ELLIPTIC 3-FOLDS

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NOTE TO THE READER

This is a preliminary version of this paper. We often speak of "when the Leray Spectral sequence degenerates at the E_2-term". We do not know whether this ever occurs, or always. However, it is not a critical part of our arguments, so we hope the reader will excuse this inaccuracy. We are working on this presently.

After this was written, E. Viehweg brought the article "On Weierstraß models" by N. Nakayama to my attention. He has independently proven some of the results of §4. Combining his results with our's, the results of §4 can be strenghthened. We will incorporate these results in the final draft of this paper..

Furthermore, after comparing his results with mine I discovered thatIhave implicitely (without stating it) assumed throughout this paper, that all models are elliptically minimal at all points. This means, when the singular locus is $\Sigma=g_2^3 - 27g_3^2$, we assume $\min(3\nu_s(g_2), 4\nu_s(g_3))<12$ for all $s\in S$. In particular, we assume this to hold throughout §4. Nakayama proved that Theorem 4.7. is still true, without this assumption. The precise statements will be incorperated in the final draft.

Elliptic 3-folds

Introduction

Although elliptic surfaces (i.e. a surface S with a holomorphic map $\pi : S \longrightarrow \Delta$ onto a curve Δ, such that the generic fibre is an elliptic curve) were known to the Italien geometers, it was Kodaira who in a series of papers ((Ko1), (Ko2), (Ko3)) extensively studied them and founded a rigerous theory. His approach was basically to view such an elliptic surface S as a 1-dimensional family of elliptic curves, that is, deformation theory. The importance of elliptic surfaces stems from the fact that *any* compact, complex, analytic surface with Kodaira dimension (or algebraic dimension) equal to one, is an elliptic surface. Also, any algebraic surface with trivial canonical bundle is a *deformation* of an elliptic surface (see (Ko3), Theorems 13 and 18).

There is good reason to believe that elliptic fibre spaces in higher dimensions will also play an important role in classification theory (see §1). Elliptic 3-folds have been studied by a number of authors, in particular, (Kaw), (Ue3), (Mi), (Fu1) and (Fu2). In this paper we continue this work, and ultimately would like to answer some of the questions Kodaira answered so effectively for surfaces.

Chapter I, in spite of its lenghth, is concerned with only one relatively simple problem: find a good model for an elliptic 3-fold. This is absolutely necessary for further work, i.e. calculating invariants, etc. The solution is so difficult because in higher dimensions one has no good theory of minimal models. Thus the discussion of Chapter I is more or less a contribution to the theory of minimal models. In this respect we note the following. In (Fu1), Fujita has shown the existance of a Zariski decomposition on elliptic 3-folds. By general theory (compare {V2} p.140) this is closely related to the question of minimal models. Here we use a more down to earth approach, explicitely showing how to get a (relatively) minimal model.

Let X be a normal, compact, complex space. X is called a (3-dimensional) *elliptic fibre space*, if there is a holomorphic map

$$\pi : X \longrightarrow S$$

onto a smooth, compact, complex analytic surface such that for all $s \in S - \Sigma$, Σ a pure divisor on S, $\pi^{-1}(s)$ is an elliptic curve. Σ is called the *singular locus* of X. If Σ has only normal crossings, then (Corollary 4.2.) X has only canonical singularities. A *family of elliptic fibre spaces* is a morphism

$$\varphi : \mathfrak{B} \longrightarrow T$$

such that each fibre $\mathfrak{B}_t$, $t \in T$, is an elliptic fibre space over a fixed surface S with singular locus $\Sigma_t \subset S$. Our first result is

Theorem 1: *Let $\mathfrak{B} \longrightarrow T$ be a $\mathbb{Q}$-Gorenstein family of elliptic fibre spaces (each $\mathfrak{B}_t$ is $\mathbb{Q}$-Gorenstein). Assume that for $t \neq t_o$, Σ_t has normal crossings. Then $\mathfrak{B}_{t_o}$ has canonical singularities.*

Armed with this, we can apply Reid's crepant resolution to get (unique) minimal models ({R2}, 0.6,0.7). The result is

Theorem 2: *Let $X \longrightarrow S$ be a $\mathbb{Q}$-Gorenstein 3-dimensional elliptic fibre space. Assume:*

α) S is projective algebraic and smooth

β) $\Sigma \subset S$ moves in a linear system on S.

Conclusion: there is a crepant partial resolution

$$g : X' \longrightarrow X$$

such that (i) $K_{X'}$ is relatively nef ($K_{X'} \cdot C < 0$ for all curves C contracted by g)

(ii) X' has only terminal singularities.

Moreover, X' may be uniquely choosen (Reid's Choice).

The relavent definitions are given in §2 and §4. This result is interesting in that X needn't be projective (of even Moishezon). Thus we have a satisfactory result for both $\kappa(X)=2$ and $a(X)=2$.

However, at least as far as calculations are concerned, we are not quite satisfied with this model. It has two drawbacks: 1)it may be singular, and 2) the projection is not flat (there may be divisors in the fibres). To remedy this we introduce in §5 *models with multiplicative reduction*. This is a model $\hat{\pi}:\hat{B} \longrightarrow \hat{S}$ covering the model $X' \longrightarrow S$ above:

$$\begin{array}{ccc} \hat{B} & \longrightarrow & \hat{S} \\ \downarrow & & \downarrow \\ X' & \longrightarrow & S \end{array}$$

which has the following properties:

(i) $\hat{B}$ is smooth (even projective algebraic)

(ii) $\hat{\pi}$ is flat and has a section

(iii) $\hat{B}$ has only singularities of type I_k

(iv) $\hat{B}$ is a group variety over $\hat{S}$.

Property (iv) is explained in §5. It is a 3-dimensional analogue of (Kol), Theorem 9.1. It is precisely this group structure which led us to consider the model $\hat{B}$. Let $B_o^{\#}$ denote the Jacobi fibering associated to $\hat{B}$. The group structure implies the existance of the following exact sequence of sheaves on S:

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}(\ell) \longrightarrow \mathcal{O}(B_o^{\#}) \longrightarrow 0.$$

Here $\mathcal{G} = R^1\pi_*\mathbb{Z}$ is the homological invariant and ℓ is the normal bundle of the section (pulled back to $\hat{S}$). This yields the following long exact sequence of cohomology groups, which is one of our main objects of study:

$$0 \longrightarrow H^0(S,\mathcal{O}(\ell)) \longrightarrow H^0(S,\mathcal{O}(B_o^{\#})) \longrightarrow H^1(S,\mathcal{G}) \longrightarrow H^1(S,\mathcal{O}(\ell)) \longrightarrow \ldots$$
$$\ldots \longrightarrow H^1(S,\mathcal{O}(B_o^{\#})) \longrightarrow H^2(S,\mathcal{G}) \longrightarrow H^2(S,\mathcal{O}(\ell)) \longrightarrow \ldots$$
$$\ldots \longrightarrow H^2(S,\mathcal{O}(B_o^{\#})) \longrightarrow H^3(S,\mathcal{G}) \longrightarrow 0.$$

Several of the groups in this sequence have geometric meanings, and the exactness of the sequence relates them to one another. For example, if we assume $K_S \otimes \mathcal{O}(-\ell)$ is positive in the sense of Kodaira (i.e. $\kappa(\hat{B})=2$), then we have

Theorem 3: *(i)* $H^0(S,\mathcal{O}(B_o^{\#})) \cong H^1(S,\mathcal{G})$

(ii) $H^1(S,\mathcal{O}(B_o^{\#})) \cong \mathcal{F}(\mathcal{J},\mathcal{G})$ *, tensored with* $\mathbb{C}$*, is a subgroup of* $H^3(\hat{B},\mathbb{C})$.

This is in marked contrast with the elliptic surface case.

In §6, we calculate several invariants of the model $\hat{B}$, including the Hodge numbers, in terms of the following data: $e(\Sigma_i)$ (the euler characteristic of the irreducible components of the singular locus Σ), the number of intersections $\Sigma_i \cap \Sigma_j$, and r=rank $H^o(S,\mathcal{O}(B_o^{\#}))$. The invariant r is arithmetical in character, and is probably very difficult to calculate.

The rest of Chapter II is concerned with thhe applications of a theorem of (HM) to elliptic fibre spaces with *trivial canonical bundle* (or more generally $c_1^{\mathbb{R}}=0$). Our main result is

Theorem 4: *There are constants γ_1, γ_2, such that*

$$\gamma_1 \le c_3(X) \le \gamma_2$$

for any Moishezon elliptic 3-fold X with $c_1^{\mathbb{R}}=0$. Moreover, $\gamma_1 \le -756$ and $\gamma_2 \ge 112$.

This confirms in part a conjecture of F. Hirzebruch, to the effect that any Moishezon 3-fold X with $c_1^{\mathbb{R}}=0$ has bounded euler number. Since the euler number is a diffeomorphism invariant, it is constant in deformation families. The following conjectures would confirm Hirzebruch's conjecture in full:

Conjecture 1: Any Moishezon 3-fold with $c_1^{\mathbb{R}}=0$ and $h^{22}>1$ is a deformation of elliptic 3-fold.

Conjecture 2: Any Moishezon 3-fold with $c_1^{\mathbb{R}}=0$ and $h^{22}=h^{11}=1$ is a deformation of a non-singular complete intersection.

Finally, in §9, we give lots of examples of Moishezon 3-folds with trivial canonical bundle, in particular the examples with euler number -756 and +112. Both of these examples have the structure of elliptic fibre spaces.

I would like to thank E. Viehweg for discussions about the contents of §4. Also I want to thank A. Todorov for pointing out the necessity of $h^{22}>1$ in the conjecture above. Finally, I acknowledge financial support of the Arbeitsamt during the preperation of this paper.

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§1.Classification Theory

1.1. Iitaka's Theorem

Let X be a compact, complex analytic (c.c.a.) N-fold. One defines the *algebraic dimension* of X as $a(X):=\mathrm{tran}_{\mathbb{C}}K(X) \le N$.

Let K_X be the canonical bundle, and

$$\varphi_{mK}:X \longrightarrow W \subset \mathbb{P}^{\dim|mK|}$$

the pluricanonical map. The *Kodaira dimension* is defined as follows:

$$\kappa(X)= \max_m \dim W \quad \text{or } -\infty \text{ if } |mK|=\phi \text{ for all } m.$$

From the definitions it follows immediately that

$$\kappa(X)\le a(X)\le N.$$

At the one extreme we have $a(X)=N$, in which case X is said to be *Moishezon*. In this case X has the function field of an (projective) algebraic variety of dimension N, so that X is birational to an algebraic variety. A Moishezon X is Kaehler iff it is projective algebraic. At the other extreme are c.c.a. N-folds with $a(X)=0$. In this case it is easily seen that the geometric genus of X is 1 iff $\kappa(X)=0$, and the geometric genus is 0 iff $\kappa(X)=-\infty$, and in this case $(p_g(X)=0)$, X is necessarily non-Kaehler.

The basic tool for studying the range $0\le\kappa(X)\le a(X)\le N-1$ is the following:

Iitaka's Thoerem: *X as above, with* $\kappa(X)>0$. *Then there exists a c.c.a.* X^* *bimeromorphic to X, such that* X^* *has the structure of fibre space*:

$$X^* \longrightarrow W$$

which has the following properties:

i) $\kappa(F)=0$ *for a generic fibre F*

ii) $\dim W=\kappa(X)$, *W algebraic and smooth*

iii) X^* *is unique up to birational equivalence.*

Furthermore, if X is smooth, $X^* \longrightarrow W$ *is bimeromorphic to the pluricanonical map.*

Definition:A normal c.c.a. N-fold X is called an *elliptic fibre space*, $:\Longleftrightarrow$ there exists a map $\pi:X\longrightarrow W$ onto a smooth c.c.a. (N-1)-fold W such that the generic fibre X_w is an elliptic curve.

From Iitaka's theorem above we get

Proposition 1.1: *If $\kappa(X)=N-1$, then there is a birational (bimeromophic) model of X, X^* such that X^* has the structure of elliptic fibre space $X^* \longrightarrow W$ with the following properties:*

i) the fibering $X^* \longrightarrow W$ is unique.

ii) W is projective algebraic.

1.2. Classification of algebraic 3-folds

We summarise the classification of algebraic 3-folds in the following table (borrowed from (V)) :

Theorem 1.2.: *Every projective smooth 3-fold $X\hat{}$ has a birational model X, such that*

$\kappa(X\hat{})$	$q(X\hat{})$	Structure of X		
3	-	general type		
2	-	$f:X \longrightarrow W$	$\dim(F)=1$, $\kappa(F)=0$	
1	-	$f:X \longrightarrow W$	$\dim(F)=2$, $\kappa(F)=0$	
0	0	??????		
0	1	the albanese map $\alpha_X:X \longrightarrow A(X)$ is an etale fibre bundle	$\dim(F)=2$, $\kappa(F)=0$	
0	2	(as above)	$\dim(F)=1$, $\kappa(F)=0$	
0	3	abelian variety		
$-\infty$	0	? ? ? ?		
$-\infty$	≥ 1	the Stein factorisation $f:X \longrightarrow W$ of the albanese map $\alpha_X:X \longrightarrow A(X)$ has the property:	$\dim(F)=2$, $\kappa(F)=-\infty$, $q(W)=q(X)$, $\kappa(W)\geq 0$	$\dim\alpha(X)=1$
			$F=P^1(\mathbb{C})$, $q(W)=q(X)$, $\kappa(W)\geq 0$	$\dim\alpha(X)=2$

Remark: Although $\kappa(X)=2$ is *sufficient* for X to be (bimeromorphic to) an elliptic fibre space, it is of course not *necessary*. There are many interesting examples of elliptic fibre spaces with $\kappa(X)=0$, and in fact help to understand 3-folds with trivial canonical bundle (see§§7-9).

1.3. Some results of Fujiki

We now consider the situation where X is a c.c.a. manifold with

$$0 \le a(X) \le N-1,$$

where $N=\dim X$. Let K(X) be the function field of X. It is well known that K(X) is an algebraic field, i.e. there exists an algebraic manifold Y, $\dim(Y)=a(X)$, such that $K(X) \cong K(Y)$.

Definition: An *algebraic reduction* of X is a meromorphic fibre space

$$f: X \longrightarrow Y$$

such that $K(X)=f^*K(Y)$.

Proposition 1.3.:([Fu],p.234) *Let $f: X \longrightarrow Y$ be an algebraic reduction. Then:*

i) *$\kappa(X_y) \le 0$ for any generic fibre of f*

ii) *$a(X)=N-1 \implies X \longrightarrow Y$ is an elliptic fibre space*

iii) *$a(X)=N-2 \implies$ every smooth fibre of f is bimeromoph ically equivalent to one of the following surfaces: 1) complex torus, 2) hyperelliptic surface, 3) K3 surface, 4) Enriques surface, 5) ruled surface of genus 1, 6) rational surfaces, 7) elliptic surface with trivial canonical bundle, 8) surface of type VII_o*

In general, not much more can be said. In case X is *Kaehler*, however, much stronger statements are possible. In fact, for this the Kaehler condition is not strictly necessary.

Definition: A c.c.a. manifold X is *in the class* $\mathscr{C}$, $:\iff$ there exists a Kaehler space $\hat{X}$ and a surjective, meromorphic map

$$g: \hat{X} \longrightarrow X.$$

Notice that $\dim \hat{X}$ may be larger than dim X.

Fujiki has derived some strong results in case X is in the class $\mathscr{C}$. The following properties are basic for $X \in \mathscr{C}$:

A) Functorial Properties.

i) If $V \subset X$ is any subvariety of X, then V is also in the class $\mathscr{C}$.

ii) any meromorphic image of X is again in the class $\mathscr{C}$.

B) Hodge decomposition.

$$H^k(X,\mathbb{C}) = \oplus_{p+q=k} H^{p,q}(X,\mathbb{C}), \qquad H^{p,q}(X,\mathbb{C}) = \overline{H^{q,p}(X,\mathbb{C})}$$

In particular the odd dimensional betti numbers are even.

C) Closedness of the Douady space $\mathfrak{D}_X$ of X. (see (Fu), (Ue4))

If X is compact in $\mathscr{C}$, then any irreducible component of $\mathfrak{D}_X$ is again compact and belongs to $\mathscr{C}$.

An application of the existance of the Douady space (for any complex space X) is the theorem that the group of automorphisms of X carries a natural complex structure, with respect to which it is a complex Lie group. If X is in $\mathscr{C}$, then property C) above implies that there are only finitely many conponents in any stability subgroup, and the identity component $Aut_o(X)$ of Aut(X) has a natural compactification.

1.4. Classification of c.c.a. 3-folds in the class $\mathscr{C}$

We describe the classification given by Fujiki in the following table.

a(X)	Structure of X
3	Moishezon
2	elliptic fibre space
1	I. $f:X \longrightarrow Y$ (algebraic reduction) is holomorphic α) $X_y \cong$ complex torus β) $X_y \cong \mathbb{P}^1$-bundle over an elliptic curve II. Quotient variety of S×C by a finite group acting diagonally on S×C, S a surface, C a curve.
0	I. Kummer II. $\mathbb{P}^1$-fibre space over a surface III. simple and k(X)=0

The relevant definitions are as follows. X is *Kummer*, iff X is the quotient of a complex torus by a finite group. X is *simple*, iff there exists no covering family $(A_t)_{t\in T}$, of proper analytic subvarieties A_t of X with $\dim A_t>0$. k(X)=0 means that there is no surjective meromorphic map of X onto a Kummer manifold. In particular then q(X)=0.

§2. Minimal models of canonical 3-folds

2.1. Relatively minimal models

Let X be a complete, non-singular variety.

Definition: (i) X is *a relatively minimal model*,:$\Longleftrightarrow$ any birational map f:X $\longrightarrow$ Y , which is everywhere defined (and Y is smooth) is actually an isomorphism. (nothing can be smoothly blown down.)

(ii) X is *an absolutly minimal model*,:$\Longleftrightarrow$ any birational map g:X $\longrightarrow$ Y (where Y is assumed smooth) is actually an isomorphism.

By Zariski's Main Theorem, the exceptional locus of any birational map f:X $\longrightarrow$ Y as in (i) is a pure *divisor*. Therefore, to check that an algebraic 3-fold is relatively minimal it suffices to check that there are no exceptional divisors (blow-ups at non-singular points or curves), and these are all contained in the canonical divisor (considering the behavior of canonical divisors under blow-ups). Relatively minimal models exist for any algebraic variety, but in general there are no absolutely minimal models. However, from the viewpoint of birational geometry, the notion of relatively minimal models is not at all well behaved, as the following theorem exemplates (copied from [Ue3]):

Theorem 2.1.: *Let X be a relatively minimal Moishezon manifold of dimension N>2. If X contains a rational curve (which may be singular), then there exists a relatively minimal model of X which is not isomorphic to X. If X contains a ruled surface (which may be singular), then in its birational class there are continuously many distinct relatively minimal models.*

Because of this, we are forced to consider *singular* models - blowing down the rational curves. Roughly speaking, the numerical expression of not having these rational curves is that the canonical bundle (or the canonical divisor) be numerically effective. This is the background for M. Reid's theory of minimal models, which we now briefly review.

2.2. Canonical singularities

Let X be a *normal* algebraic variety, ω_X its dualizing sheaf and Ω_X^N the sheaf of differentials. Then:

$$\omega_X = (\Omega_X^N)^{**} = j_*(\omega_{X_o}) = \mathcal{O}(K_X)$$

where $X_o \subset X$ is the smooth locus, K_X=Weil divisor class representing ω_X. This is a Weil divisor such that $\mathcal{O}_{X_o}(K_X) = \Omega_{X_o}^N$.

Definitions: 1) X is *locally* $\mathbb{Q}$*-factorial* :$\iff$ for any Weil divisor $D\subset X$, there is a $r\in N$, such that rD is a Cartier divisor.

2) X is $\mathbb{Q}$*-Gorenstein* :$\iff$ for some $r\in N$, rK_X is a Cartier divisor.

3) X has only *terminal* (*canonical*) singularities, :$\iff$

i) X is $\mathbb{Q}$-Gorenstien

ii) for any resolution $f:X^* \longrightarrow X$, we have

$rK_{X^*} = f^*(rK_X) + \Sigma\nu_i E_i$ with $\nu_i>0$ ($\nu_i\geq 0$), all i.

4) X is a *minimal model*, :$\iff$

i) X has only terminal singularities

ii) K_X is nef ($K_X\cdot C\geq 0$ for all effective curves $C\subset X$).

In 3) we mean by resolution one in which the exceptional locus consists only of divisors. Canonical singularities may be isolated or non-isolated. If they are non-isolated, they are locally of the form D^1x(Du-Val singularity) (D^1 a disk). In addition to the non-isolated singularities there are finitely many "dissidents", isolated canonical singularities.. Examples of these are the terminal singularities, which are in fact quotients of isolated compound Du Val (cDV) points (see (R1)) Resolving the terminal singularities introduces curves C (these are $\mathbb{P}^1$'s) with $K_X\cdot C<0$, which is why one doesn't resolve them. See (R1) for more details on canonical singularities.

2.3. Reid's Theorem

Reid's Theorem on minimal models ((R2),0.6): *Let X be a normal 3-dimensional variety such that X has only canonical singularities. Then:*

i) There is a partial resolution $f:X^* \longrightarrow X$ such that

a) K_{X^*} is relatively nef and X^* is Cohen-Macauly

b) X^* has only terminal singularities.

ii) This X^* can be choosen uniquely (Reid's choice).

Thus is K_X is nef (for example the canonical model), then X^* is a minimal model in the sense above.

2.4. Kawamata's Theorem

This can be turned around by starting with a smooth Y and trying to blow down exceptional loci by a birational map $f:Y \longrightarrow X^*$ such that K_{X^*} is nef. This is the object of theorems due to S.Mori and Kawamata. Let X be a non-singular 3-fold, $\kappa(X)>0$ (for simplicity). We look for a minimal model X_m in the category of $\mathbb{Q}$-factorial Gorenstein schemes with only terminal singularities, as follows:

1) We have a series of normal projective 3-folds

$$X=X_o, \quad X_1, \ldots . X_m$$

such that X_m has only terminal singularities, and K_{X_m} is nef.

2) for each $i=1,\ldots,m-1$ there is a map ϕ_i such that either

Case a) $\phi_i:X_i \longrightarrow X_{i+1}$ is a birational map with $\rho(X_i)=\rho(X_{i+1})+1$, in which case ϕ_i is called an *elementary contraction*.

(here ρ=Picard number)

Case b) $\phi_i:X_i \longrightarrow X_{i+1}$ is an isomorphism in codimension 1. ϕ_i is called an *elementary transformation* in this case.

The idea is the following. If K_X is not nef, let $C\subset X$ be a curve such that $K_X\cdot C<0$. Then either

Case α) C moves in a divisor D (we would like to contract this D), or

Case β) C doesn't move in a divisor.

Kawamata's Theorem (Ka): *In Case* α), *there exists a contraction. Thus* Case a) *can be completed by induction.*

Finally we remark that the existance of minimal models along these lines is still conjectural (because of Case b)).

§3. Structure of elliptic fibre spaces

In this paragraph, we gather results valid for any elliptic fibre space. We therefore let $\pi : X \longrightarrow W$ be an N-dimensional elliptic fibre space. That is, we assume the following:

1) X is a compact, normal, complex space, N=dim X
2) W is a c.c.a. manifold of dimension N-1
3) $\Sigma \subset W$, the degeneracy locus of π, is a pure divisor,

$$\Sigma = \sum_{i=1}^{k} n_i \Sigma_i$$

its decomposition into irreducible, reduced components Σ_i.

Set: $W' = W - \Sigma$.

4) $\pi_{|W'} : X' \longrightarrow W'$ is a smooth fibre bundle with fibre an elliptic curve.

3.1. Homological Invariant

We consider the *sheaf* $\mathcal{G} = R^1\pi_*\mathbb{Z}$ on W. This is a locally free sheaf on W' with stalk $H^1(X_w, \mathbb{Z}) = \mathbb{Z}\oplus\mathbb{Z}$ over a point $w \in W'$, called the *homological invariant* of the elliptic fibre space $\pi : X \longrightarrow W$. This sheaf is equivalent to the *representation* of the fundamental group $\pi_1(W', *)$ defined as follows. Let γ_1, γ_2 be a base of the stalk $\mathcal{G}_*$, where $*$ is a fixed base point on W'. By continously translating this base along a path $\beta \in \pi_1(W', *)$, γ_1 and γ_2 transform by an automorphism of the stalk, i.e.:

$$\rho : \pi_1(W', *) \longrightarrow \mathrm{Aut}(\mathbb{Z}\oplus\mathbb{Z}) = SL(2, \mathbb{Z})$$

$$\beta \longrightarrow \begin{pmatrix} a_\beta & b_\beta \\ c_\beta & d_\beta \end{pmatrix}$$

where $\beta^*\begin{pmatrix}\gamma_1 \\ \gamma_2\end{pmatrix} = \begin{pmatrix} a_\beta\gamma_1 + b_\beta\gamma_2 \\ c_\beta\gamma_1 + d_\beta\gamma_2\end{pmatrix}$. This representation is called the *monodromy representation* and the image of ρ, a subgroup of finite index, is called the *monodromy group*. The monodromy determines the type of singularities (at least over smooth points $s\in\Sigma\subset W$) or types of singular fibres, respectively. (Since we are not assuming X to be smooth, both cases are possible).

3.2. Gauß-Manin Connection

Consider the *Leray spectral sequence* of the map $\pi:X \longrightarrow W$. For elliptic fibre spaces, the sequence degenerates in many cases at the E_2 term :

$$H^p(X,\mathbb{C})=\bigoplus_q E_2^{q,p-q}$$

For example we have

$$H^2(X,\mathbb{C})=E_2^{2,0}\oplus E_2^{1,1}\oplus E_2^{0,2}$$

$$E_2^{2,0}=H^2(W,R^0\pi_*\mathbb{C})$$

$$E_2^{1,1}=H^1(W,R^1\pi_*\mathbb{C})$$

$$E_2^{0,2}=H^0(W,R^2\pi_*\mathbb{C})$$

Now consider the differential of the spectral sequence,

$$d_1^{0,1}:R^1\pi_*\mathbb{C} \longrightarrow \Omega^1_W\otimes_{\mathcal{O}_W} R^1\pi_*\mathbb{C}$$

It turns out that this is an (integrable) connection, called the *Gauss-Manin connection* (see (KO)). Since $R^1\pi_*\mathbb{C}$ is a rank 2 vector bundle on W, d_1^{01} is a differential operator with two linearly independent solutions, ω_1 and ω_2, which one assumes fulfill $^{\omega_1}/\omega_2\in \mathcal{H}$ = upper half-plane. The precise form of $d_1^{0,1}$ has been determined by Stiller ((S)):

$$\Lambda f=\frac{d^2f}{dw^2}+P(w)\frac{df}{dw}+Q(w)f=0$$

where $P(w)$ and $Q(w)\in K(W)$ are rational functions on W. Such a differential equation need not be unique, but the equation $\Lambda f=0$, *together* with a meromorphic, many-valued *quotient* $\omega = {}^{\omega_1}/\omega_2$ of two solutions determines uniquely an elliptic surface and vice-versa. Notice that this is a differential equation for *periods*, i.e. a *Picard-Fuchs* equation (see (Ka)). In fact, the (many-valued) holomorphic function ω may also be defined in the following manner:

$$\omega(w)={}^{\int_{\gamma_1,w}\theta_w}\Big/\int_{\gamma_2,w}\theta_w$$

where θ_w is the unique holomorphic 1-form on the fibre X_w and γ_1,w, γ_2,w form a base of $H_1(X_w,\mathbb{Z})$. By *analytic continuation* along a path $\beta\in W'$, ω transforms by *fractional linear transformations*:

$$\omega(\beta(w)) = \frac{a_\beta\omega(w)+b_\beta}{c_\beta\omega(w)+d_\beta},$$

where $\begin{pmatrix} a_\beta & b_\beta \\ c_\beta & d_\beta \end{pmatrix} = \rho(\beta)$ is the monodromy of $\beta\in\pi_1(W',*)$. Thus, the monodromy determined by ω is (in $PSL(2,\mathbb{Z})$ conjugate to) the *projective* monodromy representation.

3.3. Functional Invariant

Let J be the elliptic modular function on the upper half-plane $\mathcal{H}$. Then

$$\mathcal{J}(w):=J(\omega(w))$$

is a single-valued map on W, and is in fact a meromorphic function on W (see [Ko1],7.3.). $\mathcal{J}$ is called the *functional invariant* of the elliptic fibre space X. The differential equation above can be written explicitly in terms of $\mathcal{J}$:

$$\Lambda f = \frac{d^2f}{dw^2} + \frac{\left(\frac{d\mathcal{J}}{dw}\right)^2 - \mathcal{J}\left(\frac{d^2\mathcal{J}}{dw^2}\right)}{\mathcal{J}[\frac{d\mathcal{J}}{dw}]}\cdot\frac{df}{dw} + \frac{\left(\frac{d\mathcal{J}}{dw}\right)\cdot\left(\frac{31}{144}\cdot\mathcal{J} - \frac{1}{36}\right)}{\mathcal{J}^2(\mathcal{J}-1)^2}\, f=0$$

From this one sees that the differential equation has regular singular points. Actually this is true quite generally for the Gauß-Manin connection. The singularities of X lie over points $w\in W$ such that.

1) $\mathcal{J}(w)=0,1$ or ∞

2) The monodromy around w in non-trivial

(We are assuming there are no *multiple fibres*.)

The relationship between the monodromy representation ρ and the map ω is easy to see in case $-1\notin\Gamma$, $\Gamma=\mathrm{Im}(\rho)$ the monodromy group. Since ω as defined above is many-valued, it can be lifted to a single-valued, holomorphic function on $\tilde W$:=(universal cover of W') into $\mathcal{H}$:

$$\begin{array}{ccccc}
\tilde\omega:\tilde W & \longrightarrow & \mathcal{H} & & \\
\downarrow & & \downarrow & & E'_\Gamma \\
 & & & \swarrow & \cap \\
\omega':W' & \longrightarrow & {}_\Gamma\backslash\mathcal{H} = \Delta'_\Gamma & & E_\Gamma \\
\cap & & \cap \qquad \cap & \swarrow & \\
\omega\ :W & \longrightarrow & {}_\Gamma\backslash\mathcal{H}^* = \Delta_\Gamma & &
\end{array}$$

Since ω is Γ-invariant the above diagramm commutes. Let E'_Γ be the elliptic modular surface associated to Γ (here we need $-1\notin\Gamma$) on Δ'_Γ and $E_\Gamma\longrightarrow\Delta_\Gamma$ its compactification. Then ω' may be viewed as the *classifying map* for $X'\longrightarrow W'$, since, as is easily seen, X' is the (elliptically minimal) bundle over W' induced by ω'. The monodromy is now just the induced map on homotopy groups:

$$\omega':W'\longrightarrow\Delta'_\Gamma$$

$$\omega'_*:\pi_1(W',*)\longrightarrow\pi_1({}_\Gamma\backslash\mathcal{H})=\Gamma.$$

3.4. Basic Elliptic Fibre Spaces

From now on we make the following assumptions :

1) $\mathcal{J}$ has no points of indeterminacy
2) $\Sigma\subset W$ is a *normal crossings divisor.*

The first assumption is crucial; it means that at any double point of Σ, say $\Sigma_i\cap\Sigma_j$, the singular fibres (or singularities) along both components Σ_i and Σ_j must have the *same* $\mathcal{J}$-invariant. These are listed in the following table:

fibre type	II, II^*, IV, IV^*	III, III^*	I_k or I_k^*
value of $\mathcal{J}$	0	1	pole of order k

Let $\pi:X \longrightarrow W$ be an elliptic fibre space satisfying the two conditions above, with homological and functional invariants $\mathcal{G}$ and $\mathcal{J}$, respectively. Let $\tilde{W}$ be the universal cover of W'. The data $(W',\mathcal{G},\mathcal{J})$ determines an essentially *unique* elliptic fibre space $p:B'\longrightarrow W'$ possesing a *global holomorphic section*

$$\sigma:W'\longrightarrow B',$$

which is easily constructed as a quotient of $\tilde{W}\times\mathbb{C}$. Indeed, if

$$\rho:\pi_1(W')\twoheadrightarrow\Gamma\subset SL(2,\mathbb{Z})$$

is the monodromy representation, let

$$G(\mathcal{G},\mathcal{J}):=\pi_1(W',*)\rtimes_\rho\mathbb{Z}\oplus\mathbb{Z}$$

(semi-direct product). This operates in a standard fashion on $\tilde{W}\times\mathbb{C}$:

$$G(\mathcal{G},\mathcal{J})\ni(\beta,(\tilde{w},\zeta))\longmapsto(\beta(\tilde{w}),(c_\beta\omega(\tilde{w})+d_\beta)^{-1}(\zeta+m_1\omega(\tilde{w})+m_2));$$

the action is free and $G(\mathcal{G},\mathcal{J})^{\backslash\tilde{W}\times\mathbb{C}}$ is easily seen to be an elliptic fibre space $\pi':B'\longrightarrow W'$. The holomorphic section $\sigma:W'\longrightarrow B'$ is the obvious zero section which is just the image of $\tilde{W}\times\{0\}$.

Without going into details, we indicate briefly how B' can be (uniquely) compactified to a complex space B (which will be singular along Σ):

$$\begin{array}{ccc} B' & \subset & B \\ \downarrow & & \downarrow \\ W' & \subset & W. \end{array}$$

Since Σ is a normal crossings divisor, we can find local coordinates $(w_1,\dots,w_{N-1})$ on W such that $\Sigma_i=\{w_i=0\}$, $\Sigma_i\cap\Sigma_j=\{w_i=w_j=0\},\dots,$ $\Sigma_{i_1}\cap\dots\cap\Sigma_{i_{N-1}}=\{w_{i_1}=\dots=w_{i_{N-1}}=0\}$. We can cover W by coordinate patches U_i and $W-\cup_i U_i$, where

U_i-tubular neighborhood of Σ_i

U_{ij}-tubular neighborhood of $\Sigma_i \cap \Sigma_j$

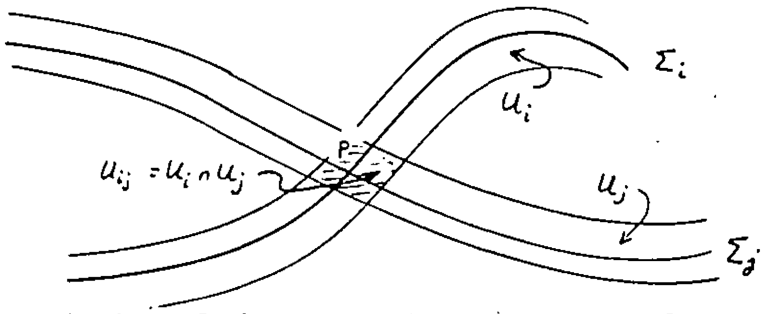
. . . .

. . . .

$U_{i_1 \ldots i_{N-1}}$-neighborhood of $\Sigma_{i_1} \cap \ldots \cap \Sigma_{i_{N-1}}$

On the cover of $(U_i - \bigcup_{j \neq i} U_{ij})$ in $\tilde{W} \times \mathbb{C}$, $G(\mathcal{G},\mathcal{J})$ has a purely codimension one fix point set with singular quotient which can be glued to B'. On the cover of $(U_{ij} - \bigcup_{k \neq i,j} U_{ijk})$ in $\tilde{W} \times \mathbb{C}$, $G(\mathcal{G},\mathcal{J})$ has a purely codimension 2 fix point set, and this is glued on to the $(U_i - \bigcup_{k \neq i} U_{ik})^{\sim}$ $\mathbb{C}/G(\mathcal{G},\mathcal{J})$ and the $(U_j - \bigcup_{k \neq j} U_{jk})^{\sim} \times \mathbb{C}/G(\mathcal{G},\mathcal{J})$ where $(..)^{\sim}$ denotes the universal cover of (..).

For example, if W is a surface, this looks as follows:



We remark that the resolution of the singuarities over singular points of Σ is by no means a trivial matter, whereas over the smooth points we can use a more or less "canonical resolution". We will discuss this in the 3-dimensional case below.

3.5. Families of elliptic fibre spaces

Let $\pi : X \longrightarrow W$, $\pi_1 : X_1 \longrightarrow W_1$ be two elliptic fibre spaces. We say X and X_1 are *elliptically bimeromorph*, iff there are bimeromorphic maps respecting the fiberings:

$$\begin{array}{ccc} X & \xrightarrow{f} & X_1 \\ \pi \downarrow & & \downarrow \pi_1 \\ W & \xrightarrow{g} & W_1 \end{array}$$

In this case the functional and homological invariants of X correspond (uniquely) to those of X_1 (see for example {Kaw},p.135). Let W be a c.c.a. manifold of dimension N-1, $\mathcal{J}$ a meromorphic function on W and $\mathcal{G}$ a locally free sheaf with generic stalk $\mathbb{Z} \oplus \mathbb{Z}$, which fulfill:

1) $\mathcal{J}$ belongs to $\mathcal{G}$, i.e. the many-valued function $\omega = J^{-1} \circ \mathcal{J}$ transforms with $\mathcal{G}$ in the sense above.
2) $\mathcal{J}$ has no points of indetermancy

3) Σ = locus of $\{w \in W | \mathcal{G}_w \neq \mathbb{Z}\oplus\mathbb{Z}\}$ is a normal crossings divisor.

Definition: the *family of elliptic fibre spaces over W with invariants $\mathcal{J}$ & $\mathcal{G}$:*

$\mathcal{F}(\mathcal{J},\mathcal{G})$ = { all equivalence classes of elliptically bimero. elliptic fibre spaces $\pi^*: X^* \longrightarrow W^*$ with homological and functional invariants corresponding (under $g: W^* \longrightarrow W$) to $\mathcal{G}$ and $\mathcal{J}$ such that:

(i) π^* is flat

(ii) X^* is elliptically minimal

(iii) X^* has no multiple fibres.

X^* elliptically minimal means there are no generically contractible divisors *in the fibres*. By the results of the last section, there is a unique (class of) basic elliptic fibre space $B \longrightarrow W$ in $\mathcal{F}(\mathcal{J},\mathcal{G})$ which has a global holomorphic section. As in [Kaw], p.135, we get

Proposition 3.1.: *Every element $X \in \mathcal{F}(\mathcal{J},\mathcal{G})$ can be constructed by reglueing the basic member $B \longrightarrow W$.*

Remark: One might define $\mathcal{F}(\mathcal{J},\mathcal{G})$ slightly differently, for example, all classes of elliptic fibre spaces $\pi^*: X^* \longrightarrow W$ over some *fixed* W. However, if one uses this definition, then to get a good model of a given X, one may have to *change families*. (See for example Miranda's flat smooth model).

3.6. Weierstraß Normal Form

Let $B \longrightarrow W$ be the basic member of a family $\mathcal{F}(\mathcal{J},\mathcal{G})$ as above. Since B has a section, it can be described by a single equation as follows. Let ℓ be the bundle along the fibres of $B \longrightarrow W$, that is the normal bundle to the section, $N_B\sigma$, viewed as a bundle on W via σ, and set $L = \ell^*$. L is a complex line bundle over W. B, viewed as an elliptic curve over the function field of W, can be given by the equation

$$y^2 = 4x^3 - g_2x - g_3$$

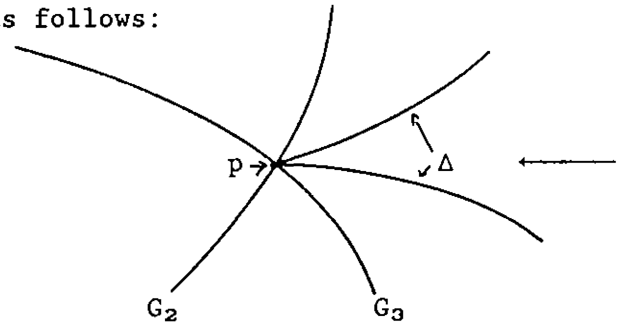
where $x \in \Gamma(L^2)$, $y \in \Gamma(L^3)$

$$g_2 \in \Gamma(W, \mathcal{O}(L^4)),\ g_3 \in \Gamma(W, \mathcal{O}(L^6)).$$

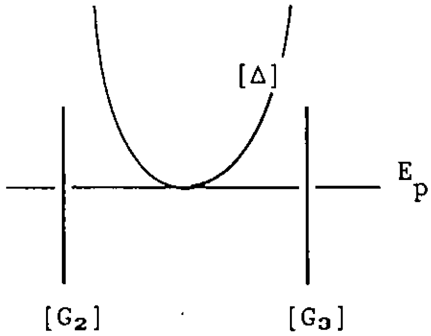
The singular locus $\Sigma \subset W$ is the divisor corresponding to the *discriminant*:

$$\Delta = g_2^3 - 27g_3^2 \in \Gamma(W, \mathcal{O}(L^{12})).$$

The $\mathcal{J}$-invariant is then: $\mathcal{J} = g_2^3/\Delta$. Let G_2, G_3, and D be the reduced divisors corresponding to g_2, g_3 and Δ, respectively. The assumption that $\mathcal{J}$ has no points of indeterminancy implies the following: If G_2 and G_3 meet, then they have a component in common. If W is a surface, this looks as follows:



$\mathcal{J}$ has point of indetermanancy at p



$\mathcal{J}$ has no point of indetermanancy

Also, the type of singularity over smooth points of Σ_i is determined by the orders of vanishing of g_2, g_3 and Δ, as in the following table:

fibre (sing.) type:	II	III	IV	IV^*	III^*	II^*	I_k	I_k^*
$\nu_W(g_2)$	≥1	1	≥2	≥3	3	≥4	0	2
$\nu_W(g_3)$	1	≥2	2	4	≥5	5	0	3
$\nu_W(\Delta)$	2	3	4	8	9	10	k	k+6

Because of this, the Weierstraß form is very convienient to work with. Also, these considerations are valid over any field, not just $\mathbb{C}$. The singular points lying over singular points of Σ can also be determined, see (Mi),Prop. 2.1.

Remark: If we are given an elliptic fibre space $X \longrightarrow W$ with W not algebraic (Moishezon), then such a representation may not be possible. Indeed, there might be no line bundle L on W such that L^4, L^6, and L^{12} all have sections.

§4. Models of elliptic 3-folds

In this paragraph we study the case where W=S is a compact, complex analytic surface. Let $B \longrightarrow S$ be the basic member of some family $\mathcal{F}(\mathcal{J},\mathcal{G})$ of elliptic fibre spaces over S with given functional and homological invariants $\mathcal{J}$ and $\mathcal{G}$, respectively. In this dimension, Kawai ([Kaw]) has proved the following

Theorem 4.1.:*(i) B is projective algebraic if S is (but of course singular along* Σ*)*

(ii) $\pi:B \longrightarrow S$ *is flat with a holomorphic section* $\sigma:S \longrightarrow B$.

4.1. Ueno's Resolution

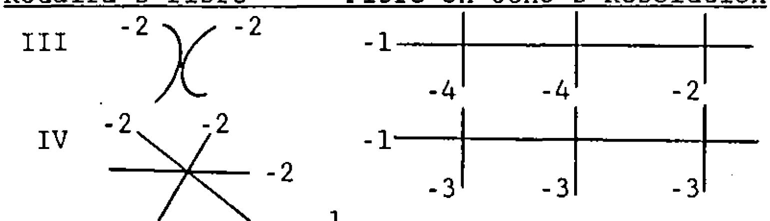
In [Ue3], Ueno has constructed an explicit resolution $\overline{B} \longrightarrow B$, which again fibers over S (that is, the resolution does not modify S):

$$\begin{array}{ccc} \overline{B} & \longrightarrow & B \\ & \overline{\pi}\searrow \quad \swarrow \pi & \\ & S & \end{array}$$

$\overline{B}$ has the following properties:

A) If $s\in\Sigma_{smooth}$, then the singularity of B over s is of the type $\mathbb{C}\times$(Du Val) and the fibre $\overline{\pi}^{-1}(s)$ on $\overline{B}$ is one of the fibres in Kodaira's list, except for the following:

<u>Kodaira's fibre</u> <u>Fibre on Ueno's Resolution</u>



B) If $s\in\Sigma_{sing}$, then the fibre $\pi^{-1}(s)$ consists of ruled surfaces.

C) The canonical divisor of $\overline{B}$ is given by the formula

$$K_{\overline{B}} = \overline{\pi}^{*}(K_W+[F]) + [G] + [H],$$

where [F], [G], [H] are *effective**, and

*$\mathbb{Q}$-divisors

a) [G] is based on fibres over double points of Σ

b) [H] is contained in fibres of type III and IV as described above.

In particular, $\bar{B}$ has the properties 1) $\bar{B}$ is not minimal ($K_{\bar{B}}$ is not nef), 2) $\bar{\pi}$ is not flat, and 3) $\bar{B}$ is not elliptically minimal. From the fact that G and H are effective, we see that they are divisors resolving *terminal singularities*. This is in fact one of the earliest occurences of terminal singularities in the literature. From this same fact we also get

Corollary 4.2.: *The basic memnber B has only canonical singularities.*

4.2. Canonical Singularities

We first rephase the corollary above.

Proposition: *If $\pi:X \longrightarrow \mathbb{C}$ is a local elliptic fibre space with $\mathbb{C}\times$(Du Val) singularity over D_1=(x=0) and D_2=(y=0), then the singularity at (0,0) is canonical.*

Or, taking into account §2, we might say a "normal crossings collision" of canonical singularities is canonical. By our assumption to the effect that $\mathcal{J}$ has no points of indeterminancy, we may also express this as follows (see (Mi), 2.1.): any hypersurface singularity of the form

$$y^2-4x^3 - s^{\alpha_1}t^{\beta_1}x - s^{\alpha_2}t^{\beta_2}$$

is canonical. We are interested in generalising this by dropping the "normal crossings" assumption.

Theorem 4.4.: *Let $\pi:X \rightarrow \Delta$ be an (affine) elliptic fibre space with a local section $\sigma:\Delta \longrightarrow X$ over a neighborhood of (0,0) in $\mathbb{C}^2$, and assume X is $\mathbb{Q}$-Gorenstein. Suppose the singular locus $\Sigma\subset U$ has an isolated singularity at the origin. The the 3-fold X has a canonical singularity over (0,0).*

proof: Let

$$\begin{array}{ccc} X' & \xrightarrow{p} & X \\ \downarrow & & \downarrow \\ \Delta' & \xrightarrow{\rho} & \Delta \subset \mathbb{C}^2 \end{array}$$

be an embedded resolution of Σ at (0,0). Then we have

$K_{\Delta'}=\rho^*(K_\Delta)+\sum E_i$, E_i=the exceptional curves of the resolution

$K_{X'}=\pi'^*(K_{\Delta'}+L')$, L'=conormal bundle of a section $\Delta' \longrightarrow X'$

$=\pi'^*(\rho^*(K_\Delta)+\sum E_i+\rho^*(L))$, L=" " " $\Delta \longrightarrow X$

$$=p^*\pi^*(K_\Delta+L) + \pi'^*(\sum E_i)$$
$$=p^*(K_X) + \pi'^*(\sum E_i),$$

which proves the theorem since the coefficients of the E_i are ≥ 0.

Remarks: 1) This may also be formulated as follows: any hypersurface singularity

$$y^2 = 4x^3 - g_2(s,t)x - g_3(s,t)$$

which is $\mathbb{Q}$-Gorenstein is actually canonical.

2) It may be possible that such an X is automatically $\mathbb{Q}$-Gorenstein.* At any rate, it would be interesting to find sufficient conditions (in terms of the types of singularities of X over the components Σ_i of Σ and the singularities of Σ at the origin) for X to be $\mathbb{Q}$-Gorenstein.

The proof above actually shows the following

Corollary 4.5.: *Let $\mathfrak{B} \longrightarrow D\subset\mathbb{C}$ be a $\mathbb{Q}$-Gorenstein family of elliptic 3-folds, i.e. each fibre $\mathfrak{B}_t$ is a $\mathbb{Q}$-Gorenstein elliptic fibre space $\mathfrak{B}_t \longrightarrow S$ over a fixed surface S with singular locus $\Sigma_t\subset S$ and $t\in D\subset\mathbb{C}$. Suppose for $t\neq t_o$, Σ_t is a normal crossings divisor. Then the central fibre $\mathfrak{B}_{t_o}$ has only canonical singularities.*

Thus, we may alow Σ to aquire *any singularities whatsoever*,** requiring only that S be smooth and $\mathfrak{B}_{t_o}$ to be $\mathbb{Q}$-Gorenstein. Thus, in some sense, the singularities are of a quite general kind.

We now give an example to show that this need *not* hold if the base surface S has singularities. This example is a Fermat cover (see §9 or (H) for details on this).

Example 4.6.: Take the arrangement $A_1^3(10)$ consisting of the 4 faces and 6 sym-metry planes of the tetrahedron in $\mathbb{P}^3(\mathbb{C})$. Delete one of the faces. The resulting arrangement has the data (with notations as in (H)):

$t_3(1)=7$	$t_6=2$	$t_{6,3}=8$	$t_{6,2}=6$
$t_2(1)=15$	$t_5=3$	$t_{5,3}=6$	$t_{5,2}=12$
$t_3=3$	$t_4=6$		

Let $X \longrightarrow \mathbb{P}^3(\mathbb{C})$ be the (singular) Fermat cover defined by the Kummer extension

*This is in fact the case, proven by N. Nakayama

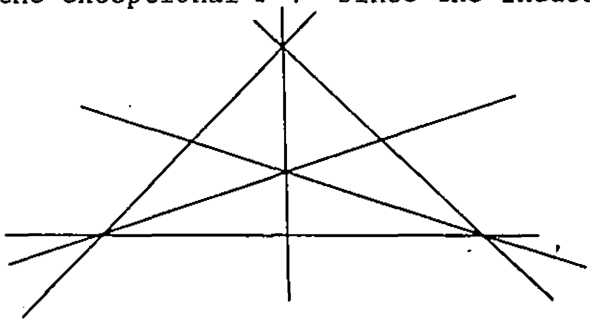
** under the assumption of elliptic minimality, see "Note to the reader".

$$\mathbb{C}(x_1/x_0, x_2/x_0, x_3/x_0)\left[(l_2/l_1)^{1/2},\dots,(l_9/l_1)^{1/2}\right]$$

of the rational function field. X is a 2^8-sheeted branched cover of $\mathbb{P}^3$, branched along the 9 hyperplanes $\{l_1=0\},\dots,\{l_9=0\}$. Let $\hat{\mathbb{P}}^3$ denote $\mathbb{P}^3$ blown up at one of the 6-fold points of the arrangement, and $\hat{X}$ the lift:

$$\begin{array}{ccc} \hat{X} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \hat{\mathbb{P}}^3 & \longrightarrow & \mathbb{P}^3 \end{array}$$

Since $\hat{\mathbb{P}}^3$ fibres over the exceptional $\mathbb{P}^2$, $\hat{X}$ fibres over the exceptional divisor covering the exceptional $\mathbb{P}^2$. Since the induced arrangement in this $\mathbb{P}^2$ is



the exceptional divisor covering it is the elliptic modular surface $\Gamma(4)$, (see Shioda [SH]), with all 16 sections (-2 curves) blown down to ordinary A_1-singularities. The fibering $\hat{X} \longrightarrow \Gamma(4)$ is elliptic. It is not difficult to see that at each A_1-singularity, 3 components of the singular locus meet. On the other hand, the fibres of the elliptic fibre space over the A_1-singularities contain singular points of $\hat{X}$ covering the 6-fold point of the arrangement we didn't blow up, and by 2.4.2. in [H] we know that these are *not* canonical.

Looking back at the proof of the above Corollary, we can see where the proof breaks down in this case. Since the *section* is singular, the conormal bundle does *not* lift naturally (i.e. $\rho^* L \neq L'$), K_Δ will not contain the E_i with positive coefficient, and the exceptional curves will occur with *negative* coefficients in the formula above.

4.3. Reid's Minimal Model

Armed with the above theorems we can use Reid's crepant resolution (§2) to get unique minimal models.

Theorem 4.7.: *Let $X \longrightarrow S$ be a 3-dimensional elliptic fibre space, and assume:*

α) S is a projective algebraic surface.

β) the singular locus Σ moves in a linear system on S.

Conclusion: there is a crepant partial resolution

$$g: X' \longrightarrow X$$

such that: *i)* $K_{X'}$ *is relatively nef*

ii) X' *has only terminal singularities.*

Moreover, X' *can be choosen uniquely (Reid's choice).*

Proof: Let $B \longrightarrow S$ be the basic member in the family to which X belongs. By α), Kawai's theorem implies B is projective algebraic, in fact *normal*. Thus we can apply Reid's resolution (§2). We get a unique minimal model $B' \longrightarrow B$ for which $K_{B'}$ is relatively nef. Now if X is reglued from pieces of B ([Kaw],p.135) by functions

$$\Lambda_j \Lambda_k^{-1} : U_j \cap U_k \longrightarrow \mathrm{Aut}(T^2) = T^2$$

we get a unique minimal model X' for X by reglueing B' by functions

$$\tilde{\Lambda}_j \tilde{\Lambda}_k^{-1} : f^{-1}(U_j) \cap f^{-1}(U_k) \longrightarrow T^2$$

where $\tilde{\Lambda}_j := \Lambda_j \circ f$. Also $K_{X'}$ will be nef if $K_{B'}$ is and the singularities on X' will be the *same* as on $K_{B'}$, q.e.d.

Remarks:1) If S is *not* algebraic we should proceed differently, but here the situation is much simpler.

A) $a(S)=1$. S is an elliptic surface $S \longrightarrow \Delta$, with no section. The only curves on S are the fibres, and they *don't meet*. The only intersections of curves are therefore intersections of components of singular fibres (in particular, normal crossings).

B) $a(S)=0$. In this case there are only finitely many curves on S (compare [Kol],§5), and the $\mathcal{J}$-invariant reduces to a constant. These possibilities could be checked explicitly.

2) Obviously one cannot expect $K_{X'}$ to be nef in general, for example if $\kappa(X)=-\infty$. If, however, K_S is nef, or more generally if K_S+L is nef, then we will get minimal models ($K_{X'}$ nef)

3) We would like to emphasis that the statement of the theorem is very strong. It settles the question of minimal models *completely* for $\kappa(X)=2$ *and* $a(X)=2$.

4) This thoerem does away with the assumption "Σ normal crossings", which, as we will see in §9, is not natural.

4.4. Miranda's flat model

Miranda has in {Mi} used a completely different approach to the problem, and we explain this breifly, as one of his small resolutions will be used in the next section. Miranda constructs a *smooth* model $B'' \longrightarrow B$ of the basic member in some family $\mathcal{F}(\mathcal{J},\mathcal{G})$, the basic member of which is $B \longrightarrow S$, where S is assumed to be an algebraic surface. B'' has the following properties:

$$\begin{array}{ccc} B'' & \longrightarrow & B \\ \downarrow & & \downarrow \\ S'' & \longrightarrow & S \end{array}$$

i) $B'' \longrightarrow S''$ is elliptically minimal over a surface S'' which is birational to S.

ii) $B'' \longrightarrow S''$ has a global holomorphic section

iii) $B'' \longrightarrow S''$ is *flat*.

B'' is constructed in 2 steps:

1st Step: Modify S along double points of Σ until the collisions are only of certain types (listed in {Mi}).

2nd Step: Resolve the remaining singularities over double points of Σ with *small resolutions*.

This approach has the disadvantage of modifying S more than necessary. We now describe one type of small resolution which will be used in the next section. Let Σ_1 and Σ_2 be two components of Σ meeting at a point $p \in S$. Suppose the singular fibre types are I_{k_1} and I_{k_2} over Σ_1 and Σ_2, respectively. There is a small resolution of the 3-fold singularity over $p \in S$ such that the resulting fibre is again a Kodaira fibre, and in fact of type $I_{k_1+k_2}$. If one of k_i is even and the other odd, then the resolution is not unique; if both are even or both are odd, then the resolution is also unique. In both cases the small resolution stays in the projective category.

§5 Multiplicative Reduction and the Group Variety

In this paragraph we introduce a *minimal model with multiplicative reduction* which is a model of special type which greatly facilitates the calculation of most invariants of an elliptic fibre space. To motivate things, we start with a review of the group structure on elliptic *surfaces*.

5.1. Analytic fibre systems of abelian groups

Let $B \longrightarrow \Delta$ be the basic *elliptic surface* in some family $\mathcal{F}(\mathcal{J},\mathcal{G})$ of elliptic fibre spaces over Δ with homological and functional invariants $\mathcal{G}$ and $\mathcal{J}$, respectively, $\Delta'=\Delta-\{a_1,\ldots a_k\}$ the open subset of Δ over which all fibres are smooth. On $B'\longrightarrow \Delta'$ there is an obvious structure of groups, in the following sense:

Definition: $B'\longrightarrow \Delta'$ has the *structure of analytic fibre system of abelian groups*, iff:

i) each fibre B'_x is an abelian group

ii) the complex structure of the fibre *as a submanifold* of B' is identical with the complex group structure.

iii) group multiplication is a holomorphic map of B'

Now let $B^{\#}=B'\cup$(the union of components of singular fibres of multiplicity 1 with all singular points deleted). Then the structure of groups on B' can be *extended* to $B^{\#}$ ((Ko2),Theorem 9.1). The group structures on the singular fibres are listed in the following table:

fibre	I_0^*	I_k	I_k^*	II, II^*	III, III^*	IV, IV^*
group	$\mathbb{C}\times\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{C}^*\times\mathbb{Z}_k$	$\mathbb{C}\times\mathbb{Z}_2\times\mathbb{Z}_2$ or $\mathbb{C}\times\mathbb{Z}_4$	$\mathbb{C}$	$\mathbb{C}\times\mathbb{Z}_2$	$\mathbb{C}\times\mathbb{Z}_3$

Notice that the group of a singular fibre is $\mathbb{C}^*$ iff the singular fibre is of type I_k. In this case B is said to have *multiplicative reduction*, since in this case the group structure is *multiplicative*. Set $B_o^{\#}=B'\cup$(those components of singular fibres which the section hits with singular points deleted). Thus a fibre of $B_o^{\#}$ is either an elliptic curve, $\mathbb{C}$ or $\mathbb{C}^*$. Now let ℓ be the bundle along the fibres of B, i.e. the normal bundle to the section. Each fibre ℓ_x is the tangent space to the group fibre $(B_o^{\#})_x$, and there is a natural exponential map

$$\exp:\ell_x \longrightarrow (B_o^{\#})_x$$

which yields a map of sheaves:

$$e:\mathcal{O}(\ell) \longrightarrow \mathcal{O}(B_o^{\#}).$$

This in turn yields an exact sequence of sheaves on Δ ([Ko2], Theorem 11.2):

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}(\ell) \longrightarrow \mathcal{O}(B_o^{\#}) \longrightarrow 0.$$

The corresonding long exact cohomology sequence is one of the most interesting objects of study of elliptic surfaces:

$$0 \longrightarrow H^0(\Delta,\mathcal{O}(B_o^{\#})) \longrightarrow H^1(\Delta,\mathcal{G}) \longrightarrow H^1(\Delta,\mathcal{O}(\ell)) \longrightarrow \ldots$$
$$\ldots \longrightarrow H^1(\Delta,\mathcal{O}(B_o^{\#})) \longrightarrow H^2(\Delta,\mathcal{G}) \longrightarrow 0$$

All of these groups have *geometric meanings*:

$H^0(\Delta,\mathcal{O}(B_o^{\#}))$= group of sections (knowledge of which allows calculation of the Picard number of B).

$H^1(\Delta,\mathcal{G})\otimes\mathbb{C} = H^1_{par}(G,\mathbb{C}^2)$, where $G=\pi_1(\Delta',*)$.

$H^1(\Delta,\mathcal{O}(\ell))\cong H^0(B,\Omega^2)$=vector space of holomorphic 2-forms on B.

$H^1(\Delta,\mathcal{O}(B_o^{\#}))\cong \mathcal{F}(\mathcal{J},\mathcal{G})$.

$H^2(\Delta,\mathcal{G})$=finite group (if $\mathcal{J}\neq$const.) of "characteristic classes" of elliptic surfaces in the family $\mathcal{F}(\mathcal{J},\mathcal{G})$.

We also remark that $H^1(\Delta,\mathcal{O}(\ell))$ can be identified with a space of mixed cusp forms (see [HM]). Therefore the map

$$H^1(\Delta,\mathcal{G}) \longrightarrow H^1(\Delta,\mathcal{O}(\ell))$$

is closely related to the theory of automorphic forms and has a very arithmetical meaning.

5.2. The covering trick

Let $B \longrightarrow S$ be the basic 3-dimensional elliptic fibre space in the family $\mathcal{F}(\mathcal{J},\mathcal{G})$. Let $B'\longrightarrow B$ be Reid's choice of minimal model as discussed in §4.

Theorem 5.1.: *There exists a finite Galois covering* $\hat{S} \longrightarrow S$ *such that the fibre product* $\hat{B}'=\hat{S}\times_\sigma B'$:

$$\begin{array}{ccc} \hat{B}' & \longrightarrow & B' \\ \downarrow & & \downarrow \\ \hat{S} & \xrightarrow{\sigma} & S \end{array}$$

has only multiplicative reduction, i.e. only singular fibres of type I_k.

Proof: Let $G=\pi_1(S-\Sigma,*)$ and $\rho:G \longrightarrow SL(2,\mathbb{Z})$ the monodromy representation. Let $\beta_1,\ldots\beta_t$ be a system of generators for G, and set

$$n_i = \text{order of the semi-simple part of } \rho(\beta_i),\ i=1,\ldots,t.$$

Then
$$\beta_1^{n_1},\ldots,\beta_t^{n_t} \in G$$

generate a *normal subgroup* $N\subset G$. N defines the covering
$$\hat{S} \longrightarrow S$$

which is a Galois cover since N is normal. The model $\hat{B}'$ can be explicitly constructed by compactifying
$$\hat{B}'_o = {}^{U\times\mathbb{C}}/_{\mathbb{Z}\oplus\mathbb{Z}\rtimes N},$$

where U={universal cover of $S-\Sigma$}, and then desingularising.

Remark: Although this is satisfactory from a theoretical standpoint, it is not from the computational. In principal at least, if we know all invariants of $\hat{B}'$ we can calculate those of B', but in practice this may be almost impossible. This is because, although we know the branching locus and branching degrees of the branched cover $\hat{S} \longrightarrow S$, it is difficult to determine the *degree* (=[G:N]) of this covering, since $G/N = Gal(\hat{S}/S)$ will not be abelian in general.

5.3. Minimal models with multiplicative reduction

In this section we define a certain type of model of elliptic fibre space, which admits also a group structure as do elliptic surfaces, and which will be used in §6 to calculate invariants and study the long exact sequence. By construction, the elliptic fibre space $\hat{B}' \longrightarrow \hat{S}$ has singularities only over double points of $\Sigma\subset S$. At these double points, we have collisions of the type I_{k_1} & I_{k_2}, and we can apply Miranda's small resolution to get a *smooth* elliptic fibre space
$$\hat{\pi}:\hat{B} \longrightarrow \hat{S}$$

which has the following properties:

α) $\hat{\pi}$ is flat and there is a section

β) singular fibres at all points are of type I_k

γ) the singular fibres over double points of Σ where two components with singular fibres of types I_{k_1} and I_{k_2} meet are of type $I_{k_1+k_2}$.

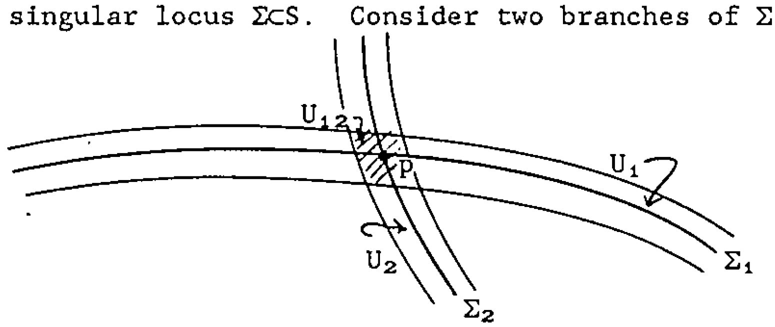
We call $\hat{B}$ a *minimal model with multiplicative reduction.*

5.4. The group structure

Let $\hat{B} \longrightarrow \hat{S}$ be a minimal model with multiplicative reduction as in the last section.

Theorem 5.2.: *$\hat{B}$ admits a unique structure of analytic system of abelian groups over $\hat{S}$.*

Proof: This is a local calculation which must only be checked at singular points of the singular locus $\Sigma \subset S$. Consider two branches of Σ which meet at $p \in S$:



Suppose the fibre type is I_{k_1} over Σ_1 and I_{k_2} over Σ_2. Then the fibre type over p is $I_{k_1+k_2}$. Let U_p be the universal covering of the open set $U_{12}-(\Sigma_1 \cup \Sigma_2)$ in the figure above, with coordinates ℓ_1, ℓ_2, and ζ (the fibre coordinate) in $\mathbb{C}$. Set:

$$\tau_1 = e^{2\pi i \ell_1}, \quad \tau_2 = e^{2\pi i \ell_2}, \quad \text{and} \quad w = e^{2\pi i \zeta}.$$

Assume $\{\tau_1 = 0\} = \Sigma_1$, $\{\tau_2 = 0\} = \Sigma_2$. According to Kodaira ([Ko2],pp.597-600) U_1 is covered by k_1 open sets $W_1^{(1)}, \ldots, W_{k_1}^{(1)}$, U_2 is covered by k_2 open sets $W_1^{(2)}, \ldots, W_{k_2}^{(2)}$, and U_{12} is covered by k_1+k_2 open sets $W_1^*, \ldots, W_{k_1+k_2}^*$, where $W_j^{(i)}$ has coordinates

$$((\tau_1, \tau_2, w))_j^{(i)} = (\tau_1, \tau_2, w(\operatorname{mod} \tau_i^{k_i}))$$

with the identifications

$$((\tau_1, \tau_2, w))_j^{(i)} = ((\tau_1, \tau_2, w\tau_i^{k-j}))_k^{(i)}$$

and W_j^* has coordinates $((\tau_1, \tau_2, w))_j := (\tau_1, \tau_2, w(\operatorname{mod} \tau_1^{k_1}\tau_2^{k_2}))$ with the identifications

$$((\tau_1, \tau_2, w))_j = ((\tau_1, \tau_2, w\tau_1^{k_1}\tau_2^{k_2}))_{k_1+k_2-j}.$$

The group structure is given by :

$$\text{over } \Sigma_1 \qquad ((0, \tau_2, w))_k^{(1)} \cdot ((0, \tau_2, v))_j^{(1)} = ((0, \tau_2, wv^{-1}))_{k-j}^{(1)}$$

$$\text{over } \Sigma_2 \qquad ((\tau_1, 0, w))_k^{(2)} \cdot ((\tau_1, 0, v))_j^{(2)} = ((\tau_1, 0, wv^{-1}))_{k-j}^{(2)}$$

$$\text{over } p \qquad ((0, 0, w))_k \cdot ((0, 0, v))_j = ((0, 0, wv^{-1}))_{k-j}$$

On the intersections, we identify

$$U_{12}\cap U_1 \qquad ((\tau_1,\tau_2,w))_{k+k_2} = ((\tau_1,\tau_2,w\tau_1^{-k}))_1^{(1)}, \quad k\in\mathbb{Z}_{k_1}$$

$$U_{12}\cap U_2 \qquad ((\tau_1,\tau_2,w))_{k+k_1} = ((\tau_1,\tau_2,w\tau_2^{-k}))_1^{(2)}, \quad k\in\mathbb{Z}_{k_2}$$

hence on $U_{12}\cap U_1$

$$\begin{aligned}((\tau_1,\tau_2,w))_{k+k_2}-((\tau_1,\tau_2,v))_{j+k_2} &= ((\tau_1,\tau_2,w\tau_1^{-k}))_1^{(1)}-((\tau_1,\tau_2,v\tau_1^{-j}))_1^{(1)}\\ &= ((\tau_1,\tau_2,wv^{-1}\tau_1^{j-k}))_1^{(1)}\\ &= ((\tau_1,\tau_2,wv^{-1}))_{k-j+k_2}\end{aligned}$$

and on $U_{12}\cap U_2$

$$\begin{aligned}((\tau_1,\tau_2,w))_{k+k_1}-((\tau_1,\tau_2,v))_{j+k_1} &= ((\tau_1,\tau_2,w\tau_2^{-k}))_1^{(2)}-((\tau_1,\tau_2,v\tau_2^{-j}))_1^{(2)}\\ &= ((\tau_1,\tau_2,wv^{-1}\tau_2^{j-k}))_1^{(2)}\\ &= ((\tau_1,\tau_2,wv^{-1}))_{k-j+k_1},\end{aligned}$$

so the group structure is an analytic extension on $U_{12}\cap U_1$ and $U_{12}\cap U_2$, q.e.d.

Argueing the same way as in (Ko3),p.4, we get

Corollary 5.3.: *We have an exact sequence of sheaves on* $\hat{S}$,

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}(\ell) \longrightarrow \mathcal{O}(\hat{B}_o^{\#}) \longrightarrow 0 \ .$$

§6. Invariants

In this paragraph we shall calculate a number of invariants of an elliptic fibre space which we assume is a minimal model with multiplicative reduction as in 5.3., by utilizing the long exact sequence coming from the exact sequence of sheaves on the base surface S derived in the corollary above. In this §6, we denote by $B \longrightarrow S$ the smooth minimal model with multiplicative reduction described in 5.3.

6.1. The long exact sequence

In 5.4. we derived the existance of the following exact sequence of sheaves on S:

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}(\ell) \longrightarrow \mathcal{O}(B_o^{\#}) \longrightarrow 0.$$

From this we get the following long exact sequence of cohomology groups:

$$0 \longrightarrow H^0(S,\mathcal{O}(\ell)) \longrightarrow H^0(S,\mathcal{O}(B_o^{\#})) \longrightarrow H^1(S,\mathcal{G}) \longrightarrow H^1(S,\mathcal{O}(\ell))$$
$$\longrightarrow. \quad \ldots \longrightarrow H^1(S,\mathcal{O}(B_o^{\#})) \longrightarrow H^2(S,\mathcal{G}) \longrightarrow H^2(S,\mathcal{O}(\ell)) \longrightarrow \ldots$$
$$\ldots \longrightarrow H^2(S,\mathcal{O}(B_o^{\#})) \longrightarrow H^3(S,\mathcal{G}) \longrightarrow 0$$

Let K_S be the canonical bundle on S. In what follows we shall assume $K_S \otimes \mathcal{O}(-\ell)$ is positive in the sense of Kodaira. This assumption is almost always fulfilled; if not, one should consider the above sequence seperately. Since $K_S \otimes \mathcal{O}(-\ell)$ is positive it follows from Serre duality that

$$H^0(S,\mathcal{O}(\ell)) \cong \overline{H^2(S,K_S \otimes \mathcal{O}(-\ell))} = 0$$

$$H^1(S,\mathcal{O}(\ell)) \cong \overline{H^1(S,K_S \otimes \mathcal{O}(-\ell))} = 0$$

Thus the long exact sequence above splits into two shorter ones,

$$0 \longrightarrow H^0(S,\mathcal{O}(B_o^{\#})) \longrightarrow H^1(S,\mathcal{G}) \longrightarrow 0$$
$$0 \longrightarrow H^1(S,\mathcal{O}(B_o^{\#})) \longrightarrow H^2(S,\mathcal{G}) \longrightarrow H^2(S,\mathcal{O}(\ell)) \longrightarrow \ldots$$
$$\ldots \longrightarrow H^2(S,\mathcal{O}(B_o^{\#})) \longrightarrow H^3(S,\mathcal{G}) \longrightarrow 0.$$

From our decription of $\mathcal{F}(\mathcal{J},\mathcal{G})$ we have

$$H^1(S,\mathcal{O}(B_o^{\#})) \cong \mathcal{F}(\mathcal{J},\mathcal{G}).$$

Also in analogy with elliptic surfaces we have

$$H^2(S,\mathcal{O}(\ell)) = H^0(S,K_S\otimes\mathcal{O}(-\ell)) = H^0(S,\pi_*K_B) = H^0(B,K_B)$$

is the vector space of holomorphic 3-forms on B. The second term in the second sequence turns up in the decomposition of $H^3(B,\mathbb{C})$ arising from the Leray spectral sequence (which we assume for the moment degenerates at the E_2-term.):

$$H^3(B,\mathbb{C}) \cong H^1(S,R^2\pi_*\mathbb{C})\oplus H^2(S,R^1\pi_*\mathbb{C})\oplus H^3(S,R^0\pi_*\mathbb{C})$$
$$= H^1(S,R^2\pi_*\mathbb{C})\oplus H^2(S,\mathcal{G}\otimes\mathbb{C})\oplus H^3(S,\mathbb{C}).$$

The second exact sequence above therefore implies that the family $\mathcal{F}(\mathcal{J},\mathcal{G})$ (which is a $\mathbb{Z}$-module), when tensored with $\mathbb{C}$, can be identified with a subgroup of $H^3(B,\mathbb{C})$. Likewise, $H^3(S,\mathcal{G})$ occurs in the Leray decomposition,

$$H^4(B,\mathbb{C}) \cong H^2(S,R^2\pi_*\mathbb{C})\oplus H^3(S,R^1\pi_*\mathbb{C})\oplus H^4(S,R^0\pi_*\mathbb{C})$$
$$= H^2(S,R^2\pi_*\mathbb{C})\oplus H^3(S,\mathcal{G}\otimes\mathbb{C})\oplus H^4(S,\mathbb{C})$$

The term $H^2(S,\mathcal{O}(B_0^\#))$ in the sequence above arouses curiosity. We have no idea what it has for a geometric meaaning.

6.2. Hodge numbers

We now proceed to some calculations. For the *geometric genus* $p_g(B)$ of B we have $p_g(B) = \dim H^2(S,\mathcal{O}(\ell))$. Since both $H^1(\mathcal{O}(\ell))$ and $H^0(\mathcal{O}(\ell))$ vanish, we can use Riemann-Roch to calculate $\dim H^2(S,\mathcal{O}(\ell))$: $p_g(B) = \dim H^2(S,\mathcal{O}(\ell))$

$$= \chi(S,\mathcal{O}(\ell)) = \frac{c_1^2(\ell)-c_1(\ell)\cdot K_S}{2} + \chi(S,\mathcal{O}_S)$$

To calculate the first term we make use of the Weierstraβ form for B (§4.6.):

$$y^2=4x^3-g_2x-g_3$$
$$g_2\in \Gamma(S,\mathcal{O}(L^4)),\quad g_3\in \Gamma(S,\mathcal{O}(L^6))$$

where $L=\ell^*$. We have for the singular locus $\Sigma\subset S$,

$$\Sigma = (\Delta),\quad \Delta = g_2^3-27g_3^2 \in \Gamma(S,\mathcal{O}(L^{12})).$$

Write Σ as a sum of *irreducible*, *reduced* components,

$$\Sigma = \sum_{i=1}^{k} n_i\Sigma_i$$

which implies the fibre over Σ_i is of type I_{n_i}. In $H^2(S,\mathbb{Z})$ we have the relation

$$c_1(L^{12}) = \sum n_i\Sigma_i.$$

Set $\Sigma_i=c_1(L_i)$, and insert this into the above:

$$12c_1(L) = \sum n_i c_1(L_i)$$

and calculate,

$$144c_1^2(L) = \sum_{i=1}^{k} n_i^2 c_1^2(L_i) + 2\sum_{i\neq j} n_i n_j c_1(L_i)c_1(L_j)$$

$$12c_1(L)\cdot K_S = \sum n_i c_1(L_i)\cdot K_S$$

so $$c_1^2(L)+c_1(L)\cdot K_S = \frac{1}{12}\left\{\sum n_i(c_1^2(L_i) + c_1(L_i)\cdot K_S)\right\} + \sum n_i\left\{\frac{n_i-12}{144}\right\} c_1^2(L_i) + \frac{1}{72}\left\{\sum n_i n_j c_1(L_i)c_1(L_j)\right\}$$

and applying adjunction,

$$= \frac{1}{12}\sum n_i e(\Sigma_i) + \sum n_i\left\{\frac{n_i-12}{144}\right\}\cdot\Sigma_i^2 + \frac{1}{72}\sum n_i n_j \Sigma_i\cdot\Sigma_j .$$

This gives a formula for the geometric genus of B. Now suppose $\kappa(B)=2$. Then, since the fibering is unique, we get for the Hodge numbers h^{01} and h^{02}

$$h^{01} = \begin{cases} q(S) & \ell \text{ not trivial} \\ q(S)+1 & \ell \text{ trivial} \end{cases}$$

$$h^{02} = \begin{cases} p_g(S) & \ell \text{ not trivial} \\ p_g(S)+q(S) & \ell \text{ trivial} \end{cases}$$

From this and the formula above we get

Theorem 6.1.: *Let* $B \longrightarrow S$ *be an elliptic model with multiplicative reduction with* $\kappa(B)=2$, *and assume* ℓ *is not trivial. Then the* arithmetic genus *of B is given by the formula*:

$$\chi(B,\mathcal{O}_B) = -\frac{c_1^2(\ell) - c_1(\ell)\cdot K_S}{2}.$$

Euler Number: Let e(B) denote the Euler-Poicare characteristic of B. Since $B' \longrightarrow S'=S-\Sigma$ is a smooth fibre bundle of elliptic curves (which have euler number =0), e(B) is just the euler number of the singular fibres. In terms of the data Σ_i,

$$e(B) = \sum_{i=1}^{k} n_i e(\Sigma_i)$$

On the other hand we have by definition

$$e(B) = 2 - 2b_1 + 2b_2 - b_3,$$
$$= 2 - 4q(B) + 4g_2(B) + 2h^{11}(B) - 2p_g(B) - 2h^{21}(B)$$

where of course b_i=ith betti number. From this we see: we need only calculate one of the numbers h^{11} and h^{21}, and the other can be calculated from e(B). We try $h^{11}(B)$. By definition,

$$h^{11}(B) = b_2 - 2g_2(B),$$

and we can try to calculate b_2 from the Leray decomposition,

$$H^2(B,\mathbb{C}) = H^0(S,R^2\pi_*\mathbb{C}) \oplus H^1(S,\mathcal{G}\otimes\mathbb{C}) \oplus H^2(S,R^0\pi_*\mathbb{C})$$

$$b_2 = b_2(S) + r + \dim H^0(S,R^2\pi_*\mathbb{C}),$$

r=rank of $H^0(S,\mathcal{O}(B_o^\#))$ is the rank of the group of sections. To calculate $H^0(S,R^2\pi_*\mathbb{C})$ we use Mayer-Vitoris. Let U be a tubular neighborhood of Σ, $S=S'\cup U$, $D=S'\cap U$ = disk bundle over Σ. Set $\mathcal{F}=R^2\pi_*\mathbb{C}$. We have the sequence:

$$0 \longrightarrow H^0(S,\mathcal{F}) \longrightarrow H^0(S',\mathcal{F}_{|S'}) \oplus H^0(U,\mathcal{F}_{|U}) \longrightarrow H^0(D,\mathcal{F}_{|D}) \longrightarrow \cdots$$

We infer readily that $\dim H^0(S,\mathcal{F}) = 1 + \sum_{i=1}^{k}(n_i-1)$, so

$$b_2 = b_2(S) + r + 1 + \sum(n_i-1).$$

and this in turn yields a formula for $h^{11}(B)$ (in terms of r)

$$h^{11} = b_2(S) - 2p_g(S) + r + 1 + \sum(n_i-1)$$

$$= h^{11}(S) + r + 1 + \sum(n_i-1) \ .$$

From this, as mentioned above, one can calculate $h^{21}(B)$, so all Hodge numbers have been calculated.

§7. A finiteness theorem

7.1. The theorem for surfaces

The formula we derived above for the geometric genus of B has as two-dimensional analogue

$$p_g(E) = \chi(E,\mathcal{O}_E) + q(E) - 1$$
$$= \frac{e(E)}{12} + g(\Delta) - 1,$$

where $E \longrightarrow \Delta$ denotes an elliptic surface. Suppose now we are given k points $a_1,\dots,a_k$ on Δ; what can we say about the possible p_g? With a little care one can derive the following *inequality*: (compare (HM))

(1) $$p_g(E) \le 2g(\Delta) - 2 + \frac{k}{2}.$$

This has the following interesting corollary:

Corollary 7.1.: *Given $a_1,\dots,a_k \in \Delta$, the set of all elliptic surfaces $E \longrightarrow \Delta$ (with section) which have singular fibres over $a_1,\dots,a_k$ and $\mathcal{J}\neq$const. is a finite set.*

7.2. N-dimensional case

It would be interesting to generalise the inequality (1) above to higher dimensions. At any rate, the corollary generalises readily:

Theorem 7.2.: *Let W be a smooth, projective (N-1)-fold. Given $\Sigma_1,\dots,\Sigma_k$ divisors on W such that $\Sigma = \sum_{i=1}^{k}\Sigma_i$ is normal crossings, the set of all elliptic N-folds $X \xrightarrow{\pi} W$ with singular fibres over the Σ_i (X smooth, say, and with section) is a finite set.*

Proof: Let $D\subset W$ be an ample divisor. Then $D^{N-2}\subset W$ is a curve which by Nakai's criterium meets each component Σ_i. The theorem follows from the corollary above applied to $\pi^{-1}(D^{N-2}) \longrightarrow D^{N-2}$, since the fibre type on each Σ_i is locally constant.

§8. A bound on the euler number

8.1. Theorem for elliptic 3-folds

Theorem 8.1.: *There are constants* γ_1, γ_2, *such that*

$$\gamma_1 \leq c_3(X) \leq \gamma_2$$

holds for any elliptic 3-fold $X \longrightarrow S$ *with* K_X *trivial.*

Proof: First we may assume $\Sigma \subset S$ has normal crossings, since modifying S until Σ is normal crossings adds fixed components to K_X. Thus, X belongs to a family $\mathcal{F}(\mathcal{J},\mathcal{G})$ and has the same singular fibres as the basic member $B \in \mathcal{F}(\mathcal{J},\mathcal{G})$. Thus we may assume $\pi : X \longrightarrow S$ admits a section $\sigma : S \longrightarrow X$, with the corresponding Weierstraß form

$$y^2 = 4x^3 - g_2 x - g_3$$

$$g_2 \in \Gamma(S, \mathcal{O}(L^3)), \qquad g_3 \in \Gamma(S, \mathcal{O}(L^4))$$

$$\Delta = g_2^3 - 27 g_3^2 \in \Gamma(S, \mathcal{O}(L^{12}))$$

and by the formula for the canonical bundle

$$\mathcal{O}_X = K_X = \pi^*(K_S \otimes L)$$

which implies : $\quad K_S \approx -L$ (linear equivalence)

and writing $L \approx \sum n_i L_i$ as above we have:

$$\sum n_i L_i \approx -K_S, \qquad c_1(L_i) = \Sigma_i$$

for any elliptic fibre space $X \longrightarrow S$ with singular fibres along Σ_i and trivial canonical bundle. There are only finitely many combinations of linear equivalence classes for Σ_i which fulfill the conditions above. Given any $\Sigma = \sum n_i \Sigma_i$ which has the right linear equivalence class, there are only finitely many possibilities for singular fibres by the theorem above.

Now notice there are only two possible birational classes for S. In fact, since $h^0(S, -3K_S)$, $h^0(S, -4K_S)$ and $h^0(S, -12K_S)$ must all be positive, it follows that S must be birational to

a) $\mathbb{P}^2$

b) $E \times \mathbb{P}^1$; E an elliptic curve.

In the second case, we have $g_1(B) > 0$, and it follows from general theorems (compare (V2), Proposition 8.2) that B is an etale fibre bundle, (i.e. no singular fibres) so $e(B)=0$. The theorem now follows from the following

Lemma 8.2.: *The number of possible types of singular fibres over Σ_i (and by our formula for $c_3(X)$, the euler number of X) is uniformly bounded for all S' birationally equivalent to S, i.e.*

$$\exists_{\delta} \quad \forall_{S' \in \mathscr{C}(S)} \left\{ \begin{array}{c} \text{\# possible singular} \\ \text{fibres on } X \end{array} \right\} \leq \delta$$

Proof: Let $S' \longleftarrow S'' \longrightarrow S$ be a sequence of blow-ups followed by blow downs. Let $D \subset S'$ be a smooth, irreducible curve meeting each Σ_i' but none of the points blown up , and let Σ_i'' denote thier proper transforms. Then the proper transform of D meets each Σ_i'', and the corollary in 7.2. can be applied to $\pi^{-1}(D)$. Thus it suffices to consider the induced fibrations over the exceptional $\mathbb{P}^1$'s. These are either generically smooth and then contribute nothing to e(X), or are $\mathbb{P}^1 \times$(Kodaira fibre). In the latter case this fibre type is determined by the components of Σ' meeting at the point blown up (compare the discussion of "collisions" in Miranda's article). This discussion applies equally well to S' and S, so the lemma is proved.

Corollary 8.3.: *Let $\pi: X \longrightarrow S$ be any elliptic 3-fold with $c_1^{\mathbb{R}}(X)=0$. Then the conclusion of 8.1. holds.*

Proof: Since $c_1^{\mathbb{R}}=0$, there is some finite covering $X' \longrightarrow X$ such that X' has trivial canonical bundle. Thus 8.1. applies.

8.2. Discussion of an N-dimensional analogue

The Lemma we have just proved applies to higher dimensional varieties X, assuming that X is *normal*. This implies that codim(singX)≥2 which means we can find a curve on X not meeting all exceptional divisors, and the corollary of 7.2. can be applied to the elliptic fibering over the curve. However, it does not seem so obvious that the other part of the argument is true in higher dimensions.

Question: *Are there at most finitely many birational equivalence classes of (N-1)-dimensional algebraic varieties with*

$$h^0(W,-3K_W), \quad h^0(W,-4K_W) \quad \text{and} \quad h^0(W,-12K_W) \quad \text{are all} \geq 1?$$

If this were so then the theorem above holds for elliptic N-folds X with trivial canonical bundle.

In another vien, the following seems quite plausible,

Question: *Let X be an N-fold with trivial canonical bundle. Does X have a deformation Y such that Y has the structure of elliptic fibre space.?*

If the answer here were affirmative as well as the question above it, the following would follow formally:

Question: *Let X be a Moishezon N-fold with trivial canonical bundle. Is the euler number of X bounded from above and below?*

The interest in this theorem is the general conjecture that there will only be finitely many deformation families of N-folds with trivial canonical bundle. This clearly would imply all of the above.

§9. Examples of 3-folds with trivial canonical bundle

In this paragraph we give many examples of smooth algebraic 3-folds with trivial canonical bundle, including the two examples with the highest (lowest) known euler numbers. We use two methods, *Fermat covers of* $\mathbb{P}^3$ and *elliptic 3-folds over* $\mathbb{P}^2$ defined with the help of a Weierstraß form.

9.1. Fermat covers

This is a construction originally due to Hirzebruch, and studied in detail in (H) for the dimension 3. Let $H_1,\dots,H_k$ be k hyperplanes in $\mathbb{P}^3$ defined by k linear forms $\ell_1,\dots,\ell_k$. The *quotients* $\ell_2/\ell_1,\dots,\ell_k/\ell_1$ define global meromorphic functions on $\mathbb{P}^3$. We can adjoin any roots of these elements to the function field of $\mathbb{P}^3$, and this *Kummer extension*

$$\mathbb{C}(x_1/x_o, x_2/x_o, x_3/x_o)[(\ell_2/\ell_1)^{1/n},\dots,(\ell_k/\ell_1)^{1/n}]$$

defines (the birational class of) an algebraic 3-fold X, which is a ramified cover of $\mathbb{P}^3$ of degree n^{k-1}, branched along the k planes $H_1,\dots,H_k$ with branching degree n along each. X has singularities where the arrangement (the union of the k planes) has singularities, by which we mean

a) more than 3 of the H_i meet at some point

or b) more that 2 of the H_i meet in some line.

X can be resolved by a smooth Y such that the following diagramm commutes:

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ \hat{\mathbb{P}}^3 & \longrightarrow & \mathbb{P}^3 \end{array}$$

where $\hat{\mathbb{P}}^3$ denotes some moniodal transformation of $\mathbb{P}^3$. To construct Y it is sufficient to name some arrangement and an $n\in\mathbb{N}$. This is what we do in this section. In (H), (atrocious!) formula for the characteristic numbers of Y were given in dependance on the combinatorial data of the arrangement and n. For Y with trivial canonical bundle, however, the only non-vanishing Chern number is the euler number, and this can often be calculated by ad-hoc methods, which we will do for the most part here.

To get trivial canonical bundle, we consider arrangements with either $k=8$ and $n=2$ or $k=6$ and $n=3$. If the arrangement is in general position, then the cover X will already be smooth, and is a smooth complete intersection (of Fermat hypersurfaces) of the types listed below. In this case also the euler numbers are well-known and easily calculated:

$k=8$,	$n=2$	$(2,2,2,2)$ in $\mathbb{P}^7$	$e(X)=-128$
$k=6$,	$n=3$	$(3,3)$ in $\mathbb{P}^5$	$e(X)=-144$

To get interesting examples, we may allow *canonical* singularities which *are not terminal* (see §2). These are singularities of the arrangements as follows:

$$n=2 \begin{cases} \text{3-fold line} \\ \text{5-fold point} \end{cases}$$

$$n=3 \quad \text{4-fold point}$$

In addition, for $n=2$, a 4-fold point is an ordinary double point (which is a terminal singularity), given by the equation

$$x^2+y^2+z^2+w^2=0$$

and we can use the *small resolutions* described by Brieskorn. These resolutions are gotten by blowing down either of the rulings of the resolving $\mathbb{P}^1\times\mathbb{P}^1$. This process retains the property of trivial canonical bundle (since the resolving set has codimension 2), but has the disadvantage that the resolution *need not be projective*. In fact, it can occur that the resolving $\mathbb{P}^1$ is homologous to zero, in which case the small resolution cannot be Kaehler, so in particular not projective.

Example 9.1: Take an arrangement of 6 planes with 1,2 or 3 4-fold points. The arrangement with 3 4-fold points, for example, is the arrangement of the facet planes of a *cube* in $\mathbb{P}^3$. Let Y^i be the (desingularisation of the singular) Fermat cover for $n=3$, with i 4-fold points. The euler number can be calculated as follows. Consider Y^i as a *degeneration* of a smooth (3,3) complete intersection Y. Over each singular point of the arrangement lie 3 singular points, each being resolved by a cubic surface with euler number 9. In local coordinates each singularity has the form:

$$x^3+y^3+z^3+w^3=0$$

which has Milnor number 16. It follows that the euler number increases by 24 per singular point, i.e. $e(Y^1)=-72$, $e(Y^2)=0$, and $e(Y^3)=72$. Y^i has the structure of elliptic 3-fold over a cubic surface. $Y^1 \longrightarrow S$ is flat (i.e. all fibres are one dimensional), but Y^2 and Y^3 are not (since they will have cubic surfaces in the fibres). It is easy to describe the elliptic fibering on Y^1. Consider the diagramm

$$\begin{array}{ccc} Y^1 & \longrightarrow & \hat{\mathbb{P}}^3 \\ \downarrow & & \downarrow \\ S & \longrightarrow & \mathbb{P}^2 \end{array} .$$

S is a cover of $\mathbb{P}^2$ of degree 27, branched over the union of 4 lines in general position in $\mathbb{P}^2$. It is easy to see that the *degeneracy locus on S* will be $p^*(h)$ (h the hyperplane class on $\mathbb{P}^2$) which is the proper transform of the plane in $\mathbb{P}^3$ through the 4-fold point and the line where H_5 and H_6 meet, where $H_5$and H_6 are the two planes of the arrangement *not passing through* the singular 4-fold point. $p^*(h)$ is the intersection of S with another cubic surface, a smooth, irreducible curve C with euler number -18. The degenerate fibre over every point of C is of type IV (in Kodaira's list). So we can check the calculation above, since

$$e(Y^1)=e(C).e(IV)=(-18)(4)=-72.$$

Example 9.2: Take an arrangement of 8 planes with 1,2 or 3 5-fold points. Let Y^i be the (smooth) Fermat cover for n=2, with i 5-fold points. On Y^i there are 4 singular points lying over each singular point of the arrangement. In local coordinates these singularities are given by te following two equations:

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 = 0$$

$$c_1z_1^2 + c_2z_2^2 + c_3z_3^2 + c_4z_4^2 + c_5z_5^2 = 0$$

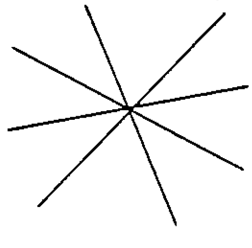
One can compute the Milnor number of this singularity to be 9. On the other hand, each singular point on Y^i is resolved by a (2,2) complete intersection in $\mathbb{P}^4$, which has euler number 8. Therefore the euler number will increase from -128 by 16 per singular point. This yields:

$$e(Y^1)= -64$$

$$e(Y^2)= 0$$

$$e(Y^3)= 64.$$

These examples also have the structure of elliptic fibre spaces. For example, Y^1 fibres over the resolving surface S, which is a (2,2) complete intersection in $\mathbb{P}^3$ (a del Pezzo surface). The degeneracy locus on S is seen to be the intersection of S wth 3 other quadrics in $\mathbb{P}^3$, a curve with 3 components, each one of which has euler number -8. These meet 3 at a time at the 16 inverse images of a point $p\in S$. The singular fibres are of type I_2 over the smooth locus of the curve C. Over the singular points (points of intersection of 3 components), the singular fibres look as follows:



This fibre is not in Kodaira's list. This is to be expected, since the degeneracy locus is not a normal crossings divisor. This exotic fibre has the euler number 5, so we can check the calculation above,

$$e(Y^1) = 3(-8-16)2 + 16\cdot 5 = -144 + 80 = -64.$$

Example 9.3: Consider an arrangement of 8 planes with one or two 3-fold lines and otherwise in general position. Let Y^i be the (smooth) Fermmat cover for n=2, covering the arrangement with i singular lines. We have

$$e(Y^1) = -48, \qquad e(Y^2) = -96$$

Y^i fibres onto a $\mathbb{P}^1$ with fibre a K3-surface. Y^1 has 96 A_1-singularities in the fibres, Y^2 has 144.

Example 9.4: This example is due to Hirzebruch. Using small resolutions of singularities covering 4-fold points for n=2, we can also achieve K_Y trivial. Let L be the arrangement consisting of the 8 facet planes of the octahedron. This arrangement has 12 4-fold points. There are $12\cdot 8 = 96$ singularities which have Milnor number 1. The small resolution therfore increases the euler number by 2/singularity, yielding

$$e(Y) = -128 + 192 = 64.$$

Example 9.5: We can combine 5-fold points and 4-fold points (n=2), using big and little resolutions, respectively, to get smooth (but mabye not projective) 3-folds with trivial canonical bundle. For example, take the 6 facet planes of the cube, add the plane at infinity and one further plane passing through 3 of the corners of the cube. This is an arrangement with 3 5-fold points and 3 4-fold points. If Y is the smooth Fermat cover (with $K_Y \neq \mathcal{O}_Y$) we have $e(Y)=160$, and blowing down the terminal $\mathbb{P}^1\times\mathbb{P}^1$'s to $\mathbb{P}^1$'s, we get the small resolution Y'. Here we have $e(Y')=112$, which is to date the highest known euler number for a 3-fold with trivial canonical bundle. Y' also has the structure of elliptic fibre space over the same (2,2) complete intersection. This example is in fact a further degeneration of example 2.

Example 9.6: Our final example of Fermat cover combines all of the above. Take the arrangement $A_1^3(10)$ and delete 2 of the symmetry planes through opposite edges of the tetrahedron. This is an arrangement with the following data (notations as in (H); $t_q(1)=$ #q-fold lines, $t_p=$ #p-fold points);

$k=8$	$t_3(1)=4$	$t_5=4$	$t_{5,3}=8$	$t_{5,2}=16$
	$t_2(1)=16$	$t_4=1$		$t_{4,2}=6$
		$t_3=4$.		

Let Y be the Fermat cover for $n=2$. Then $e(Y) = 96$, and K_Y contains only the resolving $\mathbb{P}^1\times\mathbb{P}^1$'s of the 4-fold point. Blowing down each $\mathbb{P}^1\times\mathbb{P}^1$ in one direction or the other, we get a 3-fold Y′ with trivial canonical bundle and $e(Y') = 80$. Y′ also has the structure of elliptic fibre space.

9.2. Elliptic 3-folds over $\mathbb{P}^2$ with $K_X = \mathcal{O}_X$

In this section we describe elliptic 3-folds over $\mathbb{P}^2$ with trivial canonical bundle. We do not want to blow up $\mathbb{P}^2$ to get a good model, so we are looking for elliptic 3-folds with either

a) $\Sigma \subset \mathbb{P}^2$ is irreducible

or b) The singularities over the double points of Σ have small resolutions.

From the considerations above the Weierstraβ form will be:

$$L=3H,\quad H \text{ hyperplane class on } \mathbb{P}^2$$
$$g_2\in H^0(\mathbb{P}^2,12H),\qquad g_3\in H^0(\mathbb{P}^2,18H)$$
$$\text{and}\quad \Delta = g_2^3-27g_3^2 \in H^0(\mathbb{P}^2,36H).$$

So we are looking for polynomials of degrees 12, 18 and 36, respectively. The singular fibres are determined by the order of vanishing of g_2, g_3 and Δ (see 3.6). Consider first case b), i.e. Σ is reducible,

$$\Sigma = \sum_{n\geq 1} n\Sigma_{I_n} + \sum (m+6)\Sigma_{I_m^*} + 2\Sigma_{II} + 3\Sigma_{III} + 4\Sigma_{IV} + 8\Sigma_{IV^*} + 9\Sigma_{III^*} + 10\Sigma_{II^*}$$

where Σ_X is the union of components over which the singular fibres are of type X. Small resolutions exist for the following collisions:

collision	resolving fibre	euler #
II & I_0^*	$\overset{1}{\circ}-\overset{2}{\circ}-\overset{3}{\circ}$	4
II & IV^*	$\overset{1}{\circ}-\overset{2}{\circ}-\overset{3}{\circ}-\overset{4}{\circ}-\overset{2}{\circ}$	6
II & IV	$\overset{1}{\circ}-\overset{2}{\circ}$	3
IV & I_0^*	$\overset{1}{\circ}-\overset{2}{\times}-\overset{4}{\circ}-\overset{2}{\circ}$	5
III & I_0^*	$\overset{1}{\circ}-\overset{2}{\times}-\overset{3}{\circ}-\overset{2}{\circ}-\overset{1}{\circ}$	6
I_{k_1} & I_{k_2}	$I_{k_1+k_2}$	k_1+k_2

We insert here a general consideration. Let $\mathcal{J}$ be the functional invariant of an elliptic 3-fold Y over a surface S. If $\mathcal{J}\neq$const., then $\mathcal{J}$ has zeros, therefore also poles, which implies Y has fibres of type I_k. If the singular locus Σ contains two components Σ_1 and Σ_2, such that along Σ_1 we have fibres of type I_k, and along Σ_2 we have fibres of type III, III^* or IV, IV^*,-II, II^*, then the functional invariant $\mathcal{J}$ will be completely indeterminant at the intersection points of Σ_1 and Σ_2. To get a smooth model, one would have to modify S.

Applying this general consideration to the case at hand, we see that we must have either

A) All fibres of type I_k

OR B) $\mathcal{J}$=const.

We now list the possible collisions, and the implications of the above

Fibres of types II, IV and I_o^*

Here, since $\mathcal{J}$=const., necessarily $\mathcal{J}$=0, $g_2=0$.

$$g_3 = f_{II}\cdot(f_{IV})^2\cdot(f_{I_o^*})^3$$

$$\Delta = -27(f_{II})^2\cdot(f_{IV})^4\cdot(f_{I_o^*})^6$$

with $2\deg f_{II} + 4\deg f_{IV} + 6\deg I_o^* = 36$ and the f_X are irreducible.

ex.#	$\deg f_{II}$	$\deg f_{IV}$	$\deg f_{I}*$	e(X)	ex.#	$\deg f_{II}$	$\deg f_{IV}$	$\deg f_{I}*$	e(X)
1	18	-	-	-540	19	6	3	2	-156
2	16	1	-	-456	20	5	5	1	-168
3	15	-	1	-408	21	5	2	3	-132
4	14	2	-	-384	22	4	7	-	-196
5	13	1	1	-336	23	4	4	2	-132
6	12	-	2	-300	24	4	1	4	-120
7	12	3	-	-324	25	3	-	5	-120
8	11	2	1	-276	26	3	6	1	-156
9	10	4	-	-244	27	3	3	3	-108
10	10	1	2	-240	28	2	8	-	-204
11	9	-	3	-162	29	2	5	2	-120
12	9	3	1	-228	30	2	2	4	-96
13	8	5	-	-240	31	1	7	1	-156
14	8	2	2	-192	32	1	4	3	-106
15	7	4	1	-192	33	1	1	5	-96
16	7	1	3	-182	34	-	3	4	-84
17	6	-	4	-132	35	-	6	2	-120
18	6	6	-	-216	36	-	9	-	-216
					37	-	-	6	-108

Fibres of types II & IV^*

Here again we have $g_2=0$, $g_3= f_{II}\cdot(f_{IV}*)^4$, $\Delta = (f_{II})^2\cdot(f_{IV}*)^8$.

example #	deg f_{II}	deg $f_{IV}*$	e(X)
38.	2	4	-60
39.	6	3	-108
40.	10	2	-204
41.	14	1	-348

Fibres of type III & I_o^*

example #	deg f_{III}	deg $f_{I_o}*$	euler (X)
42	2	5	-84
43	4	4	-84
44	6	3	-108
45	8	2	-156
46	10	1	-228
47	12	0	-324

Fibres of type I_k

ex. #	deg f_{I_1}	# cusps	# double points	e(X)
48	36	216	0	-756
49	36	216	36	-648
50	36	216	72	-540
51	36	216	108	-432
52	36	216	144	-324

We just explain the last table. Here we are posed with the following problem. Given a polynomial Δ of degree 36 in the projective plane, are there polynomials of degree 12 and 18, respectively, relatively prime to Δ, such that

$$\Delta=g_2^3-27g_3^2?$$

Furthermore, in order to get a *smooth* model, we must require the following (see [Mi],2.1.):

1) g_3 must be irreducible

2) where Δ is singular, we must have

α) the zero set of g_2 and g_3 meet transversally

β) g_3 is smooth there.

This problem is closely related to the problem of finding all elliptic surfaces S over $\mathbb{P}^1$ with only fibres of type I_k and $\chi(S)=3$. Indeed, restricting an elliptic 3-fold over $\mathbb{P}^2$ to some line gives an elliptic surface S over $\mathbb{P}^1$ with euler number 36, i.e. $\chi(S)=3$. For an elliptic surface over $\mathbb{P}^1$ with ≤ 3 singular fibres we have $\chi(S)\leq 2$, (see (S-H)) so the reduced discriminant must have degree ≥ 4. But we can say more about Δ:

Lemma 9.1.: Δ is singular where $g_2=g_3=0$.

Proof: We are assuming g_3 is irreducible and smooth at $g_2=g_3=0$. Therefore in local coordinates we have:

$$g_3=(x_1=0), \quad g_2=(x_2^{\nu}=0)$$

so $\Delta = x_2^{3\nu}-27x_1^2$ and Δ has a $(3\nu,2)$-cusp at $g_2=g_3=0$.

Remark: Here we are allowing the $\mathcal{J}$-invariant to have points of indeterminancy (see §3.6.).

Now let G_2, G_3 and D denote the *reduced* divisors of g_2, g_3 and Δ, respectively. Then, counting multiplicity, D must have at least 216 cusps, so by the Pluecker formula, (assuming D is irreducible for the moment)

$$g(D) = \frac{(d-1)(d-2)}{2} - 216 \geq 0 \quad \Longrightarrow \quad d \geq 24.$$

So if D is irreducible, then Δ is automatically reduced. We can refine this line of argument. Let $\Delta = \sum n_i\Delta_i$ be the decomposition of Δ into irreducible, reduced factors. Let σ_i=# cusps on Δ_i, d_i=deg Δ_i. Then

$$\sum g(\Delta_i) = \sum[\binom{d_i-1}{2}-\sigma_i] \geq 0$$

$$(1) \qquad \sum(d_i-1)(d_i-1)\geq 532.$$

We also have: $(2) \qquad \sum n_i d_i = 36$. There are only finitely many solutions to (1) & (2). The most obvious one is $k=1$, $d_1=36$, $n_1=1$, an example of which we now give.

Example 9.7: Let w,t be inhomogenous coordinates on $\mathbb{P}^2$. Consider a Fermat quadric and cubic:

$$g_2 = w^2 + t^2 + 1$$
$$g_3 = w^3 + t^3 + 1$$

Claim: $\Delta = g_2^3 - 27g_3^2$ *is an irreducible sextic with 6 cusps at the intersection points of* $(g_2=0)$ and $(g_3=0)$, *and otherwise smooth.*

Proof: Since g_2 and g_3 are irreducible, relatively prime and meet transversally, we have in local coordinates $x_1 = \{g_2=0\}$ and $x_2 = \{g_3=0\}$. Then Δ has a simple cusp at thier intersections. Since Δ has 6 cusps, it follows from the Pluecker formula that

$$\Delta \text{ is reducible} \Longleftrightarrow \Delta = \{\text{line} \cup \text{quintic}\}$$

since quadrics can have no cusps, cubics at most one, quartics at most 3 and quintics at most 6. We have:

$$\frac{\partial\Delta}{\partial x} = 3g_2^2 \frac{\partial g_2}{\partial x} - 54\cdot g_3\frac{\partial g_3}{\partial x}$$

$$\frac{\partial\Delta}{\partial y} = 3g_2^2\cdot\frac{\partial g_2}{\partial y} - 54\cdot g_3\frac{\partial g_3}{\partial y}$$

Setting $P=\frac{\partial\Delta}{\partial x} - \frac{\partial\Delta}{\partial y}$, we get

$$P = (x-y)(6g_2^2 - 162g_3(x+y) = (x-y)\cdot P_1.$$

If Δ is the union of a quintic and a line, there will be 5 singular points (=points of intersection). It is easy to see this cannot occur: If x=y, then $\frac{\partial\Delta}{\partial x} = \frac{\partial\Delta}{\partial y}$ is a polynomial of degree 5, which therefore has at most 4 zeroes on common with Δ (euclidean algorithm), unless it divides Δ. But this is absurd:

$$\Delta = f\cdot\frac{\partial\Delta}{\partial x} \Rightarrow \frac{\partial\Delta}{\partial x} = \frac{\partial f}{\partial y}\cdot\frac{\partial\Delta}{\partial x} + f\cdot\frac{\partial^2\Delta}{\partial x} \Rightarrow \frac{\partial f}{\partial y} = 1,\ f = 0 ,$$

a contradiction. So Δ is irreducible. In fact, it is *smooth* except for the 6 cusps. To see this, write

$$\frac{\partial\Delta}{\partial x} = x(6g_2^2 - 162g_3x) = x\cdot P_2$$

Then we have
$$\Delta = 1/6g_2P_2 - 27g_3^2 + 27g_2g_3x$$
$$= 1/6g_2P_2 + 27g_3(g_2x-g_3).$$

From this, if $\Delta = P_2 = 0$, then

$$g_2x = g_3$$

The reader may check that this condition implies x=0, via symmetry y=0, which does not lie on Δ=0. So Δ is smooth except for cusps.

Now consider the 6-th power map:

$$\varphi:\mathbb{P}^2 \longrightarrow \mathbb{P}^2$$
$$(z_0,z_1,z_2) \longrightarrow (z_0^6,z_1^6,z_2^6)$$

Then $\varphi^{-1}(\Delta)$, which we also denote by Δ, is a plane curve of degree 36, with 216 cusps (since the cusps do not lie on the coordinate axi) and otherwise smooth (since the sextic Δ meets the coordinate axi transversally). The Weierstraß elliptic 3-fold defined by Δ is *smooth* (compare the last remark in {Mi}, p.132). It has fibres of type I_1 over Δ, which is a curve of genus 379, which makes for the euler number -756. It follows that the elliptic 3-fold Y has $e(Y) = -756$. We may modify this by taking different irreducible g_2, g_3's. If Δ (the sextic in the plane) has λ double points ($\lambda \leq 4$) in addition to the 6 cusps, then $\varphi^{-1}(\Delta)$ will have $36 \cdot \lambda$ double points (assuming the double points do not lie on the coordinate axi). We the get an elliptic 3-fold over $\mathbb{P}^2$, which is smooth if g_2 and g_3 both vanish at the double points and g_3 is smooth there, with fibres of type I_2 over the double points of Δ.

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