ON FUNCTION SPACES ON SYMMETRIC SPACES

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ABSTRACT. Let $Y = G/H$ be a semisimple symmetric space. It is shown that the smooth vectors for the regular representation of G on $L^p(Y)$ vanish at infinity.

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1. **Vanishing at infinity**

Let G be a connected unimodular Lie group, equipped with a Haar measure dg, and let $1 \leq p < \infty$. We consider the left regular representation L of G on the function space $E_p = L^p(G)$.

Recall that $f \in E_p$ is called a *smooth vector for L* if and only if the map

$$G \to E_p, \quad g \mapsto L(g)f$$

is a smooth E_p-valued map.

Write $\mathfrak{g}$ for the Lie algebra of G and $\mathcal{U}(\mathfrak{g})$ for its enveloping algebra. The following result is well-known, see [3].

Theorem 1.1. *The space of smooth vectors for L is*

$$E_p^\infty = \{f \in C^\infty(G) \mid L_u f \in L^p(G) \text{ for all } u \in \mathcal{U}(\mathfrak{g})\}.$$

Furthermore, $E_p^\infty \subset C_0^\infty(G)$, the space of smooth functions on G which vanish at infinity.

Our concern is with the corresponding result for a homogeneous space Y of G. By that we mean a connected manifold Y with a transitive action of G. In other words

$$Y = G/H$$

with $H \subset G$ a closed subgroup. We shall request that Y carries a G-invariant positive measure dy. Such a measure is unique up to scale and commonly referred to as Haar measure. With respect to dy we form the Banach spaces $E_p := L^p(Y)$. The group G acts continuously by isometries on E_p via the left regular representation:

$$[L(g)f](y) = f(g^{-1}y) \qquad (g \in G, y \in Y, f \in E_p)\,.$$

We are concerned with the space E_p^∞ of smooth vectors for this representation. The first part of Theorem 1.1 is generalized as follows, see [3], Thm. 5.1.

Theorem 1.2. *The space of smooth vectors for L is*

$$E_p^\infty = \{f \in C^\infty(Y) \mid L_u f \in L^p(Y) \text{ for all } u \in \mathcal{U}(\mathfrak{g})\}.$$

We write $C_0^\infty(Y)$ for the space of smooth functions vanishing at infinity. Our goal is to investigate an assumption under which the second part of Theorem 1.1 generalizes, that is,

$$E_p^\infty \subset C_0^\infty(Y). \tag{1.1}$$

Notice that if H is compact, then we can regard $L^p(G/H)$ as a closed G-invariant subspace of $L^p(G)$, and (1.1) follows immediately from Theorem 1.1.

Likewise, if $Y = G$ regarded as a homogeneous space for $G \times G$ with the left$\times$right action, then again (1.1) follows from Theorem 1.1, since a left$\times$right smooth vector is obviously also left smooth.

However, (1.1) is false in general as the following class of examples shows. Assume that Y has finite volume but is not compact, e.g. $Y = \mathrm{Sl}(2,\mathbb{R})/\mathrm{Sl}(2,\mathbb{Z})$. Then the constant function $\mathbf{1}_Y$ is a smooth vector for E^p, but it does not vanish at infinity.

2. Proof by convolution

We give a short proof of (1.1) for the case $Y = G$, based on the theorem of Dixmier and Malliavin (see [2]). According to this theorem, every smooth vector in a Fréchet representation (π, E) belongs to the Gårding space, that is, it is spanned by vectors of the form $\pi(f)v$, where $f \in C_c^\infty(G)$ and $v \in E$. Let such a vector $L(f)g$, where $g \in E_p = L^p(G)$, be given. Then by unimodularity

$$[L(f)g](y) = \int_G f(x)g(x^{-1}y)\,dx = \int_G f(yx^{-1})g(x)\,dx. \tag{2.1}$$

For simplicity we assume $p = 1$. The general case is similar. Let $\Omega \subset G$ be compact such that $|g|$ integrates to $< \epsilon$ over the complement. Then, for y outside of the compact set $\mathrm{supp} f \cdot \Omega$, we have

$$yx^{-1} \in \mathrm{supp} f \Rightarrow x \notin \Omega,$$

and hence

$$|L(f)g(y)| \leq \sup|f| \int_{x \notin \Omega} |g(x)|\,dx \leq \sup|f|\,\epsilon.$$

It follows that $L(f)g \in C_0(G)$.

Notice that the assumption $Y = G$ is crucial in this proof, since the convolution identity (2.1) makes no sense in the general case.

3. Semisimple symmetric spaces

Let $Y = G/H$ be a semisimple symmetric space. By this we mean:

- G is a connected semisimple Lie group with finite center.
- There exists an involutive automorphism τ of G such that H is an open subgroup of the group $G^\tau = \{g \in G \mid \tau(g) = g\}$ of τ-fixed points.

We will verify (1.1) for this case. In fact, our proof is valid also under the more general assumption that G/H is a reductive symmetric space of Harish-Chandra's class, see [1].

Theorem 3.1. *Let $Y = G/H$ be a semisimple symmetric space, and let $E_p = L^p(Y)$ where $1 \leq p < \infty$. Then*

$$E_p^\infty \subset C_0^\infty(Y)\,.$$

Proof. A little bit of standard terminology is useful. As customary we use the same symbol for an automorphism of G and its derived automorphism of the Lie algebra $\mathfrak{g}$. Let us write $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ for the decomposition in τ-eigenspaces according to eigenvalues $+1$ and -1.

Denote by K a maximal compact subgroup of G. We will and may assume that K is stable under τ. Write θ for the Cartan-involution on G with fixed point group K and write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the eigenspace decomposition of θ. We fix a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p} \cap \mathfrak{q}$.

The simultaneous eigenspace decomposition of $\mathfrak{g}$ under $\operatorname{ad}\mathfrak{a}$ leads to a (possibly reduced) root system $\Sigma \subset \mathfrak{a}^*\backslash\{0\}$. Write $\mathfrak{a}_{\mathrm{reg}}$ for $\mathfrak{a}$ with the root hyperplanes removed, i.e.:

$$\mathfrak{a}_{\mathrm{reg}} = \{X \in \mathfrak{a} \mid (\forall \alpha \in \Sigma)\ \alpha(X) \neq 0\}\,.$$

Let $M = Z_{H\cap K}(\mathfrak{a})$ and $W_H = N_{H\cap K}(\mathfrak{a})/M$.

Recall the polar decomposition of Y. With $y_0 = H \in Y$ the base point of Y it asserts that the mapping

$$\rho : K/M \times \mathfrak{a} \to Y, \quad (kM, X) \mapsto k\exp(X)\cdot y_0$$

is differentiable, onto and proper. Furthermore, the element X in the decomposition is unique up to conjugation by W_H, and the induced map

$$K/M \times_{W_H} \mathfrak{a}_{\mathrm{reg}} \to Y$$

is a diffeomorphism onto an open and dense subset of Y.

Let us return now to our subject proper, the vanishing at infinity of functions in E_p^∞. Let us denote functions on Y by lower case roman letters, and by the corresponding upper case letters their pull backs to $K/M \times \mathfrak{a}$, for example $F = f \circ \rho$. Then f vanishes at infinity on Y translates into

$$\lim_{\substack{X\to\infty \\ X\in\mathfrak{a}}} \sup_{k\in K} |F(kM, X)| = 0\,. \tag{3.1}$$

We recall the formula for the pull back by ρ of the invariant measure dy on Y. For each $\alpha \in \Sigma$ we denote by $\mathfrak{g}^\alpha \subset \mathfrak{g}$ the corresponding root space. We note that $\mathfrak{g}^\alpha$ is stable under the involution $\theta\tau$. Define p_α, resp. q_α, as the dimension of the $\theta\tau$-eigenspace in $\mathfrak{g}^\alpha$ according to eigenvalues $+1, -1$. Define a function J on $\mathfrak{a}$ by

$$J(X) = \left| \prod_{\alpha\in\Sigma^+} [\cosh\alpha(X)]^{q_\alpha} \cdot [\sinh\alpha(X)]^{p_\alpha} \right|\,.$$

With $d(kM)$ the Haar-measure on K/M and dX the Lebesgue-measure on $\mathfrak{a}$ one then gets, up to normalization:

$$\rho^*(dy) = J(X)\, d(k,X) := J(X)\ d(kM)\, dX\,.$$

We shall use this formula to relate certain Sobolev norms on Y and on $K/M \times \mathfrak{a}$. Fix a basis $X_1, \ldots, X_n$ for $\mathfrak{g}$. For an n-tupel $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{N}_0^n$ we define elements $X^{\mathbf{m}} \in \mathcal{U}(\mathfrak{g})$ by

$$X^{\mathbf{m}} := X_1^{m_1} \cdot \ldots \cdot X_n^{m_n}\,.$$

These elements form a basis for $\mathcal{U}(\mathfrak{g})$. We introduce the L^p-Sobolev norms on Y,

$$S_{m,\Omega}(f) := \sum_{|\mathbf{m}| \le m} \left[\int_\Omega |L(X^{\mathbf{m}}) f(y)|^p \, dy \right]^{1/p}$$

where $\Omega \subset Y$, and where $|\mathbf{m}| := m_1 + \ldots + m_n$. Then $f \in E_p^\infty$ if and only if $S_{m,Y}(f) < \infty$ for all m.

Likewise, for $V \subset \mathfrak{a}$ we denote

$$S^*_{m,V}(F) := \sum_{|\mathbf{m}| \le m} \left[\int_{K\times V} |L(Z^{\mathbf{m}}) F(kM, X)|^p \, J(X)\, d(k,X) \right]^{1/p}$$

Here Z refers to members of some fixed bases for $\mathfrak{k}$ and $\mathfrak{a}$, acting from the left on the two variables, and again $\mathbf{m}$ is a multiindex.

Observe that for $Z \in \mathfrak{a}$ we have for the action on $\mathfrak{a}$,

$$[L(Z)F](kM, X) = [L(Z^k)f](k \exp(X) \cdot y_0)$$

where $Z^k := \mathrm{Ad}(k)(Z)$ can be written as a linear combination of the basis elements in $\mathfrak{g}$, with coefficients which are continuous on K. It follows that there exists a constant $C_m > 0$ such that for all $F = f \circ \rho$,

$$S^*_{m,V}(F) \le C_m S_{m,\Omega}(f) \tag{3.2}$$

where $\Omega = \rho(K/M, V) = K \exp(V) \cdot y_0$.

Let $\epsilon > 0$ and set

$$\mathfrak{a}_\epsilon := \{ X \in \mathfrak{a} \mid (\forall \alpha \in \Sigma)\ |\alpha(X)| \ge \epsilon \}\,.$$

Observe that there exists a constant $C_\epsilon > 0$ such that

$$(\forall X \in \mathfrak{a}_\epsilon) \quad J(X) \ge C_\epsilon\,. \tag{3.3}$$

We come to the main part of the proof. Let $f \in E_p^\infty$. We shall first establish that

$$\lim_{\substack{X \to \infty \\ X \in \mathfrak{a}_\epsilon}} F(eM, X) = 0\,. \tag{3.4}$$

It follows from the Sobolev lemma, applied in local coordinates, that the following holds for a sufficiently large integer m (depending only on p and the dimensions of K/M and $\mathfrak{a}$). For each compact symmetric neighborhood V of 0 in $\mathfrak{a}$ there exists a constant $C > 0$ such that

$$\begin{aligned} &|F(eM, 0)| \\ &\leq C \sum_{|\mathbf{m}| \leq m} \left[\int_{K/M \times V} |[L(Z^{\mathbf{m}})F](kM, X)|^p \, d(k, X) \right]^{1/p} \end{aligned} \tag{3.5}$$

for all $F \in C^\infty(K/M \times \mathfrak{a})$. We choose V such that $\mathfrak{a}_\epsilon + V \subset \mathfrak{a}_{\epsilon/2}$.

Let $\delta > 0$. Since $f \in E^p$, it follows from (3.2) and the properness of ρ that there exists a compact set $B \subset \mathfrak{a}$ with complement $B^c \subset \mathfrak{a}$, such that

$$S^*_{m,B^c}(F) \leq C_m S_{m,\Omega}(f) < \delta \tag{3.6}$$

where $\Omega = K \exp(B^c) \cdot y_0$.

Let $X_1 \in \mathfrak{a}_\epsilon \cap (B + V)^c$. Then $X_1 + X \in \mathfrak{a}_{\epsilon/2} \cap B^c$ for $X \in V$. Applying (3.5) to the function

$$F_1(kM, X) = F(kM, X_1 + X),$$

and employing (3.3) for the set $\mathfrak{a}_{\epsilon/2}$, we derive

$$\begin{aligned} &|F(eM, X_1)| \\ &\leq C \sum_{|\mathbf{m}| \leq m} \left[\int_{K/M \times V} |[L(Z^{\mathbf{m}})F_1](kM, X)|^p \, d(k, X) \right]^{1/p} \\ &\leq C' \sum_{|\mathbf{m}| \leq m} \left[\int_{K/M \times B^c} |[L(Z^{\mathbf{m}})F](kM, X)|^p \, J(X) \, d(k, X) \right]^{1/p} \\ &= C' S^*_{m,B^c}(F) \leq C'\delta, \end{aligned}$$

from which (3.4) follows.

In order to conclude the theorem, we need a version of (3.4) which is uniform for all functions $L(q)f$, for $q \in Q \subset G$ a compact subset.

Let $\delta > 0$ be given, and as before let $B \subset \mathfrak{a}$ be such that (3.6) holds. By the properness of ρ, there exists a compact set $B' \subset \mathfrak{a}$ such that

$$QK \exp(B) \cdot y_0 \subset K \exp(B') \cdot y_0.$$

We may assume that B' is W_H-invariant. Then, for each $k \in K$, $X \notin B'$ and $q \in Q$ we have that

$$q^{-1}k \exp(X) \cdot y_0 \notin K \exp(B) \cdot y_0, \tag{3.7}$$

since otherwise we would have

$$k\exp(X)\cdot y_0 \in qK\exp(B)\cdot y_0 \subset K\exp(B')\cdot y_0$$

and hence $X \in B'$.

We proceed as before, with B replaced by B', and with f, F replaced by $f_q = L_q f$, $F_q = f_q \circ p$. We thus obtain for $X_1 \in \mathfrak{a}_\epsilon \cap (B'+V)^c$,

$$|F_q(eM, X_1)| \leq CS^*_{m,(B')^c}(F_q) \leq CC_m S_{m,\Omega'}(f_q)$$

where $\Omega' = K\exp((B')^c)\cdot y_0$.

Observe that for each X in $\mathfrak{g}$ the derivative $L(X)f_q$ can be written as a linear combination of derivatives of f by basis elements from $\mathfrak{g}$, with coefficients which are uniformly bounded on Q. We conclude that $S_{m,\Omega'}(f_q)$ is bounded by a constant times $S_{m,Q^{-1}\Omega'}(f)$, with a uniform constant for $q \in Q$. By (3.7) and (3.6) we conclude that the latter Sobolev norm is bounded from the above by δ.

We derive the desired uniformity of the limit (3.4) for $q \in Q$,

$$\lim_{\substack{X\to\infty \\ X\in\mathfrak{a}_\epsilon}} \sup_{q\in Q} |F_q(eM, X)| = 0\,. \tag{3.8}$$

Finally we choose an appropriate set Q. Let $\epsilon > 0$ be arbitrary. There exists $X_1, \dots, X_N \in \mathfrak{a}$ such that

$$\mathfrak{a} = \bigcup_{j=1}^{N} (X_j + \mathfrak{a}_\epsilon)\,. \tag{3.9}$$

Set $a_j = \exp(X_j) \in A$ and define a compact subset of G by

$$Q := \bigcup_{j=1}^{N} Ka_j\,.$$

Then, for every $X \in \mathfrak{a}$ we have $X - X_j \in \mathfrak{a}_\epsilon$ for some j. Hence with $q = k\exp(X_j)$

$$\lim_{X\to\infty} F(kM, X) = \lim_{X\to\infty} F_q(eM, X - X_j) = 0,$$

as was to be shown. $\square$

Remark. Let $f \in L^2(Y)$ be a K-finite function which is also finite for the center of $\mathcal{U}(\mathfrak{g})$. Then it follows from [4] that f vanishes at infinity. The present result is more general, since such a function necessarily belongs to E_2^∞.

References

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