table 1. Interestingly, the models corresponding to the non-simply laced cases G_2 and F_4 have no spherical vectors, so that part of the compact symmetries are spontaneously broken. An open problem is to identify gravitational theories reducing to those in the process of dimensional reduction.

To conclude, we have pointed out the relevance of conformal quantum mechanics to the gravity dynamics at a space-like singularity. We found that a negative cosmological constant renders the Wheeler-DeWitt operator spectrum discrete. The wave function of the universe is then obtained as the spherical vector of the representation of the enhanced conformal group. We have constructed a family of conformal quantum mechanical models with symmetry enhanced to an arbitrary simple non-compact group G in the split real form. For $G = D_4$, this reproduces the conformal mechanics arising from the reduction of $2+1$ gravity, together with a decoupled field κ yet to be understood. The quantization of coadjoint orbits, not necessarily nilpotent, thus provides a wealth of extended conformal quantum mechanical models awaiting to serve in cosmological or other set-ups. Clearly, an important question is whether this construction can be extended to infinite Kac-Moody groups such as E_{10} or BE_{10}, which should control the dynamics of M-theory or the heterotic string at a space-like singularity.

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