Isometries of Intrinsic Metrics on Strictly Convex Domains

by

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Introduction

We introduce an intrinsic metric g on a complex manifold M admitting a positive, bounded strictly plurisubharmonic function τ satisfying the Monge-Ampère equation $\tau^{\alpha\bar{\beta}}\tau_{\alpha}\tau_{\bar{\beta}} = \tau$. It is shown that the holomorphic sectional curvature of the metric g along the tangential direction of each leaf of the Mōnge-Ampère foliation associated to τ is identically -1 (cf. Theorem 1, § 1). This result applies in particular to bounded strictly convex domains in $\mathbb{C}^n$ with smooth boundaries, for which it is shown that foliation-preserving isometries of the intrinsic metric are biholomorphic or anti-biholomorphic (cf. Theorem 2, § 2 and the remarks following it). As a corollary, we obtain a recent theorem of Bland-Duchamp-Kalka ([1]) that for any two bounded strictly convex domains D and $\tilde{D}$ with smooth boundaries, any biholomorphism between corresponding Kobayashi balls D_r and $\tilde{D}_r$ having the center fixed can be extended to a biholomorphism of D and $\tilde{D}$ (cf.corollary to Theorem 2). This theorem can be thought of as an anylytic continuation principle for the homogeneous Monge-Ampère equation. (Recall that a harmonic map is holomorphic everywhere if it is holomorphic on an open set). The proof of theorem 2 is based on the observation that a Monge-Ampère exhaustion of a strictly convex domain D, which in general is not smooth at the center, is however real analytic along each leaf of the Monge-Ampère foliation and induce a C^∞ exhaustion on the blow up of D at the center. Along the same line of argument, we characterize circular domains among strictly convex domains by the property that there is a biholomorphic map from the domain to a Kobayashi ball, having the center fixed (cf. Theorem 3, § 3).

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§ 1. Curvature of Intrinsic Metrics

Let M be a complex manifold with a bounded positive strictly plurisubharmonic function τ of class C^∞ which satisfies:

$$(1.1) \qquad \tau^{\alpha\bar\beta}\tau_\alpha\tau_{\bar\beta} = \tau$$

where lower indices indicate partial differentiation and $\tau^{\alpha\bar\beta}$ are entries of the inverse of the Levi matrix $(\tau_{\alpha\bar\beta})$. Summation convention will be used throughout this article. The function τ will be refered to as a Monge-Ampère function. It is well-known that condition (1.1) is equivalent to the condition

$$(1.1)' \qquad \det(u_{\alpha\bar\beta}) = 0$$

where $u = \log\tau$.

Suppose that $0 < \tau < A^2$, we define the following metrics associated to τ :

$$(1.2) \qquad h = \tau_{\alpha\bar\beta}dz^\alpha d\bar z^\beta \quad \text{and} \quad g = (A^2-\tau)^{-2}h \ .$$

We are interested in the behavior of these metrics. For simplicity we will always assume that $A = 1$ in (1.2). Consider the vector field

$$(1.3) \qquad Z = \tau^\alpha\ \partial/_{\partial z^\alpha} = \tau^{\alpha\bar\beta}\tau_{\bar\beta}\ \partial/_{\partial z^\alpha}$$

then (1.1) implies

$$(1.4) \qquad |Z|_h^2 = \tau, \ |Z|_g^2 = \tau/(1-\tau)^2 .$$

It is also well-known (and easy to see by using (1.1)') that the interior product $Z \lrcorner \partial\bar{\partial}u = \bar{Z} \lrcorner \partial\bar{\partial}u = 0$. In terms of local coordinates

$$(1.5) \qquad u_{\alpha\bar{\beta}}\tau^{\alpha} = 0 \ , \ u_{\alpha\bar{\beta}}\tau^{\bar{\beta}} = 0$$

where $\tau^{\alpha} = \tau^{\alpha\bar{\beta}}\tau_{\bar{\beta}}$ and $\tau^{\bar{\alpha}} = \tau^{\bar{\alpha}\beta}\tau_{\beta} = \tau^{\beta\bar{\alpha}}\tau_{\beta}$ is the cmmplex conjugate.

The manifold is foliated by complex curves which are the integral manifolds of the vector field Z . This foliation is refered to as the <u>Monge-Ampère foliation</u> associated to τ .

<u>Theorem 1</u>. <u>The holomorphic sectional curvatures in the direction of</u> Z <u>are given by</u>

$$K_h(Z) \equiv 0 \quad \text{and} \quad K_g(Z) \equiv -1$$

<u>for the respective metrics</u> h <u>and</u> g <u>defined by</u> (1.2).

<u>Proof</u>. The fact that $K_h(Z) = 0$ is known (cf. Wong [9]). For convenience we include here a very short proof (different from the proof in [9]).

By differentiating equation (1.1) we get

$$\tau_{\gamma} = \tau^{\alpha\bar{\beta}}\tau_{\bar{\beta}\gamma}\tau_{\alpha} + \tau^{\alpha\bar{\beta}}\tau_{\bar{\beta}}\tau_{\alpha,\gamma}$$

$$= \tau_{\gamma} + \tau^{\alpha}\tau_{\alpha,\gamma}$$

thus for any γ we have

$$(1.6) \qquad \tau^{\alpha}\tau_{\gamma,\alpha} = \tau^{\alpha}\tau_{\alpha,\gamma} = 0 .$$

Here the comma denotes covariant differentiation and note that the curvature is of type (1.1) so that successive differentiation of the same type commute.

We obtain by differentiating (1.6)

$$\tau^{\alpha}{}_{,\bar{\beta}}\tau_{\alpha,\gamma} + \tau^{\alpha}\tau_{\alpha,\gamma\bar{\beta}} = 0$$

and by contracting with τ^{γ}, we can get rid of the first term above by (1.6). We thus arrive at the following identity,

$$(1.7) \qquad \tau^{\gamma}\tau^{\alpha}\tau_{\alpha,\gamma\bar{\beta}} = 0 .$$

From the Ricci commutation formula

$$\tau_{\alpha,\gamma\bar{\beta}} - \tau_{\alpha,\bar{\beta}\gamma} = R^{\mu}_{\alpha\gamma\bar{\beta}}\tau_{\mu}$$

and from the fact that the metric tensor is covariant constant, i.e. $\tau_{\alpha,\bar{\beta}\gamma} = \tau_{\alpha\bar{\beta},\gamma} = 0$, we conclude that

$$(1.8) \qquad R^{\mu}_{\alpha\gamma\bar{\beta}}\,\tau_{\mu}\tau^{\alpha}\tau^{\gamma} = 0 .$$

In particular $K_h(Z) = \frac{1}{2}R^{\mu}_{\alpha\dot{\gamma}\bar{\beta}}\tau_{\mu}\tau^{\alpha}\tau^{\gamma}\tau^{\bar{\beta}}/|Z|_h \equiv 0$.

To compute the curvature of g , we use the fact that g is conformal to h , so that the curvature forms Ω^{α}_{β} (for g) and R^{α}_{β} (for h) are related by

$$\Omega^{\alpha}_{\beta} = R^{\alpha}_{\beta} + 2\delta^{\alpha}_{\beta}\partial\bar{\partial}\log(1-\tau)$$

(cf. Wong [9], p.270). By direct computations

$$[\log(1-\tau)]_{\mu\bar{\nu}} = (1-\tau)^{-2}[-\tau_{\mu\bar{\nu}} + (\tau\tau_{\mu\bar{\nu}} - \tau_{\mu}\tau_{\bar{\nu}})]$$

and for $u = \log\tau$,

$$u_{\mu\bar{\nu}} = \tau^{-2}(-\tau_{\mu\bar{\nu}} + \tau_{\mu}\tau_{\bar{\nu}}) \ .$$

Thus we have

$$[\log(1-\tau)]_{\mu\bar{\nu}} = (1-\tau)^{-2}(-\tau_{\mu\bar{\nu}} + \tau^2 u_{\mu\bar{\nu}})$$

and so the components of the curvature tensors are related by

$$\Omega^{\alpha}_{\beta\mu\bar{\nu}} = R^{\alpha}_{\beta\mu\bar{\nu}} + 2(1-\tau)^{-2}[-\tau_{\mu\bar{\nu}} + \tau^2 u_{\mu\bar{\nu}}]\delta^{\alpha}_{\beta}$$

$$\Omega_{\beta\bar{\gamma}\mu\bar{\nu}} = (1-\tau)^{-2}\tau_{\alpha\bar{\gamma}}\Omega^{\alpha}_{\beta\mu\bar{\nu}}$$

$$= (1-\tau)^{-2}R_{\beta\bar{\gamma}\mu\bar{\nu}} + 2(1-\tau)^{-4}\tau_{\beta\bar{\gamma}}[-\tau_{\mu\bar{\nu}} + \tau^2 u_{\mu\bar{\nu}}] \ .$$

The holomorphic sectional curvature of g is given by

$$K_g(Z) = \tfrac{1}{2}\Omega_{\beta\bar{\gamma}\mu\bar{\nu}}\tau^{\beta}\tau^{\bar{\gamma}}\tau^{\mu}\tau^{\bar{\nu}}/|Z|_g^4$$

$$= \frac{1}{2}\frac{|Z|_h^4}{(1-\tau)^2|Z|_g^4}K_h(Z) + \frac{1}{(1-\tau)^4|Z|_g^4}\tau_{\beta\bar{\gamma}}\tau^{\beta}\tau^{\bar{\gamma}}[-\tau_{\mu\bar{\nu}}\tau^{\mu}\tau^{\bar{\nu}} + \tau^2 u_{\mu\bar{\nu}}\tau^{\mu}\tau^{\bar{\nu}}] .$$

Since $K_h(Z) = 0$, $\tau_{\beta\bar{\gamma}}\tau^{\beta}\tau^{\bar{\gamma}} = |Z|_h^2 = \tau$ and $u_{\mu\bar{\nu}}\tau^{\mu}\tau^{\bar{\nu}} = 0$ (cf. (1.5)), we obtain

$$K_g(Z) = -\frac{\tau^2}{(1-\tau)^4|Z|_g^4} \equiv -1$$

where the last equality follows from (1.4).

Q.E.D.

For a bounded strictly convex domain D with C^∞ smooth boundary in C^n , the fundamental work of Lempert [2] says that for any point p in D , a unique Monge-Ampère exhaustion $\tau : \bar{D} \to [0,1]$ exists. More precisely τ is an exhaustion with the following properties:

(1.9) $\tau(z)=0$ iff $z=p$, $\tau(z)=1$ iff $Z \in \partial D$;

(1.10) $\tau \in C^0(\bar{D}) \cap C^\infty(\bar{D}\setminus\{p\})$; moreover if $\pi : \tilde{D} \to D$ is the blow up of D at p then $\pi\circ\tau \in C^\infty(\tilde{D})$;

(1.11) τ is strictly plurisubharmonic (in fact strictly convex and satisfies (1.1) on $D\setminus\{p\}$;

(1.12) $\log\tau(z) = \log|z-p|^2 + O(1)$ *near* p .

In other words, the terminology of [4], (D,τ) is a manifold of circular type with radius 1 and center p , such a manifold is biholomorphic to a complete cicular domain iff the associated Monge-Ampère foliation is holomorphic or equivalently the gradient vector field Z defined in (1.3) is a holomorphic vector field (cf. [10]). The following corollary is an immediate consequence of Theorem 1:

Corollary. *On a bounded strictly convex domain* D *in* $\mathbb{C}^n$ *with* C^∞ *smooth boundary then for any point* p *in* D , *intrinsic metrics* h *and* g *as defined in* (1.2) *exist and the conclusions of Theorem* 1 *hold for these metrics on* $D\setminus\{p\}$.

We know also from the work of Lempert that the exhaustion τ is related to the Kobayashi distance $\delta(z) = \delta_D(z;p)$ from the fixed point p by the formula

(1.13) $$\delta = \log \frac{1+\sqrt{\tau}}{1-\sqrt{\tau}} .$$

In fact the leaves of the Monge-Ampère foliation are precisely the (complex) geodesics (i.e. *extremal discs*) *of the Kobayashi metric through* p (cf. [2] and [5]). Let $\Delta \subset \mathbb{C}$ be the unit disc in $\mathbb{C}$ and $f : \Delta \to D$ with $f(0) = p$ be such an extremal disc. It is easily verified that

$$\tau\circ f(z) = |z|^2$$

(1.14) $f^*(h) = dzd\bar{z}$ *is the Euclidean metric*

$$f^*(g) = \frac{dzd\bar{z}}{(1-|z|^2)^2}$$ *is the Poincaré metric*

thus the metric g may be regarded as a generalization of the Poincaré metric. Furthermore, from the uniqueness (relative to the center p) it is also clear that biholomorphic maps of D leaving the center p fixed, are isometries of the metrics h and g .

§ 2. Isometries of Intrinsic Metrics

In this section we study the converse, namely the problem of finding conditions under which certain isometries of the intrinsic emtrics (defined in the previous section) are biholomorphic (or antibiholomorphic) maps.

Associated to the Monge-Ampère exhaustion τ with center p in a bounded strictly convex domain with C^∞ smooth boundary, there is a bounded complete circular domain G (which in fact is the Kobayashi indicatrix of D at p) in $\mathbb{C}^n$ and a homeomorphism $\Phi : \bar{G} \to \bar{D}$ with the following properties (cf. [2] and [3]):

(2.1) $\Phi(0) = p$ and Φ is a diffeomorphism from $\bar{G}\setminus\{0\}$ onto $\bar{D}\setminus\{p\}$

(2.2) $\sigma = \tau\circ\Phi$ is the Monge-Ampère exhaustion of G centered at the origin with the property that $\sigma(\lambda z) = |\lambda|^2\sigma(z)$ for all complex number with $|\lambda|\leq 1$ and for all z in G

(2.3) Φ maps discs through the origin biholomorphically onto the leaves of the Monge-Ampère foliation associated to τ .

As the leaves of the foliation are geoedesics of the Kobayashi metric, Φ is simply the exponential map from the indicatrix onto the domain. It should be emphasized that for general domains the exponential map is not well-defined because of the non-uniqueness of extremal discs (cf. [7]).

The following relationship between τ and σ in a neighbourhood of p is very useful in the sequel

$$(2.4) \qquad \tau(z) = \sigma(z-p) + 0(|z-p|^3)$$

we refer to [4] § 4 for the proof. Without loss of generality we assume from here on that $p = 0$. From (2.4) it follows that

$$(2.5) \qquad \tau_\alpha(z) = \sigma_\alpha(z) + 0(|z|^2), \; \tau_{\bar\beta}(z) = \sigma_{\bar\beta}(z) + 0(|z|^2)$$

$$(2.5)' \qquad \tau_{\alpha\bar\beta}(z) = \sigma_{\alpha\bar\beta}(z) + 0(|z|)$$

and so

$$(2.6) \qquad \det(\tau_{\alpha\bar\beta}(z)) = \det(\sigma_{\alpha\bar\beta}(z)) + 0(|z|) .$$

The same is true for the minors of the two matrices,

$$M_\tau^{\alpha\bar\beta}(z) = M_\sigma^{\alpha\bar\beta}(z) + 0(|z|) .$$

As a result we also have the expansion for the inverse

$$(2.7) \qquad \tau^{\alpha\bar\beta}(z) = \sigma^{\alpha\bar\beta}(z) + 0(|z|) .$$

Recall that σ is the Monge-Ampère exhaustion of the circular domain G with center at the origin and it has the homogeneity property given by (2.2). In fact $\sigma(z) = |z|^2 e^{g(z)}$ where g is a smooth function on $\mathbb{C}^n \setminus \{0\}$ and is constant on each complex line through the origin. It follows that $\sigma_{\alpha\bar{\beta}}(z)$ and $\sigma^{\alpha\bar{\beta}}(z)$ are bounded as $z \to 0$. In fact from the homogenity property (2.2), we have

$$(2.8) \qquad \sigma_\alpha(\lambda z) = \bar{\lambda}\sigma_\alpha(z) \quad \text{and} \quad \sigma_{\alpha\bar{\beta}}(\lambda z) = \sigma_{\alpha\bar{\beta}}(z) \; .$$

From (2.5) we conclude that

(2.9) $\tau_{\alpha\bar{\beta}}(z)$ <u>and</u> $\tau^{\alpha\bar{\beta}}(z)$ <u>are bounded as</u> $z \to 0$

(2.10) $\exists\, c > 0$ <u>such that</u> $(\tau_{\alpha\bar{\beta}}) > c\, I$ <u>and</u> $(\tau^{\alpha\bar{\beta}}) > c\, I$

where I denotes the identity matrix.

Finally we also noted that because of (1.10) the restriction of $(\tau_{\alpha\bar{\beta}})$ and $(\tau^{\alpha\bar{\beta}})$ to any leaf extends smoothly across the center p. With these remarks we now prove the main result concerning isometries of the intrinsic metrics.

<u>Theorem 2.</u> <u>Let</u> D <u>and</u> $\tilde{D}$ <u>be two bounded strictly convex domains with</u> C^∞ <u>smooth boundaries in</u> $\mathbb{C}^n$. <u>Let</u> τ (<u>resp.</u> $\tilde{\tau}$) <u>be the Monge-Ampère exhaustion of</u> D (<u>resp.</u> $\tilde{D}$) <u>centered at</u> p (<u>resp.</u> $\tilde{p}$) <u>and</u> h, g (<u>resp.</u> $\tilde{h}, \tilde{g}$) <u>be the associated intrinsic metrics defined in</u> § 1. <u>Then every</u> C^∞ <u>isometry</u> $\varphi : D \to \tilde{D}$ <u>of the metrics</u> h <u>and</u> $\tilde{h}$ (<u>resp.</u> g <u>and</u> $\tilde{g}$) <u>is either biholomorphic or anti-biholomorphic</u>.

<u>Proof</u>. First we show that φ is holomorphic or anti-biholomorphic when restricted to each leaf of the foliation. Since φ preserves the foliation, it maps a leaf L onto a leaf $\tilde{L}$ and $\varphi(p) = \tilde{p}$ where p and $\tilde{p}$ are the corresponding centers in D and $\tilde{D}$. The map φ is also an isometry of the induced metrics h and $\tilde{h}$ (resp. g and $\tilde{g}$) on L and $\tilde{L}$ respectively, which by the remarks at the end of § 1 are biholomorphically isometric to the unit disc in $\mathbb{C}$ with the Euclidean metric (resp. the Poincaré metric) and with the center p (resp. $\tilde{p}$) corresponding to the origin. An isometry of the disc with Euclidean metric (or Poincaré metric) is either holomorphic or anti-holomorphic. Without loss of generality assume that it is holomorphic, then a connectedness argument shows that φ is holomorphic along every leaf, i.e. $\bar{Z}\varphi = 0$ where Z is the gradient vector field of τ (cf. (1.3)) tangent to the leaves.

Since φ is an isometry, $\nabla d\varphi = 0$ i.e.

$$(2.11) \qquad \varphi^{\alpha}_{jk} - \Gamma^{l}_{jk}\varphi^{\alpha}_{l} + \tilde{\Gamma}^{\alpha}_{\beta\gamma}\varphi^{\beta}_{j}\varphi^{\gamma}_{k} = 0$$

for all α and for all j,k ranging over real coordinate systems. Here $\Gamma^{l}_{j,k}$ and $\tilde{\Gamma}^{\alpha}_{\beta\gamma}$ are the Christoffel symbols for h and $\tilde{h}$ respectively. We claim that $\bar{\nabla}|\bar{\partial}\varphi|^2 = \nabla|\bar{\partial}\varphi|^2 = 0$. By a direct computation we have for any l

$$\frac{\partial}{\partial z^{l}}|\bar{\partial}\varphi|^2 = \{h^{j\bar{k}}(z)\tilde{h}_{\alpha\bar{\beta}}(\varphi(z))\varphi^{\alpha}_{\bar{k}}(z)\varphi^{\bar{\beta}}_{j}(z)\}_{l}$$

$$= h^{j\bar{k}}_{l}(\tilde{h}_{\alpha\bar{\beta}}\varphi^{\alpha}_{\bar{k}}\varphi^{\bar{\beta}}_{j}) + h^{j\bar{k}}(\tilde{h}_{\alpha\bar{\beta}\gamma}\varphi^{\gamma}_{l} + \tilde{h}_{\alpha\bar{\beta}\bar{\gamma}}\varphi^{\bar{\gamma}}_{l})\varphi^{\alpha}_{\bar{k}}\varphi^{\bar{\beta}}_{j}$$

$$+ h^{j\bar{k}}\tilde{h}_{\alpha\bar{\beta}}(\varphi^{\alpha}_{\bar{k}l}\varphi^{\bar{\beta}}_{j} + \varphi^{\alpha}_{\bar{k}}\varphi^{\bar{\beta}}_{jl}) .$$

Choosing normal coordinate at z and at $\varphi(z)$, we have

$$\frac{\partial}{\partial z^l}|\bar{\partial}\varphi|^2 = h^{j\bar{k}}\tilde{h}_{\alpha\bar{\beta}}(\varphi^{\alpha}_{\bar{k}l}\varphi^{\bar{\beta}}_{j} + \varphi^{\alpha}_{\bar{k}}\varphi^{\bar{\beta}}_{jl})$$

which is zero because with normal coordinates at z , (2.8) implies that all second derivatives of φ vanish at z . Similarly for any l we have

$$\frac{\partial}{\partial \bar{z}^l}|\bar{\partial}\varphi|^2 = 0$$

proving the claim. The above computations work the same way for the metric g in place of h . In any case $|\bar{\partial}\varphi|^2$ is constant on $D\setminus\{p\}$.

Since φ is C^∞ at the center, to evaluate $\varphi^{\alpha}_{\bar{k}}(p)$, we need only evaluate its value at p along the leaf in the direction of $\partial/_{\partial z}k$. Since we have observed already that φ is holomorphic along each leaf, it is then clear that $\varphi^{\alpha}_{\bar{k}}(p) = 0$. On the other hand we have, by definition,

$$|\bar{\partial}\varphi|^2_h = \tau^{j\bar{k}}\tau_{\alpha\bar{\beta}}\varphi^{\alpha}_{\bar{k}}\varphi^{\bar{\beta}}_{j} \ , \ |\bar{\partial}\varphi|^2_g = (1-\tau)^{-2}|\bar{\partial}\varphi|^2_h \ .$$

Now form (2.6) we know that $\tau^{j\bar{k}}(z)$ and $\tau_{\alpha\bar{\beta}}(z)$ remain bounded as z approaches the center, it follows that $|\bar{\partial}\varphi|^2(z)$, which is constant on $D\setminus\{p\}$, approaches zero as $z \to p$. Clearly the only possibility is that $\bar{\partial}\varphi \equiv 0$, namely that φ is a holomorphic map.

Q.E.D.

Remark. The theorem above is true under the weaker assumption that φ is a C^∞ diffeomorphism preserving the foliations and the induced metric on corresponding leaves of the foliation. Instead of concluding that all partial derivatives of $|\bar\partial\varphi|^2$ are zero, we can only conclude that $Z|\bar\partial\varphi|^2 = \bar Z|\bar\partial\varphi|^2 = 0$, where Z is the gradient vector field and hence $|\bar\partial\varphi|^2$ is constant when restricted to a leaf. However the proof that $|\bar\partial\varphi|^2(z) \to 0$ as $z \to p$ works the same way and constancy on each leaf is enough to force $|\bar\partial\varphi|^2 \equiv 0$. With this remark we obtain, as a corollary, the following result of Bland, Duchamp and Kalka [1]:

Corollary. Let D and $\tilde D$ be two bounded strictly convex domains with C^∞ boundaries in $\mathbb{C}^n$ and let D_r and $\tilde D_r$ be the Kobayashi ball of radius r centered at $p \in D$ and $\tilde p \in \tilde D$ respectively. Then every biholomorphism $\varphi : D_r \to \tilde D_r$ sending p to $\tilde p$ is the restriction of a biholomorphism between D and $\tilde D$.

Proof. The indicatrix G of D at p, being a complete circular domain has a natural biholomorphic contraction map $\lambda : G \to G_{r(\lambda)}$, taking z to λz, where λ $0 < \lambda < 1$. The image $G_r = G_{r(\lambda)}$ is the Kobayashi ball centered at the origin with radius $r = \log \frac{1+\lambda}{1-\lambda}$. This induces a map from D to D_r via the exponential map Φ so that the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\lambda} & G_r \\ {\scriptstyle\Phi}\downarrow & \circlearrowright & \downarrow{\scriptstyle\Phi} \\ D & \cdots\cdots\cdots> & D_r \end{array}$$

We refer to this map as the contraction map (cf. [1]) . From (2.2) it is clear that the contraction map is biholomorphic when restrict to a leaf of the Monge-Ampère foliation.

By composing with the contraction map (and its inverse) it is clear that every biholomorphism $\varphi : D_r \to \tilde{D}_r$ extends to a foliation preserving diffeomorphism of D and $\tilde{D}$. Furthermore this extension is biholomorphic when restrict to a leaf and a priori preserve the induced metrics. By the previous remark, this extension is holomorphic.

Q.E.D.

Remark. Both Theorem 2 and the corollary hold in the general setting of manifolds of circular type ([4]). In fact for any manifolds of circular type with bounded exhaustion, the leaves of the Monge-Ampère foliation are Kobayashi extremal discs ([5]), hence the exponential map from its indicatrix is again well-defined.

§ 3. When is a Strictly Convex Domain Circular?

A bounded circular doamin is biholomorphic to any of its Kobayashi ball centered at the origin. In fact this property characterize circular domains among strictly convex domains. The proof presented below follows from the same line of argument as in the previous section. The theorem is probably true for more general domains. However at present our techniques, based on curvature of intrinsic metrics, work only for strictly convex domains.

Theorem 3. Let D be a bounded strictly convex domain in $\mathbb{C}^n$ with smooth boundary. The following statements are equivalent:

(a) D is biholomorphic to a complete circular domain;

(b) for some point p in D and some r, with $0 < r < \infty$, the contraction map $\varphi : D \to D_r$, where D_r denotes the the Kobayashi ball of radius r centered at p, is biholomorphic;

(c) for some p in D and $0 < r < \infty$, there exists a biholomorphic map φ from D onto the Kobayashi ball D_r of radius r centered at p, leaving the center p fixed.

Proof. The implications (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) are obvious, it remains to show (c) $\Rightarrow$ (a).

It is known that D_r is also strictly convex ([2]) . Let $\tilde{\tau}$ be the Monge-Ampère exhaustion of D_r at p . Since φ is biholomorphic and fixes p , we have from uniqueness of Monge-Ampère exhaustion centered at p , that $\tilde{\tau} = \tau\circ\varphi^{-1}$, where τ is the Monge-Ampère exhaustion for D . On the other hand, from (1.13), the Kobayashi ball is given by

$$D_r = \{z \in D | \tau(z) < \lambda^2\}$$

where $r = \log \frac{1+\lambda}{1-\lambda}$. Since $\lambda^{-2}\tau = 1$ on ∂D_r and clearly satisfies the Monge-Ampère equation, we have by uniqueness that on D_r , $\lambda^{-2}\tau = \tau\circ\varphi^{-1} = \tilde{\tau}$.

It follows that on D_r

$$\tilde{\tau}_{\bar{\beta}} = \lambda^{-2}\tau_{\bar{\beta}} \ , \ \tilde{\tau}_{\alpha\bar{\beta}} = \lambda^{-2}\tau_{\alpha\bar{\beta}} \quad \text{and} \quad \tilde{\tau}^{\alpha\bar{\beta}} = \lambda^2\tau^{\alpha\bar{\beta}}$$

hence the corresponding gradient vector fields for τ and $\tilde{\tau}$ are equal

$$(3.1) \qquad \tilde{Z} = \tilde{\tau}^{\alpha\bar{\beta}}\tilde{\tau}_{\bar{\beta}} = \tau^{\alpha\bar{\beta}}\tau_{\bar{\beta}} = Z$$

on D_r . The same is true for the Ricci forms

$$(3.2) \qquad \tilde{S} = \bar{\partial}\partial\log\det(\tilde{\tau}_{\alpha\bar{\beta}}) = \bar{\partial}\partial\log\det(\tau_{\alpha\bar{\beta}}) = S$$

on D_r . Since $\varphi : D \to D_r$ is biholomorphic, it is an isometry of the metrics $i\partial\bar{\partial}\tau$ and $i\partial\bar{\partial}\tilde{\tau}$, and so preserves the corresponding Ricci curvatures and sends the gradient vector field of one to the other, i.e.

$$S(Z,\bar{Z})(z) = \tilde{S}(\varphi_* Z, \overline{\varphi_* Z})(\varphi(z)) = \tilde{S}(\tilde{Z},\bar{\tilde{Z}})(\varphi(z)) \ .$$

By iteration we get from (3.1), (3.2) and the above

$$(3.3) \qquad S(Z,\bar{Z})(z) = S(Z,\bar{Z})(\varphi(z)) = S(Z,\bar{Z})(\varphi^k(z))$$

where φ^k denotes the composite of φ with itself k times.

From (2.6) and (2.8) we know that

$$\log \det(\tau_{\alpha\bar{\beta}}(tz)) = \log \det(\sigma_{\alpha\bar{\beta}}(z)) + O(|tz|)$$

and by differentiating the equation with respect to z we get

$$t^2 \frac{\partial^2 \log \det(\tau_{\alpha\bar{\beta}})}{\partial z_\mu \partial \bar{z}_\nu}(tz) = \frac{\partial^2 \log \det(\sigma_{\alpha\bar{\beta}})}{\partial z_\mu \partial \bar{z}_\nu}(z) + O(|t|) \ .$$

Hence the Ricci forms S_τ and S_σ are related by

$$t^2 S_\tau(tz) = S_\sigma(z) + O(|t|) \ .$$

On the other hand by (2.5), (2.7) and (2.8), the vector fields Z_τ and Z_σ are related by

$$Z_\tau(tz) = Z_\sigma(tz) + O(|t|) = tZ_\sigma(z) + O(|t|^2) \ .$$

Hence we have

$$S_\tau(Z_\tau,\bar{Z}_\tau)(tz) = S_\sigma(Z_\sigma,\bar{Z}_\sigma)(z) + 0(|t|) \ .$$

For a circular domain, the vector field Z_σ is holomorphic (in fact $Z_\sigma = z^\alpha \frac{\partial}{\partial z_\alpha}$) which is equivalent to (cf. [9]) the condition that

$$S_\sigma(Z_\sigma,\bar{Z}_\sigma) \equiv 0 \ .$$

It follows that

$$S_\tau(Z_\tau,\bar{Z}_\tau)(z) = 0(|z|)$$

on a neighborhood of the origin.

From $\tau(\varphi(z)) = \lambda^2\tau(z)$, we have

$$\tau(\varphi^k(z)) = \lambda^{2k}\tau(z)$$

and property (1.12) of τ implies that

$$\lambda^{2k}B|z|^2 \leq \tau(\varphi^k(z)) \leq \lambda^{2k}A|z|^2$$

for some positive constants A and B . It follows that

$$\lambda^{2k}\,\frac{B}{A}|z|^2 \leq |\varphi^k(z)|^2 \leq \lambda^{2k}\,\frac{A}{B}|z|^2$$

hence

$$S_\tau(Z_\tau,\bar{Z}_\tau)(\varphi^k(z)) = 0(\lambda^{2k})$$

which converges to zero as $k \to \infty$. From (3.3) we conclude that $S_\tau(Z_\tau, \bar{Z}_\sigma) \equiv 0$ or equivalently that Z_τ is holomorphic, hence D is biholomorphic to a circular domain. In fact the exponential map Φ from the indicatrix is biholomorphic ([4]).

Q.E.D.

The following corollary is an immediate consequence of the above theorem and Theorem 3 and 9.4 of [3]:

Corollary. Let D be as in Theorem 3, then the following statements are equivalent:

(a) D is biholomorphic to the unit ball;

(b) part (b) of Theorem 3 holds for two distinct points in D ;

(c) part (c) of Theorem 3 holds for two distinct points in D .

Theorem 3 and its corollary, suitably rested, hold in the general setting of manifolds of circular type. In fact, the arguments used in the proofs depend only on the properties of the Monge-Ampère exhaustion and therefore they work whenever such exhaustions exist.

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