Symmetric periodic orbits for the n-body problem: some preliminary results

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Abstract

We show the existence of some infinite families of periodic solutions of the planar Newtonian n-body problem –with positive masses– which are symmetric with respect to suitable actions of finite groups. The method is by minimizing a discretization of the action functional under symmetry constraints.

1 Introduction

Exact periodic solutions of the Newtonian n-body problem have long been a topic of interest, and not only among the specialists of celestial mechanics. The recently found "eight" choreography of Montgomery and Chenciner [8] has been the starting point of the numerical discovery by Simó and others of several interesting periodic orbits of the same kind, which are symmetric under the action of a finite group [9, 7]. In this note we simply define suitable actions of some finite groups on the configuration space and infer the existence of periodic orbits by the well-known variational approach. For details and further reading the reader is referred to [1, 4, 12, 16, 13, 8, 9, 19, 3, 2]. The most recent advances in this theory will appear in Chenciner's papers for the ICM 2002 (see also [7]). We need to consider the assumption that the local minima of the Lagrangian action for the corresponding Bolza problem is collision-free. This has been proved to be true by Marshall (in a preprint) in case the symmetry group is cyclic and the potential is Newtonian; it deserves some effort to be proved in general. But in any case we can circumvent this problem by a strong–force perturbation of the potential. The paper is roughly structured as follows: we will prove rigorously (even if omitting full details, that can be find in the cited papers) some needed results, and then use numerical simulations to verify the collision assumption when needed and to compute the orbits. We obtain some non–trivial periodic orbits in the following cases.

(a) 3-body with all equal masses: the Montgomery–Chenciner choreography (section 7.1). *(b)* 3-body with two equal masses: infinitely many periodic orbits (section 7.2 and figure 3). Among these, there are the orbits with all the three masses equal (figure 2). *(c)* 4-body with all equal masses: the method only yields some numerical examples (section 8.1), like the one in figure 4. *(d)* 4-body with two pairs of equal masses: an infinite family of solutions (section 8.3 and figure 5). *(e)* 4-body with all equal masses and an additional symmetry: an infinite family of periodic orbits (section 9.1, figures 6, 7). *(f)* 4-body with two pairs of equal masses and an additional symmetry: an infinite family (section 9.2, figure 8). *(g)* Plane choreographies ("eight" shape) with n odd equal masses (section 10, figures 9, 10).

Many of these orbits are well-known (but in general only numerically), other might not. This is an attempt —most far from being ultimate— to give a general method for proving the existence of such periodic orbits with variational methods combined with a computer–assisted proof. More general results (not confined to dihedral groups and without the collision assumption) will come in subsequent papers. Most of the ideas are borrowed from Susanna Terracini, who helped me to understand the problem and willingly shared her knowledge during several discussions on the topic. Sincere thanks are also due to Andrea Venturelli, who generously gave his much appreciated comments.

2 Preliminaries

Let $n \geq 2$ be the number of bodies in the Euclidean space $V = E^k \approx \mathbb{R}^k$ of dimension $k \geq 2$. Here we consider the n-body problem with masses m_i, $i = 1, \ldots, n$. Following the notation of [7], let $\mathcal{X}$ denote the subspace of V^n of all points $x = (x_1, x_2, \ldots, x_n)$ such that $\sum_{i=1}^n m_i x_i = 0$, i.e., with center of mass in the origin O. Let $\Delta_{i,j}$ denote the collision set in $\mathcal{X}$ of the i-th and the j-th particles, and $\Delta = \cup_{i<j} \Delta_{i,j}$ the collision set in $\mathcal{X}$. Let $U \colon \mathcal{X} \to \mathbb{R}$ be the potential function

$$U(x) = \sum_{i<j} \frac{1}{|x_i - x_j|}.$$

We can consider also a deformation U_{sf} of U in a small neighborhood of the collision set, so that U satisfies the *strong force* condition. The kinetic energy, defined on the tangent bundle of $\mathcal{X}$, is $K = \sum_{i=1}^n \frac{1}{2} m_i |\dot{x}_i|^2$, and the Lagrangian is $L = K + U$. Moreover $I = \sum_i m_i |x_i|^2$ is the moment of inertia with respect to the center of mass. Let $T^1 \subset \mathbb{R}^2$ denote the unit circle and $\Lambda = H^1(T^1, \mathcal{X})$ the Sobolev space of the L^2 loops $T^1 \to \mathcal{X}$ with L^2 derivative. Then, the critical points of the positive-defined action functional $\mathcal{A} \colon \Lambda \to \mathbb{R} \cup \{\infty\}$ are periodic orbits of the Newtonian n-body problem; the action is defined by

$$\mathcal{A}(x) = \int_{T^1} L(x(t), \dot{x}(t)) dt \tag{2.1}$$

for every loop $x = x(t) \in \Lambda$. The action functional is called *coercive* if it goes to infinity as the moment of inertia I goes to infinity. (i.e., if the H^1-norm of x goes to infinity).

Let G be a finite group, acting on a space X. The space X is then called G-*equivariant* space. We recall some standard notation. If $H \subset G$ is a subgroup of G, then $G_x = \{g \in G : gx = x\}$ is termed the *isotropy* of x, or the *fixer* of x in G. The space $X^H \subset X$ consists of all the points $x \in X$ which are fixed by H, that is $X^H = \{x \in X : G_x \supset H\}$. Given two G-equivariant spaces X and Y, an *equivariant map* $f \colon X \to Y$ is a map with the property that $f(g \cdot x) = g \cdot f(x)$ for every $g \in G$ and every $x \in X$. An equivariant map induces, by restriction to the spaces X^H fixed by subgroups $H \subset G$, maps $f^H \colon X^H \to Y^H$.

3 Symmetry constraints

We give here an introduction to symmetry constraints which is slightly different than the well-known in the literature. Let G be a finite group, τ an orthogonal representation of G on T^1 and ρ an orthogonal representation on the Euclidean space V. Furthermore, let σ be a group homomorphism $\sigma \colon G \to \Sigma_n$ from G to the symmetric group on n elements. Therefore we let G act on the space V, on the time T^1 and on the set of indexes of the masses

$\mathbf{n} = \{1, 2, \ldots, n\}$. Let $\mathbb{R}[\mathbf{n}]$ denote the n-dimensional real vector space generated by the n elements of $\mathbf{n}$, and $\mathbb{R}_0[\mathbf{n}]$ the linear subspace of elements with coordinates $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that $\sum_{i=1}^{n} m_i \lambda_i = 0$. Given $\sigma \colon G \to \Sigma_n$, G acts on $\mathbb{R}_0[\mathbf{n}]$ by the action on $\mathbf{n}$, provided that $\sigma(g)(i) = j$ implies $m_i = m_j$ for every $g \in G$ and every $i = 1, \ldots, n$. Two homomorphisms σ and σ' yield the same real representation on $\mathbb{R}_0[\mathbf{n}]$ if they are conjugated by an element of Σ_n, that is if one is obtained by permuting coordinates on the other. With an abuse of notation we call σ the representation induced by the homomorphism σ. It is equivalent to the representation given by $[N] - [1]$ (in the representation ring of Σ_n), where N is the natural representation and 1 the trivial representation. Now, since

$$\mathcal{X} \cong V \otimes_{\mathbb{R}} \mathbb{R}_0[\mathbf{n}],$$

given ρ and σ we have an orthogonal action of G on $\mathcal{X}$.

Furthermore, since G acts on T^1 and on $\mathcal{X}$, then there is the standard diagonal action on the loop space Λ, defined by $x(t) \mapsto gx(g^{-1}t)$. Let us note that the loops in Λ fixed by G are the equivariant maps $x \colon T^1 \to \mathcal{X}$. In this terminology, a *symmetry constraint* is a such action of G on Λ. Since the action functional is invariant with respect to the G-action, we have a restricted action

$$\mathcal{A}^G \colon \Lambda^G \subset \Lambda \to \mathbb{R}, \tag{3.1}$$

and the following proposition (Palais principle of symmetric criticality – see [7]).

(3.2) Lemma. *A critical point of $\mathcal{A}^G$ in Λ^G is a critical point of $\mathcal{A}$ on Λ.*

Now, the problem arises about which representations yield symmetry constraints that are sufficient to imply the existence of nontrivial (i.e., non-homographic) periodic solutions in the equivariant loops class. As shown by Chenciner in [7], we have to consider the problem of collisions, coercivity and non-triviality. We start by trying to see which conditions on τ, ρ and σ might give the desired properties.

Let $\ker \tau$, $\ker \rho$ and $\ker \sigma$ be the kernels of the corresponding homomorphisms $\rho \colon G \to O(k)$, $\tau \colon G \to O(2)$ and $\sigma \colon G \to \Sigma_n$. Without loss of generality we can assume that $\ker \tau \cap \ker \rho \cap \ker \sigma = 1$. Moreover, assume that $g \in \ker \tau \cap \ker \rho$: this implies that $g \notin \ker \sigma$. Then, if $x(t)$ is an equivariant loop, then the restriction map $x(t)^g \colon T^1 = T^{1g} \to \mathcal{X}^g$ sends every point of T^1 to $\mathcal{X}^g$; since $g \in \ker \rho$ but $g \notin \ker \sigma$, the space $\mathcal{X}^g$ consists entirely of collisions. Therefore we must have $\ker \tau \cap \ker \rho = 1$ in order to avoid necessary collisions.

Furthermore, assume that $g \in \ker \tau \cap \ker \sigma$, and thus $g \notin \ker \rho$. Again, every configuration in the orbit $x(t)^g = x(t)$ needs to belong to $\mathcal{X}^g$, which is nothing but the subspace of configurations of points in V^g, which is a linear proper subspace of V. Thus we can consider it as a sub-problem, and assume that $\ker \tau \cap \ker \sigma = 1$ as well.

Finally, consider $g \in \ker \rho \cap \ker \sigma$. Its action on T^1 can be a rotation or a reflection. In case it is a rotation, we are considering n-bodies that actually tread the same loop more than once, and clearly the problem can be solved by solving the problem concerning loops with just one iteration. So, we can assume that every element of $\ker \rho \cap \ker \sigma$ acts as a reflection on T^1. But if there are two distinct such elements $g_1 \neq g_2 \in \ker \rho \cap \ker \sigma$, then their product $g_1 g_2$ would act as a rotation on T^1, hence $g_1 g_2$ must be trivial, i.e., $g_1 = g_2^{-1} = g_2$. This implies that $\ker \rho \cap \ker \sigma$ has at most one non-trivial element, that is it is a subgroup of order at most 2. If $\ker \rho \cap \ker \sigma \neq 1$, then every loop in Λ^G can be decomposed as $\gamma\gamma^{-1}$, i.e., it is a loop that runs along a path γ from $g(0)$ to $\gamma(1)$ in 1/2 of the time, and then from $\gamma(1)$ to $\gamma(0)$ in the second half of the time interval.

(3.3) Remark. The equivariant loops $x \in \Lambda^G$ can be seen as $G/\ker\tau$-equivariant loops $T^1 \to \mathcal{X}^{\ker\tau}$. Thus we can consider the same problem related to a finite subgroup of $O(2)$ (thus a subgroup of a dihedral group) and a linear subspace $\mathcal{X}^{\ker\tau} \subset \mathcal{X}$. Moreover, if $\mathcal{X}^{\ker\tau} \subset \Delta$ then all the loops are just made of collision points. Therefore we assume that $\mathcal{X}^{\ker\tau} \not\subset \Delta$.

If for a choice of σ, ρ and τ one of the following cases occurs, we say that the action of G is *degenerate*. *(a)* $\ker\tau \cap \ker\sigma \cap \ker\rho \neq 1$. *(b)* Every loop in Λ^G has collisions. *(c)* There is a proper linear subspace S of E^k such that for every $t \in T^1$ and for every $x \in \Lambda^G$, the body $x_i(t)$ belongs to S. *(d)* For every loop $x(t)$ in Λ^G there is a loop $y(t)$ and $k \in \mathbb{Z}$, $k \neq 0, \pm 1$, such that for every $t \in T^1$ we have $x(t) = y(kt)$.

(3.4) Lemma. *If the action of G is non-degenerate, then G is a finite subgroup of $O(2) \times O(k)$ and a finite subgroup of $O(2) \times \Sigma_n$.*

Proof. Since $\ker\tau \cap \ker\rho = 1$, the homomorphism $\tau \times \rho \colon G \to O(2) \times O(k)$ has trivial kernel. The same happens to the homomorphism $\tau \times \sigma$. *q.e.d.*

For example, in case $k = 2$, this implies that G is a subgroup of the direct product of two dihedral groups, and hence metabelian. For $k = 3$, G is a subgroup of the direct product of a dihedral group and a finite subgroup of $O(3)$, hence either G is metabelian or it is an extension of a finite metabelian group with with a finite subgroup of $O(3)$. Hence the only nonsolvable group occurs if G projects onto the icosahedral group A_5 in $O(3)$.

4 Coercivity

(4.1) Lemma. *The symmetric action functional Λ^G is coercive if and only if $\mathcal{X}^G = 0$.*

Proof. Consider the group $G/\ker\tau$. It is a finite subgroup of $O(2)$, hence it is either a cyclic group or a dihedral group. Let us consider first the cyclic case. Let c be a generator of G/τ. Let $X = \mathcal{X}^{\ker\tau}$. Since $\mathcal{X}^G = X^c$, we have $X^c = 0$. Therefore X can be decomposed into irreducible components $X = \mathbb{R} + \mathbb{R} + \cdots + \mathbb{R} + \mathbb{C} + \cdots + \mathbb{C}$, where on the one-dimensional components $\mathbb{R}$ the action of c is given by $c(s) = -s$, while on the two-dimensional components we have $c(z) = e^{2\pi i/l} z$ for a suitable $l \in \mathbb{Z}$. Thus, using the same argument as in Bessi and Coti-Zelati [5], it is possible to show that Λ^G is coercive. Now consider the case G/τ is dihedral. Let h_1 and h_2 be two generators of order 2 of G/τ. Again, X can be decomposed as $X = \mathbb{R}+\mathbb{R}+\cdots+\mathbb{R}+\mathbb{C}\cdots+\mathbb{C}$, where on the one-dimensional irreducible components the action is either $r_1(s) = -s = r_2(s)$ or $r_1 = -r_2(s) = \pm s$, while on the two-dimensional irreducible components $\mathbb{C}$ is a dihedral representation. Thus, again exactly with same argument as [5] it can be shown that there is $\alpha > 0$ such that $|x|_{L^2} \leq \alpha |\dot{x}|_{L^2}$, i.e., that the action functional is coercive.

For the converse, if $\mathcal{X}^G \neq 0$ let x_0 denote a loop in Λ^G (possibly with collision) with finite action $\mathcal{A}^G$. Then $x_0 + v$, with $v \in \mathcal{X}^G$, is again a loop in Λ^G, with action $\mathcal{A}^G(x_0+v) < \mathcal{A}^G(x_0)$. But as $|v| \to \infty$ also $x_0 + v$ goes to infinity, hence $\mathcal{A}^G$ is not coercive. *q.e.d.*

5 Dihedral orbits

Consider the time circle $T^1 \subset \mathbb{R}^2$ of radius $\frac{T}{2\pi}$, where T is the period of a periodic orbit. Let h_1 and h_2 be two reflections in $\mathbb{R}^2$ that fix two lines forming an angle of $\frac{\pi}{l}$, with $l > 1$. Then the group G generated by h_1 and h_2 is the *dihedral group* D_l of order $2l$. Consider as above a k-dimensional orthogonal representation of G and an homomorphism $\sigma\colon G \to \Sigma_n$ to the symmetric group of order $n!$. This means that h_1 and h_2 act on $V = E^k$ (with a symmetry of order 2 along a plane, a line or the origin) and on the set $\{1, 2, \dots, n\}$ of indexes via the homomorphism σ. The elements $\sigma(h_i)$ need to be of order 2 in Σ_n, whenever they are not trivial. Given these data, G acts on $\mathcal{X}$ by

$$g(x_1, \dots, x_n) = (gx_{\sigma(1)}, \dots, gx_{\sigma(n)}),$$

where we mean $\sigma(i) = \sigma(g)(i)$ and on T^1. Then G acts on the loop space Λ by

$$g \cdot \gamma\colon t \to g\gamma(g^{-1}t)$$

for every $t \in T^1$ and every $\gamma \in \Lambda$. The space Λ^G consists of the equivariant loops. It is easy to see that Λ^G is homeomorphic to the space P of all the paths $\lambda\colon [0,1] \to \mathcal{X}$ with the property that $\lambda(0) \in \mathcal{X}^{h_1}$ and $\lambda(1) \in \mathcal{X}^{h_2}$. The homeomorphism is given by the restriction function $r\colon \Lambda^G \to P$. The action functional can be defined in exactly the same way on P, by integrating L along λ with a rescaled time. Let is denote it by $\mathcal{A}_P$. If L is invariant with respect to the action of G, then $2l\mathcal{A}_P r(\gamma) = \mathcal{A}^G(\gamma)$, for every $\gamma \in \Lambda^G$. Hence γ is a stationary point for $\mathcal{A}^G$ if and only if its restriction $r(\gamma)$ is stationary for $\mathcal{A}_P$. We can hence consider critical points and local minima of $\mathcal{A}_P$ in P. This is a sort of *generalized Bolza problem.*

(5.1) Lemma. *Any critical point of $\mathcal{A}_P$ in P yields a critical point of $\mathcal{A}^G$ in $\Lambda^G \subset \Lambda$, which is a critical point of Λ.*

6 Minimizing on constrained paths

More generally, assume that h_1 and h_2 are two elements of order 2 acting isometrically on E^k and $\{1, 2, \dots, n\}$. Let X_1 and X_2 be two the fixed subspaces $\mathcal{X}^{h_1}$ and $\mathcal{X}^{h_2}$ of $\mathcal{X}$ and let P denote the Sobolev space of all the paths $\gamma\colon [0,1] \to \mathcal{X}$ such that $\gamma(0) \in X_1$ and $\gamma(1) \in X_2$. That is, $P = H^1(([0,1], 0, 1), (\mathcal{X}, X_1, X_2))$ is the Sobolev space of the L^2 paths $[0,1] \to \mathcal{X}$ with L^2 derivative with the constraints at the endpoints of the interval $[0,1]$. Let $\mathcal{A}_P$ be defined on P as above.

(6.1) Lemma. *Any local minimum of $\mathcal{A}_P$ can be extended to a solution (in the weak sense) $x\colon \mathbb{R} \to \mathcal{X}$ which is periodic in a rotating frame.*

Now we have to face the problem of the possible existence of collisions in (local) minima of the action functional.

(6.2) Collision Assumption. *If G is a finite group and $\Lambda^G \subset \Lambda$ is defined as in (3.1), then all local minima of the action $\mathcal{A}^G$ in Λ^G are collision-free.*

We cannot term (6.2) a lemma, since it has not yet been proved in full generality. However, there is a certain evidence that it holds for general actions. In fact, if the group G is cyclic and acts in the standard way on T^1 (that is, yielding choreographies), then it was proved recently by Marchall (in some unpublished notes). The proof can be extended without significant change to some other group actions, but will not work in full generality. On the other hand Majer–Terracini methods on collisions singularities [17, 12, 16, 13, 15, 11, 18, 10] can be extended to the equivariant case, if n is 3 or 4 and under some further assumptions. But there are still some gaps in the proofs, so that we hope to provide a complete proof in the future. For the purpose of this note, it suffices either to consider a strong–force perturbation U_{sf} or to consider the numerical hint that the algorithm stops at a non-collision loop and determines it as a global minimum.

7 Three bodies in the plane

Now we can start to investigate which symmetry constraints yield non-trivial periodic solutions. We start with the case of 3-bodies. Recall that G now is a dihedral group with standard generators h_1 and h_2.

7.1 Three equal masses

We can assume that $m_i = 1$ for $i = 1, 2, 3$. If G acts on $\{1, 2, 3\}$ without fixed points, then it must be $\sigma(h_1) = (12)$ and $\sigma(h_2) = (23)$, up to an inner permutation of the indexes. So that the symmetric group Σ_3 is a homomorphic image of G. To determine G and its action on E^2 we consider now the cases for h_1 and h_2.

First case: both are rotations (of angle π) on E^2. Then the minimal G with this property is the dihedral group $D_3 \cong \Sigma_3$ of order 6. The space $\mathcal{X}^{h_1}$ is the space of all the configurations with $x_3 = 0$ and $x_1 = -x_2$, while $\mathcal{X}^{h_2}$ is given by all the configurations with $x_1 = 0$ and $x_2 = -x_3$. It is clear that $\mathcal{X}^{h_1} \cap \mathcal{X}_{h_2} = 0$, hence by (4.1) the minimum $x = x(t)$ in Λ^G exists and is collision-free by (6.2). Since the product $h_1 h_2$ is a rotation of $T/3$ in the time circle and acts trivially on E^2, we have that x is a choreography. It cannot be an Euler or Lagrange solution, hence it is a non-trivial choreography. It is possible to show that it has an "eight" shape. There is the natural question, whether it is the same as the Montgomery-Chenciner orbit or not.

Second case: h_1 acts on E^2 by reflection along a line and h_2 by rotation of angle π. Since the product $h_1 h_2$ acts as a reflection in E^2 and as the cyclic permutation (123) in Σ_n, G needs to be the dihedral group D_6 of order 12. The configurations in $\mathcal{X}^{h_1}$ are those such that (x_1, x_2, x_3) is a triangle symmetric with respect to the line fixed by h_1 and the configurations in $\mathcal{X}^{h_2}$ are those such that again $x_1 = 0$ and $x_2 = -x_3$. This is the action described in [8], and the corresponding solution is the figure eight choreography.

Third case: both h_1 and h_2 act on E^2 by a reflection along a line (l_1 and l_2 respectively). If $l_1 = l_2$, then the product $h_1 h_2$ acts trivially on E^2, hence the minimal G is the dihedral group D_6. The minimum exists and numerical experiments let one guess that it is the Lagrange orbit. Otherwise, l_1 and l_2 intersect with an angle π/q, with $q > 1$ integer. The minimal G is therefore D_q if q is $0 \mod 3$ and D_{3q} otherwise. Since the Lagrange orbit belongs to Λ^G it can be the minimum. We did not check whether it is a minimum for every q or not.

7.2 Two equal masses

Now assume that the first two masses are equal ($m_1 = m_2$). We again list the possible cases. Without loss of generality we can assume $\sigma(h_2) = (12)$, since at least one of h_1 and h_2 needs to act non-trivially on the indexes.

First case: $\sigma(h_1) = (12)$. If both h_1 and h_2 act rotating on E^2, then $\mathcal{X}^{h_1} = \mathcal{X}^{h_2} = \mathcal{X}^G$, and the functional is not coercive. If h_1 acts by reflection and h_2 by rotation, then again the functional is not coercive, and the same is true if they act by reflecting along the same line.

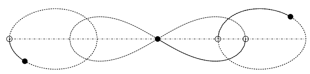

Figure 1: Two equal masses

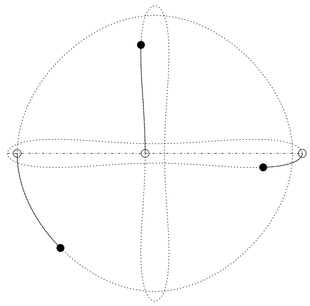

Figure 2: $m_1 = m_2 = m_3 = 1$

So it is left to check the case in which they act on E^2 by reflection along two distinct lines. In this case the functional is coercive, and Λ^G contains the Lagrange solution so that it is of no interest.

Second case: $\sigma(h_1) = ()$. Since h_1 does not move the indexes, to avoid collisions it is is necessary that h_1 does not act on E^2 with a rotation. It cannot have a trivial action, since otherwise $\mathcal{X}^{h_1} = \mathcal{X}$ and the orbit would not be dihedral, so that it will be a reflection. Now, it is left to determine the action of h_2. If h_2 is a rotation of angle π, then the action functional is not coercive.

(7.1) Remark. We can restrict the space of paths considering only paths x with a prescribed order of the configuration $x(0)$. If we look at the configurations such that $x_3(0)$ does not lie between x_1 and x_2, then $\mathcal{A}$ is coercive. In a strong-force settings a collision-free minimum need to exist. Numerical experiments show that such minima might exist even for a potential of type $1/r^a$, with $a \geq 1.3$ (see figure 1). 1. This is the solution of braid $b_1^2 b_2^{-2}$ of Moore [14], pag. 3677.

Otherwise, h_2 is a reflection along a line. If this line coincides with the line fixed by h_1, then again the functional is not coercive, since there are orbits in which x_1 and x_2 rotate in a circle very far from x_3. So we can assume that the two lines intersect with an angle $0 < \alpha \leq \pi/2$. If the angle α is equal to $\pi/2$ then again $\mathcal{A}$ is not coercive, so we assume $0 < \alpha < \pi/2$. Now the functional is coercive, and there is a minimum. If q is an integer, then the minimal group G is D_q if q is even and D_{2q} if q is odd. At $t = 2T/q$ the configuration is the same as the configuration at $t = 0$ with the two bodies interchanged and rotated by an angle of $2\pi/q$; at $t = 4T/q$ it is exactly the configuration at $t = 0$ rotated of an angle $4\pi/q$. If $q > 2$ is not an integer, then one obtains an orbit periodic with respect to a rotating frame. Unfortunately the Euler orbits belong to this class, so that the minimum can be achieved on a Euler solution. Some numerical simulations give a hint that this is not the case: orbits like the one in figure 2 can be found with constrained optimization techniques, with an action less than the action of the corresponding Euler orbit.

To prove the existence of such orbits, provided that because of (6.2) there are no collisions in a minimizing orbit, it suffices to use the following level estimates. We are going to compare the levels of the action of suitable symmetric orbits with the action of the Euler orbit. Let

$m_1 = m_2 = 1$, $m_3 = m > 0$ be the masses. Let c, r_0 and l be three positive constants (to be determined), and consider the path in P determined by the equations

$$
\begin{aligned}
x_1(t) &= le^{i\theta t} + (r_0 + ct)e^{i(\theta-\frac{\pi}{2})t} \\
x_2(t) &= le^{i\theta t} - (r_0 + ct)e^{i(\theta-\frac{\pi}{2})t} \\
x_3(t) &= -\frac{2l}{m}e^{i\theta t}.
\end{aligned}
\tag{7.2}
$$

The kinetic contribution of (1) and (2) to the action $\mathcal{A}(x)$ is

$$
K_1 + K_2 = 1/12\, c^2\pi^2 + 1/4\, r_0{}^2\pi^2 + r_0\, \theta^2 c - c r_0\, \theta\, \pi + c^2 + 1/3\, c^2\theta^2 - r_0{}^2\theta\, \pi + \\ + l^2\theta^2 - 1/3\, c^2\theta\, \pi + r_0{}^2\theta^2 + 1/4\, r_0\, \pi^2 c.
$$

The kinetic term coming from (3) is simply

$$
K_3 = 2\frac{l^2\theta}{m}.
$$

Now consider the terms corresponding to the potential. The term corresponding to the interaction between (2) and (3) is equal to

$$
U_3 = \frac{1}{2c}\log(1 + \frac{c}{r_0}).
$$

Now, the term of the interaction between (1) and (3) is bounded by

$$
U_2 \leq m\left((r_0 + c)^2 + l^2(1 + 2/m)^2\right)^{-1/2},
$$

and a similar inequality holds for the term (2) – (3):

$$
U_1 \leq \frac{m^2}{m(l - r_0) + 2l}.
$$

Let $\mathcal{A}_D = K_1 + K_2 + K_3 + U_1 + U_2 + U_3$ denote the Lagrangian action of the path (7.2). The action of the Euler solution with the body x_3 in the center of mass is

$$
\mathcal{A}_E = \frac{3}{2}\left[(1/2 + 2m)^2(\pi/2 - \theta)\right]^{1/3}.
$$

With some computations it is possible to simplify the difference as

$$
\begin{aligned}
\mathcal{A}_D - \mathcal{A}_E = {} & l^2\theta^2(1 + 2/m) + c^2 + \frac{1}{12}(3r_0^2 + c^2 + 3cr_0)(\pi - 2\theta)^2 + \frac{m^2}{m(l - r_0) + 2l} + \\
& + \frac{m}{\sqrt{(r_0 + c)^2 + l^2(1 + 2/m)^2}} + \frac{1}{2c}\log(1 + \frac{c}{r_0}) - \frac{3}{2}\left[(1/2 + 2m)^2(\pi/2 - \theta)\right]^{1/3}.
\end{aligned}
\tag{7.3}
$$

Now, let $D \subset \mathbb{R}^2$ the domain of all the pairs (m, θ) such that

$$
\inf\{\mathcal{A}_D - \mathcal{A}_E : l > r_0 > 0, l > c > 0\} < 0. \tag{7.4}
$$

The following proposition is a trivial consequence of the definition of D.

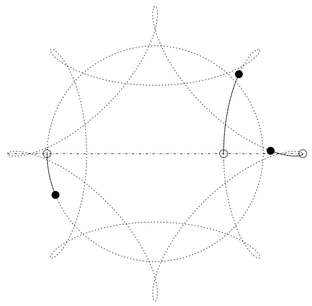

Figure 3: $m_1 = m_2 = 1$, $m = m_3 = 2$ and $\theta = \pi/8$.

(7.5) Proposition. *If $(m, \theta) \in D$ then there are l,r_0 and d such that the action of the orbit (7.2) is less than the action of the Euler orbit. Therefore non-homographic dihedral orbits exists for every $(m, \theta) \in D$.*

(7.6) Proposition. *The set D is a non-empty open subset of $\mathbb{R}^2$.*

Proof. Since inf is upper semi continuous, D is open. We only need to show that it is not empty: let $m = 2$, $\theta = \pi/8$, $l = 1$, $r_0 = 0.4$, $c = 0.3$. Evaluating for such values an approximation of (7.3) yields $-.124390105 < 0$. A candidate for the corresponding minimum can be seen in figure 3. *q.e.d.*

(7.7) Remark. Of course propositions (7.5) and (7.6) do not imply that the orbit in figure 2 exists. The minimization in that case has been done with a piecewise linear path, which at the moment cannot be reproduced symbolically, whose action is less than $\mathcal{A}_E$. The orbit in figure 2 is the same as the orbit with braid $b_1^2 b_2 b_1^{-2} b_2$, found numerically by Moore in [14].

8 Four bodies in the plane

Now we analyze in the same way the situation of 4 bodies in the plane. Let again h_1 and h_2 be the reflections in T^1 generating G.

8.1 Four equal masses

Assume that $m_i = 1$, $i = 1, \ldots, 4$. We want that $\sigma(h_1)$ and $\sigma(h_2)$ generate a subgroup of Σ_4 that acts transitively on the indexes $\{1, 2, 3, 4\}$.

First case: $\sigma(h_1) = (12)$. Then the only transitive subgroup up to inner automorphism of Σ_4 is given by the choice $\sigma(h_2) = (13)(24)$. The action of h_1 on the plane E^2 cannot be a rotation (since fixes the indexes 3 and 4). Since it cannot be trivial (otherwise at $t = 0$ a collision is not avoidable) it needs to be a reflection along a line l_1. Now, if h_2 acts by reflection along a line l_2, there are the following sub-cases. If $l_2 = l_1$, then the homographic solution of a rotating square can be a minimum (actually, apparently it is the minimum), so we do not consider this case. If l_2 and l_1 meet at an angle $\pi/4$, then the action is not coercive (there is a big square with stationary masses). On the other hand even if the angle is different from $\pi/4$ then the homographic solution of the rotating square can be a minimum, hence we do not consider this case too (even if it might be possible that the minimum is achieved by a non-homographic orbit).

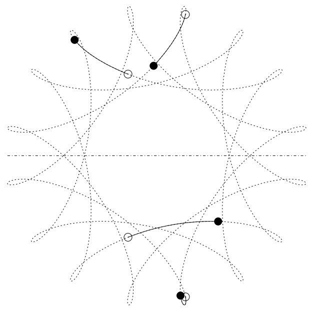

Figure 4: Four equal masses with l_1/l_2-angle $\pi/8$.

So, it is only left the case in which h_2 acts by rotation of angle π on E^2. Since $\mathcal{X}^{h_1} \cap \mathcal{X}^{h_2} \neq 0$, by lemma (4.1) the action is not coercive.

Now consider the second case $\sigma(h_1) = (12)(34)$. Then the only choice of $\sigma(h_2)$ that has has not yet been considered for a transitive action is $\sigma(h_2) = (13)(24)$. This time h_1 can act on E^2 either as a reflection or as a rotation, and the same holds for h_2. If one is a reflection and one is a rotation, then the action is not coercive, since a sequence of increasing stationary squares can have as small as possible action. If both act as rotations, then the group G is equal to the group generated by $\sigma(h_1)$ and $\sigma(h_2)$, i.e., the elementary abelian group $\mathbb{Z}_2^2$ of order 4. The resulting symmetric orbit can be two coupled Kepler orbits, hence there is no coercivity. So it is left the case in which both are reflections along lines l_1 and l_2 in E^2. If the lines $l_1 = l_2$ coincide, then again it is possible to see that the functional not is coercive, by taking two symmetric Keplerian orbits that have increasing distance (i.e., by applying lemma (4.1)).

Otherwise, if they are orthogonal, then again it is easy to see that the functional is not coercive. If they are not orthogonal then the functional is coercive, but in the class of symmetric paths there are the homographic orbits of rotating squares. Of course the question arises whether the homographic orbits achieve the minimum or not. Numerical simulations lead to think that the minimum might be achieved by non-homographic orbits, like the one depicted in figure 4. It could be of some interest to prove some estimates like those in the previous section, to actually prove or disprove their existence. This is true also for other examples listed below, and we will not rise the question again.

8.2 Three equal masses

Assume now that there are three equal masses $m_1 = m_2 = m_3 = 1$ and a fourth mass $m_4 = m$. Then the subgroup of Σ_4 generated by $\sigma(h_1)$ and $\sigma(h_2)$ needs to act transitively only on the set $\{1, 2, 3\}$. The only possibility, up to rearranging the indexes, is $\sigma(h_1) = (12)$ and $\sigma(h_2) = (13)$, like in the case of 3 bodies. Since both fix two indexes, the actions of h_1 and h_2 on E^2 need to be reflections along the lines l_1 and l_2 respectively. If the lines coincide, then a rotating triangle with the mass (4) in the center can be the minimum. If the angle is $\pi/3$, then the problem is no longer coercive, since any constant equilateral triangle with bodies (1), (2) and (3) with (4) in the center is symmetric with respect to this action. On the other hand, such a triangle when rotating at a suitable speed always belongs to the set of symmetric loops Λ^G, however the two lines intersect. We do not know whether it is a minimum in Λ^G.

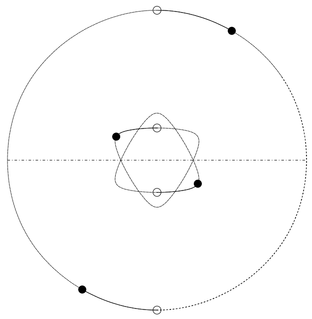

Figure 5: $m_1 = m_2 = 1$ and $m_3 = m_4 = 4$, with D_3-symmetry.

8.3 Two pairs of equal masses

Assume that $m_1 = m_2 = 1$ and $m_3 = m_4 = m$. Since at least one from $\sigma(h_1)$ and $\sigma(h_2)$ is non-trivial, we can assume that $\sigma(h_1) = (12)$ or $\sigma(h_1) = (12)(34)$.

In the first case, $\sigma(h_1) = (12)$, necessarily it must be $\sigma(h_2) = (34)$. The only possible action of h_1 and h_2 on E^2 is given by reflections along lines l_1 and l_2. If the lines coincide, then the functional is not coercive by (4.1). It is not coercive also if they are orthogonal: a square in increasing size can give a sequence going to infinity with bounded action. If the lines meet with an angle π/q, then it is coercive, but a rotating central configuration with the masses at the vertexes of a parallelogram belongs to Λ^G, and hence it can be the homographic minimum. Again, as above, the question arises whether the minimum is homographic or not.

Consider now the second case, $\sigma(h_1) = (12)(34)$. There are three possibilities for $\sigma(h_2)$: up to rearranging indexes, the trivial (), or (12) or (12)(34). Consider $\sigma(h_2) = ()$. Then h_2 must act on E^2 by reflection along a line l_2. However h_1 acts on E^2, as a rotation or as a reflection, it is possible to find a rotating collinear central configuration belonging to Λ^G. So we consider the next case, $\sigma(h_2) = (12)$. Again h_2 needs to act as a reflection along a line l_2, and rotating collinear configurations now cannot belong to Λ^G. If h_1 acts by rotation, then a rotating parallelogram belongs to Λ^G, hence we consider only the case of h_1 acting by reflection along a line l_1. If the two lines coincide or are orthogonal, then the action functional is not coercive. Otherwise it is coercive, and hence there is a minimum, which is collisionless due to theorem (6.2). Can it be homographic? No: at the time $t = 0$ (i.e., the time in T^1 fixed by h_1) the lines through (1)–(2) and (3)–(4) are parallel (both orthogonal to l_1), while at time 1 (i.e., the time in T^1 fixed by h_2) they are orthogonal. Such orbits can be described as follows: two masses in a roughly Keplerian orbit outside, and two masses in a retrograde approximate Keplerian orbit inside, like in figure 5.

Now it is left the case $\sigma(h_1) = \sigma(h_2) = (12)(34)$. Since we are assuming the action of G on T^1 to be faithfully dihedral, h_1 and h_2 must act on E^2 in different ways, so that at least one of them acts as a reflection. Let us assume that h_1 acts by reflecting along a line l_1. If h_2 acts by rotation, then the functional is not coercive; if it acts as a reflection along a line $l_2 \neq l_1$, then homographic orbits belong to Λ^G, so that this case is of minor interest.

9 Orbits with an additional central symmetry

Consider the symmetries in the previous sections. If we can find an element σ_3 in Σ_n of order 2 that fixes at most one index and commutes with $\sigma(h_1)$ and $\sigma(h_2)$, we can consider

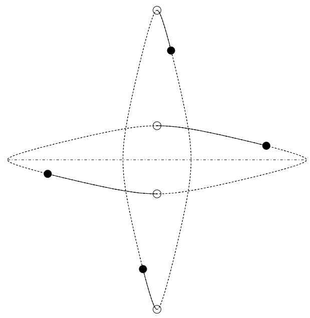

Figure 6: $m_i = 1$, with $(D_4 \times \mathbb{Z}_2)$-symmetry

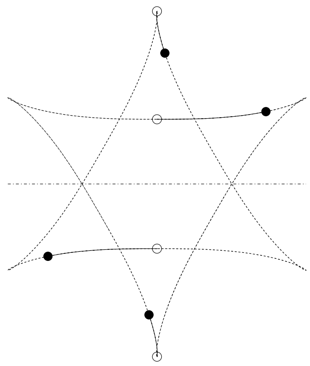

Figure 7: $m_i = 1$ with $(D_6 \times \mathbb{Z}_2)$-symmetry.

the following additional central symmetry: the symmetry group is $G \times \mathbb{Z}_2$ (where G is the group in the example under consideration), where the direct factor $\mathbb{Z}_2$ is generated by h_3; this element acts trivially on T^1, acts as a rotation of angle π in the plane E^2, and is sent to σ_3 by the homomorphism σ. This means that for every $t \in T^1$ the configuration at time t is in $\mathcal{X}^{h_3}$, that is, bodies (with the same mass) in the same cycle in σ_3 are symmetric with respect $0 \in E^2$, while the possible body with index fixed by σ_3 lies in $0 \in E^2$. If $n = 3$, then such orbits are trivial, since they need to be always collinear. So consider the case $n = 4$. We can analyze the cases exploited in section 8 to see when these conditions are fulfilled, and if the additional central symmetry h_3 yields non-homographic orbits. We omit the details of this case-by-case analysis, and exhibit only a particular family of dihedral orbits.

9.1 Four equal masses

In section 8.1 consider the case of $\sigma(h_1) = (12)(34)$ and $\sigma(h_2) = (13)(24)$, where h_1 and h_2 act on E^2 by reflection along different lines l_1 and l_2. If l_1 and l_2 are orthogonal, then the functional is not coercive, even when we add the additional symmetry $\sigma_3 = (12)(34)$.

In case l_1 and l_2 meet at an angle π/q, with $q > 2$, we can avoid the homographic solution again by the same additional symmetry σ_3. The group G acts faithfully on T^1 and is equal to the dihedral group of order $2k$, where k is the least common multiple of 2 and q. Thus we obtain an infinite family of periodic orbits in the 4-body problem with equal masses. We can see the case $q = 4$ in figure 6 and $q = 3$ in figure 7. The orbit of figure 6 is very likely the orbit found by Chen [6].

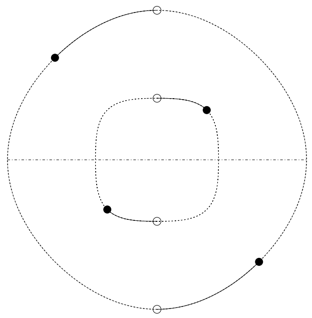

Figure 8: $m_1 = m_2 = 1$, $m_3 = m_4 = 2$, with $(D_4 \times \mathbb{Z}_2)$-symmetry.

9.2 Two pairs of equal masses

The periodic solutions of section 8.3 can be endowed with the additional symmetry given by $\sigma_3 = (12)(34)$. So we consider $\sigma(h_1) = (12)(34)$, $\sigma(h_2) = (12)$, the action of h_1 and h_2 on E^2 is by reflection along two lines l_1 and l_2 that intersect at an angle π/q with $q > 2$. We can see the case $m_1 = m_2 = 1$, $m_3 = m_4 = 2$ and $q = 4$ in figure 8.

10 Some plane choreographies for $n > 3$ bodies

As shown in [7], it is not difficult to generalize the eight-shaped choreography of Montgomery-Chenciner to the case of $n > 3$ odd bodies with equal masses. Consider the following permutations on $\{1, 2, \ldots, n\}$:

$$\begin{aligned} \sigma_1 \colon i \to n - i \quad \bmod n \\ \sigma_2 \colon i \to n - i + 1 \quad \bmod n, \end{aligned} \tag{10.1}$$

where we understand that $0 \equiv n \mod n$. That is, for $n = 3$ we have $\sigma_1 = (12)$, $\sigma_2 = (13)$; for $n = 5$ we have $\sigma_1 = (14)(23)$ and $\sigma_2 = (15)(24)$; for $n = 7$ it is $\sigma_1 = (16)(25)(34)$ and $\sigma_2 = (17)(26)(35)$. The product $\sigma_1\sigma_2$ ($\sigma_2\sigma_1$ in functional notation) sends i to $i + 1 \mod n$, i.e., $\sigma_1\sigma_2$ is the cyclic permutation $(12 \ldots n)$. Thus the subgroup generated by σ_1 and σ_2 is a dihedral group of order $2n$. We can define σ_1 and σ_2 in a geometrical way: consider a regular n-gon with consecutive vertices $(1), (2), \ldots, (n)$. Then σ_1 is the reflection with axis through the vertex (n) and σ_2 the reflection fixing the vertex (i) with $i = (n + 1)/2$. We will of course choose $\sigma(h_i) = \sigma_i$, with $i = 1, 2$, where h_i are the generators of the symmetry group G as above (it is not assumed that the homomorphism σ is a monomorphism; it will depend on the choice of the action of h_1 add h_2 on E^2).

Hence, it is only left to choose the action of h_1 and h_2 on the plane E^2. Again, h_1 and h_2 need to be of order two, hence they can be either rotations of angle π or reflections along lines l_1 or l_2. Let us consider first the case in which h_1 and h_2 are both the rotation of angle π. The group G is therefore the dihedral group of order $2n$. By lemma (4.1) the functional is coercive, hence it attains the minimum; by (6.2), the minimum is collision-free. It is only left to show that this minimum is not homographic. This is easy, since at time $t = 0$ the body (n) is in the origin $0 \in E^2$, while at time $T/(2n)$ the body in the origin is the body (i) with $i = (n + 1)/2$ (figure 9).

If h_1 acts by rotating and h_2 by a reflection, then the same results hold, only this time the group G is the dihedral group of order $4n$. Again, we have a choreography with n equal

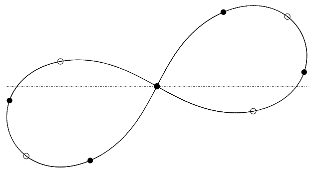

Figure 9: 5-body choreography

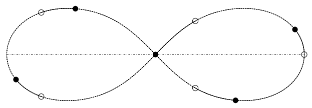

Figure 10: 5-body choreography

masses. It might be interesting to see whether it is the same as the choreography with h_i rotations or not (we have tried some simulations, obtaining something like figure 10, which is very similar to figure 9).

In case h_1 and h_2 both act by reflections, then clearly a rotating regular n-gon belongs to Λ^G (if the functional is coercive), so that the minimum can be homographic and we cannot apply the previous results.

11 Remarks

(11.1) Remark. The methods used in the paper for proving the existence of non-homographic periodic orbits are quite simple, once the collision assumption is proved, and can be extended in a straightforward way to the case of $n > 4$ bodies and non–dihedral groups. A full classification of non-degenerate actions of dihedral groups or abelian extensions of dihedral groups is not difficult, and will be the content of a forthcoming paper.

(11.2) Remark. So far, in this note we have considered explicitly only planar periodic orbits. This is not a serious restriction, since the only change needed to deal with the case E^3 of non-planar periodic orbit is to add a one-dimensional irreducible representation of G to the representation ρ (there are no 3-dimensional irreducible representation of the dihedral group). The known periodic orbits in the space (such as the Chenciner–Venturelli "hip–hop" orbit [9]) obtained by symmetry constraints use a 3-dimensional representation which can be decomposed into an irreducible 2-dimensional representation and a 1-dimensional representation.

In fact, the method is quite simple. Consider a periodic orbit arising from the method given above, with generators h_1 and h_2 of G. Then h_1 and h_2 can act on $E^2 \subset E^3$ by this action, and on the third orthonormal coordinate of E^3 as ± 1. If they act both trivially, then the problem will not be coercive. Otherwise, we can have three choices ($(+,-)$,$(-,+)$ and $(-,-)$) that will yield non-degenerate coercive actions on $\mathcal{X}$. In some cases $(+,-)$ and $(-,+)$ will yield the same periodic orbit, up to a time shift, but in general we will get three periodic orbits in the space. Numerical simulations can be done exactly as in the planar case: we obtained some interesting analogues of the "hip–hop" orbit.

(11.3) Remark. The program used for the simulations is rather simple. We consider a PL discretization of the loops, and so we obtain a finite dimensional space Λ. Then by relaxation dynamics on Λ^G (that is, a very simple gradient method) and a random method for avoiding poor progress (that is, we restart the relaxation process after a random small variation within Λ^G if the progress is not good, until it is apparent that the program is in a local minimum) we obtain an approximation of the minimum. Now we can compare the action functional on such a path (that can be computed with a reasonable precision), and compare with other known values (like homographic solutions). The language used was FORTRAN 95 with double precision arithmetic, and the NETLIB SLATEC library for the ODE solver and error-handling routines. The figures are produced with GNUPLOT run on the raw data files.

(11.4) Remark. This is a very preliminary report. Not only the collision assumption (6.2) is still present, but also the program used is quite bad designed and has a very poor performance. While writing the program I was more concerned about flexibility, robustness and simplicity, than performance or very good approximation of the solutions. Thus the algorithm is very slow and does not give a very good approximation of the orbits. Moreover, I did not compute the linear stability of the orbits, nor I used the more efficient approaches of variational optimization techniques in symmetric periodic problems, available in the literature.

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