Root Vectors in Quantum Groups

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Abstract: In this paper we give a description for the set of all root vectors in a quantum group (Theorem 4.4). For type A_n we get a clear formula for the coproduct of a root vector (Theorem 5.5).

1. Introduction

Recall some basic concepts.

1.1. Let R be an irreducible root system with simple roots α_i $(1 \le i \le n)$, $R^\vee$ and $\alpha_i^\vee$ be the corresponding dual. Then $(a_{ij})_{1\le i,j\le n}$ is a Carten matrix, where $a_{ij} =< \alpha_i^\vee, \alpha_j >$. Assume that we are given integers $d_i \in \{1,2,3\}$ $(1 \le i \le n)$ such that $d_i a_{ij} = d_j a_{ji}$. The quantum group U over $\mathbf{Q}(v)$ (v is an indeterminate) associated to (a_{ij}) is an associative algebra over $\mathbf{Q}(v)$, generated by E_i, F_i, K_i, K_i^{-1} $(1 \le i \le n)$ which satisfy the q-analog of Serre relations (see for exemple, [L2]). The algebra U is in fact a Hopf algebra, the coproduct Δ , antipode S, counit ϵ are defined as follows:

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \Delta(K_i) = K_i \otimes K_i,$$

$$S(E_i) = -K_i^{-1}E_i, \quad S(F_i) = -F_iK_i, \quad S(K_i) = K_i^{-1},$$

$$\epsilon(E_i) = \epsilon(F_i) = 0, \quad \epsilon(K_i) = 1.$$

1.2. The root vectors in U are defined through elements of the Weyl group and some automorphisms of U (see [L2]). We recall the definition.

Let W be the Weyl group of R generated by simple reflections s_i $(1 \le i \le n)$ which are defined by $s_i(\alpha) = \alpha - < \alpha, a_i^\vee > \alpha_i$, $\alpha \in R$. Foe each i the automorphism $T_{s_i} = T_i$ is defined by Lusztig as follows (see [L2]):

$$T_iE_i = -F_iK_i, \quad T_iE_j = \sum_{r+s=-a_{ij}} (-1)^r v^{-d_is} E_i^{(r)} E_j E_i^{(s)}, \quad \text{if } i \ne j,$$

$$T_iF_i = -K_i^{-1}E_i, \quad T_iF_j = \sum_{r+s=-a_{ij}} (-1)^r v^{d_is} F_i^{(s)} F_j F_i^{(r)}, \quad \text{if } i \ne j,$$

$$T_iK_j = K_iK_j^{-a_{ij}}.$$

where $E_i^{(N)} = E_i^N/[N]^!_{d_i}$, $F_i^{(N)} = F_i^N/[N]^!_{d_i}$, $[N]^!_{d_i} = [1]_{d_i}[2]_{d_i}\cdots[N]_{d_i}$, $[N]_{d_i} = \frac{v^{Nd_i}-v^{-Nd_i}}{v^{d_i}-v^{-d_i}}$, $N \ge 1$, and $[0]_{d_i} = [0]^!_{d_i} = 1$.

These automorphisms satisfy the braid relations, thus for each element $w \in W$ we can define the automorphism T_w of U as $T_{i_k}...T_{i_2}T_{i_1}$ where $s_{i_k}...s_{i_2}s_{i_1}$ is a reduced expression of w (see [L2, 3.1-2]).

1.3. The following are some simple properties about these automorphisms T_w (see [L2]):
(a1). Let $\Omega,\ \Psi : U \to U^{opp}$ be the $\mathbf{Q}$-algebra homomorphisms defined by

$$\Omega E_i = F_i, \quad \Omega F_i = E_i, \quad \Omega K_i = K_i^{-1}, \quad \Omega v = v^{-1},$$

$$\Psi E_i = E_i, \quad \Psi F_i = F_i, \quad \Psi K_i = K_i^{-1}, \quad \Psi v = v.$$

We have $\Omega T_i = T_i \Omega$ and $T_i' = T_i^{-1} = \Psi T_i \Psi$. So $\Omega T_w = T_w \Omega$ and $T_{w^{-1}}^{-1} = \Psi T_w \Psi$ for any $w \in W$.

(a2) $$T_w E_i = E_j, \quad \text{if } w(\alpha_i) = \alpha_j.$$

By (a2) and the definition of T_w we get the following equalities.

(a3) $$T_i E_j = E_j, \quad T_i F_j = F_j, \quad T_i K_j = K_j, \quad \text{if } a_{ji} = 0.$$

(a4) $$T_i^{-1} E_j = T_j E_i, \quad T_i^{-1} F_j = T_j F_i, \quad T_i^{-1} K_j = T_j K_i, \quad \text{if } a_{ji} a_{ij} = 1.$$

(a5) $$T_i^{-1} E_j = T_j T_i E_j, \quad T_i^{-1} F_j = T_j T_i F_j, \quad T_i^{-1} K_j = T_j T_i K_j, \quad \text{if } a_{ij} a_{ji} = 2.$$

If $a_{ij} a_{ji} = 3$, then we have

(a6) $$T_i^{-1} E_j = T_j T_i T_j T_i E_j, \quad T_i^{-1} F_j = T_j T_i T_j T_i F_j, \quad T_i^{-1} K_j = T_j T_i T_j T_i K_j,$$

(a7) $$T_j^{-1} T_i^{-1} E_j = T_i T_j T_i E_j, \quad T_j^{-1} T_i^{-1} F_j = T_i T_j T_i F_j, \quad T_j^{-1} T_i^{-1} K_j = T_i T_j T_i K_j,$$

We also have

(a8) $T_i^2 E_i = v^{2d_i} K_i^{-2} E_i.$

(a9) $T_i^2 E_j = (1 - v^{-2d_i}) F_i K_i T_i(E_j) - v^{-d_i} E_j \qquad \text{if } a_{ij} = -1.$

If $a_{ij} = -2$, then

(a10) $$T_i^2 E_j = v^{-2}(1 - v^{-2})(1 - v^{-4}) F_i^{(2)} K_i^2 T_i(E_j) - v^{-1}(1 - v^{-2}) F_i K_i T_j^{-1}(E_i) + v^{-2} E_j.$$

If $a_{ij} = -3$, then

(a11) $$T_i^2 E_j = v^{-6}(1 - v^{-2})(1 - v^{-4})(1 - v^{-6}) F_i^{(3)} K_i^3 T_i(E_j)$$

$$-v^{-3}(1-v^{-2})(1-v^{-4})F_i^{(2)}K_i^2T_iT_j(E_i)+v^{-2}(1-v^{-2})F_iK_iT_j^{-1}(E_i)-v^{-3}E_j.$$

1.4. For any positive root $\alpha \in R^+$ (the set of positive roots in R), if $w^{-1}(\alpha)=\alpha_i$ ($w \in W$) is a simple root in R, then we set $E_{\alpha,w}=T_w(E_i)$ (resp. $E_{-\alpha,w}=F_{\alpha,w}=\Omega E_{\alpha,w}=T_w(F_i)$) and call it a root vector in U of root α (resp. $-\alpha$).

The definition of root vectors looks very simple, however even for some simple questions, such as how many are there root vectors of a given root, the relations between root vectors, etc., we know little. Though there are several formulas concerned with the coproducts of root vectors (see [AJS, KR, LS]), there are no closed formula for these coproducts in general. It seems also no explicit formula for the antipode of a root vector at hand. Of course, everything becomes simple when $\alpha=\alpha_i$ is a simple root: there is only one root vector in U of root α which is E_i by (a2), the coproduct and the antipode of E_i are given by definition in 1.1. Sometimes we write E_{α_i} instead of E_i.

In this paper we give a description for the set of all root vectors in U (section 4) and give a clear formula for the coproduct of a root vector in a quantum group of type A_n (section 5). We only discuss root vectors of positive roots since through the homomorphism Ω all results can be transfered to those concerned with the root vectors of negative roots.

2. Some Facts on Root System and Weyl Group

2.1. In this section we prove some results concerned with roots and Weyl groups, on which our main results depend heavily.

First we recall some facts about root systems. We number the set $\mathcal{D}=\{\alpha_1,\alpha_2,...,\alpha_n\}$ of all simple roots of R as in [B, Planche V – IX] when R is of exceptional type and as in [B, Planche I – IV] composed with $i\to n+1-i$ when R is of classical type. Then we have

Type A_n ($n\ge 1$): $R^+=\{\alpha_{ij}=\sum_{i\le m\le j}\alpha_m \mid 1\le i\le j\le n\}$.

Type B_n ($n\ge 3$): $R^+=\{\alpha_{ij}=\sum_{i\le m\le j}\alpha_m,\ \beta_{kl}=2\sum_{1\le m\le k}\alpha_m+\sum_{k<m\le l}\alpha_m \mid 1\le i\le j\le n\ ,1\le k<l\le n\}$.

Type C_n ($n\ge 2$): $R^+=\{\alpha_{ij}=\sum_{i\le m\le j}\alpha_m,\ \beta_{kl}=\alpha_1+2\sum_{1<m\le k}\alpha_m+\sum_{k<m\le l}\alpha_m,\ \gamma_k=\alpha_1+2\sum_{1<m\le k}\alpha_m \mid 1\le i\le j\le n\ ,1<k<l\le n\}$.

Type D_n ($n\ge 4$): $R^+=\{\alpha_{ij}=\sum_{i\le m\le j}\alpha_m,\ \alpha'_{1k'}=\alpha_1+\sum_{2<m\le k'}\alpha_m,\ \beta_{kl}=\alpha_1+\alpha_2+2\sum_{2<m\le k}\alpha_m+\sum_{k<m\le l}\alpha_m \mid 1\le i\le j\le n$, and $i=2$ when $j=2$; $2<k<l\le n$, $2<k'\le n\}$.

Type G_2: $R^+=\{\alpha_1,\alpha_2,\alpha_1+\alpha_2,2\alpha_1+\alpha_2,3\alpha_1+\alpha_2,3\alpha_1+2\alpha_2\}$. For details of other types see [B, Planche V-VIII].

Realizing R (resp. $R^\vee$) as a subset of an Euclidean space as in [B, Planche I-IX], we then may define the length of a root in R (resp. $R^\vee$), so we have the concept of long roots and short roots in R (resp. $R^\vee$). For a root α in R we denote its dual in $R^\vee$ by $\alpha^\vee$.

Let α, β be roots in R^+, the following facts are either standard or easy to check.

(b1). If α is a short root, then $| < \alpha, \alpha_k^\vee > | \leq 1$ for any simple root α_k in R.

(b2). $< \alpha, \alpha_k^\vee > > 0$ for some simple root α_k.

(b3). α and β (resp. $\alpha^\vee$ and $\beta^\vee$) have the same length if and only if $\beta = w(\alpha)$ (resp. $\beta^\vee = w^\vee(\alpha^\vee)$) for some $w \in W$ (resp. $w^\vee \in W^\vee$, see (b6) for the definition of $W^\vee$).

(b4). α is a long (resp. short) root if and only if $\alpha^\vee$ is a short (resp.long) root in $R^\vee$.

(b5). Assume that α, β have the same length, then $\alpha \leq \beta$ if and only if $\alpha^\vee \leq \beta^\vee$, where $\leq$ is the usual partial order in the root lattice $\mathbf{Z}R$ (resp. $\mathbf{Z}R^\vee$).

Convention: the notation $\alpha < \beta$ (resp. $\alpha^\vee < \beta^\vee$) means that $\alpha \leq \beta$ (resp. $\alpha^\vee \leq \beta^\vee$) but $\alpha \neq \beta$ (resp. $\alpha^\vee \neq \beta^\vee$). We also use the symbol $\leq$ for the Bruhat order in W or $W^\vee$. We extend $<,>$ to $\mathbf{Z}R \times \mathbf{Z}R^\vee$ as usual.

(b6). Let $W^\vee$ be the Weyl group of $R^\vee$ generated by simple reflections $s_i^\vee$ ($i \in [1,n]$) which are defined by $s_i^\vee(\alpha^\vee) = \alpha^\vee - < \alpha_i, \alpha^\vee > \alpha_i^\vee$, $\alpha \in R^\vee$, then the map $s_i \to s_i^\vee$ defines an isomorphism between the Weyl groups W and $W^\vee$.

For $w \in W$ we denote its corresponding element in $W^\vee$ by $w^\vee$, then $\ell(w) = \ell(w^\vee)$, moreover $w(\alpha)^\vee = w^\vee(\alpha^\vee)$, here ℓ is the standard length function on W or $W^\vee$.

Lemma 2.2. Given a root $\alpha \in R^+$. If $s_k\alpha < \alpha$, $s_{k'}\alpha < \alpha$, and $\alpha, \alpha_k, \alpha_{k'}$ are linearly independent, then $s_k s_{k'} = s_{k'} s_k$.

Proof: The set $(\mathbf{Z}\alpha + \mathbf{Z}\alpha_k + \mathbf{Z}\alpha_{k'}) \cap R$ is an irreducible root system with simple roots $\alpha_k, w(\alpha), \alpha_{k'}$, where w is the longest element in the group $< s_k, s_{k'} >$. The lemma follows from the fact that its Dynkin diagram is not a cycle.

2.3. For any root α in R^+, let $h'(\alpha)$ =height of $\alpha - 1$ if α is a short root, height of $\alpha^\vee - 1$ if α is a long root. We denote $\mathcal{D}_\alpha$ the set $\{\alpha_k \in \mathcal{D} \mid \alpha_k$ and α have the same length, $\alpha_k \leq \alpha\}$. Note that the root system R_α generated by $\mathcal{D}_\alpha$ is irreducible. It is plain to check the following properties concerned with $h'(\alpha)$ by using 2.1 (b1-6).

(c1). $h'(\alpha) = 0$ if and only if α is a simple root in R.

(c2). If $\alpha_k \in \mathcal{D}$, $w \in W$ such that $w(\alpha_k) = \alpha$, then $\ell(w) \geq h'(\alpha)$, moreover $\ell(w) > h'(\alpha)$ if $\alpha_k \in \mathcal{D} - \mathcal{D}_\alpha$

(c3). For a simple reflection s in W, if $0 \leq s(\alpha) < \alpha$, then $h'(\alpha) = h'(s(\alpha)) + 1$.

For an element $(w, \alpha_k) \in \mathcal{H} = \{(u, \alpha_l) \in W \times \mathcal{D} \mid u(\alpha_l) \in R^+\}$, we call it **shortable** if there exist w_1, $u_1 \in W$ such that $w = w_1 \cdot u_1$ and $u_1(\alpha_k) \in \mathcal{D}$, $\ell(u_1) \geq 1$, $u_1 \in < s, t >$ for some simple reflections $s, t \in W$; we also call $\ell(w)$ its length. Here we use the convention: for $x, x_1, x_2, ..., x_m \in W$, we write $x = x_1 \cdot x_2 \cdot \cdots \cdot x_m$ if $x = x_1 x_2 \cdots x_m$, and $\ell(x) = \ell(x_1) + \ell(x_2) + \cdots + \ell(x_m)$.

Let (w, α_k), $(u, \alpha_l) \in \mathcal{H}$, we write $(w, \alpha_k) \sim (u, \alpha_l)$ if there exists $u_1 \in W$ such that $w = u \cdot u_1$ and $u_1(\alpha_k) = \alpha_l$. The relation $\sim$ generates an equivalence relation in $\mathcal{H}$, we denote it also by $\sim$. The equivalence class containing (w, α_k) is denoted by $\widetilde{(w, \alpha_k)}$. The set of all equivalence classes in $\mathcal{H}$ is denoted by $\widetilde{\mathcal{H}}$.

Proposition 2.4. Let $\alpha \in R^+$, we have

(i). For any $\alpha_k \in \mathcal{D}_\alpha$, there exists a unique $w \in W$ such that $w(\alpha_k) = \alpha$ and $\ell(w) = h'(\alpha)$. we denote it by $w_{\alpha,k}$.

(ii). Assume that α is not a simple root in R, then for $\alpha_k \in \mathcal{D}_\alpha$, $w \in W$, $w = w_{\alpha,k}$ if and only if for any reduced expression $s_{j_i} s_{j_{i-1}} ... s_{j_1}$ of w, we have

$$s_{j_i} s_{j_{i-1}} ... s_{j_1}(\alpha_k) > s_{j_{i-1}} ... s_{j_1}(\alpha_k) > s_{j_1}(\alpha_k) > \alpha_k,$$

where $i = h'(\alpha)$.

(iii). Let s be a simple reflection in W, $k \in \mathcal{D}_\alpha$, then $s w_{\alpha,k} \leq w_{\alpha,k}$ if and only if $\alpha_k \leq s(\alpha) < \alpha$; $w_{\alpha,k} s \leq w_{\alpha,k}$ if and only if $\alpha_k < s(\alpha_k) \leq \alpha$.

(iv). Let s, t be simple reflections in W such that $s w_{\alpha,k} \leq w_{\alpha,k}$, $t w_{\alpha,k} \leq w_{\alpha,k}$ (resp. $w_{\alpha,k} s \leq w_{\alpha,k}$, $w_{\alpha,k} t \leq w_{\alpha,k}$), then $st = ts$.

Proof: We assume that α is a short root, thanks to 2.1(b3-6), it is sufficient to prove the proposition under the assumption.

(i). Now assume that $\alpha_k \in \mathcal{D}_\alpha$. First we prove that there exists $w \in W$ such that $w(\alpha_k) = \alpha$, and $\ell(w) = h'(\alpha)$. Using 2.1(b1-3) we know that there exist some $\alpha_{k'} \in \mathcal{D}_\alpha$, $w' \in W$ such that $w'(\alpha_{k'}) = \alpha$ and $\ell(w') = h'(\alpha)$. If $k = k'$, We set $w = w'$. If $k \neq k'$, then we can find a sequence $\alpha_{k'} = \alpha_{k_1}$, α_{k_2}, ..., $\alpha_{k_l} = \alpha_k$ in $\mathcal{D}_\alpha$ such that $a_{k_m, k_{m+1}} = -1$ $(m \in [1, l-1])$.

We show that $w' s_{k_2} \leq w'$. Let $s_{j'_i} s_{j'_{i-1}} ... s_{j'_1}$ $(i = h'(\alpha))$ be a reduced expression of w'. Since $\ell(w') = h'(\alpha)$ and $\alpha_k \in \mathcal{D}_\alpha$, using 2.3(c2) and the definition of $h'(\alpha)$ we know there is some $m \in [1, i]$ such that $s_{j'_m} = s_{k_2}$ and $s_{j_h} \neq s_{k_2}$ if $1 \leq h < m$. If $w' s_{k_2} \not\leq w'$, then $m \geq 2$. We can assume that m is minimal in all possibilities, then there exists a subsequence $s_{j'_m} = t_p$, t_{p-1}, ..., t_1, $t_0 = s_{k'}$ $(p \geq 2)$ of $s_{j'_m}$, $s_{j'_{m-1}}$, ..., $s_{j'_1}$, $s_{j'_0} = s_{k'}$ such that $t_q t_{q-1} \neq t_{q-1} t_q$ for any $q \in [1, p]$. Combine this and our assumption on k_2 we know that the Dynkin diagram of R contains a cycle which is impossible. So we have $w' s_{k-2} \leq w'$.

Let $w_2 = w' s_{k_2} s_{k'}$, then $w_2(\alpha_{k_2}) = \alpha$, moreover $\ell(w_2) = h'(\alpha)$ since $\ell(w_2) \leq h'(\alpha)$ by the above argument. Continue this process, finally, we get an element $w \in W$ such that $w(\alpha_k) = \alpha$, $\ell(w) = h'(\alpha)$.

We need to prove the uniqueness of w.

We use induction on $h'(\alpha)$. When $h'(\alpha) \leq 2$, it is easy to check the uniqueness. Now suppose that $h'(\alpha) \geq 3$. Let $w'' \in W$ be such that $w''(\alpha_k) = \alpha$, $\ell(w'') = h'(\alpha)$. Choose two simple reflections s, t of W such that $sw \leq w$, $tw'' \leq w''$. By 2.3(c2), the definition of $h'(\alpha)$

and our assumption on α, we have $s(\alpha) < \alpha$, $t(\alpha) < \alpha$. If $s = t$, the induction hypothesis implies that $sw = sw''$. If $s \neq t$, note that $h'(\alpha) \geq 3$, using 2.2, we see that $st = ts$, therefore by 2.4(c2-3) and the definition of $h'(\alpha)$ we get $\alpha_k \leq st(\alpha) < s(\alpha)$, $t(\alpha) < \alpha$ and $h'(st(\alpha)) = h'(\alpha) - 2 = h'(s(\alpha)) - 1 = h'(t(\alpha)) - 1$. According to induction hypothesis, there exists a unique element $u \in W$ such that $u(\alpha_k) = st(\alpha)$, $\ell(u) = h'(\alpha) - 2$, and $sw = tu$, $tw'' = su$. So we get $w = w'' = stu$. This completes the proof of (i).

(ii). It follows from the uniquenss of $w_{\alpha,k}$ and the definition of $h'(\alpha)$.

(iii). Using (ii) we see that the results hold.

(iv). Assume that $s = s_k \neq s_{k'} = t$, $sw_{\alpha,k} \leq w_{\alpha,k}$, $tw_{\alpha,k} \leq w_{\alpha,k}$. Let u be the longest element in $< s,t >$, then $\ell(w) = \ell(u) + \ell(uw)$, so $\alpha, \alpha_k, \alpha_{k'}$ are linearly independent. By 2.2 we see that $st = ts$. Another assertion follows from (ii) and the fact that the Dynkin diagram contains no cycles.

Theorem 2.5. (i). For each equivalence class $\widetilde{(w,\alpha_k)}$ in $\mathcal{H}$, there exists unique shortest element (u,α_l) in $\widetilde{(w,\alpha_k)}$. Furthermore, we have $w = u \cdot u_1$ for some $u_1 \in W$.

(ii). For two elements $(w,\alpha_k),(u,\alpha_l) \in \mathcal{H}$, choose arbitrary $(w_1,\alpha_{k_1}),(u_1,\alpha_{l_1}) \in \mathcal{H}$ such that $w_1^{-1}w = w_1^{-1} \cdot w$, $u_1^{-1}u = u_1^{-1} \cdot u$ and $w(\alpha_k) = w_1(\alpha_{k_1})$, $u(\alpha_l) = u_1(\alpha_{l_1})$, then $(w,\alpha_k) \sim (u,\alpha_l)$ if and only if $(w_1,\alpha_{k_1}) \sim (u_1,\alpha_{l_1})$. In particular, if w_1 is a shortest element such that $w_1^{-1}w = w_1^{-1} \cdot w$, and $w_1^{-1}w(\alpha_k)$ is a simple root α_{k_1}, then (w_1,α_{k_1}) is the unique shortest element in $\widetilde{(w_1,\alpha_{k_1})}$. We also denote $\widetilde{(w_1,\alpha_{k_1})}$ by $\widetilde{(w,\alpha_k)}^*$.

We need the following result.

Lemma 2.6. If $w(\alpha_k) = \alpha_l$ and $\ell(w) \geq 1$, then (w,α_k) is shortable (see 2.3 for definition).

One can prove the lemma by using the method in [L1, 1.8].

Proof of 2.5. (i). Let (u,α_l) be an element in $\widetilde{(w,\alpha_k)}$ with minimal length. We shall prove that $w = u \cdot u_1$ for some $u_1 \in W$, this forces that (u,α_l) is the unique shortest element in $\widetilde{(w,\alpha_k)}$. Let $(u',\alpha_{l'}),(w',\alpha_{k'}) \in \widetilde{(w,\alpha_k)}$ be such that $u' = u \cdot u_1'$, $u' = w' \cdot w_1'$, where $u_1' \in W$ and w_1' is one of the following elements: s_i, $\alpha_{ik'} = 0$; $s_{k'}s_i$, $\alpha_{ik'}a_{k'i} = 1$; $s_is_{k'}s_i$, $\alpha_{ik'}a_{k'i} = 2$, $s_is_{k'}s_is_{k'}s_i$, $\alpha_{ik'}a_{k'i} = 3$. Because (u,α_l) is an element in $\widetilde{(w,\alpha_k)}$ of minimal length, we get $u_1' = x \cdot w_1'$ for some $x \in W$, thus $w' = u \cdot x$. According to the definition of $\sim$ and 2.6 we see that there exists $u_1 \in W$ such that $w = u \cdot u_1$.

(ii). Suppose that $(w,\alpha_k) \sim (u,\alpha_l)$. It is no harm to assume that (u,α_l) is the shortest element in $\widetilde{(w,\alpha_k)}$. By (i) we know that $w_1^{-1}u = w_1^{-1} \cdot u$, $u_1^{-1}u = u_1^{-1} \cdot u$ and $w_1(\alpha_{k_1}) = u(\alpha_l) = u_1(\alpha_{l_1})$. Let $u_0 \in W$ be such that $u_0us_l = u_0 \cdot us_l = w_0$, the longest element of W. Then $u_0 = x_1 \cdot w_1 = x_2 \cdot u_1$ for some $x_1, x_2 \in W$. Since $u_0u(\alpha_l) = \alpha_m \in \mathcal{D}$, we get $(w_1,\alpha_{k_1}) \sim (u_0^{-1},\alpha_m) \sim (u_1,\alpha_{l_1})$. The "only if" part is similar when one notes that $w^{-1}w_1 = w^{-1} \cdot w_1$, $u^{-1}u_1 = u^{-1} \cdot u_1$.

The theorem is proved.

Part (ii) of 2.5 gives a way to compute the shortest elements in $\mathcal{H}$.

3. Several Lemmas

3.1. In this section we give several lemmas concerned with the automorphisms T_i. We refer to [L3].

Let $s_{k_1}s_{k_2}s_{k_3}\dots s_{k_{\nu-1}}s_{k_\nu}$ be a reduced expression of the longest element w_0 of W. For any $c=(c_1,c_2,\dots,c_\nu)\in\mathbf{N}^\nu$, $r=(r_1,\dots,r_n)\in\mathbf{Z}^n$, we set

$$E^c=E_{k_1}^{c_1}T_{k_1}(E_{k_2}^{c_2})T_{k_1}T_{k_2}(E_{k_3}^{c_3})\dots T_{k_1}T_{k_2}\dots T_{k_{\nu-1}}(E_{k_\nu}^{c_\nu}),\quad F^c=\Omega(E^c).$$

$$G^c=E_{k_1}^{c_1}E_{k_2}^{c_2}T_{k_2}(E_{k_3}^{c_3})T_{k_2}T_{k_3}(E_{k_4}^{c_4})\dots T_{k_2}T_{k_3}\dots T_{k_{\nu-1}}(E_{k_\nu}^{c_\nu}),\quad H^c=\Omega(G^c),$$

$$K^r=K_1^{r_1}\cdots K_n^{r_n}.$$

Let U^+ is the subalgebra of U generated by all E_i. The following two lemmas are due to Lusztig (see [L3, 2.4])

Lemma 3.2. We fix $i\in[1,n]$. Let $O_i=\{\xi\in U^+\mid F_i\xi-\xi F_i\in K_i^{-1}U^+\}$. Let O_i' be the $\mathbf{Q}(v)$-subalgebra of U^+ generated by the elements $T_i(E_j)$, $T_iT_j(E_i)$, $T_iT_jT_i(E_j)$, $T_iT_jT_iT_j(E_i)$ for j such that $a_{ij}a_{ji}=3$, the elements $T_i(E_j)$, $T_iT_j(E_i)$ for j such that $a_{ij}a_{ji}=2$, the elements $T_i(E_j)$ for j such that $a_{ij}a_{ji}=1$, and by E_j for $j\neq i$. Choose a reduced expression $s_{k_1}s_{k_2}s_{k_3}\dots s_{k_{\nu-1}}s_{k_\nu}$ of w_0 be such that $k_1=i$. Let O_i'' be the $\mathbf{Q}(v)$-subspace of U^+ spanned by the elements E^c (defined in 3.1) for various $c=(c_1,\dots,c_\nu)\in\mathbf{N}^\nu$ such that $c_1=0$. We have $O_i=O_i'=O_i''=U^+\cap T_i(U^+)$.

Proof: It is clear that O_i is a $\mathbf{Q}(v)$-subalgebra of U^+. It is easy to check that the generators of O_i' are contained in O_i. It follows that $O_i'\subset O_i$.

By using the method in the proof [L1, 1.8] we see that $O_i''\subset O_i'$. As the same way of the proof of $R_i\subset R_i''$ in [L3, 2.4] (notations in loc.cit) we get $O_i\subset O_i''$. The lemma is proved.

Lemma 3.3. We fix $i\in[1,n]$. Let $P_i=\{\xi\in U^+\mid F_i\xi-\xi F_i\in K_i^{-1}U^+\}$. Let P_i' be the $\mathbf{Q}(v)$-subalgebra of U^+ generated by the elements $T_i'(E_j)$, $T_i'T_j'(E_i)$, $T_i'T_j'T_i'(E_j)$, $T_i'T_j'T_i'T_j'(E_i)$ for j such that $a_{ij}a_{ji}=3$, the elements $T_i'(E_j)$, $T_i'T_j'(E_i)$ for j such that $a_{ij}a_{ji}=2$, the elements $T_i'(E_j)$ for j such that $a_{ij}a_{ji}=1$, and by E_j for $j\neq i$. Choose a reduced expression $s_{k_1}s_{k_2}s_{k_3}\dots s_{k_{\nu-1}}s_{k_\nu}$ of w_0 be such that $k_1=i$. Let P_i'' be the $\mathbf{Q}(v)$-subspace of U^+ spanned by the elements G^c (defined in 3.1) for various $c=(c_1,\dots,c_\nu)\in\mathbf{N}^\nu$ such that $c_1=0$. We have $P_i=P_i'=P_i''=U^+\cap T_i'(U^+)$.

The proof is similar.

3.4. For $\lambda\in\mathbf{N}R^+$, we denote U_λ the set of all elements $\xi\in U$ such that $K_i\xi K_i^{-1}=v^{d_i<\alpha_i^\vee,\lambda>}\xi$. Let $U_\lambda^+=U^+\cap U_\lambda$.

Lemma 3.5. Let $Q_i=O_i\cap P_i=\{\xi\in U^+\mid F_i\xi=\xi F_i\}$. We have $s_i(\lambda)\geq\lambda$ if $Q_i\cap U_\lambda^+\neq\{0\}$.

Proof: Let U_A be the $A = \mathbf{Q}[v]$-subalgebra of U generated by all E_j, F_j, K_j, K_j^{-1}. Regard $\mathbf{Q}$ as a $\mathbf{Q}[v]$-algebra by specializing v to 1. Thus we can get the $\mathbf{Q}$-algebra

$$U_1 = U_A \otimes_A \mathbf{Q} / < K_1 - 1, K_2 - 1, ..., K_n - 1 >,$$

which is just the universal enveloping algebra of the simple Lie algebra corresponding to the Cartan matrix (a_{ij}). Let $f_i, U_1^+, U_{1,\lambda}^+$, be the images of F_i, U^+, U_λ^+, respectively. According to the commutation relations between root vectors in U_1 and PBW Theorem one can check easily that the subalgebra $Q_{1,i} = \{x \in U_1^+ \mid f_i x = x f_i\}$ is generated by e_α ($\alpha \in R^+$) such that $\alpha - \alpha_i \notin R$, where e_α is a root vector in U_1^+ of root α. Note that $\alpha - \alpha_i \notin R$ implies that $s_i(\alpha) \geq \alpha$, we see that $Q_{1,i} \cap U_{1,\lambda}^+ \neq \{0\}$ implies that $s_i(\lambda) \geq \lambda$. Our assertion follows from this and that $Q_{1,i} \cap U_{1,\lambda}^+ \neq \{0\}$ if $Q_i \cap U_\lambda^+ \neq \{0\}$. The lemma is proved.

3.6. Remark: By 3.2 and 3.3 we know that $Q_i = O_i \cap P_i = U^+ \cap T_i(U^+) \cap T_i'(U^+)$. It is likely that Q_i is the $\mathbf{Q}(v)$-subalgebra of U^+ generated by the elements $T_k T_i(E_j)$ for j, k with $a_{ij}a_{ji} > 0$, $a_{ik}a_{ki} = 1$, and by E_j for $j \neq i$.

4. Root Vectors

4.1. In this section we describe the set of all root vectors of a given root. The main result is Theorem 4.4.

Given a positive root α in R^+. Let Y_α be the set of all root vectors of root α. $\widetilde{\mathcal{H}}_\alpha = \{\widetilde{(w, \alpha_k)} \in \widetilde{\mathcal{H}} \mid w(\alpha_k) = \alpha\}$. Fix a reduced expression $s_{j_i} s_{j_{i-1}} ... s_{j_1}$ of w_{α, j_0}, $\alpha_{j_0} \in \mathcal{D}_\alpha$, Let $Y'_\alpha = \{T_{\alpha,k,a}(E_{j_0}) \mid a \in I_\alpha\}$, where $i = h'(\alpha)$, $T_{\alpha,j_0,a} = T_{j_i}^{a_i} T_{j_{i-1}}^{a_{i-1}} ... T_{j_1}^{a_1}$, $a = (a_i, a_{i-1}, ..., a_1) \in \{1, -1\}^{h'(\alpha)} = I_\alpha$. When $h'(\alpha) = 0$, we set $I_\alpha = \{e\}$ and $T_{w,e} = \mathrm{id}_U$, where e is the neutral element of W.

Set $Y = \bigcup_{\alpha \in R^+} Y_\alpha$, $Y' = \bigcup_{\alpha \in R^+} Y'_\alpha$.

Lemma 4.2. Keep the notations in 4.1.

(i). Y'_α is independent of the choice of the reduced expression and the choice of j_0, so only depends on α.

(ii). The elements $T_{\alpha,j_0,a}(E_{j_0})$, $a \in I_\alpha$ are linearly independent over $\mathbf{Q}(v)$. In particular, the set Y'_α contains $2^{h'(\alpha)}$ elements.

Proof: (i). Using 2.4(iii) and induction on $h'(\alpha)$ we see that Y'_α is independent of the choice of the reduced expression. According to the proof of 2.4(i) and 1.3(a4) we know that Y'_α doesnot depends on the choice of k.

(ii). If each $j \in [1, n]$ appears in the sequence $j_i, j_{i-1}, ..., j_1, j_0$ at most two times, then we can choose the reduced expression such that $j_i, j_{i-1}, ..., j_{p+1}$ is a subsequence (disregard order) of $j_p, j_{p-1}, ..., j_1, j_0$ for some p. Thus for any $a \in I_\alpha$, $T_{j_p}^{a_p} T_{j_{p-1}}^{a_{p-1}} ... T_{j_1}^{a_1}(E_{j_0}) \in U^+$, $T_{j_i}^{a_i} T_{j_{i-1}}^{a_{i-1}} ... T_{j_q}^{a_q}(F_{q-1}) \in U^- = \Omega(U^+)$ for any $q \geq p+2$, since $j_i, j_{i-1}, ..., j_{p+1}$ or $j_i, j_{i-1}, ...,$ j_{p+1} are pairwise different. Combine these and using induction on i we see that in the expression

$$T_{j_i} T_{j_{i-1}}^{a_{i-1}} ... T_{j_1}^{a_1}(E_{j_0}) = \sum_{\substack{c',c \in \mathbf{N}^\nu \\ r \in \mathbf{Z}^n}} \rho_{c',r,c} F^{c'} K^r E^c, \quad \rho_{c',r,c} \in \mathbf{Q}(v),$$

(resp.

$$T_{j_i}^{-1} T_{j_{i-1}}^{a_{i-1}} ... T_{j_1}^{a_1}(E_{j_0}) = \sum_{\substack{c',c \in \mathbf{N}^\nu \\ r \in \mathbf{Z}^n}} \rho'_{c',r,c} H^{c'} K^r G^c, \quad \rho'_{c',r,c} \in \mathbf{Q}(v),)$$

if $\rho_{c',r,c} \neq 0$ (resp. $\rho'_{c',r,c} \neq 0$), then $E^c \in O_{j_i}$ (resp. $G^c \in P_{j_i}$), where $F^{c'}, E^c, G^{c'}, H^c, K^r$ are defined as in 3.1, we choose the reduced expression of w_0 such that $k_1 = j_i$. According to 2.4(ii) we see that

(*) $\qquad s_{j_i} s_{j_{i-1}} ... s_{j_r}(\alpha_{j_{r-1}}) \geq s_{j_{i-1}} ... s_{j_r}(\alpha_{j_{r-1}})$ for any $1 \leq r \leq i-1$.

Therefore if $\rho_{c',r,c} \neq 0$ (resp. $\rho'_{c',r,c} \neq 0$), then $E^c \in U_\lambda^+$ (resp. $G^c \in U_\lambda^+$) for some $\lambda \in \mathbf{N}R^+$ such that $s_{j_i}(\lambda) < \lambda$. Using 3.5 we see that if

$$\sum_{a \in I_\alpha} \rho_a T_{\alpha, j_0, a}(E_{j_0}) = 0, \quad \rho_a \in \mathbf{Q}(v),$$

then

$$\sum_{\substack{a \in I_\alpha \\ a_i = 1}} \rho_a T_{\alpha, j_0, a}(E_{j_0}) = 0, \qquad \sum_{\substack{a \in I_\alpha \\ a_i = -1}} \rho_a T_{\alpha, j_0, a}(E_{j_0}) = 0.$$

Using induction we know that $\rho_a = 0$ for all $a \in I_\alpha$. Thus we have proved (ii) for type A_n, B_n, C_n, D_n, G_2.

In general we argue as follows.

Let

$$T_{j_i}^{a_i} T_{j_{i-1}}^{a_{i-1}} ... T_{j_1}^{a_1}(E_{j_0}) = \xi_a + \xi'_a,$$

where

$$\xi_a = \sum_{\substack{c',c \in \mathbf{N}^\nu \\ r \in \mathbf{Z}^n \\ E^c \in O_{j_i}}} \rho_{c',r,c} F^{c'} K^r E^c, \quad \xi'_a = \sum_{\substack{c',c \in \mathbf{N}^\nu \\ r \in \mathbf{Z}^n \\ E^c \notin O_{j_i}}} \rho'_{c',r,c} F^{c'} K^r E^c, \qquad \text{if } a_i = 1,$$

$$\xi_a = \sum_{\substack{c',c \in \mathbf{N}^\nu \\ r \in \mathbf{Z}^n \\ G^c \in P_{j_i}}} \rho_{c',r,c} H^{c'} K^r G^c, \quad \xi'_a = \sum_{\substack{c',c \in \mathbf{N}^\nu \\ r \in \mathbf{Z}^n \\ G^c \notin P_{j_i}}} \rho'_{c',r,c} H^{c'} K^r G^c, \qquad \text{if } a_i = -1,$$

$\rho_{c',r,c} \in \mathbf{Q}(v)$, $\rho'_{c',r,c} \in \mathbf{Q}(v)$.

Note that

(**) The image of $T_{j_i}^{a_i} T_{j_{i-1}}^{a_{i-1}} ... T_{j_r}^{a_r}(F_{j_{r-1}})$ $(1 \le r \le i)$ in U_1^- (see the proof of 3.5) is not zero,

and $\alpha_{j_i}, s_{j_i} s_{j_{i-1}} ... s_{j_r}(\alpha_{j_{r-1}})$, $1 \le r \le i$ are pairwise different. Using induction on i and the fact (*) it is not difficult to check that if $\rho_{c',r,c} \neq 0$, $E^c \in O_{j_i} \cap U_\lambda$ (resp. $\rho'_{c',r,c} \neq 0$, $G^c \in P_{j_i} \cap U_\lambda$), then $s_{j_i}(\lambda) < \lambda$, and that the set $\{\xi_a \mid a_i = 1\}$ (resp. $\{\xi_a \mid a_i = -1\}$) is $\mathbf{Q}(v)$-linearly independent. By these and 3.5 we see that (ii) is true.

4.3. Remark: By (*) and (**) in the proof of 4.2 we know that if $T_{j_r}^{a_r} T_{j_{r-1}}^{a_{r-1}} ... T_{j_1}^{a_1}(E_{j_0}) \notin U^+$ for some $r \le i$, then $T_{\alpha, j_0, a}(E_{j_0}) \notin U^+$.

Theorem 4.4. Keep the notations in 4.1. Let $\alpha \in R^+$, then

(i). $\Psi(Y_\alpha) = Y_\alpha$. In particular, $\Psi(Y) = Y$.

(ii). $Y_\alpha \subset Y'_\alpha \cap U^+$. In particular, the set Y is linearly independent over $\mathbf{Q}(v)$.

(iii). The map $\Phi : \widetilde{(w, \alpha_k)} \to T_w E_k$ defines a bijection between $\widetilde{\mathcal{H}}$ and Y, moreover $\Phi(\widetilde{\mathcal{H}}_\alpha) = Y_\alpha$.

(iv). $\Phi(\widetilde{(w, \alpha_k)}^*) = \Psi \cdot \Phi(\widetilde{(w, \alpha_k)})$.

Proof: Let $E = T_w(E_l) \in Y_\alpha$.

(i). Choose $u \in W$ be such that $u^{-1}w = u^{-1} \cdot w$ and $u^{-1}w(\alpha_l) = \alpha_{l'}$ for some l', according to 1.3(a1-2) we get $\Psi(E) = T_u(E_{l'}) \in Y_\alpha$.

(ii). We have $h'(s_j w(\alpha_l)) < h'(\alpha)$. Use induction hypothesis we see that there exist $a_{i-1}, ..., a_1 \in \{1, -1\}$, such that $T_u E_l = T_{j_{i-1}}^{a_{i-1}} ... T_{j_1}^{a_1}(E_{j_0})$, where $u = s_{j_i} w$. Terefore $T_w(E_l) = T_{j_i} T_{j_{i-1}}^{a_{i-1}} ... T_{j_1}^{a_1}(E_{j_0})$, if $\ell(w) = \ell(s_j w) + 1$; $T_w(E_l) = T_{j_i}^{-1} T_{j_{i-1}}^{a_{i-1}} ... T_{j_1}^{a_1}(E_{j_0})$, if $\ell(w) = \ell(s_j w) - 1$. Thus $E \in Y'_\alpha \cap U^+$.

(iii). By 1.3(a2) we know that Φ is well defined and is surjective. We use induction on $h'(\alpha)$ to prove that Φ is injective. If $\Phi(\widetilde{(w, \alpha_k)}) = \Phi(\widetilde{(u, \alpha_l)})$. Let $w' = s_{j_i} w$, $u' = s_{j_i} u$. Using (i),(ii), 1.3(a1) and 2.5(ii) we may assume that $w' \le w$, $u' \le u$. By induction hypothesis we have $\widetilde{(w', \alpha_k)} = \widetilde{(u', \alpha_l)}$, using 2.5(i) we get $\widetilde{(w, \alpha_k)} = \widetilde{(u, \alpha_l)}$.

(iv). It follows from the proof of (i).

The theorem is proved.

Remark: (i). It is likely that $Y = Y' \cap U^+$.

(ii). For any $v_0 \in \mathbf{C}^*$, we regard $\mathbf{Q}(v_0)$ as a $A = \mathbf{Q}[v]$-algebra by specializing v to v_0. Let $U_{v_0} = U_A \otimes_A \mathbf{Q}(v_0)$. If $v^{2d} \neq 1$ for any $1 \le d \le \max\{d_i\}$, the same argument show that 4.2-4 are true for U_{v_0}. If $v_0^2 = 1$, then for each $\alpha \in R$, there is a unique (up to ± 1) root vector of root α.

Corollary 4.5. Notations are as in 4.1. Let $E = T_{\alpha,j_0,a}(E_{j_0}) \in Y'_\alpha$, $a = (a_i, a_{i-1}, ..., a_1)$, $i = h'(\alpha)$, then

(i). $E \in Y_\alpha$ if and only if $\Psi(E) \in Y_\alpha$; if $a_i = 1$, then $E \in Y_\alpha$ if and only $T_{j_{i-1}}^{a_{i-1}} ... T_{j_1}^{a_1}(E_{j_0}) = T_u(E_l) \in Y$ for some $u \in W$, $l \in [1,n]$ and $s_{j_i} u \geq u$.

(ii). For any $1 \leq m \leq i$, $T_{j_m}^{a_m} T_{j_{m-1}}^{a_{m-1}} ... T_{j_1}^{a_1}(E_{j_0})$ is a root vector if $E \in Y_\alpha$ (i.e. E is a root vector).

(iii). If $T_{j_p}^{a_p} T_{j_{p-1}}^{a_{p-1}} ... T_{j_1}^{a_1}(E_{j_0})$ is not a root vector for some $1 \leq p < i$, then E is not a root vector, i.e., $E \notin Y_\alpha$.

Proof: (i). The first assertion follows from 4.4(i). The second follows from the proof of 4.4(ii).

(ii). Suppose that $E = T_w(E_l)$, $w \in W$, as in the proof of 4.3(ii) we see $T_{w'}(E_l) = T_{j_m}^{a_m} T_{j_{m-1}}^{a_{m-1}} ... T_{j_1}^{a_1}(E_{j_0})$, where $w' = s_{j_{m+1}} s_{j_{m+2}} \cdots s_{j_i} w$.

(iii). It follows from (ii).

For any $E \in Y$, we shall denote the shortest elements in $\Phi^{-1}(E)$, $\Phi^{-1}(\Psi(E))$ by (w_E, α_{k_E}), $(w_E^*, \alpha_{k_E^*})$ respectively.

Corollary 4.6. Let α, j_i be as in 4.1 and let $E \in Y_\alpha$, then

(i). $s_{j_i} w_E \leq w_E$ if and only if $s_{j_i} w_E^* \geq w_E^*$.

(ii). $w_E, w_E^* \in W_\alpha$, $\alpha_{k_E}, \alpha_{k_E^*} \in \mathfrak{D}_\alpha$, where W_α is the subgroup of W generated by these simple reflections s_m such that $\alpha_m \leq \alpha$.

(iii). We have $w_E^{-1} w_E^* = w_E^{-1} \cdot w_E^*$ and $w_E^{-1} w_E^*(\alpha_{k_E^*}) = (\alpha_{k_E})$.

Proof: (i). Let $a \in I_\alpha$ be such that $E = T_{\alpha,j_0,a}(E_{j_0})$ (notations as in 4.1). By 4.4(ii) and its proof we see that $s_{j_i} w_E \leq w_E$ if and only if $a_i = 1$. Since $\Psi(E) = T_{j_i}^{-a_i} \cdots T_{j_1}^{-a_1}(E_{j_0})$, we know that our assertion is true.

(ii). From the proof of 2.5(ii) we see that $w_E \in W_\alpha$ if and only if $w_E^* \in W_\alpha$. Thus we may assume that $a_i = 1$ to prove (ii). In this case, according to 4.5(i), 4.4(iii) and 2.5(i), it is obvious that we have $w_E = s_{j_i} w_{E'}$, where $E' = T_{j_{i-1}}^{a_{i-1}} \cdots T_{j_1}^{a_1}(E_{j_0})$. Thus we can use induction on $h'(\alpha)$ to prove the result since $h'(s_{j_i}(\alpha)) = h'(\alpha) - 1$.

(iii). It follows from the proof of 2.5(ii).

By means of Ψ we can describe the antipode $S(E)$ for a root vector $E \in Y_\alpha$.

Theorem 4.7. For any $E \in Y_\alpha, \alpha = m_1\alpha_1 + m_2\alpha_2 + ... + m_n\alpha_n \in R^+$, we have $S(E) = \rho_\alpha K_\alpha^{-1} \Psi(E)$, where

$$\rho_\alpha = (-1)^{m_1+m_2+...+m_n} \prod_{k=1}^{n} v^{m_k(m_k-1)d_k} \prod_{k=1}^{n-1} v^{m_k d_k (m_{k+1} a_{k,k+1} + ... + m_n a_{k,n})},$$

$$K_\alpha = K_1^{m_1} K_2^{m_2} \cdots K_n^{m_n}.$$

Note that we have $\Psi(E) \in Y_\alpha$.

Proof: It follows from $K_i^{-1}E_iK_j^{-1}E_j = v^{d_ia_{i,j}}K_i^{-1}K_j^{-1}E_iE_j = v^{d_ja_{j,i}}K_i^{-1}K_j^{-1}E_iE_j$ and the definitions of S, Ψ.

Proposition 4.8. We have $\#Y_\alpha \leq 2^{h'(\alpha)}$. The equality holds if and only if $j_i, j_{i-1}, ..., j_1, j_0$ (notations as in 4.1) are pairwise different.

Proof: The first part is obvious.

Thanks to 4.5(i) and 4.6(ii) we see the "if" part of the second assertion is true.

Assume that $j_m = j_{m'}$ for some different m, m'. Using 4.5(iii) we can suppose that α, R is one of the following cases: $\alpha_1+\alpha_2+2\alpha_3+\alpha_4,\ D_4$; $2\alpha_1+2\alpha_2+\alpha_3,\ B_3$; $\alpha_1+2\alpha_2+\alpha_3,\ C_3$; $3\alpha_1+2\alpha_2,\ G_2$; $2\alpha_1+\alpha_2,\ G_2$. Then it is easy to check that the following elements are not in U^+ by using 1.3(a8-11): $T_3^{-1}T_1T_2T_4(E_3),\ D_4$; $T_2T_3^{-1}T_1^{-1}(E_2),\ B_3$; $T_2T_3^{-1}T_1^{-1}(E_2),\ C_3$; $T_2T_1^{-1}(E_2),\ G_2$; $T_1T_2^{-1}(E_1),\ G_2$. In particular, they are not vector roots. The proposition is proved.

4.9. Remark: Let $\alpha = m_1\alpha_1 + m_2\alpha_2 + ... + m_n\alpha_n \in R^+$, using PBW Theorem and 4.8 we see that U_α^+ is spanned by Y_α if all $m_k \leq 1$. It seems that U_α^+ is not spanned by Y_α if $m_k \geq 2$ for some $k \in [1, n]$.

5. An Example, Type A_n

5.1. It is easy to say a little more for type A_n. In this section we shall assume that R is of type A_n and fix $\alpha = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ $(i \leq j)$. We choose all d_k to be 1. We have

(i). $h'(\alpha) = j - i$.

(ii). $\mathcal{D}_\alpha = \{\alpha_i, \alpha_{i+1}, ..., \alpha_j\}$.

(iii). $w_{\alpha,k} = s_js_{j-1}\cdots s_{k+1}s_is_{i+1}\cdots s_{k-1},\ i \leq k \leq j$.

(iv). $W_\alpha = < s_i, s_{i+1}, ..., s_j >$.

(v). We have $\#Y_\alpha = \#Y'_\alpha = 2^{j-i}$. So $\#Y = 2^{n+1} - n - 2$.

(vi). Let $E = T_j^{a_j}T_{j-1}^{a_{j-1}}\cdots T_{i+1}^{a_{i+1}}(E_i), (a_j, ..., a_{i+1}) \in I_\alpha$, then we have

$$(i) \quad E = \begin{cases} v^{-1}E_iE' - E'E_i, & \text{if } a_{i+1} = 1\ , \\ v^{-1}E'E_i - E_iE', & \text{if } a_{i+1} = -1. \end{cases}$$

$$(ii) \quad E = \begin{cases} v^{-1}E''E_j - E_jE'', & \text{if } a_j = 1\ , \\ v^{-1}E_jE'' - E''E_j, & \text{if } a_j = -1, \end{cases}$$

where $E' = T_j^{a_j}T_{j-1}^{a_{j-1}}\cdots T_{i+2}^{a_{i+2}}(E_{i+1})$, $E'' = T_{j-1}^{a_{j-1}}\cdots T_{i+1}^{a_{i+1}}(E_i)$. Moreover, $E_jE = v^{\pm a_j}EE_j$, $E_iE = v^{\mp a_i}EE_i$.

Proof: (i-v) is obvious by results in sections 2 and 4. Now we prove (vi).

(i) is obvious. Note that $E = T_i^{-a_{i+1}} T_{i+1}^{-a_{i+2}} \cdots T_{j-1}^{-a_j}(E_j)$, we get (ii). The remain part of (vi) can be easily deduced from the definition relations of U.

Let O_{ij} be the set of monomials $E_i,\ E_{i+1},\ \cdots, E_j$ such that in any of which E_k $(i \le k \le j)$ appears exactly once. It is obvious that $O_{ij} = \{E_jE, EE_j \mid E \in O_{i,j-1}\}$ (we define $O_{i,j-1}$ similarly), so there are at most 2^{j-i} elements in O_{ij}. But each element $E \in Y_\alpha$ is a $\mathbf{Q}(v)$-linear combination of elements in O_{ij}, thus (v) implies that O_{ij} has exactly 2^{j-i} elements which are linearly independent over $\mathbf{Q}(v)$ (one also can get this from PBW Theorem).

Using (vi) and induction on $j-i$ it is easy to see that the determinate of the transformation matrix from the set Y_α to the set O_{ij} is $\pm(v^{-2}-1)^{(j-i)2^{j-i-1}}$.

We give some properties for $(w_E, \alpha_{k_E}), E \in Y_\alpha$. We need the following lemma.

Lemma 5.2. Given $(w, \alpha_k) \in \mathcal{H}$ and let $t_q t_{q-1} \cdots t_2 t_1$ be a reduced expression of w. If

$$t_p t_{p-1} t_{p-2} \cdots t_1(\alpha_k) < t_{p-1} t_{p-2} \cdots t_1(\alpha_k) \ge t_{p-2} \cdots t_1(\alpha_k) \ge \cdots \ge t_1(\alpha_k) \ge \alpha_k$$

for some $1 < p \le q$, then (w, α_k) is shortable.

Proof: (w, α_k) is obvious shortable when there exists some simple reflection s in $\mathcal{R}(w) = \{s_i \mid ws_i \le w,\ i \in [i,n]\}$ such that $s(\alpha_k) = \alpha_k$. Suppose that there exists no s in $\mathcal{R}(w)$ such that $s(\alpha_k) = \alpha_k$, then $\#\mathcal{R}(w)$ =1 or 2. When $\#\mathcal{R}(w)$ =1, it is easy to see that $w = u \cdot s_k s_{k-1}$ or $w = u \cdot s_k s_{k+1}$ for some $u \in W$, so (w, α_k) is shortable. When $\#\mathcal{R}(w)$ =2, we have $\mathcal{R}(w) = \{s_{k-1}, s_{k+1}\}$, and $w = w_1 s_k \cdot s_{m_1} s_{m_1-1} \cdots s_{k+2} s_{k+1} s_{n_1} s_{n_1-1} \cdots s_{k-2} s_{k-1}$ for some $m_1 > k$, $n_1 < k$, where $w_1 s_k$ is the shortest element in the coset wW_k', W_k' is the subgroup of W generated by those s_i such that $i \ne k$. Our assumption on $\mathcal{R}(w)$ implies that $w_1 = w_2 s_k \cdot s_{m_2} s_{m_2-1} \cdots s_{k+2} s_{k+1} s_{n_2} s_{n_2-1} \cdots s_{k-2} s_{k-1}$ or $w_2 s_k \cdot s_{m_2} s_{m_2-1} \cdots s_{k+2} s_{k+1}$ or $s_{n_2} s_{n_2-1} \cdots s_{k-2} s_{k-1}$ for some $m_2 > k$, $n_2 < k$, where $w_2 s_k$ is the shortest element in the coset $w_1 W_k'$. If $m_2 \ge m_1$ or $n_2 \le n_1$, we have $w = u \cdot s_k s_{k-1}$ or $w = u \cdot s_k s_{k+1}$ for some $u \in W$, so the assertion is true. If $m_2 < m_1$ and $n_2 > n_1$, we continue this process, finally we see that $w = u \cdot s_k s_{k-1}$ or $w = u \cdot s_k s_{k+1}$ for some $u \in W$, which is what we need.

Proposition 5.3. Let $E = T_j^{a_j} T_{j-1}^{a_{j-1}} \cdots T_{i+1}^{a_{i+1}}(E_i) \in Y_\alpha$, $(a_j, a_{j-1}, ..., a_{i+1}) \in I_\alpha$. Then

(i). $w_E = s_i w_{E'}$ if $a_{i+1} = -1$, and $w_E = s_j w_{E''}$ if $a_j = 1$, where

$$E' = T_j^{a_j} T_{j-1}^{a_{j-1}} \cdots T_{i+2}^{a_{i+2}}(E_{i+1}),\quad E'' = T_{j-1}^{a_{j-1}} \cdots T_{i+1}^{a_{i+1}}(E_i).$$

(ii). $w_E = s_k s_{k+1} ... s_j w_G$ if $a_j = a_{j-1} = ... = a_{k+1} = -1$, $a_k = 1$, $j > k > i$, where $G = T_{j-1}^{-1} \cdots T_k^{-1} T_{k-1}^{a_{k-1}} \cdots T_{i+1}^{a_{i+1}}(E_i)$.

(iii). $w_E = u_E \cdot w_{\alpha, k_E}$ for some $u_E \in W_{\alpha-\alpha_i-\alpha_j}$ (if $\alpha - \alpha_i - \alpha_j \notin R^+$ we set $W_{\alpha-\alpha_i-\alpha_j} = \{e\}$). We have $w_E = w_{\alpha,k_E}$ when $k_E = i$ or j.

(iv). $\#\{E \in Y_\alpha \mid k_E = k\} = C_{j-i}^{k-i}$. Note that C_{j-i}^{k-i} is also the number of different reduced expressions of w_{α, l_E}.

(v). Set $Y_{\alpha,k} = \{E \in Y_\alpha \mid l_E = k\}$ $(i \le k \le j)$, then $\Psi(Y_{\alpha,k}) = Y_{\alpha,j-k+i}$.

Proof: (i). Note that we also have $E = T_i^{-a_{i+1}} T_{i+1}^{-a_{i+2}} \cdots T_{j-1}^{-a_j}(E_j)$, we see that (i) was already proved in the argument of 4.6(ii).

(ii). Let $w = s_k s_{k+1} ... s_j w_G$ and let $w_E = s_h s_{h+1} ... s_j w_1$, $i < h < j$. Then $T_w(E_k) = E$ for some $k \in [i, j-1]$ (in fact $k = k_G$). Since $w, w_E \in W_\alpha$, by 2.5(i) we can find some $x \in W_\alpha$ such that $w = w_E \cdot x$. But $w(\alpha_k) = \alpha$, we necessarily have $x \in W_{\alpha-\alpha_j}$. This forces that $k = h$. We then have $T_{w_1}(E_{k_E}) = T_{w_G}(E_k)$. Therefore $w_1 = w_G$ since w_E is the shortest element in $\Phi^{-1}(E)$. (ii) is proved.

(iii). If $k_E = i$ or j, by 5.2 we see that $w_E = w_{\alpha,k_E}$. If $k_E \ne j$, by the proof of (ii) we see that $w_E = s_h s_{h+1} \cdots s_j w_G$, $k_E = k_G$ for some $h \in [i+1, j], G \in Y_{s_j(\alpha)}$. Using induction hypothesis we know that $w_G = u_G \cdot w_{s_j(\alpha),k_G}$ for some $u_G \in W_{s_j(\alpha)-\alpha_i-\alpha_{j-1}}$. So we have $s_j u_G = u_G s_j$. Note that $s_j w_{s_j(\alpha),k_G} = w_{\alpha,k_E}$, we see (iii) is true in this case.

From the proof of (ii) it is easy to see that $k_E = k$ if and only if $\#\{m \in [i+1, j] \mid a_m = -1\} = k - i$. Thus we get (v), and (iv) follows from 5.1(v).

The proposition is proved.

Remark: In general 5.2 is not true. For type D_4, let $w = s_3 s_1 s_2 s_4 s_3 s_1 s_2 s_4$, then (w, α_3) is the shortest element in $\widetilde{(w, \alpha_3)}$, but $w(\alpha_3) < s_3 w(\alpha_3)$, so 5.2 is false for type D_4.

5.4. We shall give a clear formula for the coproduct of a root vector. We need some preparation.

Let α be as in 5.1. For any $\beta \in \mathbf{N}R^+$, let $c(\beta)$ be the number of connected components of β. When $\beta \le \alpha$, $c(\beta)$ is just the minimal number of roots in R^+ whose sum is β.

Let $E = T_j^{a_j} T_{j-1}^{a_{j-1}} \cdots T_{i+1}^{a_{i+1}}(E_i) = T_i^{-a_{i+1}} T_{i+1}^{-a_{i+2}} \cdots T_{j-1}^{-a_j}(E_j)$ be a root vector in Y_α. Let $\beta \in \mathbf{N}R^+$ be such that $\beta \le \alpha$. If $\beta = 0$ we set $E_\beta = 1$, $K_\beta = 1$, if $\beta = \alpha_k + \alpha_{k+1} + \cdots + \alpha_l$ $(i \le k \le l \le j)$ we set $E_\beta = T_l^{a_l} T_{l-1}^{a_{l-1}} \cdots T_{k+1}^{a_{k+1}} E_k$, $K_\beta = K_l K_{l-1} \cdots K_{k+1} K_k$, if $\beta_1, \cdots, \beta_{c(\beta)}$ are connected components of β and $\beta = \beta_1 + \cdots + \beta_{c(\beta)}$, we set $E_\beta = E_{\beta_1} ... E_{\beta_{c(\beta)}}$, $K_\beta = K_{\beta_1} ... K_{\beta_{c(\beta)}}$. E_β, K_β are well defined since for different connected components β_h, β_m we have $E_{\beta_h} E_{\beta_m} = E_{\beta_m} E_{\beta_h}$, $K_{\beta_h} K_{\beta_m} = K_{\beta_m} K_{\beta_h}$.

We define X_E inductively as follows: If $j - i \le 2$, we set

$$X_E = \{\gamma \in \mathbf{N}R^+ \mid \gamma \le \alpha, \quad w_E^{-1}(\gamma) \ge 0\}.$$

Assume that $X_{E'}$ is well defined for $E' = T_j^{a_j} T_{j-1}^{a_{j-1}} \cdots T_{i+2}^{a_{i+2}}(E_{i+1}) \in Y_{\alpha'}$, $\alpha' = \alpha - \alpha_i$, when $a_{i+1} = 1$, we set

$$X_E = \{\gamma + \alpha_i, \ \gamma' \mid \gamma, \gamma' \in X_{E'}, \quad \alpha' - \gamma' \ge \alpha_{i+1}\};$$

when $a_{i+1} = -1$, we set

$$X_E = \{\gamma + \alpha_i, \ \gamma' \mid \gamma, \gamma' \in X_{E'}, \quad \gamma \ge \alpha_{i+1}\}.$$

Now we can state our second main result.

Theorem 5.5. (i). Let $\alpha,\ E$ be as in 5.4, then

$$\Delta(E)=\sum_{\gamma\in X_E}(v^{-1}-v)^{c(\alpha-\gamma)+c(\gamma)-1}K_\gamma E_{\alpha-\gamma}\otimes E_\gamma.$$

(ii). $S(E)=(-1)^{i-j+1}v^{i-j}K_\alpha^{-1}\Psi(E)$.

Proof: When $j=i$, it follows from the definition of the coproduct. Now assume that $j>i$. Let $E'=T_j^{a_j}T_{j-1}^{a_{j-1}}\cdots T_{i+2}^{a_{i+2}}(E_{i+1})\in Y_{\alpha'},\ \alpha'=\alpha-\alpha_i$. We use induction on $j-i$.

If $a_{i+1}=1$, then (see 5.1(vi)) $E=v^{-1}E_iE'-E'E_i$, By induction hypothesis we get

$$(1)\qquad \Delta(E)=v^{-1}(E_i\otimes 1+K_i\otimes E_i)(\sum_{\substack{\gamma'\in X_{E'}\\ \beta'=\alpha'-\gamma'}}(v^{-1}-v)^{c(\beta')+c(\gamma')-1}K_{\gamma'}E_{\beta'}\otimes E_{\gamma'})$$

$$-(\sum_{\substack{\gamma'\in X_{E'}\\ \beta'=\alpha'-\gamma'}}(v^{-1}-v)^{c(\beta')+c(\gamma')-1}K_{\gamma'}E_{\beta'}\otimes E_{\gamma'})(E_i\otimes 1+K_i\otimes E_i).$$

If $\gamma'\geq\alpha_{i+1}$, then we have

(2). $E_iK_{\gamma'}=vK_{\gamma'}E_i,\ E_{\beta'}E_i=E_iE_{\beta'}$.

$v^{-1}E_iE_{\gamma'}-E_{\gamma'}E_i=E_{\gamma'+\alpha_i},\ E_{\beta'}K_i=K_iE_{\beta'},\ c(\gamma'+\alpha_i)=c(\gamma')$.

If $\beta'\geq\alpha_{i+1}$, then we have

(3). $v^{-1}E_iE_{\beta'}-E_{\beta'}E_i=E_{\beta'+\alpha_i},\ K_{\gamma'}E_i=E_iK_{\gamma'},\ c(\beta'+\alpha_i)=c(\beta')$.

$E_iE_{\gamma'}=E_{\gamma'}E_i=E_{\gamma'+\alpha_i},\ E_{\beta'}K_i=vK_iE_{\beta'},\ c(\gamma'+\alpha_i)=c(\gamma')+1$.

If $a_{i+1}=-1$, then (see 5.1(vi)) $E=v^{-1}E'E_i-E_iE'$. By induction hypothesis we get

$$(4)\qquad \Delta(E)=v^{-1}(\sum_{\substack{\gamma'\in X_{E'}\\ \beta'=\alpha'-\gamma'}}(v^{-1}-v)^{c(\beta')+c(\gamma')-1}K_{\gamma'}E_{\beta'}\otimes E_{\gamma'})(E_i\otimes 1+K_i\otimes E_i)$$

$$-(E_i\otimes 1+K_i\otimes E_i)(\sum_{\substack{\gamma'\in X_{E'}\\ \beta'=\alpha'-\gamma'}}(v^{-1}-v)^{c(\beta')+c(\gamma')-1}K_{\gamma'}E_{\beta'}\otimes E_{\gamma'}).$$

If $\gamma'\geq\alpha_{i+1}$, then we have

(5). $E_iK_{\gamma'}=vK_{\gamma'}E_i,\ E_{\beta'}E_i=E_iE_{\beta'}=E_{\beta'+\alpha_i},\ c(\beta'+\alpha_i)=c(\beta')+1$.

$v^{-1}E_{\gamma'}E_i-E_iE_{\gamma'}=E_{\gamma'+\alpha_i},\ E_{\beta'}K_i=K_iE_{\beta'},\ c(\gamma'+\alpha_i)=c(\gamma')$.

If $\beta'\geq\alpha_{i+1}$, then we have

(6). $v^{-1}E_{\beta'}E_i - E_iE_{\beta'} = E_{\beta'+\alpha_i},\ K_{\gamma'}E_i = E_iK_{\gamma'},\ c(\beta'+\alpha_i) = c(\beta')$.

$E_iE_{\gamma'} = E_{\gamma'}E_i,\ E_{\beta'}K_i = vK_iE_{\beta'}$.

Combine (1-6) and the definition of X_E we see (i) is true.

(ii). It follows from 4.7.

The theorem is proved.

Remark: For other types it is not difficult to get the formula $\Delta(E)$ for $E \in Y_\alpha$ when the $j_i, j_{i-1}, ..., j_1, j_0$ are pairwise different (see 4.1 for notations).

5.6. We shall write E_{ij} for the root vector $T_jT_{j-1}\cdots T_{i+1}(E_i)$. In particular we have $E_{ii} = E_i$. The set $\{E_{ij} \mid 1 \le i \le j \le n\}$ first appears in [J] and corresponds to the reduced expression $s_ns_{n-1}s_ns_{n-2}s_{n-1}s_n\cdots s_1s_2\cdots s_{n-2}s_{n-1}s_n$ of the longest element of W (see [L2]). In this subsection we list some formulas concerned with $E_{ij},\ F_{ij} = \Omega(E_{ij}),\ K_{ij} = T_jT_{j-1}\cdots T_{i+1}(K_i)$, one can prove them by direct computations or see [L1, R] for some of them.

The indices i, j, k, l always indicate numbers in $[1, n]$, and M, N always indicate non-negative positives, we also assume that $i \le j$ and $k \le l$.

$$\text{(d0).}\qquad E_{ij}E_{kl} = \begin{cases} E_{kl}E_{ij}, & \text{if } j < k-1 \text{ or } k < i \le j < l,\\ vE_{kl}E_{ij}, & \text{if } k < i < j = l,\\ v^{-1}E_{kl}E_{ij}, & \text{if } i = k \le j < l \text{ or } i < k \le j = l,\\ vE_{il} + vE_{kl}E_{ij}, & \text{if } j = k-1,\\ E_{kl}E_{ij} + (v^{-1}-v)E_{il}E_{kj}, & \text{if } i < k \le j < l. \end{cases}$$

we set $E_{ij}^{(N)} = E_{ij}^N/[N]!$, $F_{ij}^{(N)} = F_{ij}^N/[N]!$, where $[N]! = \prod_{i=1}^N \frac{v^i - v^{-i}}{v - v^{-1}}$ if $N \ge 1$, $[0]! = 1$. Let c be an integer, we set

$$\begin{bmatrix} K_{ij},\ c \\ N \end{bmatrix} = \prod_{r=1}^N \frac{K_{ij}v^{c-r+1} - K_{ij}^{-1}v^{-c+r-1}}{v^r - v^{-r}}.$$

(d1). $\quad E_{ij}^{(M)}E_{kl}^{(N)} = E_{kl}^{(N)}E_{ij}^{(M)}\quad$ if $j < k-1$ or $k < i \le j < l$.

(d2). $\quad E_{ij}^{(M)}E_{kl}^{(N)} = v^{MN}E_{kl}^{(N)}E_{ij}^{(M)}\quad$ if $k < i < j = l$.

(d3). $\quad E_{ij}^{(M)}E_{kl}^{(N)} = v^{-MN}E_{kl}^{(N)}E_{ij}^{(M)}\quad$ if $i = k \le j < l$, or $i < k \le j = l$.

$$\text{(d4).}\qquad E_{ij}^{(M)}E_{kl}^{(N)} = \sum_{\substack{p\ge0,q\ge0\\p+q=N\\q+r=M}} v^{rp+q}E_{kl}^{(p)}E_{il}^{(q)}E_{ij}^{(r)}\quad \text{if } j = k-1.$$

$$(d5). \qquad E_{ij}^{(M)}E_{kl}^{(N)} = \sum_{0\le t\le M,N} v^{-\frac{t(t-1)}{2}}(v^{-1}-v)^t[t]!E_{kj}^{(t)}E_{kl}^{(N-t)}E_{ij}^{(M-t)}E_{il}^{(t)}$$

if $i<k\le j<l$.

$$(e0). \quad E_{ij}F_{kl} = \begin{cases} F_{kl}E_{ij}, & \text{if } j<k \text{ or } k<i\le j<l, \\ F_{kl}E_{ij}+v^{-1}K_{k,j}^{-1}E_{i,k-1}, & \text{if } i<k\le j=l, \\ F_{kl}E_{ij}-F_{j+1,l}K_{ij}^{-1}, & \text{if } i=k\le j<l, \\ F_{kl}E_{ij}+\left[{K_{ij};\,0 \atop 1}\right], & \text{if } i=k,\ j=l, \\ F_{kl}E_{ij}+v^{-1}(v-v^{-1})F_{j+1,l}K_{k,j}^{-1}E_{i,k-1}, & \text{if } i<k\le j<l. \end{cases}$$

$$(e1). \qquad E_{ij}^{(M)}F_{kl}^{(N)} = F_{kl}^{(N)}E_{ij}^{(M)} \qquad \text{if } j<k \text{ or } k<i\le j<l.$$

$$(e2). \qquad E_{ij}^{(M)}F_{kl}^{(N)} = \sum_{0\le t\le M,N} v^{t(N-t-1)}F_{kj}^{(N-t)}K_{kj}^{-t}E_{ij}^{(M-t)}E_{i,k-1}^{(t)} \quad \text{if } i<k\le j=l.$$

$$(e3). \qquad E_{ij}^{(M)}F_{kl}^{(N)} = \sum_{0\le t\le M,N} (-1)^t v^{t(M-t)}F_{j+1,l}^{(t)}F_{kl}^{(N-t)}K_{ij}^{-t}E_{ij}^{(M-t)} \quad \text{if } i=k\le j<l.$$

$$(e4). \qquad E_{ij}^{(M)}F_{ij}^{(N)} = \sum_{0\le t\le M,N} F_{ij}^{(N-t)}\begin{bmatrix} K_{ij};\ 2t-M-N \\ t \end{bmatrix}E_{ij}^{(M-t)}$$

$$(e5). \qquad E_{ij}^{(M)}F_{kl}^{(N)} = \sum_{0\le t\le M,N} v^{-\frac{t(2N+t-1)}{2}}(v-v^{-1})^t[t]!F_{kl}^{(N-t)}F_{j+1,l}^{(t)}K_{kj}^{-t}E_{ij}^{(M-t)}E_{i,k-1}^{(-t)}$$

if $i<k\le j<l$.

We have $X_{E_{ij}} = \{0, \alpha_{ii},\ \alpha_{i,i+1},\ \dots,\ \alpha_{ij}\}$ (see 2.1 for notations), so we get

(f0). $\Delta(E_{ij}) = E_{ij}\otimes 1 + K_{ij}\otimes E_{ij} + (v^{-1}-v)\sum_{i\le k<j} K_{ik}E_{k+1,j}\otimes E_{ik}$.

$$(f1). \qquad \Delta(E_{ij}^{(M)}) = \sum_{\substack{m_0,m_i,m_{i+1}\dots,m_j\ge 0 \\ m_0+m_i+m_{i+1}+\dots+m_j=M}} \xi_{\mathbf{m}}K_{\mathbf{m}}E_{\mathbf{m}}\otimes E'_{\mathbf{m}},$$

where $\mathbf{m}=(m_0,m_i,m_{i+1},\dots,m_j)$, $K_{\mathbf{m}} = K_{ii}^{m_i}K_{i,i+1}^{m_{i+1}}\cdots K_{ij}^{m_j}$,

$$\xi_{\mathbf{m}} = v^{-m_0(M-m_0)}\prod_{r=i}^{j-1}(v^{-1}-v)^{m_r}[m_r]!v^{\frac{m_r(m_r-1)}{2}},$$

$$E_{\mathbf{m}} = E_{j,j}^{(m_{j-1})} E_{j-1,j}^{(m_{j-2})} \cdots E_{i+1,j}^{(m_i)} E_{i,j}^{(m_0)}, \qquad E'_{\mathbf{m}} = E_{ii}^{(m_i)} E_{i,i+1}^{(m_{i+1})} \cdots E_{ij}^{(m_j)}.$$

(g0). $$S(E_{ij}) = (-1)^{i-j+1} v^{i-j} K_{ij} \Psi(E_{ij}).$$

(g1). $$S(E_{ij}^{(M)}) = (-1)^{M(i-j+1)} v^{M(i-j)+M(M-1)} K_{ij}^M \Psi(E_{ij}^{(M)}).$$

Note that $\Psi(E_{ij}) = T_i T_{i+1} \cdots T_{j-1}(E_j)$ is also a root vector.

Apply Ω one can get more formulas.

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