The length of a shortest geodesic loop at a point.

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Abstract

In this paper we prove that given a point $p \in M^n$, where M^n is a closed Riemannian manifold of dimension n, the length of a shortest geodesic loop $l_p(M^n)$ at this point is bounded above by $2nd$, where d is the diameter of M^n. We also prove that the length of a shortest geodesic loop $\alpha(M^n)$ on a closed Riemannian manifold M^n is bounded above by $6nFillRadM^n$, where $FillRadM^n$ denotes the filling radius of M^n. Moreover, we show that on a closed simply connected Riemannian manifold M^n with non-trivial second homotopy group either there exist at least three geodesic loops of length less than or equal to $2d$ at each point of M^n, or the length of a shortest closed geodesic on M^n is bounded from above by 4d. We believe that the last result can be generalized for an arbitrary closed Riemannian manifold, although we will not show that in this paper.

Introduction and main results.

Let M^n be a closed Riemannian manifold of dimension n. In 1983 M. Gromov asked whether one can bound above the length of a shortest closed geodesic $l(M^n)$ on M^n by $c(n)vol(M^n)^{\frac{1}{n}}$, where $vol(M^n)$ is the volume of M^n and $c(n)$ is a constant that depends on the dimension of M^n only. A similar question can be asked about the relationship between $l(M^n)$ and the diameter of a manifold d. The fact that on each manifold there exists a closed geodesic was shown by L. Lusternik and A. Fet. A similar argument shows that there exists a geodesic loop at each point of a closed Riemannian manifold. So, one can also ask if there exists a constant $k(n)$ such that for each point $p \in M^n$, the length of a shortest geodesic loop $l_p(M^n)$ at this point is bounded above by $k(n)d$, and, in particular, whether $l_p(M^n) \leq 2d$. Note, that it is quite easy to see that $l_p(M^n) \leq 2d$ in case of a closed

Riemannian manifold that is not simply connected. Note also, that for no constant $C(n)$ we can bound above $l_p(M^n)$ by $C(n)vol(M^n)^{\frac{1}{n}}$ for every $p \in M^n$. For example, consider a prolate ellipsoid E, that is an ellipsoid generated by an ellipse rotating around its major axis. Let us denote its polar radius by R. Let $p \in E$ be the north pole of E. Then all geodesics and, thus, geodesic loops passing through p are ellipses, (see fig. 1). Therefore, the ratio $\frac{l_p(E)}{\sqrt{A(E)}}$ will approach infinity as R goes to infinity, and the smaller semiaxis is fixed.

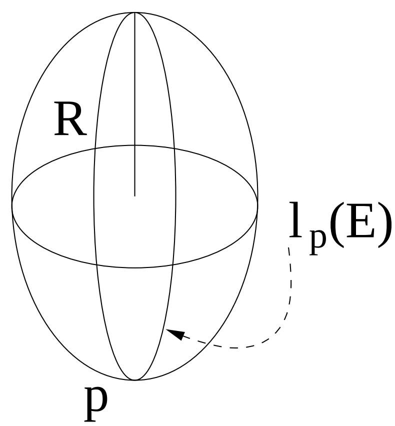

Figure 1: Prolate Ellipsoid.

Let M^n be a closed Riemannian manifold of dimension n. Here is the main result of our paper.

Theorem 0.1 *Let q denote the smallest integer for which $\pi_q(M^n) \neq \{0\}$. Then for each $p \in M^n$ there exists a geodesic loop based at p of length $\leq 2qd$, where d is the diameter of M^n. In particular, the length of a shortest geodesic loop based at p is $\leq 2nd$.*

A related problem is the probelm of estimating the length of a shortest geodesic loop, $\alpha(M^n)$ on the closed Riemannian manifold M^n. Here the base point is not fixed. The first such curvature-free estimates were obtained in 2004 and are due to S. Sabourau, who established that $\alpha(M^n)$ is bounded above by $c(n)vol(M^n)^{\frac{1}{n}}$ for some constant $c(n)$ that was not explicitely calculated in his paper [S2]. He also demonstrated that $\alpha(M^n) \leq \frac{(8 \cdot 3^n - 2)d}{3}$. In this paper we obtain an estimate for $\alpha(M^n)$ in terms of the Filling Radius of M^n. The following definition is due to M. Gromov, (see [G]).

Definition 0.2 *Filling Radius. Let M be a Riemannian manifold topologically imbedded into an arbitrary metric space X. Then Filling Radius $FillRad(M \subset X)$ is the infimum of $\varepsilon > 0$, such that M bounds in the ε-neighborhood $N_\varepsilon(M)$. Filling Radius of an abstract Riemannian manifold is $FillRad(M \subset X)$, where $X = L^\infty(M)$, i.e. the Banach space of bounded Borel functions f on M, and the embedding of M into X is a map that to each point p of M assigns a distance function $p \longrightarrow f_p = d(p,q)$, (see [G]).*

Theorem 0.3 *Let M^n be a closed Riemannian manifold. Then the length of a shortest geodesic loop, $\alpha(M^n)$ on M^n is bounded above by $6nFillRadM^n$.*

The volume inequality then follows from the previous theorem and from the volume upper bound for the Filling Radius due to Gromov.

Corollary 0.4 *Let M^n be a closed Riemannian manifold M^n. Then the length of a shortest geodesic loop $\alpha(M^n) \leq 6(n+1)n^{n+1}(n+1)!^{\frac{1}{2}} vol(M^n)^{\frac{1}{n}}$, where $vol(M^n)$ is the volume of M^n.*

Proof. This corollary follows from the above theorem and from the Gromov's estimate for the filling radius of M^n in terms of the volume of M^n, namely $FillRadM^n \leq (n+1)n^n(n+1)!^{\frac{1}{2}} vol(M^n)^{\frac{1}{n}}$, (see [G]).

□

This corollary provides an explicit value for the constant $c(n)$. We believe that this value is better than the one that can be obtained after some computations using the methods of [S2].

At present there do not exist similar curvature-free upper bounds for the length of a shortest closed geodesic $l(M^n)$ in the general case of a closed Riemannian manifold M^n, though such bounds do exist for stationary 1-cycles, ([NR2]) and minimal surfaces, ([NR3]) as well as for some topological types of Riemannian manifolds, namely, 2-dimensional sphere, ([C], [M], [S1], [NR1], [R1], [R2]), and 1-essential manifolds, ([G]). (Gromov's estimate generalizes results of many people, who worked on estimating systoles in case of surfaces, namely, C. Loewner, P. Pu, R. Accola, C. Blatter, C. Bavard, Ju. Burago and V. Zalgaller, J. Hebda and others (see [BZ], [CK])). Thus one of the central problems in this subject remains to find upper bounds of similar nature for $l(M^n)$. With this goal in mind, we will prove the following theorem.

Theorem 0.5 *Let M^n be a simply connected closed Riemannian manifold with $\pi_2(M^n) \neq \{0\}$. Then either the length of a shortest closed geodesic is bounded above by $4d$, or at each point of M^n there exist three distinct geodesic loops based at that point of length bounded above by $2d$.*

One can view this theorem in the following way: unless there are three geodesic loops of length $\leq 2d$ based at each point of M^n, which seems to be unlikely for many Riemannian manifolds, there exists a closed geodesic of length $\leq 4d$. Our methods can be used to generalize this theorem to arbitrary closed Riemannian manifolds, though not with constant 4.

1 The proof of Theorem 0.1.

Here are some useful observations: We will begin with the following lemma.

Lemma 1.1 *Let M^n be a Riemannian manifold. Let $p, q \in M^n$. Let $\gamma_1(t), \gamma_2(t)$ be two curves connecting the point p to the point q of lengths l_1, l_2 respectively. Consider the curve $\gamma_2 * -\gamma_1$, that is a product of γ_2 and $-\gamma_1$. This curve is a loop based at p. If this loop is contractible to p by a path homotopy along the curves of length $\leq l_1 + l_2$ then there is a path homotopy $h_\tau(t), \tau \in [0,1]$, such that $h_0(t) = \gamma_1(t), h_1(t) = \gamma_2(t)$ and the length of curves during this homotopy is bounded above by $2l_1 + l_2$. (Note, that by a path homotopy we mean homotopy that fixes the end points of a curve).*

Proof. Let $\tilde{h}_\tau(t)$ be a homotopy that connects $\gamma_2 * -\gamma_1$ with a point p, (see fig. 2 (a) and (b)). Then let us consider the following homotopy $\gamma_1 \sim \tilde{h}_{1-\tau} * \gamma_1 \sim \gamma_2 * -\gamma_1 * \gamma_1 \sim \gamma_2$, (see fig. 2 (a)-(g)). The length of curves during this homotopy is $\leq 2l_1 + l_2$.

□

(A similar argument is used by C.B. Croke to prove Lemma 3.1 in [C].)

In order to prove our theorems, we will also need the following observation.

Observation. *Let M^n be a complete Riemannian manifold. Let $p \in M^n$. Suppose that the length of a shortest geodesic loop $l_p(M^n)$ based at p is greater than L. Then given any piecewise differentiable closed curve $\gamma : [0,1] \longrightarrow M^n$, of length $\leq L$ such that $\gamma(0) = \gamma(1) = p$ there exists a length decreasing path homotopy that connects this curve with p. Moreover, this homotopy depends continuously on a curve γ. In other words the space of loops of length $\leq L$ based at p is contractible.*

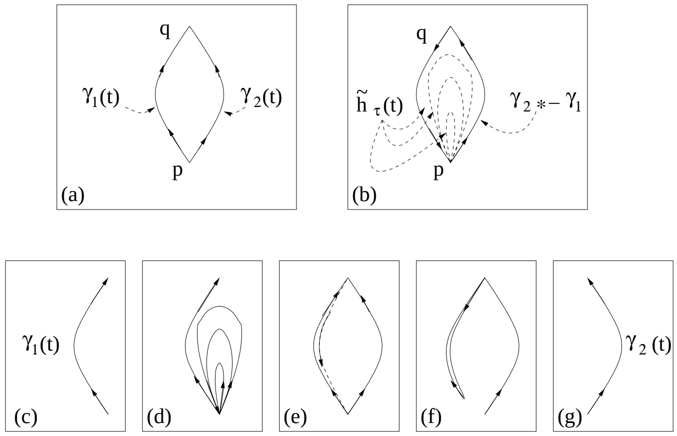

Figure 2: Illustration of the proof of Lemma 1.1.

Let us first provide a short explanation of the proof of Theorem 0.1. The proof of Theorem 0.3 will be similar.

In order to do that, let us first consider a manifold with $\pi_1(M^n) \neq \{0\}$, this is the case of Theorem 0.1 in which $q = 1$. Here we can easily show that the length of a shortest closed geodesic loop at p is bounded above by $2d$ for any $p \in M^n$. For let us consider any non-contractible map $f : S^1 \longrightarrow M^n$. Suppose S^1 is partitioned (triangulated) into very small segments, so that the diameter of each edge in the induced triangulation on $f(S^1)$ is smaller than some $\delta > 0$. Let D^2 be the standard disc that is triangulated as a cone over S^1. Assume that for some $p \in M^n$ the length of a shortest geodesic loop $l_p > 2d + \delta$. We will show in that case we can extend $f : S^1 \longrightarrow M^n$ to D^2, thus reaching a contradiction with the fact that this map is non-contractible. The extension procedure will be inductive to skeleta of D^2. 0-skeleton of D^2 consists of one additional simplex, namely, the center of the disc that we will denote by $\tilde{p}$. We will let $f(\tilde{p}) = p$. Next to extend to 1-skeleton, consider an arbitrary edge of the form $[\tilde{p}, \tilde{v}_i]$, where $\tilde{v}_i$ is the vertex of triangulation of S^1. We will assign to this edge a minimal geodesic segment $[p, v_i]$ connecting the point p with $v_i = f(\tilde{v}_i)$. Next to extend to 2-skeleton, consider a 2-simplex $[\tilde{p}, \tilde{v}_i, \tilde{v}_{i+1}]$. The boundary of this simplex is mapped to a closed curve of length $\leq 2d + \delta$, consisting of two minimizing geodesic segments and an edge $[v_i, v_j]$ of length $\leq \delta$. This curve passes through p. Let us apply a curve shortening process with a fixed p. Since we have assumed there is no geodesic loops of length $\leq 2d + \delta$ based at p, this curve is contractible to p. Thus, we can assign to the above 2-simplex, surface generated by the homotopy contracting this curve to p. Therefore, we have succeeded at extending $f : S^1 \longrightarrow M^n$ to D^2, which contradicts our assumption about non-contractibility of f.

This shows that there must be a geodesic loop of length $\leq 2d + \delta$ based at p. We conclude by letting δ approach 0.

At the next step, for the sake of simplicity, assume that $\pi_1(M^n) = \{0\}$, but $\pi_2(M^n) \neq \{0\}$. This is the case of Theorem 0.1, in which $q = 2$.

Let $f : S^2 \longrightarrow M^n$ be a non-contractible map from the standard 2-dimensional sphere to M^n. Suppose S^2 is endowed with a fine triangulation in such a way that the diameter of any simplex in the induced triangulation of $f(S^2)$ is smaller than some $\delta > 0$. Furthermore, suppose that D^3 is a disc that is triangulated as the cone over S^2. Assume that $l_p(M^n) > 4d$ for some $p \in M^n$. We will extend the map $f : S^2 \longrightarrow M^n$ to D^3, thus reaching a contradiction. The procedure will be inductive to skeleta of D^3. We will begin by extending f to 0-skeleton of D^3 that consists of a single additional point $\tilde{p}$ at the center of the disc. We will let the image of $\tilde{p}$ be the given point $p \in M^n$. Next, let us extend to 1-skeleton as follows: we will assign to an edge $[\tilde{p}, \tilde{v}_i]$ that connects the center of the disc with the vertex $\tilde{v}_i$ a minimal geodesic segment $[p, v_i]$ connecting the point p with the vertex $v_i = f(\tilde{v}_i)$. Next we extend to 2-skeleton. Consider an arbitrary 2-simplex $\tilde{\sigma}_i = [\tilde{p}, \tilde{v}_{i_1}, \tilde{v}_{i_2}]$. Its boundary $\partial\tilde{\sigma}_i^2 = [\tilde{p}, \tilde{v}_{i_1}] - [\tilde{p}, \tilde{v}_{i_2}] + [\tilde{v}_{i_1}, \tilde{v}_{i_2}]$ is mapped to a closed curve of length $\leq 2d + \delta$. Assuming that the length of a shortest geodesic loop based at p is greater than $2d+\delta$, this curve can be contracted to a point by a length-decreasing homotopy that fixes p, i.e. all the curves in the homotopy will pass through p.

We will map the 2-simplex to the surface generated by this homotopy, that will be denoted as σ_i^2. Note that we should not be able to extend map f any further. That means that there exists a 3-simplex $\tilde{\sigma}_i^3$, such that $f : \partial\tilde{\sigma}_i^3 \longrightarrow M^n$ is not contractible. On the other hand, let $\tilde{\sigma}_i^3 = [\tilde{p}, \tilde{v}_{i_1}, \tilde{v}_{i_2}, \tilde{v}_{i_3}]$. Then $\partial\tilde{\sigma}_i^3 = \Sigma_{j=0}^3 (-1)^j [\tilde{v}_{i_0}, ...\hat{\tilde{v}}_{i_j}, ..., \tilde{v}_{i_3}]$, where $\tilde{v}_{i_0} = \tilde{p}$. Let us denote $[\tilde{p}, \tilde{v}_{i_j}] = \tilde{e}_j$ and $[p, v_{i_j}]$ as e_j. Since $[v_{i_1}, v_{i_2}, v_{i_3}]$ can be made arbitrarily small we will treat it here as a point q for the sake of simplicity of the exposition, (see the Remark below). Note also, that assuming that there is no geodesic loops based at p of length $\leq 4d$ we can contract $f : \partial\tilde{\sigma}_i^3 \longrightarrow M^n$ to a point as follows:

1. By Lemma 1.1 there is a path homotopy between e_1 and e_2 that passes through curves $e_{\tau_{12}}$, where $1 \leq \tau_{12} \leq 2$ of length $\leq 3d$. This homotopy, we claim, can be used to construct a homotopy between the above sphere and

a point.
2. We will define $S^2_{\tau_{12}}$ as follows: consider the two points p and q joined by two geodesic segments e_2, e_3 and the curve $e_{\tau_{12}}$, (see fig. 3 (a)). Assuming that there is no geodesic loops based at p of length $\leq 4d$, both curves $e_2 * -e_{\tau_{12}}$ and $e_{\tau_{12}} * -e_3$ are contractible to point p without length increase. Let us call the discs obtained during this homotopy $(D^2_2)_{\tau_{12}}$, (see fig. 3 (b)) and $(D^2_3)_{\tau_{12}}$ respectively. They change continuously with τ_{12}. Then $S^2_{\tau_{12}}$ is obtained by the obvious gluing of the three discs: $\sigma^2_{i_0,i_2,i_3}$, $(D^2_2)_{\tau_{12}}$ and $(D^2_3)_{\tau_{12}}$ along their boundaries. Note that when $\tau_{12} = 1$, $S^2_{\tau_{12}}$ is the original sphere and when $\tau_{12} = 2$ it is a sphere construced as follows: we begin with two points p and q, join them with three segments two of which coincide: e_2, e_2, e_3. Next obtain three discs: one of which is degenerate and constructed by contracting a curve $e_2 * -e_2$ along itself, and the other two coincide, but have opposite orientation: one is obtained by contracting $e_2 * -e_3$ and the second one, by contracting $e_3 * -e_2$, (see fig. 3 (c)). So, obviously, the sphere that we obtain consists of two identical discs but with opposite orientation glued along their boundary, and is contractible along itself. Thus, we obtain a homotopy between the above sphere and a point and, therefore, reach a contradiction.

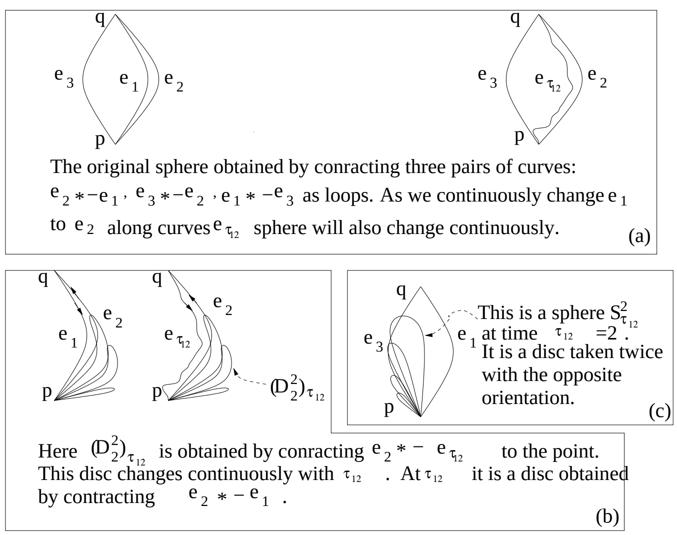

Figure 3: Construction of $S^2_{\tau_{12}}$

Remark. Let us consider a sphere in the manifold M^n obtained by taking a small 2-simplex $[v_{i_1}, v_{i_2}, v_{i_3}]$ and a point p, connecting p with each

v_{i_j} by a minimal geodesic segment $e_j, j = 1, 2, 3$, and finally, by contracting each of the closed curves $e_j + [v_{i_j}, v_{i_{j \bmod 3+1}}] - e_{j \bmod 3+1}$, where $j = 1, 2, 3$ to the point p as loops, (see fig. 4). Denote this sphere by S_0. We claim that for all practical purposes $[v_{i_1}, v_{i_2}, v_{i_3}]$ can be treated as a point q. Simply take a point $q \in [v_{i_1}, v_{i_2}, v_{i_3}]$ connect it with each v_{i_j} in $[v_{i_1}, v_{i_2}, v_{i_3}]$ by a short segment σ_j, (see fig. 4 (b)), $j = 1, 2, 3$ of length $\leq \delta$. Then instead of curves e_j we can just consider the new curves $e_j^* = e_j + \sigma_j$ of length $\leq d + \delta$, (see fig. 4 (c)). Note that each of the digons of the form $e_{j \bmod 3+1}^* * - e_j^*$ is contractible to p as loops without the length increase, (see fig. 4 (d)). Therefore, one can apply Lemma 1.1 to show that e_j^* is path homotopic to $e_{j \bmod 3+1}^*$ and the length of curves in this path homotopy is bounded by $3d + 3\delta$. These three homotopies give rise to three discs. Gluing them together results in a sphere that we will denote by S_1. It is easy to see that spheres S_0 and S_1 are homotopic, when δ is small enough. The intermediate spheres S_t are depicted on fig. 5. Therefore, if S_1 is not contractible whenever S_0 is not contractible. We can eventually let δ go to 0.

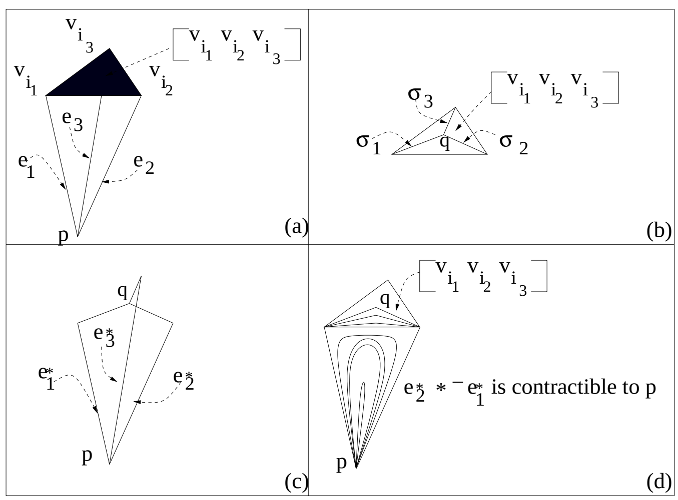

Figure 4: Small 2-simplex can be ignored.

Now, let us present the proofs of theorems.

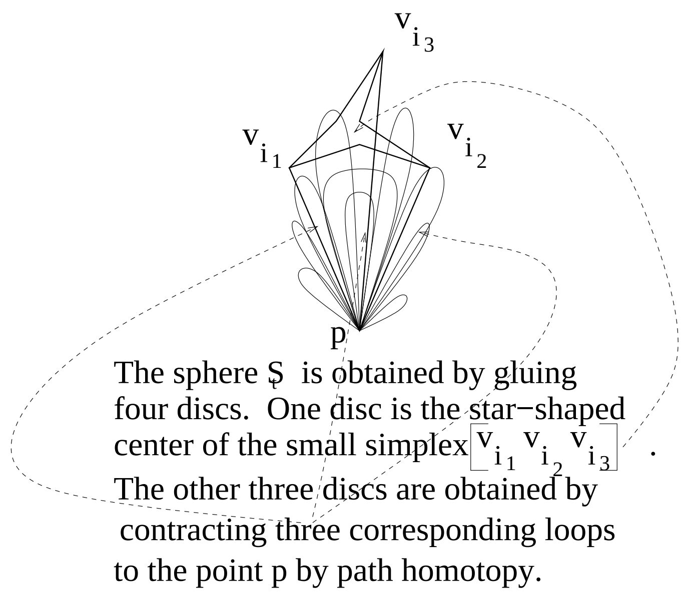

Figure 5: Spheres S_t.

2 Proof of Theorem 0.1.

Before giving the proof of Theorem 0.1 let us describe the main ideas. Let M^n be a closed Riemannian manifold, and suppose that $q > 0$ is the smallest natural number, such that $\pi_q(M^n)$ is not trivial. We consider a non-contractible sphere $f : S^q \longrightarrow M^n$ and show that, assuming there is no "short" geodesic loops, it can be filled by a disc. To construct this disc we use the following bootstrap procedure of constructing spheres and discs of progressively growing dimensions: One begins with two points p and q joined by k segments. Now, to construct a sphere of dimension $s < k$, one selects $s+1$ segments. The sphere is constructed by a natural gluing of $s+1$ s-discs. These discs are glued as the simplices in the boundary of $s+1$-dimensional simplex, where one of the simplices degenerates to a point.

Each such disc corresponds to s segments that are selected out of the given $s+1$ segments, and is generated by a family of $s-1$-dimensional spheres that start with a sphere that is constructed from on those s-segments on the previous step of induction and ends with a point.

Proof of Theorem 0.1. Let $f : S^q \longrightarrow M^n$ be a non-contractible map. Assume S^q is triangulated into fine simplexes, and that $f(S^q)$ has induced triangulation, such that diameter of any simplex in this triangulation is smaller than δ. Let D^{q+1} be triangulated as a cone over S^q. Assuming that the length of a shortest geodesic loop based at $p \in M^n$ is greater than $2qd$ we will extend our map to D^{q+1}, thus reaching a contradiction. We will first extend to 0, 1, 2, and 3-skeleta, as described above in section 1. Let us denote the image of a 3-simplex $\tilde{\sigma}^3_{i_0,\dots,i_3} = [\tilde{v}_{i_0}, \tilde{v}_{i_1}, ..., \tilde{v}_{i_3}]$, where $\tilde{v}_{i_0} = \tilde{p}$ by $\sigma^3_{i_0,\dots,i_3}$.

Now suppose we want to extend our map to 4-skeleton. Let us consider an arbitrary simplex $\tilde{\sigma}_{i_0,i_1,i_2,i_3,i_4} = [\tilde{p}, \tilde{v}_{i_1}, \tilde{v}_{i_2}, \tilde{v}_{i_3}, \tilde{v}_{i_4}]$. Its boundary is mapped to the following 3-sphere $\Sigma^4_{j=0}(-1)^j \sigma_{i_0,..,\hat{i}_j,\dots,i_4}$. Now let us construct the following homotopy contracting this sphere to a point. Again, without loss of generality, assume that simplex $[v_{i_1}, ..., v_{i_4}]$ is so small that it can, for our purposes, be treated as a point, that we will denote by q. Each of the four edges $[p, v_{i_j}]$ will be denoted by e_j. We know that e_1 is homotopic to e_2 by a path homotopy along the curves $e_{\tau_{12}}$, $1 \le \tau_{12} \le 2$ of length $\le 3d$, (see Lemma 1.1). Let us "move" e_1 to e_2 and construct a homotopy of the 3-sphere that will "follow" this move. That is for each τ_{12} we want to construct a sphere $S^3_{\tau_{12}}$ that continuously depends on τ_{12}. This sphere will be made of four discs glued together. These discs are glued as four simplices in the boundary of the 4-simplex, where the fifth simplex degenerates to a point.

Disc $(D^3_1)_{\tau_{12}}$ will stay constantly equal to $\sigma^3_{i_2,i_3,i_4}$.

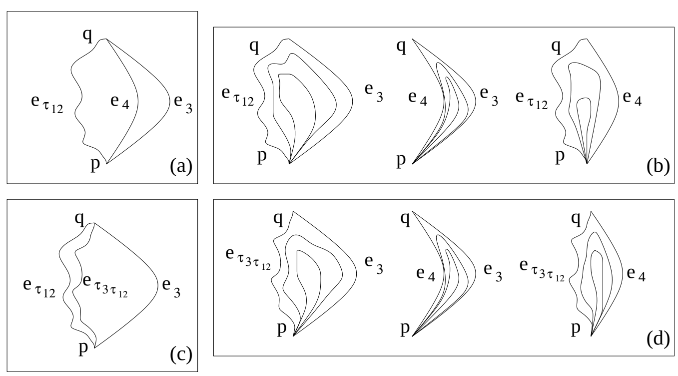

Figure 6: Constructing $(D^3_2)_{\tau_{12}}$.

$(D^3_2)_{\tau_{12}}$ is constructed as follows: take two points p, q connected by three segments: $e_{\tau_{12}}, e_3, e_4$, (see fig. 6 (a)). We know that in this situation, we can construct a sphere $S^2_{\tau_{12}}$ and also to continuously deform it to a point as

follows:
1. We construct $S^2_{\tau_{12}}$ by taking three loops $e_3 * -e_{\tau_{12}}$, $e_4 * -e_3$, $e_{\tau_{12}} * -e_4$ and contracting them to p by a length decreasing path homotopy, (see fig. 6 (b)). Here we use the assumption that the length of a shortest geodesic loop at p is greater than $2qd$, and, thus, greater than $4d$. So each of the loops is contractible to p without the length increase.
2. Now, by Lemma 1.1 there exists a path homotopy that connects $e_{\tau_{12}}$ with e_3 along the curves $e_{\tau_{3\tau_{12}}}$, $1 \leq \tau_{3\tau_{12}} \leq 2$ of length $\leq 5d$. This is due to the fact, that the loop $e_{\tau_{12}} * -e_3$ is contractible to p without the length increase, (see fig. 6 (c)).
3. As $e_{\tau_{12}}$ moves to e_3, we use the fact that the length of a geodesic loop is also greater than $6d$ to construct a family of 2-dimensional spheres $S^2_{\tau_{3\tau_{12}}}$ that continuously depends on $\tau_{3\tau_{12}}$ and that coincides with $S^2_{\tau_{12}}$, when $\tau_{3\tau_{12}} = 1$. That is we repeat Step 1, but with $e_{\tau_{3\tau_{12}}}$ replacing $e_{\tau_{12}}$. Note also, that when $\tau_{3\tau_{12}} = 2$, we obtain a degenerate sphere, consisting of a 2-disc taken twice with the opposite orientation, that can be contracted to a point. This family of spheres corresponds to a 3-disc $(D^3_2)_{\tau_{12}}$. Note that at $\tau_{12} = 1$ it is $\sigma^3_{i_1,i_3,i_4}$ and at $\tau_{12} = 2$ it is $-\sigma^2_{i_2,i_3,i_4}$.
4. The other two discs $(D^3_3)_{\tau_{12}}$ and $(D^3_4)_{\tau_{12}}$ are obtained in a similar way.
5. The sphere $S^3_{\tau_{12}}$ is obtained by the obvious gluing. Furthermore, S^3_1 is the original sphere and S^3_2 is a sphere that is obtained by gluing $\sigma^3_{i_2,i_3,i_4}$ and $-\sigma^3_{i_2,i_3,i_4}$, and so it is contractible to a point. We will map $\tilde{\sigma}^4_{i_0,\ldots,i_4}$ to the disc generated by this family of 3-spheres. Let us denote this disc by $\sigma^4_{i_0,\ldots,i_4}$.

Now suppose we have extended the map f to the k-skeleton of D^{q+1} in a similar fashion and now we want to extend it to $(k+1)$-skeleton. Consider an arbitrary $(k+1)$-simplex $\tilde{\sigma}^{k+1}_{i_0,\ldots,i_{k+1}}$. Its boundary is mapped to $\Sigma^{k+1}_{j=0}(-1)^j \sigma^k_{i_0,\ldots,\hat{i}_j,\ldots,i_{k+1}}$. As before, we assume that $\sigma_{i_1,\ldots,i_{k+1}}$ can be treated as a point denoted by q and we denote edges $[p, v_{i_j}]$ as e_j. We know that there is a path homotopy between e_1 and e_2 that passes through the curves $e_{\tau_{12}}$ of length $\leq 3d$. We can extend this homotopy to the homotopy between the underlying map of the boundary of a simplex $\tilde{\sigma}^{k+1}_{i_0,\ldots,i_{k+1}}$ and a k-sphere that will then be contracted to a point. This homotopy can be explained as follows. Its image is a $q+1$-dimensional disc $\sigma^{k+1}_{i_0,\ldots,i_{k+1}}$ that is generated by the family of spheres $S^k_{\tau_{12}}, 1 \leq \tau_{12} \leq 2$, such that $S^k_1 = \partial\sigma^k_{i_0,\ldots,i_{k+1}} = f(\partial\tilde{\sigma}^k_{i_0,\ldots,i_{k+1}})$ and S^k_2 is a sphere that is contractible along itself. This family of spheres $S^k_{\tau_{12}}$ is constructed by taking two points p, q joining them by $e_{\tau_{12}}, e_2, \ldots, e_{k+1}$ and repeating the whole process of constructing k-sphere based on two vertices and $k+1$ curves connecting them, but with $e_{\tau_{12}}$ replacing e_1. (We learned to construct such k-spheres on the previous step of induction). As the length of $e_{\tau_{12}}$ can exceed the length of e_1 by $2d$, so the length of curves in all of the homotopies can increase by $2d$ as well. At this step we use the assumption that $l_p(M^n) > 2qd > 2kd$. Recall that the family of spheres $S^k_{\tau_{12}}$ is constructed by gluing of k discs. The disc $(D^k_1)_{\tau_{12}}$ will be constantly equal to $\sigma^k_{i_2,...,i_{k+1}}$. And, of course, $(D^k_2)_{\tau_{12}} = D^k_2(\tau_{12})$ is constructed using the previous step of an inductive construction: we begin with the two points p, q joined by k segments: $e_{\tau_{12}} = e(\tau_{12}), e_3, ..., e_{k+1}$. The disc is constructed by constructing a family of spheres $S^{k-1}_{\tau_3\tau_{12}}$ that start with a sphere $S^{k-1}_{\tau_{12}}$ and with a sphere that is easily contractible to a point, which we already learned to do at the previous stage, etc.

Thus, we can continue until we extend to $(q+1)$-skeleton of D^{q+1}, reaching a contradiction.

□

Proof of Theorem 0.3. Assume that $\alpha(M^n) > 6nFillRadM^n$. The definition of the filling radius implies that M^n bounds in the $(FillRadM^n + \delta)$-neighborhood of M^n in $L^\infty(M^n)$. Let W denote the chain that fills M^n in this neighborhood of M^n. That is, $M^n = \partial W$, if M^n is orientable, and $M^n = \partial W$ mod 2, if M^n is not orientable. Without any loss of generality we can assume that W is a polyhedron, (see [G]).

Suppose that W together with M^n is endowed with a fine triangulation, i.e. diameter of any simplex in this triangulation is $\leq \delta$. We are going to construct a singular $(n+1)$-chain on M^n, such that the boundary of that chain is homologous to the boundary of W, thus reaching a contradiction. We will construct this chain by induction with respect to the dimension of skeleta of W. That is to each k-simplex of W we will assign a singular k-chain on M^n. Let us begin with the 0-skeleton of W. To each vertex $\tilde{v}_i$ of W we will assign a vertex v_i of M^n, such that $d(v_i, \tilde{v}_i) = d(\tilde{v}_i, M^n) \leq FillRadM^n + \delta$. Now suppose that v_i, v_j come from the vertices $\tilde{v}_i, \tilde{v}_j$ of some simplex in W. Then $d(v_i, v_j) \leq 2FillRadM^n + 3\delta$. Thus, to extend to 1-skeleton, we will assign to any 1-simplex $[\tilde{v}_i, \tilde{v}_j] \subset W \backslash M^n$ a minimal geodesic that connects v_i and v_j of length $\leq 2FillRadM^n + 3\delta$. We can see that the boundary of each 2-simplex in W is mapped to a closed curve of length $\leq 6FillRadM^n + 9\delta$. (We are assuming that all simplices in M^n are already short).

Let $\tilde{\sigma}^2_{i_0,i_1,i_2} = [\tilde{v}_{i_0}, \tilde{v}_{i_1}, \tilde{v}_{i_2}]$ be an arbitrary 2-simplex. Next, we are going to extend to 2-skeleton of W. Since, we assumed that there are no "short" geodesic loops on M^n the curve that corresponds to the boundary of this simplex in M^n is contractible as loop without the length increase to vertex v_{i_0}. We will map $\tilde{\sigma}^2_{i_0,i_1,i_2}$ to the surface generated by this homotopy. Now to extend to 3-skeleton, consider an arbitrary 3-simplex $\tilde{\sigma}^3_{i_0,i_1,i_2,i_3}$. We know that its boundary is mapped to $\Sigma^3_{j=0}\sigma^2_{i_0,\ldots,\hat{i}_j,\ldots,i_3}$. Take $\sigma^2_{i_0,i_1,i_2}$ in this boundary. By Lemma 1.1 there exists a path homotopy connecting the curve $[v_{i_0}, v_{i_2}]$ with the curve $[v_{i_0}, v_{i_1}] + [v_{i_1}, v_{i_2}]$ passing through the curves $e(\tau_{012})$ of length $\leq 8FillRadM^n + 11\delta$. Similarly, there is a path homotopy connecting the curve $[v_{i_1}, v_{i_3}]$ with the curve $[v_{i_1}, v_{i_2}] + [v_{i_2}, v_{i_3}]$ along the curves $e(\tau_{123})$ also of length $\leq 8FillRadM^n + 11\delta$, (see fig. 7). Here we use the assumption that the length of a geodesic loop is greater than $6FillRadM^n + \varepsilon$, where ε is some multiple of δ.

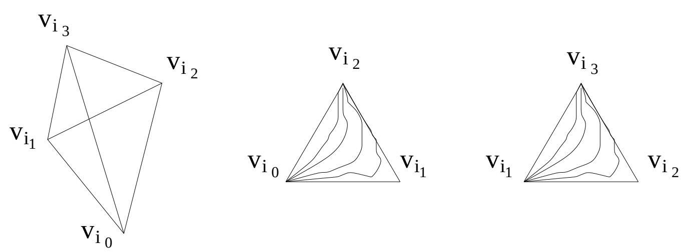

Figure 7: Homotopies between $[v_{i_0}, v_{i_2}]$ and $[v_{i_0}, v_{i_1}] + [v_{i_1}, v_{i_2}]$ and between $[v_{i_1}, v_{i_3}]$ and $[v_{i_1}, v_{i_2}] + [v_{i_2}, v_{i_3}]$

Now we can construct homotopy connecting the spherical cycle $\Sigma^3_{j=0}(-1)^j\sigma^2_{i_0,\ldots,\hat{i}_j,\ldots,i_3}$ with a point, (see fig. 8).

1. Let us begin by moving the curve $[v_{i_0}, v_{i_2}]$ to the curve $[v_{i_0}, v_{i_1}] + [v_{i_1}, v_{i_2}]$ along the curves $e(\tau_{012})$. We can generate a family of spheres $S^2(\tau_{012})$ that begin with the given spherical cycle, and that are constructed as follows. We replace $[v_{i_0}, v_{i_2}]$ by $e(\tau_{012})$. $S^2(\tau_{012})$ will be obtained by gluing of the four discs. Two discs: $\sigma^2_{i_1,i_2,i_3}$ and $\sigma^2_{i_0,i_1,i_3}$ will stay constant. The other two discs will be obtained by contracting loops $e(\tau_{012}) + [v_{i_2}, v_{i_1}] + [v_{i_1}, v_{i_0}]$ and $[v_{i_0}, v_{i_3}] + [v_{i_3}, v_{i_2}] - e(\tau_{012})$ to a point v_{i_0}, (see fig. 8 (b)).

Let us describe $S^2(2)$. This is a sphere that is obtained by gluing of the two constant discs $\sigma^2_{i_1,i_2,i_3}$ and $\sigma^2_{i_0,i_1,i_3}$, one degenerate disc, that is obtained by contracting a curve $[v_{i_0}, v_{i_1}] + [v_{i_1}, v_{i_2}] - [v_{i_1}, v_{i_2}] - [v_{i_0}, v_{i_1}]$ along itself, and, finally a disc that is obtained by contracting a curve $[v_{i_0}, v_{i_3}] + [v_{i_3}, v_{i_2}] + [v_{i_2}, v_{i_1}] + [v_{i_1}, v_{i_0}]$ to the point v_{i_0}, (see fig. 8 (c)).

2. Let us next move the curve $[v_{i_1}, v_{i_3}]$ to the curve $[v_{i_1}, v_{i_2}] + [v_{i_2}, v_{i_3}]$ along the curves $e(\tau_{123})$, $1 \leq \tau_{123} \leq 2$. We can generate a family of spheres $S^2(\tau_{123})$, such that $S^2(\tau_{123} = 1)$ coincides with $S^2(\tau_{012} = 2)$. Now

$S^2(\tau_{123} = 2)$ is a sphere that consists of one disc taken twice with the opposite orientation, (see fig. 8 (d), (e)), and, thus, is contractible. (Here we use the fact that our assumption implies that all "short" loops β based at any point p can be contracted to p by a length-decreasing homotopy, and this homotopy continuously depends on β. The fact that these two discs coincide up to orientation is due to the fact that they both are obtained by contracting the broken line $[v_{i_1}, v_{i_0}] + [v_{i_0}, v_{i_3}] + [v_{i_3}, v_{i_2}] + [v_{i_2}, v_{i_1}]$ as a loop based at v_{i_0}.) Therefore, we obtain a (possibly singular) disc $\sigma^3_{i_0,\ldots,i_3}$, which will be assigned to $\tilde{\sigma}^3_{i_0,\ldots,i_3}$

The proof then becomes analogous to that of Theorem 0.1. We continue the extension procedure until we reach $(q+1)$-skeleton of W. In the proof of Theorem 0.1 we "collpased" k-spheres built out of $k+1$ segments connecting two points p and q by connecting two of these segments by a homotopy until they became a degenerate set of $k+1$ segments. These homotopies of k-spheres gave rise to $k+1$ discs that were then glued into $(k+1)$-dimensional spheres, etc. Now our k-spheres are formed out of 1-skeletons of $(k+1)$-dimensional simplices. In order to "collapse" the 1-skeleton we collapse edges $[v_{i_0}, v_{i_1}], [v_{i_2}, v_{i_3}], [v_{i_4}, v_{i_5}], \ldots, [v_{i_k}, v_{i_{k+1}}]$ for even k and edges $[v_{i_0}, v_{i_1}], [v_{i_2}, v_{i_3}], \ldots, [v_{i_{k-1}}, v_{i_k}], [v_{i_k}, v_{i_{k+1}}]$ for odd k, (the edge $[v_{i_0}, v_{i_1}]$ is contracted to $[v_{i_0}, v_{i_2}] + [v_{i_2}, v_{i_1}]$, $[v_{i_2}, v_{i_3}]$ is contracted to $[v_{i_2}, v_{i_4}] + [v_{i_4}, v_{i_3}]$, etc. by path homotopies. We always use the verthex with the smallest number as a base point for loops and apply Lemma 1.1. For odd k. $[v_{i_k}, v_{i_{k+1}}]$ is path homotopic to $[v_{i_k}, v_{i_0}] + [v_{i_0}, v_{i_{k+1}}]$. We will leave out the ackward details of the computation leading to the upper bound, for the sake of exposition.

□

Now we are going to prove Theorem 0.5.

Proof of Theorem 0.5. Once again, let us begin with a non-contractible map $f : S^2 \longrightarrow M^n$, where S^2 is the standard 2-sphere endowed with a fine triangulation. Let $p \in M^n$. We will try to extend this map to D^3 triangulated as a cone over S^2, which, of course, is impossible. The procedure will be inductive to skeleta of D^3. We will begin as usual, by extending to 0-skeleton. This is done by assigning to the center of the disc, $\tilde{p}$ a given point p. Next we extend to 1-skeleton, by assining to an edge $[\tilde{p}, \tilde{v}_i]$ a minimal geodesic segment $[p, v_i]$, of length smaller than d. Next we extend to 2-skeleton. Consider an arbitrary 2-simplex $\tilde{\sigma}_{i_0,i_1,i_2}$. Its boundary is mapped to a closed curve of length $\leq 2d + \delta$. This curve is

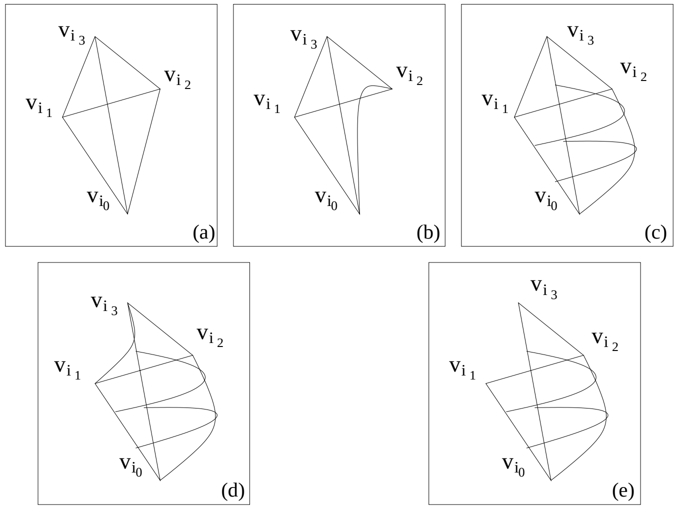

Figure 8: Contracting $\Sigma_{j=0}^{3}(-1)^j\sigma^2_{i_0,\ldots,\hat{i}_j,\ldots,i_3}$ to a point.

either contractible to p by a path homotopy, or there exists a geodesic loop based at p of length $\leq 2d$. In such a case we will release the point and will let the curve contract to a point by a regular (not path) homotopy, (see fig. 9). In either of these cases, the image of this simplex will be disc generated by the homotopy connecting the curve with a point. We will denote it as σ_{i_0,i_1,i_2}.

It is impossible to extend $f : S^2 \longrightarrow M^n$ to 3-skeleton of D^3. Therefore, there exists a 3-simplex $\tilde{\sigma}_{i_0,i_1,i_2,i_3}$ such that the map $f : \partial\tilde{\sigma}_{i_0,i_1,i_2,i_3} \longrightarrow M^n$ is a non-contractible sphere. Let us consider this sphere. It consists of three "big" discs: $-\tilde{\sigma}_{i_0,i_2,i_3}$, $\tilde{\sigma}_{i_0,i_1,i_3}$, $-\tilde{\sigma}_{i_0,i_1,i_2}$, and a "small" one σ_{i_1,i_2,i_3}. The "small" one is so small that it can be regarded as a point q for all practical purposes. The rest of the discs were obtained by contracting their corresponding boundaries to a point. Moreover, those three discs were either generated by path homotopy that connects the boundary to a point, or by a homotopy that was a path homotopy until we encountered a critical geodesic loop, and that then became a regular homotopy, (see fig. 9).

Let us consider the following three cases.

(1) The boundary of each face gets "stuck" on a distinct geodesic loop based at p of length $\leq 2d + \delta$.

(2) The boundary of one of the simplices is contractible to p via path homotopy;

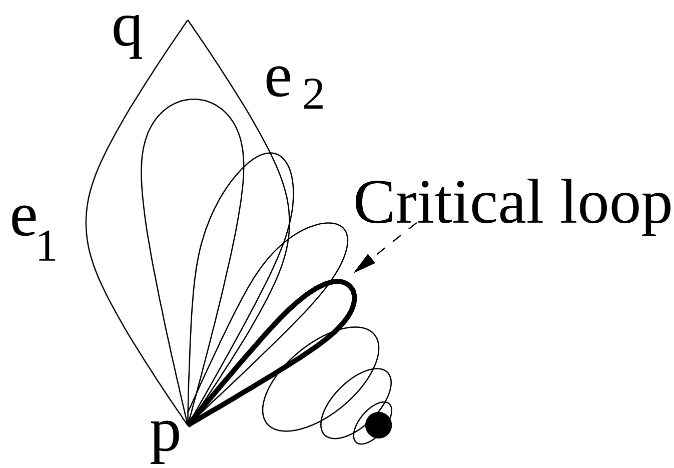

Figure 9: Extending to 2-skeleton.

(3) None of the boundaries are contractible to p via length-decreasing path homotopy, but at least two of the geodesic loops that obstruct this coincide.

In the first case, we are done. We have three distinct loops based at p of length $\leq 2d + \delta$. We just need to let δ go to 0.

In the second case, without loss of generality, assume that $e_2 * -e_1$ is contractible to a point p with a length-decreasing path homotopy. Then, by Lemma 1.1, we know that e_1 is path homotopic to e_2 through curves $e_{\tau_{12}}$ of length less than or equal to $3d$. Assume $e_1 * -e_3$ is contractible to a point q_1 along the curves $\gamma(\tau)$ and that $e_2 * -e_3$ is contractible to a point q_2 along the curves $\alpha(\tau)$ of length $\leq 2d$, (see fig. 10 (a)).

Therefore, we can construct the following homotopy in the space ΛM^n of closed curves. Here is a loop in ΛM^n. $q_1 \sim \gamma(1-\tau) \sim e_1 * -e_3 \sim e_{\tau_{12}} * -e_3 \sim e_2 * -e_3 \sim \alpha(\tau) \sim q_2 \sim q_1$, (see fig. 10 (b)-(d)). This loop corresponds to the non-contractible sphere $f : \partial\tilde{\sigma}^3_{i_0,\dots,i_3} \longrightarrow M^n$, as it was obtained from the above sphere by a sweep-out. Therefore, it is a non-contractible loop that passes through curves of length $\leq 4d$. Therefore, there exists a closed geodesic of length $\leq 4d$.

Finally, in the third case, we will construct a non-contractible loop in the space ΛM^n as follows.

Let us assume that $e_1 * -e_2$ and $e_2 * -e_3$ get "stuck" on the same loop α_1, (see fig. 11 (a)), $e_1 * -e_3$ gets "stuck" on the loop α_2, which might or might not coincide with α_1. Those loops are then contractible to points

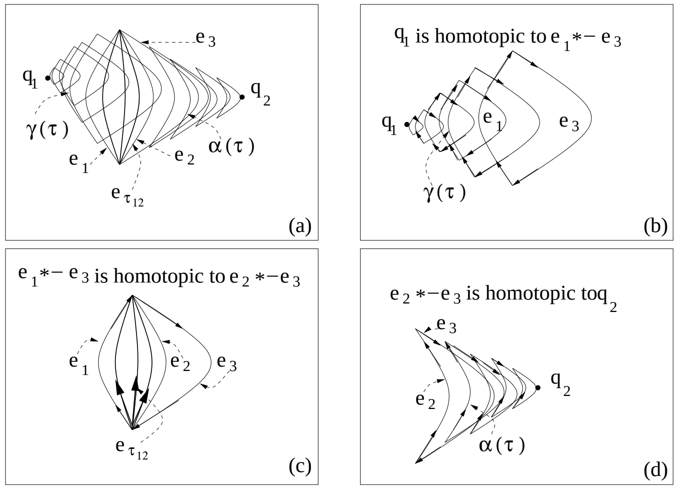

Figure 10: Loop in the space ΛM.

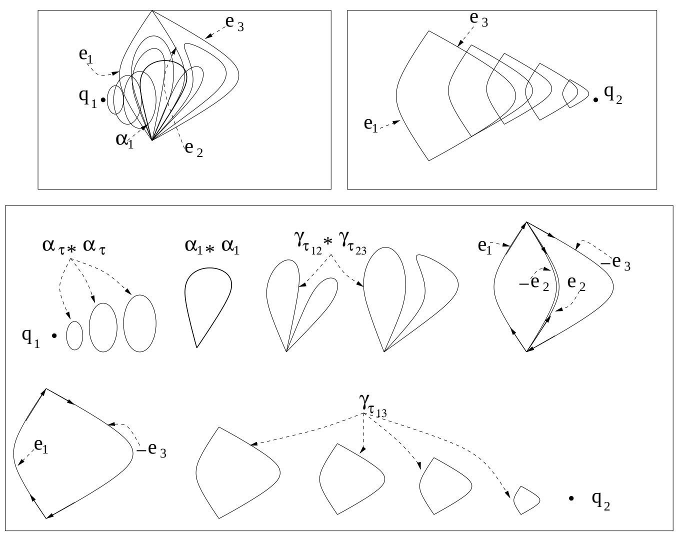

Figure 11: Loop in the space ΛM.

$\tilde{q}_1$ and $\tilde{q}_2$ respectively, (see fig. 11 (a) and (b)). Denote the curves in the homotopy that connects α_1 with q_1 by α_τ, $1 \leq \tau \leq 2$. Further, denote the curves in the homotopy that connects $e_3 * -e_1$ and q_2 by $\gamma_{\tau_{13}}$, $1 \leq \tau_{13} \leq 2$. Finally, denote the curves in the homotopy that connects $e_1 * -e_2$ and α_1 by $\gamma_{\tau_{12}}$ and the curves in the homotopy that connects $e_2 * -e_3$ and α_1 by $\gamma_{\tau_{23}}$, $1 \leq \tau_{12}, \tau_{23} \leq 2$.

We will now describe a non-contractible loop in the space ΛM^n, (see fig. 11 (c)). It will be a sweep-out of the non-contractible sphere $f : \partial\tilde{\sigma}^3_{i_0,\dots,i_3} \longrightarrow M^n$ by short loops.

$q_1 \sim \alpha_\tau * \alpha_\tau$, (that is we go around $\alpha_{1\tau}$ twice).

$\alpha_\tau * \alpha_\tau \sim \alpha_1 * \alpha_1 \sim e_{\tau_{12}} * e_{\tau_{23}} \sim e_1 * -e_2 * e_2 * -e_3 \sim e_1 * -e_3 \sim \gamma_{\tau_{13}} \sim q_2 \sim q_1$.

$\square$

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