Admissible measures for standard L–functions and nearly holomorphic Siegel modular forms

A. A. Panchishkin*

Institut Fourier, B.P.74, 38402 St.–Martin d'Hères, FRANCE;
e-mail: panchish@mozart.ujf-grenoble.fr, FAX: 33 (0) 4 76 51 44 78

§0. Introduction

0.1. Let p be a prime number (we often assume $p \geq 5$).

The purpose of this paper is to give a new conceptual construction of admissible measures (in the sense of Amice-Vélu) attached to a standard L-function of a Siegel cusp eigenform. For this purpose we use the theory of p-adic integration in spaces of nearly holomorphic Siegel modular forms (in the sense of Shimura, see [Shi2000]) over an $\mathcal{O}$-algebra A where $\mathcal{O}$ is the ring of integers in a finite extension K of $\mathbb{Q}_p$. Often we simply assume that $A = \mathbb{C}_p$.

We view such modular forms as certain formal expansions. In a previous paper [PaAD] we studied the action of certain differential operators on Siegel-Eisenstein distributions and on theta distributions. The important property of these arithmetical differential operators is their commutation with Hecke operators (under an appropriate normalization). Their action produces natural families of distributions with values in spaces of nearly holomorphic Siegel modular forms.

Denote by $\mathcal{M} = \mathcal{M}(A)$ an A-module (or simply a $\mathbb{C}_p$-vector space) of nearly holomorphic Siegel modular forms. In the present paper we apply to these distributions the canonical projector $\pi_\alpha : \mathcal{M} \to \mathcal{M}^\alpha$ onto the primary (characteristic) A-submodule associated to a non-zero eigenvalue $\alpha \in A^\times$ of the Frobenius operator $U = \Pi^+(p) = U_p$ (the

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Atkin-Lehner operator in the Siegel modular case; $\mathcal{M}^\alpha$ is the maximal A-submodule on which $U - \alpha I$ is nilpotent). This operator acts similar to the trace operator lowering the level of modular forms. On the other hand, U is invertible on $\mathcal{M}^\alpha$ if $\alpha \in A^\times$ so that one can glue its action on forms of various levels. In this way one obtains the desired distributions with values in a finite dimensional vector space starting from naturally defined distributions with values in spaces of modular forms (like Siegel-Eisenstein distributions, theta distributions etc.). In order to obtain from them numerically valued distributions interpolating critical values attached to standard L-functions of Siegel modular forms one applies a suitable linear form coming from the Petersson scalar product (using the Andrianov identity of Rankin type for the standard L-function).

In our previous works [PaAdm], [PaLNM] we constructed the non–Archimedean interpolation of these special values fixing a non zero Satake p–parameter $\alpha_0(p)$ of the cusp eigenform f. It follows that the normalized critical values $\mathcal{D}(s, f, \chi)$ of the standard L-function can be explicitely rewritten in terms of certain $\mathbb{C}_p$–valued integrals of admissible p–adic measures (over a profinite group of the type $\mathbb{Z}_p^\times$ of p–adic units) provided the character χ is non–trivial. More explicit description of these special values was given by M.Courtieu in his PhD thesis (Institut Fourier, 2000) using detailed study of the action of the Shimura differential operator on Siegel modular forms.

In the present paper we give a conceptual explanation of these p–adic properties satisfyied by the special values of the standard L-function $\mathcal{D}(s, f, \chi)$, where f is a Siegel cusp form of an even degree m and of weight $k > 2m+2$, χ is a varying Dirichlet character. We show that these admissible measures can be lifted to arithmetical nearly holomorphic Siegel modular forms studied by G.Shimura [Shi2000]. This lifting is given by a universal sequence $\Phi_s^\pm(\chi)$ of distributions with values in arithmetical nearly holomorphic Siegel modular forms (for critical pairs (s, χ), see Proposition 5.4). It would be interesting to extend these lifting results to Siegel cusp eigenforms of odd degree, using the method of Böcherer-Schmidt [BöSch].

0.2. Arithmetical nearly holomorphic Siegel modular forms admit two different descriptions:

• as formal power series $g = \sum_{\xi \in B_m} a(\xi, R_{i,j}) q^\xi \in \bar{\mathbb{Q}}[\![q^{B_m}]\!][R_{i,j}]$ such that for all $R = (R_{i,j})_{i,j=1,\cdots,m} = (4\pi \mathrm{Im}(z))^{-1} = (2\pi i(\bar{z} - z))^{-1}$ and $z \in \mathbb{H}_m$ in the Siegel upper half plane of degree m the series converges to a $\mathcal{C}^\infty$-Siegel modular form of a given weight k and character ψ;

• as certain $\mathcal{C}^\infty$-Siegel modular forms g taking values in $\bar{\mathbb{Q}}$ at all CM-points (up to a factor independnt of a concrete form).

Throughout the paper we fix embeddings

$$i_\infty : \bar{\mathbb{Q}} \to \mathbb{C}, \qquad i_p : \bar{\mathbb{Q}} \to \mathbb{C}_p,$$

and we shall often view algebraic numbers (via these embeddings) as both complex and p–adic numbers, where $\mathbb{C}_p = \widehat{\bar{\mathbb{Q}}}_p$ is the Tate field (the completion of a fixed algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$), which is endowed with a unique norm $|\cdot|_p$ such that $|p|_p = p^{-1}$.

0.3. For a fixed positive integer $N \in \mathbb{N}$ consider the profinite group

$$Y = \varprojlim_v Y_v, \quad \text{where} \quad Y_v = (\mathbb{Z}/Np^v\mathbb{Z})^\times.$$

There is a natural projection $y_p : Y \to \mathbb{Z}_p^\times$.

Definition 0.4. a) For $h \in \mathbb{N}, h \geq 1$ let $\mathcal{C}^h(Y, A)$ denote the A-module of *locally polynomial functions* of degree $< h$ of the variable $y_p : Y \to \mathbb{Z}_p^\times \hookrightarrow A^\times$; in particular,

$$\mathcal{C}^1(Y, A) \subset \mathcal{C}^{loc-const}(Y, A)$$

(the A-submodule of *locally constant functions*). We adopt the notation $\Phi(\mathcal{U}) := \Phi(\chi_\mathcal{U})$ for the characteristic function $\chi_\mathcal{U}$ of an open subset $\mathcal{U} \subset Y$. Let also denote $\mathcal{C}^{loc-an}(Y, A)$ the A-module of *locally analytic functions* and $\mathcal{C}(Y, A)$ the A-module of *continous functions* so that

$$\mathcal{C}^1(Y, A) \subset \mathcal{C}^h(Y, A) \subset \mathcal{C}^{loc-an}(Y, A) \subset \mathcal{C}(Y, A).$$

b) For a given positive integer h an *h-admissible measure* on Y with values in M is an A-module homomorphism

$$\tilde{\Phi} : \mathcal{C}^h(Y, A) \to M$$

such that for all $a \in Y$ and for $v \to \infty$

$$\left| \int_{a+(Np^v)} (y_p - a_p)^j d\tilde{\Phi} \right|_{p,M} = o(p^{-v(j-h)}) \quad \text{for} \quad \text{all} \quad j = 0, 1, \dots, h-1,$$

where $a_p = y_p(a)$. We adopt the notation $(a)_v = a + (Np^v)$ for both an element of Y_v and the corresponding open compact subset of Y.

0.5. We explain in Section 3, how to construct an h-admissible measure $\widetilde{\Phi}^\alpha : \mathcal{C}^h(Y, A) \to \mathcal{M}(A)$ out of a sequence of distributions

$$\Phi_j : \mathcal{C}^1(Y, A) \to \mathcal{M}$$

with values in an A-module $M = \mathcal{M}(A)$ of nearly holomorphic modular forms over A (for all $j \in \mathbb{N}$ with $j \leq h-1$, where A is an $\mathcal{O}$-algebra, and $\alpha \in A^\times$ is a fixed non-zero eigenvalue of the Frobenius operator $U = U(p) = \Pi^+(p)$ acting on

$$g = \sum_{\xi \in B_m, \underline{n} \geq 0} a(\xi, \underline{n}) R^{\underline{n}} q^\xi \in A[\![q^{B_m}]\!][R_{i,j}],$$

by

$$g|U(p) = \sum_{\xi \in B_m, \underline{n} \geq 0} a(p\xi, \underline{n})(pR)^{\underline{n}} q^\xi$$

(over the complex numbers $R = (4\pi)^{-1} \mathrm{Im}(z)^{-1}$ and this notation corresponds to

$$\begin{aligned} f|U(p)(z) = & \sum_{{}^t u = u \in \mathrm{M}_m(\mathbb{Z}) \bmod p} f|_k \begin{pmatrix} 1_m & u \\ 0_m & p1_m \end{pmatrix} = \\ p^{-\kappa m} & \sum_{{}^t u = u \in \mathrm{M}_m(\mathbb{Z}) \bmod p} f((z+u)/p), \\ \mathrm{Im}\,((z+u)/p) & = \mathrm{Im}(z/p) = \mathrm{Im}(z)/p = (4\pi)^{-1}(pR)^{-1}. \end{aligned}$$

(we use the Petersson-Andrianov notation for the action of matrices). Then we consider (for an $\alpha \in A^\times$) the canonical projection operator $\pi_\alpha : \mathcal{M}(A) \to \mathcal{M}(A)^\alpha$. We define an A-linear map

$$\widetilde{\Phi}^\alpha : \mathcal{C}^h(Y, A) \to \mathcal{M}$$

on local monomials y_p^j by

$$\int_{(a)_v} y_p^j \, d\widetilde{\Phi}^\alpha = \pi_\alpha(\Phi_j((a)_v))$$

where $\Phi_j : \mathcal{C}^1(Y, A) \to \mathcal{M}(A)$ are certain $\mathcal{M}(A)$-valued distributions on Y for all $j = 0, 1, \ldots, h-1$.

0.6. The standard zeta function of a Siegel cusp eigenform.

In order to describe the main result of the paper let

$$f = \sum_{\xi > 0} a(\xi) e_m(\xi z)$$

be a Siegel cusp form of the even degree m of weight k on the congruence subgroup

$$\Gamma_0(C) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp_m(\mathbb{Z}) \mid c \equiv 0 \bmod C \right\}$$

with a Dirichlet character ψ mod C. Here z belongs to the Siegel upper half plane

$$\mathbb{H}_m = \left\{ z \in GL_m(\mathbb{C}) | z =^t z, \mathrm{Im}(z) > 0 \right\}$$

and we adopt the standard notation $e_m(z) = \exp(2\pi i \mathrm{tr}(z))$.

Suppose that f is an eigenfunction of the global Hecke algebra

$$\mathcal{L}^{(m)}(C) = \otimes_{q \nmid C} \mathcal{L}^{(m)}_{(q)}(C)$$

with the eigenvalue given by a homomorphism $\lambda_f : \mathcal{L}^{(m)}(C) \to \mathbb{C}$, i.e.

$$f \mid X = \lambda_f(X) f \text{ for all } X \in \mathcal{L}^{(m)}_{(q)}(C).$$

Let

$$\alpha_0(q), \alpha_1(q), ..., \alpha_m(q)$$

be the $(m+1)$–tuple of Satake q–parameters of λ_f, which uniquely determine λ_f. It is known [An79] that the relation

$$\alpha_0^2(q)\alpha_1(q)...\alpha_m(q) = \psi(q)^m q^{m(k-(m+1)/2)}$$

holds. Recall that the standard zeta function of f with a Dirichlet character χ mod N is defined as the Euler product

$$D(s,f,\chi)=\prod_{q\not| C} D^{(q)}(s,f,\chi), \tag{0.1}$$

with

$$D^{(q)}(s,f,\chi)^{-1}=$$
$$\left(1-\frac{\chi(q)\psi(q}{q^s}\right)\prod_{i=1}^{m}\left(1-\frac{\chi(q)\psi(q)\alpha_i(q)}{q^s}\right)\left(1-\frac{\chi(q)\psi(q)\alpha_i^{-1}(q)}{q^s}\right),$$

(the product is absolutely convergent for $\mathrm{Re}(s)>1+m$). Recall the three types of the normalized zeta function: for $\kappa=(m+1)/2$, $s\in\mathbf{C}$

$$\begin{aligned}&\mathcal{D}^*(s,f,\chi)=\\&(2\pi)^{-m(s+k-\kappa)}\Gamma((s+\delta)/2)\prod_{j=1}^{m}\Gamma(s+k-j)\mathcal{D}(s,f,\chi),\end{aligned} \tag{0.2}$$

$$\begin{aligned}&\mathcal{D}^-(s,f,\chi)=\Gamma((s+\delta)/2)^{-1}\mathcal{D}^*(s,f,\chi)=\\&(2\pi)^{-m(s+k-\kappa)}\prod_{j=1}^{m}\Gamma(s+k-j)\mathcal{D}(s,f,\chi),\end{aligned} \tag{0.3}$$

$$\begin{aligned}&\mathcal{D}^+(s,f,\chi)=\frac{2\,i^\delta\Gamma(s)\cos(\pi(s-\delta)/2)}{(2\pi i)^s}\mathcal{D}^-(s,f,\chi)=\\&\frac{i^\delta\pi^{(1-2s)/2}}{\Gamma((1+\delta-s)/2)}\mathcal{D}^*(s,f,\chi),\end{aligned} \tag{0.4}$$

where $\delta=0$ or 1 according as $\psi\chi(-1)=(-1)^\delta$ with $f\in\mathcal{S}_k^m(C,\psi)$ be an eigenfunction of the global Hecke algebra $\mathcal{L}^m(C)=\otimes_{q\not| C}\mathcal{L}_q^m(C)$. Define

$$h=[2\mathrm{ord}_p\alpha_0(p)]+1,$$

Assume that $a(\xi_0)\neq 0$ for some $\xi_0>0$ with $\det(2\xi_0)=1$.

Moreover, we make the essential assumption (*) that

$$a_0(\xi_0)\neq 0 \tag{$*$}$$

for a Fourier coefficient $a_0(\xi_0)$ of the U-eigenform $f_0(z)=\sum_{\xi>0}a_0(\xi)e_m(\xi z)$ defined in Section 4 and in §1, (1.47), chapter 2 of [PaLNM]. The importance of the above non–

vanishing condition was pointed out to the author by S. Böcherer. It turns out that it actually may not be satisfied by certain Eisenstein series (note that we deal here with cusp forms). However it is not known in general, whether there exist cusp forms violating the non–vanishing condition.

Main Theorem 0.7. *Let $f \in \mathcal{S}_m^k(N,\psi)$ be a Siegel cusp eigenform of even degree m and of weight $k > 2m+2$ satisfying the above nonvanishing condition (*). Define $h = [2\mathrm{ord}_p(\alpha_0(p))] + 1$ then for any positive integer $c > 1$, $(c, Np) = 1$ there exist two $\mathbb{C}_p$-valued h-admissible measures $\mathcal{D}^{c\pm}$ on Y such that:*

(i) For all pairs (s,χ) with $s \in \mathbb{Z}$, $0 \le s \le k-m-2$ and a primitive non-trivial Dirichlet character $\chi \in X_S^{\mathrm{tors}}$ of conductor $C_\chi = p^v$ one has

- $$\int_Y \chi x_p^s \, d\mathcal{D}^{c+} = \frac{G_{C_\chi}(1_m,\chi) C_\chi^{m(2s+2k-m)/4}}{(iC_\chi)^{m^2/2}\alpha_0(C_\chi^2)} \frac{C_{\bar\chi\psi}^{s+1}}{G(\bar\chi\psi)}(1-(\bar\chi\psi)^2(c)c^{-2s-2}) \cdot \\ \cdot \prod_{q|N} \frac{(1-(\chi\bar\psi)_0(q)q^s)}{(1-(\bar\chi\psi)_0(q)q^{-s-1})} \frac{\mathcal{D}^+(s+1,f,\bar\chi)}{\langle f,f\rangle_N}.$$
- $$\int_Y \chi x_p^s \, d\mathcal{D}^{c-} = \frac{G_{C_\chi}(1_m,\bar\chi) C_\chi^{m(2-2s+2k-m)/4}}{(iC_\chi)^{m^2/2}\alpha_0(C_\chi^2)}(1-(\bar\chi\bar\psi)^2(c)c^{-2s-2}) \cdot \\ \cdot \frac{\mathcal{D}^-(-s,f,\chi)}{\langle f,f\rangle_N}.$$

These integrals are algebraic numbers $\bar{\mathbb{Q}}$ viewed as elements of the Tate field $\mathbb{C}_p$ (via the fixed embedding $i_p : \bar{\mathbb{Q}} \to \mathbb{C}_p$).

(ii) If $\mathrm{ord}_p(\alpha_0(p)) = 0$ (i.e. f is p-ordinary), then the measures $\mathcal{D}^{c\pm}$ are bounded $\mathbb{C}_p$-analytic functions.

Content of the paper: in Section 1 we recall some properties and definitions concerning nearly holomorphic Siegel modular forms. Then in Section 2 we describe algebraic differential operators and their action on Siegel-Eisenstein distributions. Then we give in Section 3 a general result on admissible measures with values in nearly holomorphic Siegel modular forms. In Section 4 we describe the definition of the standard zeta functions $\mathcal{D}(s,f,\chi)$, where f is a Siegel cusp form of the degree m and of the weight k, and we give their critical values. Then in Section 5 we construct certain algebraic linear forms on arithmetical nearly holomorphic Siegel modular forms. In Section 6 we prove the Main theorem using congruences between nearly holomorphic modular forms. We give a more detailed formulation of our main results in Theorem 6.9 containing also analytic properties of the p-adic Mellin transforms of $\mathcal{D}^{c\pm}$, and their uniqueness property.

§1. Nearly holomorphic Siegel modular forms.

§2. Algebraic differential operators and Siegel-Eisenstein distributions.

§3. A general result on admissible measures with values in nearly holomorphic Siegel modular forms.

§4. The standard L function of a Siegel cusp eigenform and its critical values.

§5. Algebraic linear forms on arithmetical nearly holomorphic Siegel modular forms.

§6. Congruences between nearly holomorphic modular forms and proof of the Main theorem.

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In Bonn I understood how to use the Frobenius operator in order to prove congruences between nearly holomorphic Siegel modular forms, and then I prepared the manuscript consulting the PhD Thesis of M.Courtieu (where only the holomorphic forms where used).

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§1. Nearly holomorphic Siegel modular forms

1.1. We use an algebraic version of nearly holomorphic Siegel modular forms over an $\mathcal{O}$-algebra A where $\mathcal{O}$ is the ring of integers in a finite extension K of $\mathbb{Q}_p$. We fix embeddings $i_\infty\colon \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $i_p\colon \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ ($\mathbb{C}_p = \hat{\bar{\mathbb{Q}}}_p$ is the Tate field). We assume that A is a closed subring of $\mathbb{C}_p$ containing $i_p(\mathbb{Q}_p)$.

Recall some notations and definitions concerning Siegel modular forms.

Let A be a commutative ring with identity, then $\mathrm{M}_{r,s}(A)$ denote the set of all $(r \times s)$-matrices with coefficients in A. For $z \in \mathrm{M}_r(\mathbb{C})$ put $e_r(z) = e(\mathrm{tr}(z))$ with $e(u) = \exp(2i\pi u)$ for $u \in \mathbb{C}$. We denote by ${}^t z \in \mathrm{M}_{r,s}(A)$ the matrix, which is transpose to $z \in \mathrm{M}_{r,s}(A)$, and write $\xi[\eta]$ for ${}^t\eta\xi\eta$. For an invertible matrix ξ we put $\xi^\star = {}^t\xi^{-1}$. If ξ is a hermitian matrix then we write $\xi \geq 0$ or $\xi > 0$ according as ξ is non negative or positive definite.

Let $\mathbb{H}_m$ denote the Siegel upper half plane on the degree m,

$$\mathbb{H}_m = \{z \in \mathrm{M}_m(\mathbb{C}) \quad | \quad {}^t z = z = x + iy, \quad y > 0\},$$

so that $\mathbb{H}_m$ is a complex analytic variety whose dimension is denoted by $\langle m \rangle = m(m+1)/2$.

Let the symbol A_m denote the lattice of all half integral symetric matrices in the vector space $V = V_m = \{y \in \mathrm{M}_m(\mathbb{R}) \ | \ {}^t y = y\}$. This lattice is dual to the latice $L = \mathrm{M}_m(\mathbb{Z}) \cap V$ with respect to the pairing given by $(u, v) \longmapsto e_m(uv)$. Let $G = \mathrm{GSp}_{2m}$ be the algebraic subgroup of GL_{2m} defined by

$$G_A = \{\gamma \in \mathrm{GL}_{2m}(A) \quad | \quad {}^t\gamma J_m \gamma = \nu(\gamma) J_m, \nu(\gamma) \in A^\times\},$$

for any commutative ring A, where

$$J_{2m} = \begin{pmatrix} 0_m & -1_m \\ 1_m & 0_m \end{pmatrix}.$$

The elements of G_A are characterized by the conditions

$$b^t a - a^t b = d^t c - c^t d = 0_m, \quad d^t a - c^t b = 1_m.$$

The multiplier ν defines a homomorphism $\nu : G_A \to A^\times$ so that $\nu(\gamma)^{2m} = \det(\gamma)^2$ and $\ker(\nu)$ is denoted by $\mathrm{Sp}_m(A)$. We also put

$$G_\infty = G_\mathbb{R}, G_\infty^+ = \{\gamma \in G_\infty \mid \nu(\gamma) > 0\}, G_\mathbb{Q}^+ = G_\infty^+ \cap G_\mathbb{Q}.$$

The group G_∞^+ acts transitively on the upper half plane $\mathbb{H}_m$ by the rule

$$z \longmapsto \gamma(z) = (az+b)(cz+d)^{-1} \left(\gamma = \begin{pmatrix} a\ b \\ c\ d \end{pmatrix} \in G_\infty^+, \quad z \in \mathbb{H}_m\right)$$

so that the scalar matrices acts trivially, and $\mathbb{H}_m$ can be identified with a homogeneous space of the group $\mathrm{Sp}_m(\mathbb{R})$. Let K_m denote the stabilizer of the point $i1_m \in \mathbb{H}_m$ in the group $\mathrm{Sp}_m(\mathbb{R})$,

$$K_m = \{\gamma \in \mathrm{Sp}_m(\mathbb{R}) \mid \gamma(i1_m) = i1_m\},$$

then there is a bijection $\mathrm{Sp}_m(\mathbb{R})/K_m \simeq \mathbb{H}_m$ and $K_m = \mathrm{Sp}_m(\mathbb{R}) \cap \mathrm{SO}_{2m}$. The group K_m is a maximal compact subgroup of the Lie group $\mathrm{Sp}_m(\mathbb{R})$ and it can be identified with the group $\mathrm{U}(m)$ of all unitary $m \times m$-matrices via the map $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \longmapsto a + ib$.

For a function $f : \mathbb{H}_m \longrightarrow \mathbb{C}$ of the form

$$f = \sum_{\xi \in A_m} c(\xi) e_m(\xi z)$$

and for an integer k we use the notation of Petersson for the action of $\gamma \in \mathrm{GSp}_{2m}$ on f:

$$(f|_k\gamma)(z) = \det(\gamma)^{k-\kappa} \det(cz+d)^{-k} f(\gamma(z)) \quad (\text{where} \quad \kappa = (m+1)/2),$$

we put also for $N \in \mathbb{N}$:

$$\begin{aligned}
f|U(N)(z) &= \sum_{\xi \in A_m} c(N\xi) e_m(\xi z) \\
&= \sum_{{}^t u = u \in \mathrm{M}_m(\mathbb{Z}) \bmod N} f|_k \begin{pmatrix} 1_m & u \\ 0_m & N1_m \end{pmatrix}, \\
f|V(N)(z) &= \sum_{\xi \in A_m} c(\xi) e_m(N\xi z) \quad = \quad f(Nz) \\
&= N^{m(\kappa-k)} f|_k \begin{pmatrix} N1_m & 0_m \\ 0_m & 1_m \end{pmatrix}, \\
f|W(N)(z) &= \det(\sqrt{N} z)^{-k} f(-(Nz)^{-1}) \\
&= N^{m(\kappa-k/2)} f|_k W(N), \quad W(N) = \begin{pmatrix} 0_m & -1_m \\ N1_m & 0_m \end{pmatrix}.
\end{aligned}$$

There are the following relations for these operators:

$$(f|W(N))|W(N) = (-1)^{mk} f, \; f|W(N'N) = N^{mk/2} f|W(N')V(N).$$

We adopt also the notations

$$f^\rho = \sum_{\xi \in A_m} \overline{c(\xi)} e_m(\xi z), \quad Y = \{y \in \mathrm{M}_m(\mathbb{R}) \quad | \quad {}^t y = y > 0\},$$

$$B_m = \{h \in A_m \mid h \geq 0\}, \quad C_m = \{h \in A_m \mid h > 0\}.$$

The bloc notation $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}_{2m}(\mathbb{Z})$ defines the congruence subgroup

$$\Gamma_0^m(N) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{2m}(\mathbb{Z}) | c \equiv 0 \bmod N \right\}$$

of the Siegel modular group $\Gamma^m = \mathrm{Sp}_{2m}(\mathbb{Z})$, and its normal subgroup

$$\Gamma_1^m(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{2m}(\mathbb{Z}) \middle| \det(d) \equiv 1 \bmod N, c \equiv 0 \bmod N \right\} \subset \Gamma_0^m(N).$$

A classical Siegel modular form $f \in \mathcal{M}_k(N, \psi)$ of weight k and character ψ for $\Gamma_0^m(N)$ is a holomorphic function $f\colon \mathbb{H}_m \to \mathbb{C}$ such that for every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0^m(N)$ one has

$$f(\gamma(z)) = \psi(\det d) \det(cz + d)^k f(z)$$

(for $m > 1$ the regularity at ∞ is automatically satisfied by Koecher).

The Fourier expansion of such f uses the semi-group B_m:

$$f(z) = \sum_{\xi \in B_m} a(\xi) q^\xi \in \mathbb{C}[\![q^{B_m}]\!]$$

where the symbols

$$q^\xi = \exp(2\pi i \mathrm{tr}(\xi z)) = \prod_{i=1}^m q_{ii}^{\xi_{ii}} \prod_{i<j} q_{ij}^{2\xi_{ij}} \subset \mathbb{C}[\![q_{11}, \ldots, q_{mm}]\!][q_{ij}, q_{ij}^{-1}]_{i,j=1,\cdots,m}$$

are used (with $q_{ij} = \exp(2\pi(\sqrt{-1} z_{i,j})))$; they form a multiplicative semi-group so that one may consider f as a formal q-expansion and one can introduce Siegel modular forms over A as certain elements of $A[\![q^{B_m}]\!]$. Let $\tilde{\mathcal{M}}_k(N, \psi)$ denote the complex vector space of $\mathcal{C}^\infty$-Siegel modular forms of weight k and character ψ for $\Gamma_0^m(N)$.

For a non negative integer r and variables q, $R = (R_{i,j})_{i,j=1,\ldots,m}$ consider the ring $A[\![q^{B_m}]\!][R_{i,j}]$ (over the complex numbers this notation corresponds to $q^\xi = \exp(2\pi i \mathrm{tr}(\xi z))$,

$R = (4\pi \mathrm{Im}(z))^{-1}$). Consider the following A-submodules

$$P_r(A) = \Big\{ g = \sum_{\xi \in B_m} a(\xi, R_{i,j}) q^\xi \in A[\![q^{B_m}]\!][R_{i,j}] \Big| \\ a(\xi, R_{i,j}) \in A[R_{i,j}], \deg_R a(\xi, R_{i,j}) \le r \Big\} \subset A[\![q^{B_m}]\!][R_{i,j}]$$

Consider also for any natural number a the A-modules

$$Q_{r,a} = \Big\{ g = \sum_{\xi \in B_m} a(\xi, R_{i,j}) q^{a^{-1}\xi} \Big\},$$

and put $Q_r = \bigcup_{a \ge 1} Q_{r,a}$.

Definition 1.2. Let $\mathcal{M}^m_{r,k}(N, \psi, A)$ denote the A-submodule of $A[\![q^{B_m}]\!][R_{i,j}]$ generated by all $F = \sum\limits_{\xi \in B_m} a(\xi, R) q^\xi$ such that all $a(\xi, R) \in A[R_{i,j}]_{i,j=1,\ldots,m}$ are polynomials of degree $\le r$ over $A \subset \mathbb{C}_p$ which belong to

$$i_p(\bar{\mathbb{Q}})[R_{i,j}]_{i,j=1,\ldots,m}$$

and

$$F = \sum_{\xi \in B_m} i_p^{-1}(a(\xi, R)) \Big|_{R=(4\pi Y)^{-1}} q^\xi \in \mathcal{M}^m_{r,k}(N), \psi, \bar{\mathbb{Q}}) \subset \mathcal{M}^m_{r,k}(N), \psi, \mathbb{C})$$

is a $\mathcal{C}^\infty$-complex function on $\mathbb{H}_m$ satisfying the following two conditions

$$\forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^m(N) \quad F|_k \gamma = \psi(\det(d)) F \tag{1.1}$$

$$\forall \gamma \in Sp_{2m}(\mathbb{Z}) \ F|_k \gamma \in Q_r \tag{1.2}$$

(a complex nearly holomorphic modular form of type r and weight k over $\bar{\mathbb{Q}}$ in the sense of Shimura, the second condition is necessary only for $m = 1$).

Put

$$\mathcal{M}^m_{r,k}(A) = \bigcup_{v \ge 1} \mathcal{M}^m_{r,k}(Np^v; A) \tag{1.3}$$

(an A-module of nearly holomorphic Siegel modular forms of type r and level Np^v ($v \ge 0$) with coefficients in A).

1.3. We may view a given cusp form

$$f = \sum_{\xi \in B_m} a(\xi; f) q^\xi \in A[\![q^{B_m}]\!],$$

as an element of $\mathcal{M}^m_{r,k}(A) \subset A[\![q^{B_m}]\!][R_{i,j}]$ in the formal q-expansion ring $A[\![q^{B_m}]\!][R_{i,j}]$).

Let $f_0 = \sum\limits_{\xi\in B_m, \underline{n}\geq 0} a(\xi, \underline{n}; f_0) R^{\underline{n}} q^\xi \in A[\![q^{B_m}]\!][R_{ij}]$ denote an eigenfunction of U with the eigenvalue $\alpha \in A$, $(U(f_0) = \alpha f_0)$ so that for any fixed $\xi = p^m \cdot \xi_0$ we have

$$p^{v|\underline{n}|} a(p^v \xi_0, \underline{n}, ; f_0) = \alpha^m a(\xi_0, \underline{n}; f_0) \in A,$$

with $|\underline{n}| = \sum_{ij} n_{ij}$, $(pR)^{v\underline{n}} = p^{v|\underline{n}|} R^{v\underline{n}}$.

1.4. Define the action of the operator $U = U(p)$ on

$$g = \sum_{\xi\in B_m, \underline{n}\geq 0} a(\xi, \underline{n}) R^{\underline{n}} q^\xi \in A[\![q^{B_m}]\!][R_{i,j}],$$

by

$$g|U(p) = \sum_{\xi\in B_m, \underline{n}\geq 0} a(p\xi, \underline{n}) (pR)^{\underline{n}} q^\xi$$

(over the complex numbers $R = (4\pi)^{-1}\mathrm{Im}(z)^{-1}$ and this notation corresponds to

$$\begin{aligned} f|U(p)(z) = & \sum_{{}^t u = u\in \mathrm{M}_m(\mathbb{Z}) \bmod p} f|_k \begin{pmatrix} 1_m & u \\ 0_m & p1_m \end{pmatrix} = \\ p^{-\kappa m} & \sum_{{}^t u = u\in \mathrm{M}_m(\mathbb{Z}) \bmod p} f((z+u)/p), \\ & \mathrm{Im}\left((z+u)/p\right) = \mathrm{Im}(z/p) = \mathrm{Im}(z)/p = (4\pi)^{-1}(pR)^{-1} \end{aligned}$$

An element $\alpha \in A$ is called a U- characteristic value on $\mathcal{M} = \mathcal{M}^m_{r,k}(A)$ attached to a cusp form if there exists a cusp form f_0 which is an eigenfunction of U with the eigenvalue $\alpha \in A$ $(U(f_0) = \alpha f_0)$.

Definition 1.5. Let $\alpha \in A$. a) The *α-proper submodule* $\mathcal{M}^{(\alpha)} \subset \mathcal{M}$ of the operator U is the following A-submodule

$$\mathcal{M}^{(\alpha)} = \mathcal{M}^{(\alpha)}(A) = \mathrm{Ker}(\alpha I - U) \subset \mathcal{M} = \mathcal{M}(A).$$

b) The *α-characteristic (primary) submodule* of $\mathcal{M}^\alpha = \mathcal{M}^\alpha(A) \subset \mathcal{M}$ of the operator U is the maximal A-submodule on which $U - \alpha I$ is nilpotent.

$$\mathcal{M}^\alpha(A) = \bigcup_{r\geq 1} \mathrm{Ker}(U - \alpha I)^r \subset \mathcal{M}(A).$$

c) For each fixed level Np^v put

$$\mathcal{M}^{(\alpha)}(Np^v) = \mathcal{M}^{(\alpha)}(Np^v; A) = \mathcal{M}^{(\alpha)} \cap \mathcal{M}(Np^v),$$

$$\mathcal{M}^{\alpha}(Np^v) = \mathcal{M}^{\alpha}(Np^v; A) = \mathcal{M}^{\alpha} \cap \mathcal{M}(Np^v).$$

Proposition 1.6. *Suppose that $\alpha \in A^\times$ is an invertible element of the algebra A then*
a) the A-linear map U^v takes $\mathcal{M}(Np^{v+1})$ to $\mathcal{M}(Np)$;
b) assume that $\mathcal{M}^\alpha(Np^v; A)$ is a locally free A-module of finite rank then for all $v \geq 1$ the A-linear map U is invertible on $\mathcal{M}^\alpha(Np^v) \subset \mathcal{M}(Np^v)$;
c) assume that $\mathcal{M}^\alpha(Np^v; A)$ is a locally free A-module of finite rank then for all $v \geq 1$ $\mathcal{M}^\alpha(Np^v; A) = \mathcal{M}^\alpha(Np; A)$ is independent of v.

Proof of 1.6: The statement a) is verified using a known relation of U^v with the trace operator on modular forms of weight k:

$$g|\mathrm{Tr}_{Np}^{Np^{v+1}} = \sum_{\gamma \in \Gamma_0(Np^{v+1})\backslash \Gamma_0(Np)} g|_k \gamma,$$

then

$$g|U^v = p^{\frac{vm(k-m-1)}{2}} g|W_{Np^{v+1}} \mathrm{Tr}_{Np}^{Np^{v+1}} W_{Np},$$

where $W_M = \begin{pmatrix} 0 & -1 \\ M & 0 \end{pmatrix}$ is the standard involution.

This operator $\mathrm{Tr}_{Np}^{Np^{v+1}}$ acts on the space $\widetilde{\mathcal{M}}_k^m(Np^{v+1}, \psi)$ by

$$\mathrm{Tr}_{Np}^{Np^{v+1}}(g) = \sum_{{}^t u = u \in \mathrm{M}_m(\mathbb{Z}) \pmod{p^v}} g|_k \begin{pmatrix} 1_m & 0_m \\ Npu & 1_m \end{pmatrix}.$$

One can write down the action of this operator on Fourier expansions using the operators $U(p^v)$ acting by

$$F|U(p^v)(z) = (p^v)^{-\frac{m(m+1)}{2}} \sum_{{}^t u = u \in \mathrm{M}_m(\mathbb{Z}) \pmod{p^v}} F((z+u)/p^v).$$

One uses the following matrix identity:

$$\begin{pmatrix} 1_m & 0_m \\ Npu & 1_m \end{pmatrix} =$$
$$(-Np^{v+1})^{-1} \begin{pmatrix} 0_m & -1_m \\ Np^{v+1}1_m & 0_m \end{pmatrix} \begin{pmatrix} 1_m & -u \\ 0_m & p^v 1_m \end{pmatrix} \begin{pmatrix} 0_m & -1_m \\ Np1_m & 0_m \end{pmatrix},$$

which implies the relation:

$$\mathrm{Tr}_{Np}^{Np^{v+1}}(g) = p^{-\frac{vm(k-m-1)}{2}} g|W(Np^{v+1})U(p^v)W(Np).$$

In order to prove b) notice that the operator U acts on the A-module of finite rank $\mathcal{M}^\alpha(Np^v; A) \subset \mathcal{M}(Np^v; A)$ and its determinant is in $A^\times$ hence U is invertible on $\mathcal{M}^\alpha(Np^v; A)$.

The statement c) then directly follows from a) and b).

Definition 1.7. Suppose that $\alpha \in A^\times$ is an invertible element of the algebra A.
a) Define the *α-characteristic projection*

$$\pi_{\alpha,v} : \mathcal{M}(Np^v) \to \mathcal{M}^\alpha(Np^v) \quad (v \geq 1)$$

as the canonical projector to the α-characteristic submodule

$$\mathcal{M}^\alpha(Np^v) = \bigcup_{r \geq 1} \mathrm{Ker}(U - \alpha I)^r$$

of the U-operator with the kernel $\bigcap_{r \geq 1} \mathrm{Im}(U - \alpha I)^r$;
b) For any $g \in \mathcal{M}$ put

$$\pi_\alpha(g) = U^{-v}\left[\pi_{\alpha,1} U^v(g)\right]$$

($\pi_\alpha(g)$ is well defined if v is sufficiently large so that $g \in \mathcal{M}(Np^{v+1})$.

§2. Algebraic differential operators and Siegel-Eisenstein distributions

2.1. Consider the differential operator Δ_m (the *Maass differential operator*) of degree m, acting on complex C^∞-functions on $\mathbb{H}_m$, defined by the equality:

$$\Delta_m = \det(\tilde{\partial}_{ij}), \qquad \tilde{\partial}_{ij} = 2^{-1}(1+\delta_{ij})\partial/\partial_{ij} \tag{2.1}$$

For any $f \in \mathcal{C}^\infty(\mathbb{H}_m, \mathbb{C})$, one defines also the operators

$$M_k f(z) = \det(z-\bar{z})^{\kappa-k}\Delta\left[\det(z-\bar{z})^{k-\kappa+1}f(z)\right],$$

$$\delta_k f(z) = \left(-\frac{1}{4\pi}\right)^m \det(y)^{-1}M_k f(z).$$

Due to H. Maass [Maa71], for an integer k and a Dirichlet character ψ modulo N, the differential operator δ_k acts on $\mathcal{C}^\infty$-Siegel modular forms, $\delta_k : \widetilde{\mathcal{M}}_k^m(N,\psi) \to \widetilde{\mathcal{M}}_{k+2}^m(N,\psi)$. One defines then the Shimura differential operator as the composition

$$\delta_k^{(r)} = \delta_{k+2r-2} \circ \cdots \circ \delta_k.$$

In order to describe explicitely the action of $\delta_k^{(r)}$ on Fourier expansions consider, for an integer $r \geq 0$ and for a complex number β, the polynomial

$$R_m(z;r,\beta) = (-1)^{mr}e^{\mathrm{tr}(z)}\det(z)^{r+\beta}\Delta_m^r\left[e^{-\mathrm{tr}(z)}\det(z)^{-\beta}\right]. \tag{2.2}$$

According to its definition the degree of the polynomial $R_m(z;r,\beta)$ is equal to mr and the term of the highest degree coincides with $\det(z)^r$. We have also that for $\beta \in \mathbb{Q}$ the polynomial $R_m(z;r,\beta)$ has rational coefficients.

Notice also that for $m = 1$ one has

$$R_1(z,r,\beta) = \sum_{t=0}^{r}\binom{r}{t}z^{r-t}\beta(\beta+1)\cdots(\beta+t-1).$$

M.Courtieu has established in his PhD thesis the following more explicit expression for the function $R_m(z;r,\beta)$ for arbitrary m (see [Cour], Chapter 3 and Theorem 2.6 of [PaAD]):

$$R_m(z;r,\beta) = \sum_{t=0}^{r}\binom{r}{t}\det(z)^{r-t}\sum_{|L|\leq mt-t}R_L(\beta)\lambda_{l_1}(z)\cdots\lambda_{l_t}(z), \tag{2.3}$$

where L runs over all the multi-indices $0 \le l_1 \le \cdots \le l_t \le m$, such that $|L| = l_1 + \cdots + l_t \le mt - t$, the coefficients $R_L(\beta) \in \mathbb{Z}[1/2][\beta]$ are polynomials in β of degree $(mt - |L|)$ and with coefficients in the ring $\mathbb{Z}[1/2]$. The polynomials $\lambda_i(z)$ are defined by

$$\det(t1_m + z) = \sum_{i=0}^{m} \lambda_i(z) t^{m-i}$$

($\lambda_i(z)$ is the sum of all diagonal minors of size $i \times i$ of the matrix z).

2.2. Let $f \in \mathcal{M}_k^m(N, \psi)$ be a Siegel modular form whose Fourier expansion is given by $f(z) = \sum_{\xi \in B_m} c(\xi) e_m(\xi z)$, then we proved (Theorem 3.2 of [PaAD]) that the nearly holomorphic Siegel modular form that $\delta_k^{(r)} f(z)$ is given then by the following formal power series expansion

$$\begin{aligned}
\delta_k^{(r)} f(z) &= \sum_{\xi \in B_m} c(\xi) \sum_{t=0}^{r} \binom{r}{t} \det(\xi)^{r-t} \\
&\qquad \sum_{|L| \le mt-t} R_L(\kappa - k - r) \operatorname{tr}\left({}^t\rho_{m-l_1}(R)\rho^\star_{l_1}(\xi)\right) \cdots \operatorname{tr}\left({}^t\rho_{m-l_t}(R)\rho^\star_{l_t}(\xi)\right) q^\xi \\
&= \sum_{\xi \in B_m} c(\xi) \sum_{t=0}^{r} \binom{r}{t} \det(\xi)^{r-t} \sum_{|L| \le mt-t} R_L(\kappa - k - r) Q_L(R, \xi) q^\xi \\
&= \sum_{\xi \in B_m} c(\xi) Q(R, \xi) q^\xi \in \mathbb{Q}(f)[\![q^{B_m}]\!][R_{i,j}]_{i,j=1,\cdots,m},
\end{aligned} \tag{2.4}$$

where $\mathbb{Q}(f)$ is the subfield of $\mathbb{C}$ generated by the Fourier coefficients of f, L runs over all the multi-indices $0 \le l_1 \le \cdots \le l_t \le m$ as above,

$$\begin{aligned}
Q(R, \xi) = Q(R, \xi; k, r) &= \sum_{t=0}^{r} \binom{r}{t} \det(\xi)^{r-t} \sum_{|L| \le mt-t} R_L(\kappa - k - r) Q_L(R, \xi)), \\
Q_L(R, \xi) &= \operatorname{tr}\left({}^t\rho_{m-l_1}(R)\rho^\star_{l_1}(\xi)\right) \cdots \operatorname{tr}\left({}^t\rho_{m-l_t}(R)\rho^\star_{l_t}(\xi)\right)).
\end{aligned}$$

For $r = 1$ this gives:

$$\delta_k f(z) = \sum_{\xi \in B_m} c(\xi) \sum_{l=0}^{m} (-1)^{m-l} c_{m-l}(k + 1 - \kappa) \operatorname{tr}\left({}^t\rho_{m-l}(R) \cdot \rho^\star_l(\xi)\right) q^\xi,$$

where

$$c_{m-l}(k+1-\kappa) = \prod_{j=0}^{m-l-1} (k + 1 - \kappa + j/2) = \frac{\Gamma_{m-l}(k + 1/2)}{\Gamma_{m-l}(k-1)/2)} \quad (l \le m - 1),$$

2.3. Siegel-Eisenstein series. We recall the definition of the Siegel-Eisenstein series$E_k(z,s,\chi)$ which come up in the integral representation of Rankin type for the standard L-function. We introduce some additional notation. Let us fix a positive integer m then a pair of integral matrices c, $d \in \mathrm{M}_m(\mathbb{Z})$ is called primitive if any matrix with rational coefficients $\gamma \in \mathrm{M}_m(\mathbb{Q})$ such that $\gamma c, \gamma d \in \mathrm{M}_m(\mathbb{Z})$ is integral: $\gamma \in \mathrm{M}_m(\mathbb{Z})$. The pair (c,d) is said symmetric if $c\cdot{}^t d = d\cdot{}^t c$. Two primitive symmetric pairs (c_1,d_1) and (c_2,d_2) are said equivalent if $(c_1,d_1) = (uc_2, ud_2)$ for a matrix $u \in \mathrm{SL}_m(\mathbb{Z})$. Let D_m denote the set of equivalence classes of symmetric couples of coprime matrices. Now let k, N be positive integers, s a complex number and χ a Dirichlet character modulo N such that $\chi(-1) = (-1)^k$. For $z \in \mathbb{H}_m$ the Siegel-Eisenstein series is defined by

$$E(z,s;k,\chi,N) = E(z,s) = \det(y)^s \sum \chi(\det(d))\det(cz+d)^{-k-|2s|}, \tag{2.5}$$

where the summation is taken over all $(c,d) \in D_m$ with the condition $c \equiv 0 \pmod N$ and we adopt the convenient notation by Deligne and Ribet [De-Ri]:

$$z^{-k-|2s|} \stackrel{\mathrm{def}}{=} z^{-k}|z|^{-2s} \text{ for } z \in \mathbb{C}^*$$

The series (2.5) is absolutely convergent for $k+2\mathrm{Re}(s) > m+1$ and it admits a meromorphic analytic continuation over the whole complex s-plane. Put $j(\alpha,z) = \det(cz+d)$ for $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $z \in \mathbb{H}_m$, then it follows from a known description of D_m that

$$E(z,s;k,\chi,N) = \det(y)^s \sum_{\alpha \in P\cap\Gamma\backslash\Gamma} \chi(\det(d)j(\alpha,z)^{-k-|2s|}, \tag{2.6}$$

where $\Gamma = \Gamma_0^m(N)$, $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, and P denotes the subgroup of $P \subset G_{\infty+}$, consisting of elements α with the condition $c = 0$. The series (2.6) define $\mathcal{C}^\infty$-modular forms $E_k(z,s,\chi) \in \widetilde{\mathcal{M}}_m^k(N,\bar\chi)$ and if $s = 0$ and $k > m+1$, these forms are holomorphic: $E_k(z,\chi) \in \mathcal{M}_k^m(N,\bar\chi)$. This series has a Fourier expansion of the type

$$E_k(z,\chi) = \sum_{\xi \in B_m} b_k(\xi) e_m(\xi z).$$

Following the works of H. Maass (see [Maa], chap. 19), the series $E_k(z,s,\chi)$ are related to each other for different values of k and s via the Shimura differential operator:

$$E_{k+2}(z,s-1,\chi) = (-4\pi)^m \frac{\Gamma_m(k+s)}{\Gamma_m(k+s+1)} \delta_k E_k(z,s,\chi). \tag{2.7}$$

One obtains by iteration of (2.7) r times

$$E_{k+2r}(z,s-r,\chi)=(-4\pi)^{mr}\frac{\Gamma_m(k+s)}{\Gamma_m(k+s+r)}\delta_k^{(r)}E_k(z,s,\chi)$$

where $\delta_k^{(r)}=\delta_k,\cdots,\delta_{k+2r-2}$. It follows from (2.3) that the series $E_{k+2r}(z,-r,\chi)$ is a nearly holomorphic Siegel modular form with the following Fourier expansion:

$$E_{k+2r}(z,-r,\chi)=\det(-y)^{-r}\frac{\Gamma_m(k)}{\Gamma_m(k+r)}\sum_{\xi\in B_m}b_k(\xi)R(4\pi\xi y;r,\kappa-k-r)e_m(\xi z).$$

2.4. Normaized Siegel-Eisenstein series. In order to obtain arithmetical nearly holomorphic forms whose Fourier coefficients depend nicely on the parameter $s=-r\leq 0$ we introduce normalized Siegel-Eisenstein series starting from the series (2.5). Such series where studied in detail by Shimura (voir [Shi83]. Keeping the same notation as above define

$$E_k^\star(z,s,\chi)=E_k(-z^{-1},s,\chi)\det(z)^{-k}. \tag{2.8}$$

(the involuted Siegel-Eisenstein series): $E_k^\star(z,s,\chi)=E_k(z,s,\chi)|_kJ_m$. The series (2.8) admit the following Fourier expansion

$$E_k^\star(z,s,\chi)=\sum_{\xi\in N^{-1}A_m}b(\xi,y,s,\chi)e_m(\xi z), \tag{2.9}$$

where the coefficients $b(\xi,y,s,\chi)$ are explicitely given in in [Shi83] (see also [PaLNM] (chap. 2, prop. 3.4.)) in terms of the confluent hypergeometric function.

We introduce here other three types of normalized Siegel – Eisenstein series in order to give a precise statement on their holomorphy properties with respect to the variable s, the properties of positivity of matrices ξ by which their Fourier coefficients are indexed, and also algebraic properties of these Fourier coefficients:

$$\begin{aligned}&G^*(z,s)=G^*(z,s;k,\chi,N)=\\&N^{m(k+s)}\widetilde{\Gamma}(k,s)L_N(k+2s,\chi)\prod_{i=1}^{[m/2]}L_N(2k+4s-2i,\chi^2)E^*(Nz,s),\end{aligned} \tag{2.10}$$

with

$$E^*(Nz,s) = E(-(Nz)^{-1},s)\det(Nz)^{-k} = N^{-km/2}E|W(N),$$

$$\widetilde{\Gamma}(k,s) = \tag{2.11}$$

$$i^{mk}2^{-m(k+1)}\pi^{-m(s+k)} \times \begin{cases} \Gamma_m(k+s)\Gamma(s+(k-\frac{m}{2}+\mu)/2), & \text{if } m \text{ is even;} \\ \Gamma_m(k+s), & \text{otherwise.} \end{cases}$$

and

$$L_N(s,\chi) = \prod_{q \nmid N}(1-\chi(q)q^{-s})^{-1}$$

is the Dirichlet series associated to the character χ and $\mu = \varepsilon(m/2+k)$, with $\varepsilon(r) \in \{0,1\}$ such that $\varepsilon(r) \equiv r \pmod 2$ for any integer $r \in \mathbb{N}$. If the character χ^2 is non trivial, the function $G_k^\star(z,s,\chi)$ is an entire function of the complex variable s (see [Fe], Th. 9.1, p. 49, and [PaLNM], Chap. 3, Th. 3.6).

Proposition 2.5. *Let $N > 1$ a positive integer such that $2|N$ then*
(a) If $\chi^2 \neq 1$ the series $G_k^\star(z,s,\chi)$ is an entire function of the variable s.
(b) If χ^2 is trivial then

(b_1) if either $2k \geq m$ and m odd or $2k \geq m$ and m is even, but $(m/2)+k$ is odd (i.e. $\mu = \varepsilon((m/2)+k) = 1$), then the function $G^*(z,s)$ is an entire function of the variable s;

(b_2) if $2k \geq m$ and both numbers m and $(m/2)+k$ are even (i.e. $\mu = \varepsilon(m) = 0$) then the function $G^*(z,s)$ is an entire function of the variable s with the exclusion of a possible simple pole at the point $s = (m+2-2k)/4$;

(b_3) if $m > 2k \geq 0$ then the function $G^*(z,s)$ is an entire function of the variable s with possible exclusion of simple poles at those points s for which $2s$ is an integer and $[(m-2k+3)/2] \leq 2s \leq (m+1-2k)/2$;

(b_4) if $k = 0$ then the function $G^*(z,s)$ has a simple pole at the point $s = (m+1)/2$ iff χ is trivial, and in this case we have that the function

$$\mathrm{Res}_{s=(m+1)/2} G^*(z,s;0,1,N)$$

of the variable z is a non-zero constant.

Proof of 2.5: (see [Fe], Th. 9.1, p. 49).

If m is odd then we put $G^+(z,s,\chi) = G^-(z,s,\chi) = G^*(z,s,\chi)$.

If m is even then we define

$$G^-(z,s) = \Gamma((k+2s-(m/2)+\mu)/2)^{-1}G^*(z,s), \tag{2.12}$$

$$\begin{aligned} G^+(z,s) = \frac{i^\mu \pi^{\kappa-k-2s}}{\Gamma((1-k-2s+(m/2)+\mu)/2)} G^*(z,s) = \\ \frac{2i^\mu \Gamma(k+2s-(m/2))\cos(\pi(k+2s-(m/2)-\mu)/2)}{(2\pi)^{k+2s-(m/2)}} G^-(z,s). \end{aligned} \tag{2.13}$$

where $\mu = \varepsilon((m/2)+k)$, $\kappa = (m+1)/2$.

The Fourier expansions of $G_k^\star(z,s,\chi)$ and $G_k^\pm(z,s,\chi)$ are given by

$$G_k^\star(z,s,\chi) = \sum_{\xi \in A_m} b^\star(\xi,y,s,\chi)e_m(\xi z),$$
$$G_k^\pm(z,s,\chi) = \sum_{\xi \in A_m} b^\pm(\xi,y,s,\chi)e_m(\xi z).$$

The Fourier coefficients $b^\star(\xi,y,s,\chi)$, $b^\pm(\xi,y,s,\chi)$ are explicitely given in [PaLNM], (Chap. 2, 3.5) and in Theorem 2.6 below, and we show that for all crtical paire (s,χ) these series define arithmetical nearly holomorphic Siegel modular forms.

Next we define an additional quadratic Dirichlet character χ_ξ depending on $\xi \in A_m$ and defined only for even rank r. Namely, for $\xi = 0$ let $\chi_\xi = \chi_0$ be trivial; for $\xi \neq 0$ we know that for some matrix $u \in \mathrm{GL}_m(\mathbb{Q})$

$${}^t u\xi u = \begin{pmatrix} \xi_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \textit{with } \det \xi_1 \neq 0, \tag{2.14}$$

then let χ_ξ denote the quadratic character attached to the quadratic field $\mathbb{Q}(\sqrt{\det \xi_1})/\mathbb{Q}$ (this definition does not depend on choice of a matrix u).

Theorem 2.6. *Let m is even, $2k > m$. Then:*

(a) If $2s$ is an integer, $s \leq 0, k+2s \geq 1+(m/2)$ there is the following Fourier expansion

$$G^+(z,s) = \sum_{A_m \ni \xi > 0} b^+(\xi,y,s,\chi)e_m(\xi z), \tag{2.15}$$

where for $s > (m+2-2k)/4$ *in (2.15) non-zero terms only occur for positive definite* $\xi > 0$, *and for all* s *from (a) with* $\xi > 0, \xi \in A_m$ *the following identity holds*

$$b^+(\xi, y, s, \chi) = W^+(y, \xi, s) L_N^+(k + 2s - (m/2), \chi\chi_\xi) M(\xi, \chi, k + 2s),$$

with

$$L^+(s, \chi) = \frac{2i^\delta \Gamma(s) \cos(\pi(s-\delta)/2)}{(2\pi)^s} L_N(s, \chi)$$

is the normalized Dirichlet L-function, $\delta = 0$ *or* 1 *according to* $\chi(-1) = (-1)^\delta$, *the factor* $M(\xi, \chi, k+2s)$ *is a finite Euler product*

$$M(\xi, \chi, k+2s) = \prod_{q \in P(\xi)} M_q(\xi, \chi(q) q^{-k-2s})$$

with $M_q(\xi, t) \in \mathbb{Z}[t]$, *extended over primes* q *in the set* $P(\xi)$ *of prime divisors of the number* N *and of all elementary divisors of the matrix* ξ*,and the character* χ_ξ *is given by (2.14).*

$$W^+(y, \xi, s) = 2^{-m\kappa} \det \xi^{k+2s-\kappa} \det(4\pi y)^s R(4\pi\xi y; -s; \kappa - k + s),$$

provided s *is an integer, where* $R(y; n, \beta)$ *is defined by (2.2), and* $b^+(\xi, y, s, \chi) = 0$ *otherwise (if* $s \notin \mathbb{Z}$*).*

(b)If $2s$ *is an integer with* $k + 2s \le m/2, k + s \ge \kappa$ *then there is the following Fourier expansion*

$$G^-(z, s) = \sum_{A_m \ni \xi \ge 0} b^-(\xi, y, s, \chi) e_m(\xi z), \tag{2.16}$$

and for all s *from (b) with* $\xi > 0, \xi \in A_m$ *the following identity holds*

$$b^-(\xi, y, s, \chi) = W^-(y, \xi, s) L_N^-(k + 2s - (m/2), \chi\chi_\xi) M(\xi, \chi, k + 2s),$$

where

$$L^-(s, \chi) = L_N(s, \chi),$$

$$W^-(y, \xi, s) = 2^{-m\kappa} \det(4\pi y)^{\kappa-k-s} R(4\pi\xi y; k + s - \kappa, -s),$$

provided $s + k - \kappa$ *is an integer, and* $b^-(\xi, y, s, \chi) = 0$ *otherwise.*

Proof of 2.6.: (see [PaLNM], Chap. 2, Th. 3.8).

Let us now write down the action of the Shimura differential operator on these normailzed Siegel-Eisenstein series.

Lemma 2.7. *The action of the Shimura differential operator on $E_k^\star(z,s,\chi)$ is given by:*

$$\delta_k E_k^\star(z,s,\chi) = \frac{1}{(-4\pi)^m}\frac{\Gamma_m(k+s+1)}{\Gamma_m(k+s)}E_{k+2}^\star(z,s-1,\chi), \tag{2.17}$$

$$\delta_k E_k^\star(Nz,s,\chi) = \frac{N^m}{(-4\pi)^m}\frac{\Gamma_m(k+s+1)}{\Gamma_m(k+s)}E_{k+2}^\star(Nz,s-1,\chi). \tag{2.18}$$

Proof of 2.7: We use the known property $\delta_k(f|_k\gamma) = (\delta_k f)|_{k+2}\gamma$ for all $\gamma \in \mathrm{Sp}_{2m}(\mathbb{R})$ and $k \in \mathbb{N}$. Taking into account $E_k^\star(z,s,\chi) = E_k(z,s,\chi)|_k J_m$, one immediately obtains (2.17) from the relation (2.7) satisfied by $E_k(z,s,\chi)$. In order to prove (2.18), one changes the variable $z' = Nz$. One obtains then

$$\delta_k(z)E_k^\star(Nz,s,\chi) = \det(N1_m)\delta_k(z')E_k^\star(z',s,\chi),$$

and (2.18) follows.

Corollary 2.8. *One has the following relations :*

$$\begin{aligned}
\delta_k(G_k^\star(z,s,\chi)) &= G_{k+2}^\star(z,s-1,\chi),\\
\delta_k(G_k^+(z,s,\chi)) &= G_{k+2}^+(z,s-1,\chi),\\
\delta_k(G_k^-(z,s,\chi)) &= G_{k+2}^-(z,s-1,\chi).
\end{aligned} \tag{2.19}$$

Proof of 2.8: One shows all the three equalities in the same way, so let us consider only the first one. One applies the Shimura operator to $G_k^\star(z,s,\chi)$. The only factors in the formula (2.10) for $G_k^\star(z,s,\chi)$ which are not invariant with respect to the change of variables $k \mapsto k+2$, $s \mapsto s-1$, are the following:

$$N^{m(s+k)} i^{mk} 2^{-m(k+1)} \pi^{-m(s+k)} \Gamma_m(k+s).$$

These factors come up essentially in the factor $\widetilde{\Gamma}(k,s)$ (see (2.11)). One applies then the Shimura operator to $G_k^\star(z,s,\chi)$ using (2.18). A simplification of the powers of N, π, 2, and i, and of the factor $\Gamma_m(k+s)$ gives $\delta_k(G_k^\star(z,s,\chi)) = G_{k+2}^\star(z,s-1,\chi)$.

2.9. Distributions with values in nearly holomorphic Siegel modular forms. Consider a profinite group $Y = \varprojlim_v Y_v$ for example

$$Y = \varprojlim_v Y_v, \quad \text{where} \quad Y_v = (\mathbb{Z}/Np^v\mathbb{Z})^\times.$$

There is a natural projection $y_p : Y \to \mathbb{Z}_p^\times$. Let A be a commutative ring, and let M be an A-module.

Let $\mathcal{C}^{loc-const}(Y, A)$ denote the A-module of locally constant functions. A distribution on Y with values in M is an A-module homomorphism

$$\Phi : \mathcal{C}^{loc-const}(Y, A) \to M.$$

We adopt the notations $\Phi(\mathcal{U}) := \Phi(\chi_{\mathcal{U}})$ for the characteristic function $\chi_{\mathcal{U}}$ of an open subset $\mathcal{U} \subset Y$, and $(a)_v = a + (Np^v)$ for both an element of Y_v and the corresponding open compact subset of Y.

Recall an algebraic version of nearly holomorphic Siegel modular forms over an $\mathcal{O}$-algebra A where $\mathcal{O}$ is the ring of integers in a finite extension K of $\mathbb{Q}_p$. We assume that A is equipped with an augmentation $e_A : A \to \mathcal{O}$, and fix embeddings $i_\infty\colon \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $i_p\colon \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$.

Let us fix another posivite integer N' and let denote by $g = \sum\limits_{\xi \in B_m} a(\xi, R) q^\xi \in \mathcal{M}_{r,k}^m(A)$ an element of

$$\mathcal{M}_{r,k}^m(A) = \mathcal{M}_{r,k}^m(A; N') = \bigcup_{v \geq 1} \mathcal{M}_{r,k}^m(N'p^v; A)$$

(A-module of nearly holomorphic Siegel modular forms of type r and level $N'p^v$ ($v \geq 0$) with coefficients in A). Let $M = \mathcal{M}_{r,k}^m(A)$ and let $\Phi : \mathcal{C}^{loc-const}(Y, A) \to M$ be a distribution on Y with values in M. Assume that A contains $(\varphi(Np^v))^{-1}$ and all the values of Dirichlet characters $\chi \bmod Np^v$ viewed as homomorphisms $Y \to A^\times$ then obviously

$$\Phi(\chi) = \sum_{a \bmod Np^v} \chi(a) \Phi((a)_v), \quad \Phi((a)_v) = \frac{1}{\varphi(Np^v)} \sum_{\chi \bmod Np^v} \chi(a)^{-1} \Phi((a)_v). \tag{2.20}$$

Proposition 2.10 (Normaized Siegel-Eisenstein distributions).

a) Let $N > 1$ a positive integer such that $2|N$ and let k, s be integers such that $2k > m$ and $\kappa - k \leq s \leq 0$ then there exist distributions $G_{k,s}^\star$, $G_{k,s}^+$, $G_{k,s}^-$ on Y with values in $\mathcal{M}_{r,k}^m(A)$ such that for any character $\chi \bmod Np^v$

$$G_{k,s}^\star(\chi) = G_k^\star(z, s, \chi), \quad G_{k,s}^+(\chi) = G_k^+(z, s, \chi), \quad G_{k,s}^-(\chi) = G_k^-(z, s, \chi). \tag{2.21}$$

where $r = m \cdot \max(-s, k - \kappa + s)$.

b) Suppose that $\kappa - k + 1 \le s \le 0$ then the action of the Shimura differential operators on $G^{\star}_{k,s}$, $G^{+}_{k,s}$, $G^{-}_{k,s}$ is given by

$$\begin{aligned} \delta_k(G^{\star}_{k,s}(\chi)) &= G^{\star}_{k+2,s-1}(\chi), \\ \delta_k(G^{+}_{k,s}(\chi)) &= G^{+}_{k+2,s-1}(\chi), \\ \delta_k(G^{-}_{k,s}(\chi)) &= G^{-}_{k+2,s-1}(\chi). \end{aligned} \tag{2.22}$$

c) Moreover if $t \ge 0$ is an integer such that $\kappa - k + t \le s \le 0$ then the action of $\delta_k^{(t)}$ on $G^{\star}_{k,s}$, $G^{+}_{k,s}$, $G^{-}_{k,s}$ is given by

$$\begin{aligned} \delta_k^{(t)}(G^{\star}_{k,s}(\chi)) &= G^{\star}_{k+2t,s-t}(\chi), \\ \delta_k^{(t)}(G^{+}_{k,s}(\chi)) &= G^{+}_{k+2t,s-t}(\chi), \\ \delta_k^{(t)}(G^{-}_{k,s}(\chi)) &= G^{-}_{k+2t,s-t}(\chi). \end{aligned} \tag{2.23}$$

Proof of Proposition 2.10: One deduces a) from (2.20) taking into account that for all integrs k, s such that $2k > m$ and $\kappa - k \le s \le 0$ the Fourier coefficients $b^{\star}(\xi, y, s, \chi)$, $b^{\pm}(\xi, y, s, \chi))$ are polynomials of the variables $R_{i,j}(, j = 1, \ldots, m)$, where $R = (4\pi \mathrm{Im}(z))^{-1}$, of degree $\le r = m \cdot \max(-s, k - \kappa + s)$ which belong to $\bar{\mathbb{Q}}[R_{i,j}]_{i,j=1,\ldots,m}$. They are explicitely given in Theorem 2.6 above (2.15), (2.16), and in [PaLNM], (Chap. 2, 3.5) . Indeed let us consider the natural representation $(0 \le r \le m)$

$$\begin{array}{cccc} \rho_r & : & \mathrm{GL}_m(\mathbb{C}) & \longrightarrow & \mathrm{GL}(\Lambda^r \mathbb{C}^m) \\ & & z & \longmapsto & \rho_r(z) \end{array}$$

of the group $\mathrm{GL}_m(\mathbb{C})$ on the vector space $\Lambda^r \mathbb{C}^m$ with respect to the basis

$$\{e_{i_1} \wedge \cdots \wedge e_{i_r} \ : \ i_1 < \cdots < i_r\}.$$

Thus $\rho_r(z)$ is a matrix of size $\binom{m}{r} \times \binom{m}{r}$ composed of the subdeterminants of z of degree r. Put

$$\rho_r^{\star}(z) = \det(z) \rho_{m-r}({}^t z)^{-1} \quad (r = 0, 1, \cdots, m),$$

in other words, $\rho_{m-r}^{\star}(z)$ is the matrix representing the action of z on $\Lambda^{m-r} \mathbb{C}^m$ with respect to the basis dual to the above basis of $\Lambda^r \mathbb{C}^m$. Then the representations ρ_r and $\rho_r^{\star}$ turn out to be polynomial representations so that for each $z \in \mathrm{M}_m(\mathbb{C})$ the linear operators $\rho_r(z)$, $\rho_r^\star(z)$ are well defined. Then by Lemma 2.4 and (3.8) of [PaAD] we have that if $r, k-\kappa-r \geq 0$,

$$\begin{aligned}\det(4\pi y)^{-r} R_m(4\pi\xi y; r, \kappa-k-r) = \sum_{t=0}^{r} \binom{r}{t} \det(\xi)^{r-t} \\ \sum_{|L|\leq mt-t} R_L(\kappa-k-r)\mathrm{tr}\left({}^t\rho_{m-l_1}(R)\rho^\star_{l_1}(\xi)\right)\cdots \mathrm{tr}\left({}^t\rho_{m-l_t}(R)\rho^\star_{l_t}(\xi)\right)),\end{aligned} \tag{2.24}$$

where L runs over all the multi-indices $0 \leq l_1 \leq \cdots \leq l_t \leq m$, such that $|L| = l_1+\cdots+l_t \leq mt-t$, the coefficients $R_L(\beta) \in \mathbb{Z}[1/2][\beta]$ are polynomials in β of degree $(mt-|L|)$ and with coefficients in the ring $\mathbb{Z}[1/2]$, and the polynomials $\lambda_i(z)$ are defined by

$$\det(t1_m + z) = \sum_{i=0}^{m} \lambda_i(z) t^{m-i}$$

($\lambda_i(z)$ is the sum of all diagonal minors of size $i \times i$ of the matrix z).
It follows that in (2.15) one has for an integer $r = -s \geq 0$:

$$\begin{aligned}\det(4\pi y)^{s} R(4\pi\xi y; -s; \kappa-k+s) = \sum_{t=0}^{r} \binom{r}{t} \det(\xi)^{r-t} \\ \sum_{|L|\leq mt-t} R_L(\kappa-k-r)\mathrm{tr}\left({}^t\rho_{m-l_1}(R)\rho^\star_{l_1}(\xi)\right)\cdots \mathrm{tr}\left({}^t\rho_{m-l_t}(R)\rho^\star_{l_t}(\xi)\right)),\end{aligned} \tag{2.25}$$

and in (2.16) one has for an integer $r' = k-\kappa+s \geq 0$:

$$\begin{aligned}\det(4\pi y)^{\kappa-k-s} R(4\pi\xi y; k+s-\kappa, -s) = \sum_{t=0}^{r'} \binom{r'}{t} \det(\xi)^{r'-t} \\ \sum_{|L|\leq mt-t} R_L(\kappa-k-r')\mathrm{tr}\left({}^t\rho_{m-l_1}(R)\rho^\star_{l_1}(\xi)\right)\cdots \mathrm{tr}\left({}^t\rho_{m-l_t}(R)\rho^\star_{l_t}(\xi)\right)).\end{aligned} \tag{2.26}$$

These are polynomials of degree $\leq m\cdot\max(-s, k-\kappa+s)$ with rational coefficients , hence the Fourier coefficients $b^\star(\xi, y, s, \chi)$, $b^\pm(\xi, y, s, \chi))$ define for each ξ distributions with values in $\mathbb{Q}[R_{i,j}]_{i,j=1,\ldots,m}$ (evaluated at χ).

In order to deduce b) and c) (by iteration) it suffices to use then Corollary 2.8.

2.11. Convolutions of distributions with values in nearly holomorphic Siegel modular forms. Consider again the profinite group

$$Y = \lim_{\substack{\leftarrow \\ v}} Y_v, \quad \text{where} \quad Y_v = (\mathbb{Z}/Np^v\mathbb{Z})^\times.$$

Let $M = \mathcal{M}^m_{r,k}(A)$ and let $\Phi_i : \mathcal{C}^{loc-const}(Y, A) \to M$ be two distributions on Y with values in $M_i = \mathcal{M}^m_{r_i,k_i}(A)$, $(i = 1, 2)$. Assume that A contains $(\varphi(Np^v))^{-1}$ and all the values of Dirichlet characters χ mod Np^v viewed as homomorphisms $Y \to A^\times$ then obviously

$$\Phi_i(\chi) = \sum_{a \bmod Np^v} \chi(a)\Phi_i((a)_v), \quad \Phi_i((a)_v) = \frac{1}{\varphi(Np^v)} \sum_{\chi \bmod Np^v} \chi(a)^{-1}\Phi_i(\chi). \tag{2.27}$$

Let us fix a Dirichlet character ψ on Y with values in A then there exists a unique distribution $\Phi = \Phi_1 *_\psi \Phi_2$ on Y with values in $M = \mathcal{M}^m_{r_1+r_2,k_1+k_2}(A)$ such that for all Dirichlet characters χ on Y with values in A one has $\Phi(\chi) = \Phi_1(\chi) \cdot \Phi_2(\psi\chi^{-1})$. One defines Φ on basic open subsets by

$$\Phi((a)_v) = \sum_{y \in Y_v} \psi(y)\Phi_1(ya)\Phi_2(y). \tag{2.28}$$

for any v such that ψ is defined $\mathrm{mod} Np^v$.

Indeed,

$$\Phi((a)_v) = \sum_{y \in Y_v} \psi(y)\Phi_1(ya)\Phi_2(y) =$$
$$\frac{1}{(\varphi(Np^v))^2} \sum_{y \in Y_v} \psi(y) \sum_{\chi' \bmod Np^v} \chi'(ya)^{-1}\Phi_1(\chi') \sum_{\chi'' \bmod Np^v} \chi''(y)^{-1}\Phi_2(\chi'')$$

hence

$$\Phi(\chi) = \sum_{a \in Y_v} \chi(a) \sum_{y \in Y_v} \psi(y)\Phi_1(ya)\Phi_2(y) = \frac{1}{(\varphi(Np^v))^2} \sum_{a \in Y_v} \chi(a) \sum_{y \in Y_v} \psi(y)$$
$$\sum_{\chi' \bmod Np^v} \chi'(ya)^{-1}\Phi_1(\chi') \sum_{\chi'' \bmod Np^v} \chi''(y)^{-1}\Phi_2(\chi'') = \Phi_1(\chi) \cdot \Phi_2(\psi\chi^{-1}) =$$
$$\frac{1}{(\varphi(Np^v))^2} \sum_{a \in Y_v} (\chi\chi'^{-1})(a) \sum_{y \in Y_v} (\psi\chi'^{-1}\chi''^{-1})(y) \sum_{\chi' \bmod Np^v} \Phi_1(\chi') \sum_{\chi'' \bmod Np^v} \Phi_2(\chi'')$$
$$= \Phi_1(\chi) \cdot \Phi_2(\psi\chi^{-1})$$

because of the orthogonality relations which give $\chi' = \chi, \chi'' = \psi\chi^{-1}$.

§3. A general result on admissible measures with values in nearly holomorphic Siegel modular forms.

3.1. Consider a profinite group $Y = \varprojlim_v Y_v$ for example

$$Y = \varprojlim_v Y_v, \quad \text{where} \quad Y_v = (\mathbb{Z}/Np^v\mathbb{Z})^\times.$$

There is a natural projection $y_p : Y \to \mathbb{Z}_p^\times$. Let A be a normed topological ring over $\mathbb{Z}_p$ with the norm $|\cdot| = |\cdot|_{p,A}$, and M be a normed A-module with the norm $|\cdot|_{p,M}$.

Definition 3.2. a) For $h \in \mathbb{N}, h \geq 1$ let $\mathcal{C}^h(Y, A)$ denote the A-module of *locally polynomial functions* of degree $< h$ of the variable $y_p : Y \to \mathbb{Z}_p^\times \hookrightarrow A^\times$; in particular,

$$\mathcal{C}^1(Y, A) \subset \mathcal{C}^{loc-const}(Y, A)$$

(the A-submodule of *locally constant functions*). We adopt the notation $\Phi(\mathcal{U}) := \Phi(\chi_\mathcal{U})$ for the characteristic function $\chi_\mathcal{U}$ of an open subset $\mathcal{U} \subset Y$. Let also denote $\mathcal{C}^{loc-an}(Y, A)$ the A-module of *locally analytic functions* and $\mathcal{C}(Y, A)$ the A-module of *continous functions* so that

$$\mathcal{C}^1(Y, A) \subset \mathcal{C}^h(Y, A) \subset \mathcal{C}^{loc-an}(Y, A) \subset \mathcal{C}(Y, A).$$

b) For a given positive integer h an *h-admissible measure* on Y with values in M is an A-module homomorphism

$$\widetilde{\Phi} : \mathcal{C}^h(Y, A) \to M$$

such that for fixed $a \in Y$ and for $v \to \infty$

$$\left| \int_{a+(Np^v)} (y_p - a_p)^j d\widetilde{\Phi} \right|_{p,M} = o(p^{-v(j-h)}) \quad \text{for all} \quad j = 0, 1, \ldots, h-1,$$

where $a_p = y_p(a)$. We adopt the notation $(a)_v = a + (Np^v)$ for both an element of Y_v and the corresponding open compact subset of Y.

3.3. We wish now to construct an h-admissible measure $\widetilde{\Phi}^\alpha : \mathcal{C}^h(Y, A) \to \mathcal{M}_{r^*}(A)$ out of a sequence of distributions

$$\Phi_j^\alpha : \mathcal{C}^1(Y, A) \to \mathcal{M}_{r^*}$$

with values in an A-module $M = \mathcal{M}_{r^*}(A)$ of nearly holomorphic modular forms over A of type r^* (for all $j \in \mathbb{N}$ with $j \le h-1$ where $r^* \in \mathbb{N}$ be a natural number, and A an algebra over $\mathcal{O}$ as in Section 1). For this purpose we define

$$\widetilde{\Phi}^\alpha : \mathcal{C}^h(Y, A) \to \mathcal{M}_{r^*}$$

by

$$\int_{(a)_v} y_p^j \, d\widetilde{\Phi}^\alpha = \Phi_j^\alpha((a)_v) = \pi_\alpha(\Phi_j((a)_v)$$

where $\Phi_j : \mathcal{C}^1(Y, A) \to \mathcal{M}_{r^*}(A)$ are certain $\mathcal{M}_{r^*}(A)$-valued distributions on Y for all $j = 0, 1, \ldots, h-1$, and $\Phi_j^\alpha((a)_v)$ their α-characteristic projections given by

$$\Phi_j^\alpha((a)_v) = U^{-v'}\left[\pi_{\alpha,1} U^{v'} \Phi_j((a)_v)\right] = \pi_{\alpha,v'+1}(\Phi_j((a)_v)$$

for any $v' \ge \varkappa v$. Note first of all that the definition

$$\int_{(a)_v} y_p^j \, d\widetilde{\Phi}^\alpha = \Phi_j^\alpha((a)_v) = U^{-\varkappa v}\left[\pi_{\alpha,1} U^{\varkappa v} \Phi_j((a)_v)\right]$$

of the linear form $\widetilde{\Phi}^\alpha : \mathcal{C}^h(Y, A) \to \mathcal{M}_{r^*}(A)$ is undependent on the choice of the level: for any $v' \ge \varkappa v$ we have by Proposition 1.6 the following comutative diagram

$$\begin{array}{ccc}
\mathcal{M}_{r^*}(Np^{v'+1}; A) & \xrightarrow{\pi_{\alpha,v'+1}} & \mathcal{M}^\alpha_{r^*}(Np^{v'+1}; A) \\
U^{v'}\downarrow & & \downarrow\wr \;\; U^{v'} \\
\mathcal{M}_{r^*}(Np; A) & \xrightarrow{\pi_{\alpha,1}} & \mathcal{M}^\alpha_{r^*}(Np; A)
\end{array}$$

in which the right vertical arrow is an A-isomorphism by Proposition 1.6 b), and by linear algebra the A-linear endomorphism U commutes with the characteristic projectors $\pi_{\alpha,v'+1}$, $\pi_{\alpha,1}$ hence

$$U^{-\varkappa v}\left[\pi_{\alpha,1} U^{\varkappa v} \Phi_j((a)_v)\right] = U^{-v'}\left[\pi_{\alpha,1} U^{v'} \Phi_j((a)_v)\right] = \pi_\alpha(\Phi_j((a)_v)).$$

Theorem 3.4. *Let $\alpha \in A$ be an element of A whose absolute value $|\alpha|_p$ satisfies $0 < |\alpha|_p < 1$ and put $h = [v_p(\alpha)] + 1$. Suppose that there exists a positive integer $\varkappa$ such that for any $(a)_v \subset Y$ the following two conditions are satisfied:*

$$\Phi_j\big((a)_v\big) \in \mathcal{M}_{r^*}(Np^{\varkappa v}), \qquad (level)$$

$$\left| U^{\varkappa v}\Big(\sum_{j'=0}^{j}\binom{j}{j'}(-a_p^0)^{j-j'}\Phi_{j'}\big((a)_v)\big)\Big)\right|_p \le Cp^{-vj} \qquad (growth)$$

for all $j = 0, 1, \ldots, \varkappa h - 1$.

Then there exists an h^**-admissible measure* $\widetilde{\Phi}^\alpha : \mathcal{C}^h(Y, A) \to \mathcal{M}_{r^*}$ *with* $h^* = h\varkappa$ *such that for any* $((a)_v) \in Y$

$$\int_{(a)_v} y_p^j \, d\widetilde{\Phi}^\alpha = \Phi_j^\alpha((a)_v)$$

for all $j = 0, 1, \ldots, h^* - 1$ *where*

$$\Phi_j^\alpha((a)_v) = U^{-\varkappa v}\left[\pi_{\alpha,1}U^{\varkappa v}\Phi_j((a)_v)\right].$$

and

$$\pi_{\alpha,v} : \mathcal{M}_{r^*}(Np^v) \to \mathcal{M}_{r^*}^\alpha(Np^v) \qquad (v \ge 1)$$

denotes the α*-characteristic projector to the* α*-characteristic submodule of the* U*-operator*

$$\mathcal{M}_{r^*}^\alpha(Np^v) = \bigcup_{n\ge 1} \mathrm{Ker}(U - \alpha I)^n$$

(with the kernel $\bigcap_{n\ge 1} \mathrm{Im}(U - \alpha I)^n$*, see definition 1.5) (where*

$$U^{\varkappa v}\Phi_j((a)_v) \in \mathcal{M}_{r^*}^\alpha(Np) = \mathcal{M}_{r^*}^\alpha(Np; A)$$

because of the inclusion

$$U^{\varkappa v-1}(\mathcal{M}_{r^*}(Np^{\varkappa v}; A)) \subset \mathcal{M}_{r^*}(Np; A) \quad (\text{for all} \quad v \ge 1)).$$

(see 1.6 a))

Proof of 3.5: We need to check the h^*-growth condition for the linear form

$$\widetilde{\Phi}^\alpha : \mathcal{C}^{h^*}(Y, A) \to \mathcal{M}_{r^*}$$

(given by the condition of the theorem 3.4), which says that for fixed $a, b \in Y$ and for $v \to \infty$

$$\left| \int_{(a)_v} (y_p - a_p)^j \, d\widetilde{\Phi}^\alpha \right|_{p,\mathcal{M}_{r^*}} = o(p^{-v(j-h\varkappa)})$$

for all $j = 0, 1, \dots, h\varkappa - 1$ where $a_p = y_p(a)$.

Let us develop the definition of $\widetilde{\Phi}^\alpha$ using the binomial formula:

$$\int_{(a)_v} (y_p - a_p)^j \, d\widetilde{\Phi}^\alpha =$$
$$\sum_{j'=0}^{j} \binom{j}{j'} (-a_p)^{j-j'} \Phi_{j'}^\alpha((a)_v) = \alpha^{-v\varkappa}\cdot$$
$$\alpha^{v\varkappa} \cdot U^{-v\varkappa} \left[\pi_{\alpha,1} U^{\varkappa v} \Big(\sum_{j'=0}^{j} \binom{j}{j'} (-a_p)^{j-j'} \Phi_{j'}\big((a)_v\big) \Big) \right]$$

First we note that all the operators

$$\alpha^{v\varkappa} \cdot U^{-v\varkappa} = \big(\alpha^{-1} U\big)^{-v\varkappa} = \big(I + \alpha^{-1} Z\big)^{-v\varkappa} = \sum_{t=0}^{n-1} \binom{-v\varkappa}{t} \big(\alpha^{-1} Z\big)^t$$

are uniformely bounded for $v \to \infty$ by a positive constant C_1 (where $U = \alpha I + Z$ and $Z^n = 0$ for $n = \mathrm{rk}_A \mathcal{M}_{r^*}(Np; A)$) because n does not depend on v and the binomial coefficients $\binom{-v\varkappa}{t}$ are all $\mathbb{Z}_p$-integers.

On the other hand by the condition (*growth*) of the theorem (for the distributions Φ_j)

$$\left| U^{\varkappa v} (\sum_{j'=0}^{j} \binom{j}{j'} (-a_p)^{j-j'} \Phi_{j'}((a)_v)) \right|_{p, \mathcal{M}_{r^*}} \le C p^{-vj}$$

for all $j = 0, 1, \dots, \varkappa h - 1$. If we apply to this estimate the previous bounded operators $\pi_{\alpha,1} U^{\varkappa v}$ we get the following inequality

$$\left| \int_{(a)_v} (y_p - a_p)^j \, d\widetilde{\Phi}^\alpha \right|_{p, \mathcal{M}_{r^*}} \le C \cdot C_1 |\alpha^{-v\varkappa}|_p \cdot p^{-vj} = o(p^{-v(j - h\varkappa)})$$

because of the estimate

$$|\alpha^{-v\varkappa}|_p = \big(p^{v_p(\alpha)}\big)^{v\varkappa} = o(p^{vh\varkappa})$$

§4. The standard L-function of a Siegel cusp eigenform and its critical values.

4.1. The standard L function. Let $f \in \mathcal{M}_k^m(C,\psi)$ be an eigenfunction of all Hecke operators, and let

$$(\alpha_0, \alpha_1, \cdots \alpha_m) = (\alpha_{0,f}(q), \alpha_{1,f}(q), \cdots \alpha_{m,f}(q)) \in [(\mathbb{C}^\times)^{m+1}]^{W_m} \tag{4.1}$$

be *the Satake q-parameters* of the modular form f (for all primes q not dividing the level C).

The standard L function $\mathcal{D}(s,f,\chi)$ of $f \in \mathcal{M}_m^k(C,\psi)$ is the product $\mathcal{D}(s,f,\chi) = \prod_{q \not| N} \mathcal{D}^{(q)}(s,f,\chi)$ with

$$\mathcal{D}^{(q)}(s,f,\chi) = (1-\chi(q)\psi(q)q^{-s})^{-1} R_{f,q}(\chi(q)\psi(q)q^{-s})^{-1},$$

that is

$$\mathcal{D}(s,f,\chi) = \\ = \prod_q \left\{ \left(1 - \frac{\chi(q)\psi(q)}{q^s}\right) \prod_{i=1}^m \left(1 - \frac{\chi(q)\psi(q)\alpha_i(q)^{-1}}{q^s}\right)\left(1 - \frac{\chi(q)\psi(q)\alpha_i(q)}{q^s}\right)\right\}^{-1}, \tag{4.2}$$

where χ is a Dirichlet character mod M. Analytic properties of the standard zeta functions were investigated by A. N. Andrianov and V. L. Kalinin [An79] in the case of even degree m, and S. Böcherer [Bö85] extended these results to the case of arbitrary degree using a different approach (see also [Bö-Sch] where a p-adic version of his construction was developped). For $m = 1$ and a normalized cusp eigenform $f(z) = \sum_{n=0}^\infty a(n)e(nz) \in \mathcal{M}_1^k(C,\psi)$ we have that

$$\mathcal{D}(s,f,\chi) = L_{2,f}(s+k-1,\chi),$$

where

$$L_{2,f}(s,\chi) = L_{CM}(2s-2k+2,\chi^2\psi^2)\sum_{n=1}^\infty \chi(n)a(n^2)n^{-s}$$

is the symmetric square of the modular form f.

4.2. Theta series (see [An79], [St2]). Let $F \in 2C_m$ be an even symmetric positive definite matrix, and q_0 its level (i.e. the smallest positive integer such that $q_0F^{-1} \in 2C_m$, and χ

a Dirichlet character modulo Q (not necesarily primitive). Put $\nu = 1$ or 0 and define the theta function

$$\theta(\chi) = \theta_F^{(\nu)}(z;\chi) = \sum_{\xi \in \mathrm{M}_m(\mathbb{Z})} \chi(\det\xi)\det\xi^{\nu} e_m(zF[\xi]/2). \tag{4.2}$$

Proposition 4.3. *(a) If*

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^m(q_0 Q^2),$$

then the following transformation formula holds

$$\theta_F^{(\nu)}(\gamma(z);\chi) = \chi(\det d)\chi_{Q^2F}^{(m)}(\gamma)\det(cz+d)^{(m/2)+\nu}\theta_F^{(\nu)}(z;\chi), \tag{4.3}$$

where $\chi_F^{(m)}(\gamma)$ is a root of unity of the eighth degree, and if m is even, then

$$\chi_F^{(m)}(\gamma) = \left(\frac{(-1)^{m/2}\det(F)}{\det d}\right). \tag{4.4}$$

(b) Let $J(M)$ denote for $M > 0$ the matrix $\begin{pmatrix} 0_m & -1_m \\ M1_m & 0_m \end{pmatrix}$. If χ is primitive modulo Q then the action of the involution $J(Q^2q_0)$ on (4.3) is given by

$$\begin{aligned} &\theta_F^{(\nu)}(J(Q^2q_0)z;\chi) = \\ &\quad \chi(-1)^m Q^{m\nu} G_Q(1_m,\chi)\det(F)^{(m/2)+\nu}[\det(-iz)]^{(m/2)+\nu}\theta_{\hat{F}}^{(\nu)}(z;\overline{\chi}), \end{aligned} \tag{4.5}$$

with $\hat{F} = q_0F^{-1}$ and

$$G_Q(\xi,\chi) = \sum_{h \in \mathrm{M}_m(\mathbb{Z}) \bmod Q} \chi(\det h)\, e_m({}^t\xi h/Q)$$

is the Gauss sum of degree m of the character χ.

4.4. The Rankin zeta function. Let f and g be two holomorphic modular forms of weights k and l on the congruence subgroup $\gamma_0^m(N) \subset \Gamma^m$, and with Dirichlet characters ψ and ω. More precisely we suppose that

$$f(z) = \sum_{\xi \in B_m} a(\xi) e_m(\xi z) \in \mathcal{S}_m^k(N,\psi), \tag{4.6}$$

$$g(z) = \sum_{\xi \in C_m} b(\xi) e_m(\xi z) \in \mathcal{M}_m^l(N, \psi), \tag{4.7}$$

with B_m being the set of half integral non negative matrices of size $m \times m$, and $C_m \subset B_m$ the subset of all positive definite matrices, see §1. Define an equivalence relation on B_m by $\xi_1 \sim \xi_2$ iff $\xi_1 = {}^t u \xi_2 u$ for some matrix $u \in \mathrm{SL}_m(\mathbb{Z})$. Then the *Rankin zeta function* (the convolution of two Siegel modular forms) is defined as the series:

$$L(s, f, g) = \sum_{\xi \in \tilde{C}_m} \varepsilon(\xi)^{-1} a(\xi) b(\xi) \det(\xi)^{-s} \quad (s \in \mathbb{C}), \tag{4.8}$$

where $\varepsilon(\xi) = |\mathrm{Aut}(\xi)|$ is the order of the unit group

$$\mathrm{Aut}(\xi) = \left\{\eta \in \mathrm{SL}_m(\mathbb{Z}) \mid {}^t \eta \xi \eta = \xi[\eta] = \xi\right\}.$$

This series is well defined due to basic properties of Fourier coefficients of Siegel modular forms; $\tilde{C}_m = C_m \bmod \sim$ denotes the orbit space with respect to $\sim$ and $\mathrm{Re}(s)$ is supposed to be large enough.

Proposition 4.5. (Integral representation for the Rankin zeta function) (see [St2], proposition 6).

For s with $\mathrm{Re}(s) \gg 0$ there is the following integral representation.

$$(4\pi)^{-ms} \Gamma_m(s) L(s, f, g) = \langle f^\rho, g E(z, s - k + (m+1)/2; k - l, \psi\omega, N) \rangle_N \tag{4.10}$$

where the Petersson inner product,

$$f^\rho(z) = \sum_{\xi \in B_m} \overline{a(\xi)} e_m(\xi z) \in \mathcal{S}_m^k(N, \psi),$$

and $\Gamma_m(s)$ denotes the Γ-function of degree m, i.e.

$$\Gamma_m(s) = \pi^{m(m-1)/4} \prod_{j=0}^{m-1} \Gamma(s - (j/2)).$$

4.6. The standard zeta function $\mathcal{D}(s, f, \chi)$ as the Rankin convolution. Now let

$$f(z) = \sum_{\xi \in C_m} \overline{a(\xi)} e_m(\xi z) \in \mathcal{S}_m^k(N, \psi)$$

be a cusp form of even degree m which is an eigenfunction of the Hecke algebra $\mathcal{L}_q^m(N)$ for q not dividing N, and let

$$(\alpha_0, \alpha_1, \cdots, \alpha_m) = (\alpha_0(q), \alpha_1(q), \cdots, \alpha_m^q)) \in [(A^\times)^{m+1}]^{W_m}$$

be the corresponding $(m+1)$-tuple of the Satake q-parameters of f. Fix a matrix $\xi_0 \in C_m$ such that $a(\xi_0) \neq 0$ and consider the primitive quadratic Dirichlet character χ_{ξ_0} definied for positive integers d with $(d, 2\det(2\xi_0)) = 1$ by the formula

$$\chi_{\xi_0}(d) = \left(\frac{(-1)^{m/2}\det(2\xi_0)}{d}\right),$$

and when $\det(2\xi_0)$ is odd, then by definition $\chi_{\xi_0}(2) = 1$ or -1 according as to which of the two following quadratic forms

$$x_1x_2 + x_3x_4 + \ldots x_{m-1}x_m$$

or

$$x_1x_2 + \ldots + x_{m-3}x_{m-2} + x_{m-1}^2 + x_{m-1}x_m + x_m^2$$

our quadratic form ξ_0 is equivalent over the field $\mathbf{F}_2$ of two elements. Assume also that χ mod M is chosen so that $(-1)^\nu = \chi(-1)$. The main result of the A.N.Andrianov's work [An6] can be stated as a certain identity expressing the standard zeta function $\mathcal{D}(s, f, \chi)$ as a Rankin zeta function, namely the convolution of the given form f and a theta function with the Dirichlet character χ mod M. The precise statement of the result is given in the following proposition.

Proposition 4.7. *Under the notation and assumptions as above for the sufficiently large values of* $\mathrm{Re}(s)$ *the following identity holds*

$$\begin{aligned} &a(\xi_0)R(s, f, \chi) = \\ &\quad 2^{-1}\det \xi_0^{(s+k-1+\nu)/2} L((s+k-1+\nu)/2, f, \theta_{2\xi_0}^{(\nu)}(z; \chi)), \end{aligned} \tag{4.11}$$

where the function $R(s, f, \chi)$ *is defined by the following equality*

$$\mathcal{D}(s, f, \chi) = L(s + (m/2), \psi\chi_{\xi_0}\chi) \prod_{i=0}^{m-1} L(2s + 2i, \psi^2\chi^2)R(s, f, \chi), \tag{4.12}$$

and it is assumed that the modulus M of the character χ is divisible by all prime divisors of the number $N\det 2\xi_0$, where $\theta^{(\nu)}_{2\xi_0}(z;\chi)$ is the theta function as in4.2.

More explicitly, the right hand side of (4.11) can be represented as the series

$$\begin{gathered}(\det\xi_0)^{(s+k-1+\nu)/2}\sum_{\xi}\chi(\det\xi)\det\xi^{\nu}a(\xi_0[\xi])(\det\xi_0[\xi])^{-(s+k-1+\nu)/2}= \\ \sum_{\xi}\chi(\det\xi)a({}^t\xi\xi_0\xi)(\det\xi)^{-(s+k-1+\nu)/2},\end{gathered}\tag{4.13}$$

where the summation is taken over the set of equivalence classes $\xi\in\mathrm{SL}_m(\mathbb{Z})\backslash\mathrm{M}_m^+(\mathbb{Z})$ of the form $\mathrm{SL}_m(\mathbb{Z})\xi$.

Now we set q_0 to be equal to the level of the quadratic form with the matrix $2\xi_0$ (see 4.2). In order to get the integral representation of the standard zeta function we apply to it the result of 2.6. For this purpose we put in the notation of 2.6 $l=(m/2)+\nu$, $\omega=\chi\chi_{\xi_0}$ with χ being a Dirichlet character modulo M, and take the number Cq_0M^2 as N. We note also that both parts of (2.27) and (2.26) converge absolutely for $\mathrm{Re}(s)>m$ (see also [An79], p. 133).

Proposition 4.8 (Integral representation of the standard zeta function). *With the notation and assumption as above for* $\mathrm{Re}(s)$ *the following equality holds:*

$$\begin{gathered}2a(\xi_0)(\det\xi_0)^{-(s+k-1+\nu)/2}(4\pi)^{-ms}\times\Gamma_m((s+k-1+\nu)/2)\,R(s,f,\chi)= \\ \langle f^{\rho},\theta^{(\nu)}_{2\xi_0}(z;\chi)E(z,(s-k+m+\nu)/2)\rangle_{NM^2q_0},\end{gathered}\tag{4.14}$$

where the Siegel – Eisenstein series in the right hand side

$$E(z,s)=E(z,s;k-\nu-(m/2),\chi\chi_{\xi_0}\psi,NM^2q_0)$$

is defined by (2.6) .

4.9. Algebraic properties of the special values of normalized distributions. Consider only the case of even m. In order to give a precise statements on algebraic properties of the standard zeta functions and of the corresponding distributions, it is convenient to make some additional normalization of these values because these properties look different for the integral points to the left and to the right of the critical line $\mathrm{Re}(s) = \frac{1}{2}$ (the same effect happens in the case of the Riemann zeta function).

Recall the following three types of the normalized zeta function: for $\kappa = (m+1)/2$, $s \in \mathbb{C}$

$$\begin{aligned}&\mathcal{D}^*(s, f, \chi) = \\ &(2\pi)^{-m(s+k-\kappa)}\Gamma((s+\delta)/2)\prod_{j=1}^{m}\Gamma(s+k-j)\mathcal{D}(s, f, \chi),\end{aligned} \tag{4.15}$$

$$\begin{aligned}&\mathcal{D}^-(s, f, \chi) = \Gamma((s+\delta)/2)^{-1}\mathcal{D}^*(s, f, \chi) = \\ &(2\pi)^{-m(s+k-\kappa)}\prod_{j=1}^{m}\Gamma(s+k-j)\mathcal{D}(s, f, \chi),\end{aligned} \tag{4.16}$$

$$\begin{aligned}&\mathcal{D}^+(s, f, \chi) = \frac{2\, i^\delta \Gamma(s)\cos(\pi(s-\delta)/2)}{(2\pi i)^s}\mathcal{D}^-(s, f, \chi) = \\ &\frac{i^\delta \pi^{(1-2s)/2}}{\Gamma((1+\delta-s)/2)}\mathcal{D}^*(s, f, \chi),\end{aligned} \tag{4.17}$$

where $\delta = 0$ or 1 according as $\psi\chi(-1) = (-1)^\delta$ with $f \in \mathcal{S}_k^m(C, \psi)$ be an eigenfunction of the global Hecke algebra $\mathcal{L}^m(C) = \otimes_{q \not| C}\mathcal{L}_q^m(C)$ with the eigenvalue given as a homomorphism $\lambda_f : \mathcal{L}^m(C) \to \mathbb{C}$.

Now we turn to the normalizations (4.16) and (4.17). Their convenience is illustrated by the following result on algebraicity of the special values of (4.16) and (4.17) which is proven in this section.

Theorem 4.10 (Algebraic properties of the special values of the standard zeta function). *Assume that the cusp eigenform $f \in \mathcal{S}_k^m(C, \psi)$ satisfies the condition $a(\xi_0) = 1$ for some $\xi_0 \in A_m, \xi_0 > 0$. Then*

a) For all integers s satisfying $1 \le s \le k - \delta - m$ and $s \ne 1$ if the character $\chi^2\psi^2$ is trivial we have that

$$\langle f,\ f\rangle^{-1}\,\mathcal{D}^+(s, f, \chi) \in K = \mathbb{Q}(f, \lambda_f, \psi, \chi), \tag{4.18}$$

where $K = \mathbb{Q}(f, \lambda_f, \psi, \chi)$ denote the field generated by Fourier coefficients of f, by the eigenvalues $\lambda_f(X)$ of Hecke operators X on f, and by the values of the characters χ and ψ. Moreover,

$$\mathcal{D}^+(s, f, \chi) = 0 \quad \text{for} \quad s \not\equiv \delta(\mathrm{mod} 2).$$

b) For all integers s satisfying $1-k+\delta+m \le s \le 0$ we have that

$$\langle f,\ f\rangle^{-1}\mathcal{D}^{-}(s,f,\chi) \in K, \tag{4.19}$$

and

$$\mathcal{D}^{-}(s,f,\chi)=0 \quad \text{for} \quad s \equiv \delta(\mathrm{mod} 2), \quad s \in \mathbf{Z}, s \le 0.$$

Proof of the theorem 4.10 (see [PaLNM], Chapter 3). One uses the identity (2.29) of chapter 2 of [PaLNM], which says that for all non-trivial Dirichlet characters χ mod p^{v}:

$$\begin{aligned} &2a(\xi_0)((4\pi)^m \det(\xi_0))^{-(s+k-1+\nu)/2}\Gamma_m((s+k-1+\nu)/2)\,\mathcal{D}(s,f,\chi) = \\ &\qquad L_N(s+\frac{m}{2},\psi\chi_{\xi_0}\chi)\prod_{i=0}^{(m/2)-1} L_N(2s+2i,\psi^2\chi^2))\times \\ &\qquad \times\langle f^{\rho}(z),\theta^{(\nu)}_{2\xi_0}(z;\chi)E(z,(s-k+m+\nu)/2)\rangle_N, \end{aligned} \tag{4.20}$$

$$E(z,s)=E(z,s;k-(m/2)-\nu,\psi\chi_{\xi_0}\chi,N)$$

is the Siegel – Eisenstein series of the weight $k-(m/2)-\nu$, the level $N=4q_0CM^2$, and the Dirichlet character $\psi\chi_{\xi_0}\chi$ is viewed mod$N$.

§5. Algebraic linear forms on arithmetical nearly holomorphic Siegel modular forms.

5.1. Convolutions of theta distributions and Eisenstein distributions with values in nearly holomorphic Siegel modular forms. Consider again the profinite group

$$Y = \lim_{\substack{\leftarrow \\ v}} Y_v, \quad \text{where} \quad Y_v = (\mathbb{Z}/Np^v\mathbb{Z})^\times.$$

Let $M = \mathcal{M}^m_{r,k}(A)$ and let $\Phi_i : \mathcal{C}^{loc-const}(Y, A) \to M$ be two distributions on Y with values in $M_i = \mathcal{M}^m_{r_i,k_i}(A)$, $(i = 1, 2)$. Assume that A contains $(\varphi(Np^v))^{-1}$ and all the values of Dirichlet characters χ mod Np^v viewed as homomorphisms $Y \to A^\times$ then obviously

$$\Phi_i(\chi) = \sum_{a \bmod Np^v} \chi(a)\Phi_i((a)_v), \quad \Phi_i((a)_v) = \frac{1}{\varphi(Np^v)} \sum_{\chi \bmod Np^v} \chi(a)^{-1}\Phi_i(\chi).$$

Let us fix a Dirichlet character ψ on Y with values in A then there exists a unique distribution $\Phi = \Phi_1 *_\psi \Phi_2$ on Y with values in $M = \mathcal{M}^m_{r_1+r_2,k_1+k_2}(A)$ such that for all Dirichlet characters χ on Y with values in A one has $\Phi(\chi) = \Phi_1(\chi) \cdot \Phi_2(\psi\chi^{-1})$ ($\Phi_1 *_\psi \Phi_2$ is called the *twisted convolution of* Φ_1 *and* Φ_2.

Let us put $M = p^v$, $N = 4q_0C$ and take as Φ_1 the theta distribution on Y defined on Dirichlet characters χ mod M by

$$\Phi_1(\chi) = \theta^{(\nu)}_{2\hat{\xi}_0}(\chi)|V(C) \tag{5.1}$$

and let us take as $\Phi^\pm_{2,s}$ the normalized Siegel-Eisenstein distribution (2.21) defined on Dirichlet characters ω by

$$\begin{aligned} &\Phi^\pm_{2,s}(\omega) = G^\pm(z, (s-k+m+\nu)/2); k-(m/2)-\nu, \omega, NM') = \\ &G^\pm_{'s-k+m+\nu)/2),k-(m/2)-\nu}(\omega), NM') = \delta^{(r^\pm)}\Psi^\pm_{2,s}(\omega) \end{aligned} \tag{5.2}$$

(we use it for the Dirichlet character $\omega = \psi\chi_{\xi_0}\bar{\chi}$) where $M' = M^2$,

$$\Psi^+_{2,s}(\omega) = G^+_{m/2+s,0}(\omega),$$
$$\Psi^-_{2,s}(\omega) = G^-_{m/2+1-s,s-(1/2)}(\omega)$$

are two *holomorphic* Siegel-Eisenstein series.

Next define $\Phi_s^{\pm} = \Phi_1 *_{\psi\chi_{\xi_0}} \Phi_{2,s}^{\pm}$ (on basic open subsets one has

$$\Phi_s^{\pm}((a)_v) = \sum_{y \in Y_v} \psi\chi_{\xi_0}(y)\Phi_1(ya)\Phi_{2,s}^{\pm}(y). \tag{5.3}$$

for any v such that $\psi\chi_{\xi_0}$ is defined $\bmod Np^v$).

We show that for all critical pairs (s, χ) as Theorem 4.10 the function $\Phi_s^{\pm}(\chi)$ is an arithmetical nearly holomorphic Siegel modular form in $\mathcal{M}_{r,k}^m(\Gamma_1^m(Np^v), \bar{\mathbb{Q}})$, and for an appropriate linear form $\mathcal{L}_{f,\alpha}$ on $\mathcal{M}_{r,k}^m(\Gamma_1(Np^v), \bar{\mathbb{Q}})$ (defined over $\bar{\mathbb{Q}}$) the value $\mathcal{L}_{f,\alpha}(\Phi_s^{\pm}(\chi))$ coincides, up to a multiplicative constant in $\bar{\mathbb{Q}}^{\times}$, with the corresponding value $\mathcal{D}_{s,\chi}^{\pm}$ of the normalized distribution constructed in [PaLNM], Chapter 3 (which produces the critical value $\mathcal{D}^{\pm}(s, f, \bar{\chi})$)).

Let us consider (as in Proposition 1.6) the canonical projection operator

$$\pi_\alpha : \mathcal{M}_{r,k}^m(\mathbb{C}) \to \mathcal{M}_{r,k}^m(\Gamma_1^m(Np); \mathbb{C})^{\alpha}$$

attached to a non-zero eigenvalue α of $U = \Pi^+(p)$ acting on

$$\mathcal{M}_{r,k}^m(N, p; \mathbb{C}) = \bigcup_{v \geq 1} \mathcal{M}_{r,k}^m(\Gamma_1^m(Np^v); \mathbb{C}) :$$

for any $g \in \mathcal{M} = \mathcal{M}_{r,k}^m(\mathbb{C})$ put

$$\pi_\alpha(g) = U^{-v}\left[\pi_{\alpha,1} U^v(g)\right]$$

($\pi_\alpha(g)$ is well defined if v is sufficiently large so that $g \in \mathcal{M}^m(Np^{v+1})$).

Proposition 5.2. *Let $\alpha \in \bar{\mathbb{Q}} \subset \mathbb{C}$ be a non-zero eigenvalue of $U = \Pi^+(p)$ on $\mathcal{M}_{r,k}^m(\mathbb{C})$ associated with a cusp eigenform $f \in \mathcal{S}_k(\Gamma_0^m(N), \psi)$ and let $f_0 = f_{0,\alpha}$ be a corresponding eigenfunction ($f_0|U = \alpha f_0$), let us define $f^0 = f_0^{\rho}|W_{N_0}$, $f_0^{\rho} = \sum_{\xi} \overline{a_\xi(f_0)} q^{\xi}$. Then*

a) $U^* = W_{N_0}^{-1} U W_{N_0}$, the adjoint operator of U in the hermitian vector space $\mathcal{S}_{r,k}(\Gamma_1^m(N_0), \mathbb{C})$ with respect to the Petersson scalar product.

b) One has $f^0|U^* = \bar{\alpha} f^0$, and for all "good" Hecke operators $T \in \mathcal{L}(Np^v)$ one has $Tf^0 = \lambda_f(T) f^0$.

c) The linear form $\ell_{f,\alpha} : g \mapsto \langle f^0, g\rangle$ *on* $\mathcal{M}^m_{r,k}(\Gamma_1^m(N_0),\mathbb{C})$ *vanishes on* $\mathrm{Ker}\pi_{\alpha,1}$*, where* $\pi_{\alpha,1} : \mathcal{M}^m_{r,k}(\Gamma_1^m(N_0),\mathbb{C}) \to \mathcal{M}^m_{r,k}(\Gamma_1^m(N_0),\mathbb{C})^\alpha$ *(the projector onto the* α*-primary subspace with the kernel* $\mathrm{Ker}\pi_{\alpha,0} = \mathrm{Im}(U-\alpha I)^n$*) hence*

$$\langle f^0, g\rangle = \langle f^0, \pi_{\alpha,1}(g)\rangle = \langle f^0, \pi_\alpha(g)\rangle.$$

d) If $g \in \mathcal{M}^m(Np^{v+1},\overline{\mathbb{Q}}) = \mathcal{M}^m(N_0p^v,\overline{\mathbb{Q}})$ *and* $\alpha \neq 0$*, one has*

$$\langle f^0, g^\alpha\rangle = \langle f^0, \pi_\alpha(g)\rangle = \alpha^{-v}\langle f^0, g \mid U^v\rangle$$

where

$$g^\alpha = \pi_\alpha(g) = (U^\alpha)^{-v}\pi_{\alpha,1}(g \mid U^v) \in \mathcal{M}^{m\,\alpha}(Np)$$

is the α*-primary part of* g *(the image of* g *under the canonical projection* π_α*):*

e) Let us define

$$\mathcal{L}_{f,\alpha}(g) = \frac{\langle f^0, \alpha^{-v} g|U^v\rangle_{N_0}}{\langle f^0, f_0\rangle_{N_0}} = \frac{\langle f^0, \pi_\alpha(g)\rangle_{N_0}}{\langle f^0, f_0\rangle_{N_0}},$$

hence $\mathcal{L}_{f,\alpha} : \mathcal{M}^m_{r,k}(N,p;\mathbb{Q}) \to \overline{\mathbb{Q}}$ *is the linear form* $\mathcal{L}_{f,\alpha}$ *on* $\mathcal{M}^m_{r,k}(N,p;\mathbb{C})$ *which is defined over* $\overline{\mathbb{Q}}$ *and factorizes through the finite dimensional projection*

$$\pi_\alpha : \mathcal{M}^m_{r,k}(N,p;\mathbb{C}) \to \mathcal{M}^m_{r,k}(N,p;\mathbb{C})^\alpha \subset \mathcal{M}^m_{r,k}(\Gamma_1^m(Np);\mathbb{C});$$

There exists a unique $\mathbb{C}_p$*-linear form* $\ell_{f,\alpha} \in (\mathcal{M}^m_{r,k}(N_0,\mathbb{C}_p)^\alpha)^*$ *such that*

$$\ell_{f,\alpha}(i_p(\pi_\alpha(g))) = i_p\left(\mathcal{L}_{f,\alpha}(g)\right) = i_p\left(\frac{\langle f^0, \alpha^{-v}U^v(g)\rangle_{N_0}}{\langle f^0, f_0\rangle_{N_0}}\right)$$

($g \in \mathcal{M}^m_{r,k}(N,p;\bar{\mathbb{Q}})$ *with coefficients in* $\bar{\mathbb{Q}}$*).*

Proof of Proposition 5.2 a) See [Miy], Th. 4.5.5 for $m=1$ (the general case is very similar).

b) By definition, $f^0|U^* = f_0^\rho \mid W_{Np}W_{Np}^{-1}UW_{Np} = \bar\alpha f_0^\rho \mid W_{Np} = \bar\alpha f^0$.

c) For any function

$$g_1 = (U-\alpha I)^n g \in \mathrm{Ker}\pi_{\alpha,0} = \mathrm{Im}(U-\alpha I)^n$$

one has

$$\langle f^0, g_1\rangle = \langle f^0, (U-\alpha I)^n g\rangle = \langle (U^* - \bar{\alpha} I) f^0, (U-\alpha I)^{n-1} g\rangle = 0$$

hence for $g_1 = g - \pi_{\alpha,0}(g)$ we get

$$\langle f^0, g\rangle = \langle f^0, \pi_{\alpha,0}(g) + (g - g^\alpha)\rangle = \langle f^0, \pi_{\alpha,0}(g)\rangle + \langle f^0, g_1\rangle = \langle f^0, \pi_{\alpha,0}(g)\rangle$$

d) Let us use directly the equality $(U^*)^v f^0 = \bar{\alpha}^v f^0$ of b):

$$\alpha^v \cdot \langle f^0, g^\alpha\rangle = \langle (U^*)^v f^0, U^{-v}\pi_{\alpha,1}(g \mid U^v)\rangle = \langle f^0, \pi_{\alpha,1}(g \mid U^v)\rangle = \langle f^0, g \mid U^v\rangle$$

by c) because $g \mid U^v \in \mathcal{M}^m(Np)$.

e) Note that $\mathcal{L}_{f,\alpha}(f_0) = 1$, $f_0 \in \mathcal{M}^m(Np; \overline{\mathbb{Q}})$; consider the complex vector space

$$\mathrm{Ker}\mathcal{L}_{f,\alpha} = \langle f^0\rangle^\perp = \{g \in \mathcal{M}^m(Np; \mathbb{C}) \mid \langle f^0, g\rangle = 0\}$$

which admits a $\overline{\mathbb{Q}}$-rational basis because it is stable by the action of for all "good" Hecke operators $T \in \mathcal{L}(Np^v)$:

$$\langle f^0, g\rangle = 0 \Longrightarrow \langle f^0, g|T\rangle = \langle f^0|T^*, g\rangle = 0$$

and one obtains such a basis by the diagonalisation of the action of all the T (a commutative family of normal operators) and e) follows.

5.3. Our next task is to compute the value

$$\begin{aligned}
&\mathcal{L}_{f,\alpha}(\Phi_s^\pm(\chi)) = \mathcal{L}_{f,\alpha}(\Phi_1(\chi)\cdot\Phi_{2,s}^\pm(\psi\chi_{\xi_0}\chi^{-1})) = \\
&\mathcal{L}_{f,\alpha}(\theta^{(\nu)}_{2\hat{\xi}_0}(\chi)|V(C)G^\pm(z,(s-k+m+\nu)/2; k-(m/2)-\nu, \psi\chi_{\xi_0}\overline{\chi}, NM')) = \\
&\frac{\alpha^{-2v}\langle f^0, (\Phi_1(\chi)\cdot\Phi_{2,s}^\pm(\psi\chi_{\xi_0}\chi^{-1}))|U^{2v}\rangle_{Np}}{\langle f^0, f_0\rangle_{Np}} \\
&\frac{\alpha^{-2v}\langle f^0, (\theta^{(\nu)}_{2\hat{\xi}_0}(\chi)|V(C)G^\pm(z, \frac{s-k+m+\nu}{2}; k-\frac{m}{2}-\nu, \psi\chi_{\xi_0}\overline{\chi}, NM'))|U^{2v}\rangle_{Np}}{\langle f^0, f_0\rangle_{Np}} \\
&\frac{\alpha^{-2v}\langle f_0^\rho|W_{Np}, (\theta^{(\nu)}_{2\hat{\xi}_0}(\chi)|V(C)G^\pm(z, \frac{s-k+m+\nu}{2}; k-\frac{m}{2}-\nu, \psi\chi_{\xi_0}\overline{\chi}, NM'))|U^{2v}\rangle_{Np}}{\langle f^0, f_0\rangle_{Np}}
\end{aligned} \tag{5.4}$$

where χ mod N_0p^v is a Dirichlet character of conductor $C_\chi = p^v$. Let us unfold the integral (5.4) using the relation (given in Proposition 1.6) of U^{2v} with the trace operator

$$g|U^{2v} = p^{vm(k-m-1)}g|W_{Np^{2v+1}}\mathrm{Tr}_{Np}^{Np^{2v+1}}W_{Np}:$$

$$\mathcal{L}_{f,\alpha}(\Phi_s^{\pm}(\chi)) = \frac{(-1)^{mk}p^{vm(k-m-1)}}{\alpha^{2v}}$$

$$\frac{\langle f_0^\rho, (\Phi_1(\chi)\cdot\Phi_{2,s}^{\pm}(\psi\chi^{-1}))|W_{Np^{2v+1}}\mathrm{Tr}_{Np}^{Np^{2v+1}}\rangle_{Np}}{\langle f^0, f_0\rangle_{Np}}$$

$$= \frac{(-1)^{mk}p^{vm(k-m-1)}}{\alpha^{2v}} \tag{5.5}$$

$$\frac{\langle f_0^\rho, (\theta_{2\hat{\xi}_0}^{(\nu)}(\chi)|V(C)G^{\pm}(z, \frac{s-k+m+\nu}{2}; k-\frac{m}{2}-\nu, \psi\chi_{\xi_0}\overline{\chi}, Np^{2v+1}))|W_{Np^{2v+1}}\rangle_{Np^{2v+1}}}{\langle f^0, f_0\rangle_{Np}}$$

$$= \frac{(-1)^{mk}p^{vm(k-m-1)}}{\alpha^{2v}\langle f^0, f_0\rangle_{Np}}\cdot\langle f_0^\rho, \Phi_1(\chi)|W_{Np^{2v+1}}\Phi_{2,s}^{\pm}(\psi\chi_{\xi_0}\chi^{-1})|W_{Np^{2v+1}}\rangle_{Np^{2v+1}}$$

Let us compute separately

$$\Phi_1(\chi)|W_{Np^{2v+1}} = \theta_{2\hat{\xi}_0}^{(\nu)}(\chi)|V(C)W_{Np^{2v+1}} \tag{5.6}$$

and

$$\Phi_{2,s}^{\pm}(\psi\chi_{\xi_0}\chi^{-1})|W_{Np^{2v+1}} = G^{\pm}(z, \frac{s-k+m+\nu}{2}; k-\frac{m}{2}-\nu, \psi\chi_{\xi_0}\overline{\chi}, NM')|W_{Np^{2v+1}} \tag{5.7}$$

using the action of the main involution on theta series, and using the definitions (2.12) and (2.13) of the normalized Siegel-Eisenstein series.

Now let us put $N_0M' = 4q_0C_\chi^2N'$ with $N' = NM_0^{\tilde{m}-1}M'C_\chi^{-2}$ so that the following transformation formula holds:

$$\begin{aligned}\theta_{2\hat{\xi}_0}^{(\nu)}(\chi)|W(N_0M') &= N'^{m(m+2\nu)/4}\theta_{2\hat{\xi}_0}^{(\nu)}(\chi)|W(4q_0C_\chi^2)V(N')\\ &=(NM_0^{\tilde{m}-1}M')^{m(m+2\nu)/4}\frac{G_{C_\chi}(1_m,\chi)}{(iC_\chi)^{m^2/2}C_\chi^{-m(m+2\nu)/2}}\cdot\\ &\quad\cdot\det(q_0^{-1/2}\xi_0)^{m/2+\nu}\theta_{2\xi_0}^{(\nu)}(\bar{\chi})|V(N')\end{aligned} \tag{5.8}$$

Recall the definition of the normalized Siegel-Eisenstein series of the level $N_0M' = Np^{2v+1}$:

$$G^-(z,s) = \Gamma((k+2s-(m/2)+\mu)/2)^{-1}G^*(z,s),$$

$$G^{+}(z,s) = \frac{i^{\mu}\pi^{\kappa-k-2s}}{\Gamma((1-k-2s+(m/2)+\mu)/2)}G^{*}(z,s) =$$

$$\frac{2i^{\mu}\Gamma(k+2s-(m/2))\cos(\pi(k+2s-(m/2)-\mu)/2)}{(2\pi)^{k+2s-(m/2)}}G^{-}(z,s).$$

where $\mu = \varepsilon((m/2)+k)$, and

$$G^{*}(z,s) = G^{*}(z,s;k,\chi,N_0M') =$$
$$(N_0M')^{m(k+s)}\widetilde{\Gamma}(k,s)L_{N_0}(k+2s,\chi)\prod_{i=1}^{[m/2]}L_{N_0}(2k+4s-2i,\chi^2)E^{*}(N_0M'z,s), \tag{5.9}$$

with

$$E^{*}(N_0M'z,s) = E(-(N_0M'z)^{-1},s)\det(N_0M'z)^{-k} = (N_0M')^{-km/2}E|W(N_0M'),$$
$$\widetilde{\Gamma}(k,s) =$$
$$i^{mk}2^{-m(k+1)}\pi^{-m(s+k)} \times \begin{cases} \Gamma_m(k+s)\Gamma(s+(k-\frac{m}{2}+\mu)/2), & \text{if } m \text{ is even;} \\ \Gamma_m(k+s), & \text{otherwise.} \end{cases} \tag{5.10}$$

and

$$L_N(s,\chi) = \prod_{q\not| N}(1-\chi(q)q^{-s})^{-1}$$

is the Dirichlet series associated to the character χ and $\mu = \varepsilon(m/2+k)$, with $\varepsilon(r)\in\{0,1\}$ such that $\varepsilon(r)\equiv r \pmod 2$ for any integer $r\in\mathbb{N}$.

Let us introduce the Gamma-factors $\Gamma^{\pm}(k,s)$ by the equality

$$\begin{aligned} G^{\pm}(z,s) &= G^{\pm}(z,s;k,\chi,N_0M') \\ &= (N_0M')^{m(k+s)}\Gamma^{\pm}(k,s) \\ &\qquad L_{N_0}(k+2s,\chi)\prod_{i=1}^{[m/2]}L_{N_0}(2k+4s-2i,\chi^2)E^{*}(N_0M'z,s) \\ &= (N_0M')^{m((k/2)+s)}\Gamma^{\pm}(k,s) \\ &\qquad L_{N_0}(k+2s,\chi)\prod_{i=1}^{[m/2]}L_{N_0}(2k+4s-2i,\chi^2)E|W(N_0M'), \end{aligned} \tag{5.11}$$

so that the change of variables $s := (s-k+m+\nu)/2$, $k := k-(m/2)-\nu$ in (5.11) gives

$k+2s := s+(m/2) = (s+2m)/2$ and

$$\begin{aligned} &G^{\pm}\left(z, \frac{s-k+m+\nu}{2}; k-\frac{m}{2}-\nu, \psi\chi_{\xi_0}\overline{\chi}, N_0M'\right)|W(N_0M') \\ &= (N_0M')^{m((s+2m)/4}\Gamma^{\pm}(k-\frac{m}{2}-\nu, \frac{s-k+m+\nu}{2}) \\ &\quad L_{N_0}((2s+m)/2, \psi\chi_{\xi_0}\overline{\chi})\prod_{i=1}^{[m/2]} L_{N_0}(2s+m-2i, \overline{\chi}^2\psi^2) \\ &E(z, (s-k+m+\nu)/2; k-(m/2)-\nu, \psi\chi_{\xi_0}\overline{\chi}, N_0M'), \end{aligned} \tag{5.12}$$

and substitute it to (4.14) with χ replaced by the Dirichlet character $\overline{\chi}$

$$\begin{aligned} &2a(\xi_0)(\det\xi_0)^{-(s+k-1+\nu)/2}(4\pi)^{-ms}\times\Gamma_m((s+k-1+\nu)/2)\,R(s,f,\overline{\chi}) = \\ &\langle f_0^{\rho}, \theta_{2\xi_0}^{(\nu)}(z;\overline{\chi})E(z,(s-k+m+\nu)/2; k-(m/2)-\nu, \psi\chi_{\xi_0}\overline{\chi})\rangle_{NM^2q_0}, \end{aligned} \tag{5.13}$$

where

$$\begin{aligned} &a(\xi_0)R(s,f,\overline{\chi}) = \\ &\quad 2^{-1}\det\xi_0^{(s+k-1+\nu)/2}L((s+k-1+\nu)/2, f_0, \theta_{2\xi_0}^{(\nu)}(z;\overline{\chi})). \end{aligned} \tag{5.14}$$

The function $R(s,f,\overline{\chi})$ is related to $\mathcal{D}(s,f,\overline{\chi})$ by the following equality

$$\mathcal{D}(s,f,\overline{\chi}) = L(s+(m/2), \psi\chi_{\xi_0}\overline{\chi})\prod_{i=0}^{(m/2)-1} L(2s+2i, \psi^2\overline{\chi}^2\psi^2)R(s,f,\overline{\chi}) \tag{5.15}$$

therefore (5.13) takes the form

$$\begin{aligned} &2a(\xi_0)(\det\xi_0)^{-(s+k-1+\nu)/2}(4\pi)^{-ms}\times\Gamma_m((s+k-1+\nu)/2)\mathcal{D}(s,f,\overline{\chi}) = \\ &L(s+(m/2), \psi\chi_{\xi_0}\overline{\chi})\prod_{i=0}^{(m/2)-1} L(2s+2i, \psi^2\overline{\chi}^2\psi^2) \\ &\langle f_0^{\rho}, \theta_{2\xi_0}^{(\nu)}(z;\overline{\chi})E(z,(s-k+m+\nu)/2; k-(m/2)-\nu, \psi\chi_{\xi_0}\overline{\chi})\rangle_{NM^2q_0}, \end{aligned} \tag{5.16}$$

We see that the integral in (5.5) transforms to the following

$$\begin{aligned} &\langle f_0^{\rho}, \Phi_1(\chi)|W_{Np^{2v+1}}\Phi_{2,s}^{\pm}(\psi\chi^{-1})|W_{Np^{2v+1}}\rangle_{Np^{2v+1}} = \\ &\langle f_0^{\rho}, \theta_{2\hat{\xi}_0}^{(\nu)}(\chi)|V(C)W_{N_0M'} \\ &\quad G^{\pm}(z, \frac{s-k+m+\nu}{2}; k-\frac{m}{2}-\nu, \psi\chi_{\xi_0}\overline{\chi}, N_0M')|W_{N_0M'}\rangle_{N_0M'} \end{aligned} \tag{5.17}$$

and (5.17) becomes

$$\langle f_0^\rho, \Phi_1(\chi)|W_{Np^{2v+1}}\Phi^\pm_{2,s}(\psi\chi_{\xi_0}\chi^{-1})|W_{Np^{2v+1}}\rangle_{Np^{2v+1}} =$$
$$(NM_0^{\tilde m-1}M')^{m(m+2\nu)/4}\frac{G_{C_\chi}(1_m,\chi)}{(iC_\chi)^{m^2/2}C_\chi^{-m(m+2\nu)/2}}\det(q_0^{-1/2}\xi_0)^{m/2+\nu}$$
$$(N_0M')^{m((s+2m)/4}\Gamma^\pm(k-\frac{m}{2}-\nu, \frac{s-k+m+\nu}{2}) \tag{5.18}$$
$$L_{N_0}((2s+m)/2, \psi\chi_{\xi_0}\overline{\chi})\prod_{i=1}^{[m/2]}L_{N_0}(2s+m-2i, \overline{\chi}^2\psi^2)$$
$$\langle f_0^\rho, \theta^{(\nu)}_{2\xi_0}(\bar\chi)|V(N')E(z, \frac{s-k+m+\nu}{2}; k-\frac{m}{2}-\nu, \psi\chi_{\xi_0}\overline{\chi}, N_0M')\rangle_{N_0M'}$$

Let us now compare the value of the linear form (5.5) expliciely given by (5.18), with the integral representation (5.16) for the standard L-function:

$$\mathcal{L}_{f,\alpha}(\Phi^\pm_s(\chi)) =$$
$$= \frac{(-1)^{mk}p^{vm(k-m-1)}}{\alpha^{2v}\langle f^0, f_0\rangle_{Np}}\cdot\langle f_0^\rho, \Phi_1(\chi)|W_{Np^{2v+1}}\Phi^\pm_{2,s}(\psi\chi_{\xi_0}\chi^{-1})|W_{Np^{2v+1}}\rangle_{Np^{2v+1}}$$

$$= \frac{(-1)^{mk}p^{vm(k-m-1)}}{\alpha^{2v}\langle f^0, f_0\rangle_{Np}}$$
$$(NM_0^{\tilde m-1}M')^{m(m+2\nu)/4}\frac{G_{C_\chi}(1_m,\chi)}{(iC_\chi)^{m^2/2}C_\chi^{-m(m+2\nu)/2}}\det(q_0^{-1/2}\xi_0)^{m/2+\nu}$$
$$(N_0M')^{m((s+2m)/4}\Gamma^\pm(k-\frac{m}{2}-\nu, \frac{s-k+m+\nu}{2}) \tag{5.19}$$
$$L_{N_0}((2s+m)/2, \psi\chi_{\xi_0}\overline{\chi})\prod_{i=1}^{[m/2]}L_{N_0}(2s+m-2i, \overline{\chi}^2\psi^2)$$
$$\langle f_0^\rho, \theta^{(\nu)}_{2\xi_0}(\bar\chi)|V(N')E(z, \frac{s-k+m+\nu}{2}; k-\frac{m}{2}-\nu, \psi\chi_{\xi_0}\overline{\chi}, N_0M')\rangle_{N_0M'}$$

Let us remove from the Euler products $\mathcal{D}(s, f, \overline{\chi})$ and $L(s, \chi)$ all the factors corresponding to the prime divisors of N_0:

$$2a(\xi_0)(\det\xi_0)^{-(s+k-1+\nu)/2}(4\pi)^{-ms}\times\Gamma_m((s+k-1+\nu)/2)\mathcal{D}_{N_0}(s, f, \overline{\chi}) =$$
$$L_{N_0}(s+(m/2), \psi\chi_{\xi_0}\overline{\chi})\prod_{i=0}^{(m/2)-1}L_{N_0}(2s+2i, \psi^2\overline{\chi}^2\psi^2) \tag{5.20}$$
$$\langle f_0^\rho, \theta^{(\nu)}_{2\xi_0}(z;\overline{\chi})E(z, (s-k+m+\nu)/2; k-(m/2)-\nu, \psi\chi_{\xi_0}\overline{\chi})\rangle_{NM^2q_0},$$

Let χ be a non-trivial Dirichlet character modulo Np^v of conductor C_χ such that $C_\chi|p^v)$.

Consider fhe complex valued distributions $\mathcal{L}_{f,\alpha}(\Phi_s^\pm(\chi))$ defined for the critical pairs (s,χ) as in Theorem 4.10 by

$$\mathcal{L}_{f,\alpha}(\Phi_s^\pm(\chi)) = \mathcal{L}_{f,\alpha}(\Phi_1(\chi)\cdot\Phi_{2,s}^\pm(\psi\chi_{\xi_0}\chi^{-1})).$$

Proposition 5.4. *Assume that the cusp eigenform* $f \in \mathcal{S}_k^m(C,\psi)$ *satisfies the condition* $a(\xi_0)=1$ *for some* $\xi_0 \in A_m, \xi_0 > 0$. *Then*

a) For all integers s *satisfying* $1 \le s \le k-\delta-m$ *and* $s \ne 1$ *if the character* $\chi^2\psi^2$ *is trivial we have that the value*

$$\mathcal{L}_{f,\alpha}(\Phi_s^+(\chi)) \in K = \mathbb{Q}(f,\lambda_f,\psi,\chi),$$

where $K=\mathbb{Q}(f,\lambda_f,\psi,\chi)$ *denotes the field generated by Fourier coefficients of* f, *by the eigenvalues* $\lambda_f(X)$ *of Hecke operators* X *on* f, *and by the values of the characters* χ *and* ψ. *Moreover,*

$$\mathcal{L}_{f,\alpha}(\Phi_s^+(\chi)) = 0 \quad \text{for} \quad s \not\equiv \delta (\mathrm{mod} 2).$$

b) For all integers s *satisfying* $1-k+\delta+m \le s \le 0$ *we have that*

$$\mathcal{L}_{f,\alpha}(\Phi_s^-(\chi)) \in K,$$

and

$$\mathcal{L}_{f,\alpha}(\Phi_s^-(\chi)) = 0 \quad \text{for} \quad s \equiv \delta(\mathrm{mod} 2), \quad s\in\mathbb{Z}, s\le 0.$$

c) Suppose moreover that N *is divisible by all of the prime divisors of* $2\det(2\xi_0)$ *then there is the following equality*

$$\mathcal{L}_{f,\alpha}(\Phi_s^\pm(\chi)) = t_\pm \frac{G_{C_\chi}(1_m,\chi)}{(iC_\chi)^{m^2/2}} \frac{C_\chi^{m(2s+2k-2-m)/4}}{\alpha_0(C_\chi)^2} D^\pm(s,f,\bar\chi),$$

where $t_\pm$ *is a multiplicative constant in* $\bar{\mathbb{Q}}^\times$, *independent of the critical pair* (s,χ), $D^\pm(s,f,\bar\chi)$ *denotes the normalized standard* L*-function as in Theorem 4.10,*

$$G_{C_\chi}(1_m,\chi) = \sum_{\xi'\in \mathrm{M}_m(\mathbf{Z}) \pmod{C_\chi}} \chi(\det\xi')e_m({}^t\xi\xi'/C_\chi),$$

is the Gauss sum of degree m *of* χ.

Proof of Proposition 5.4: is directly implied by the equalities (5.19) and (5.20) taking into account Theorem 4.10 on the critical calues of the standard L function.

§6. Congruences between nearly holomorphic modular forms and proof of the Main theorem.

6.1. Regularized distributions with values in nearly holomorphic Siegel modular forms. We now outline the method of the proof of Main Theorem 0.7 and show the existence of the h–admissible measures attached to the standard zeta functions $\mathcal{D}(s,f,\chi)$. This proof is based on congruences for the Fourier coefficients of nearly holomorphic Siegel modular forms. Consider again the profinite group

$$Y = \varprojlim_{v} Y_v, \quad \text{where} \quad Y_v = (\mathbb{Z}/Np^v\mathbb{Z})^\times.$$

and consider fhe complex valued distributions $\mathcal{L}_{f,\alpha}(\Phi_s^{\pm}(\chi))$ defined for the critical pairs (s,χ) as in Theorem 4.10 and Proposition 5.4 by

$$\mathcal{L}_{f,\alpha}(\Phi_s^{\pm}(\chi)) = \mathcal{L}_{f,\alpha}(\Phi_1(\chi)\cdot \Phi_{2,s}^{\pm}(\psi\chi_{\xi_0}\chi^{-1})). \tag{6.1}$$

The proof is based on a regularization of the distributions $\mathcal{L}_{f,\alpha}(\Phi_s^{+}(\chi))$ (for $1 \le s \le k-\nu-m$) and $\mathcal{L}_{f,\alpha}(\Phi_s^{-}(\chi))$ (for $1-k+\nu+m \le s \le 0$) (with values in $\bar{\mathbb{Q}}$).

Recall that by Proposition 5.2 the linear form $\mathcal{L}_{f,\alpha}$ factorizes through the canonical projection π_α:

$$\mathcal{L}_{f,\alpha}(\Phi_s^{\pm}(\chi)) = \mathcal{L}_{f,\alpha}(\pi_\alpha(\Phi_s^{\pm}(\chi))).$$

Consider a positive integer c such that $(c,N_0)=1$ and $N_0 = 4q_0NM_0^{\tilde{m}-1}$. One supposes always that m is even and that $k > 2m+2$. Hence there exist regularized distributions Φ_s^{c+}, Φ_s^{c-} on Y with values in the module

$$\mathcal{M}_{r,k}^m(\bar{\mathbb{Q}}) = \bigcup_{v\ge 1} \mathcal{M}_{r,k}^m(\Gamma_1^m(Np^v);\bar{\mathbb{Q}}):$$

of arithmetical nearly holomorphic Siegel modular forms. These regularized distributions are uniquely determined by their values as follows:

$$\begin{aligned} \Phi_s^{c+}(\chi) =& C_{\bar{\chi}\psi}^s G(\bar{\chi}\psi)^{-1}\left(1-(\bar{\chi}\psi)^2(c)c^{-2s}\right)\Phi_s^{+}(\chi)\cdot \\ &\cdot \prod_{q|N_0}\left[(1-(\chi\bar{\psi})_0(q)q^{s-1})(1-(\bar{\chi}\psi)_0(q)q^{-s})^{-1}\right], \\ \Phi_s^{c-}(\chi) =& \left(1-(\bar{\chi}\bar{\psi})^2(c)c^{2(s-1)}\right)\Phi_s^{-}(\bar{\chi}). \end{aligned} \tag{6.2}$$

Next consider the $\mathbb{C}_p$-linear forms $\mathcal{D}^{c\pm}: \mathcal{C}^{k-m}(Y;\mathbb{C}_p) \to \mathbb{C}_p$ defined on the local monomials x_p^j for $j = 0, 1, \cdots, k-m-1$ by:

$$\begin{aligned} \int_{a+(M)} x_p^j \mathcal{D}^{c+} &= \int_{a+(M)} \mathcal{L}_{f,\alpha}(\Phi_{j+1}^{c+}) = \int_{a+(M)} \mathcal{L}_{f,\alpha}(\pi_\alpha(\Phi_{j+1}^{c+})), \\ \int_{a+(M)} x_p^j \mathcal{D}^{c-} &= \int_{a+(M)} \mathcal{L}_{f,\alpha}(\Phi_{-j}^{c-}) = \int_{a+(M)} \mathcal{L}_{f,\alpha}(\pi_\alpha(\Phi_{-j}^{c-})). \end{aligned} \tag{6.3}$$

It follows that there exist the $\mathbb{C}_p$-linear maps $\widetilde{\Phi}_\alpha^{c\pm}: \mathcal{C}^{k-m}(Y;\mathbb{C}_p) \to \mathcal{M}_{r,k}^m(\mathbb{C}_p)$ defined on the local monomials x_p^j for $j = 0, 1, \cdots, k-m-1$ by:

$$\begin{aligned} \int_{a+(M)} x_p^j \widetilde{\Phi}_\alpha^{c\pm} &= \int_{a+(M)} \pi_\alpha(\Phi_{j+1}^{c+})), \\ \int_{a+(M)} x_p^j \widetilde{\Phi}_\alpha^{c\pm} &= \int_{a+(M)} \pi_\alpha(\Phi_{-j}^{c-}). \end{aligned} \tag{6.4}$$

In order to prove that $\mathbb{C}_p$-linear forms $\mathcal{D}^{c\pm}: \mathcal{C}^{k-m}(Y;\mathbb{C}_p) \to \mathbb{C}_p$ define in fact certain admissible measures we observe that

$$\mathcal{D}^{c\pm} = \mathcal{L}_{f,\alpha}(\widetilde{\Phi}_\alpha^{c\pm})$$

where the linear forms $\widetilde{\Phi}_\alpha^{c\pm}$ depend on the choice of α (but not on the choice of f), and in fact they take values in a finite dimensional $\mathbb{C}_p$-vector space (the image of the finite dimensional projection:

$$\pi_\alpha : \mathcal{M}_{r,k}^m(N,p;\mathbb{C}) \to \mathcal{M}_{r,k}^{m}{}^{\alpha}(N,p;\mathbb{C}) \subset \mathcal{M}_{r,k}^m(\Gamma_1^m(Np);\mathbb{C}),$$

see Proposition 5.2 (d)) hence it suffuces to prove the admissibility of the linear maps $\widetilde{\Phi}_\alpha^{c\pm}$ with values in spaces of arithmetical nearly holomorphic Siegel modular forms. For this purpose we use a general result on admissible measures with values in nearly holomorphic Siegel modular forms given in Theorem 3.4.

We have for the positive critical values

$$\begin{aligned} \int_{a+(M)} (x-a)^r d\mathcal{D}^{c+} &= \sum_{j=0}^r \binom{r}{j} (-a)^{r-j} \int_{a+(M)} d\mathcal{D}_{j+1}^{c+} \\ &= \sum_{j=0}^r \binom{r}{j} (-a)^{r-j} \frac{1}{\varphi(M)} \sum_{\chi \bmod M} \chi^{-1}(a) \mathcal{D}_{j+1}^{c+}(\chi) \\ &= \sum_{j=0}^r \binom{r}{j} (-a)^{r-j} \frac{1}{\varphi(M)} \sum_{\chi \bmod M} \chi^{-1}(a) \mathcal{L}_{f,\alpha}(\Phi_{j+1}^{c+}(\chi)). \end{aligned} \tag{6.5}$$

similarly for $\mathcal{D}^{c-}$ we have

$$\begin{aligned}\int_{a+(M)}(x-a)^r d\mathcal{D}^{c-} &= \sum_{j=0}^{r}\binom{r}{j}(-a)^{r-j}\int_{a+(M)} d\mathcal{D}^{c-}_{-j}\\ &=\sum_{j=0}^{r}\binom{r}{j}(-a)^{r-j}\frac{1}{\varphi(M)}\sum_{\chi \bmod M}\chi^{-1}(a)\mathcal{D}^{c-}_{-j}(\chi)\\ &=\sum_{j=0}^{r}\binom{r}{j}(-a)^{r-j}\frac{1}{\varphi(M)}\sum_{\chi \bmod M}\chi^{-1}(a)\mathcal{L}_{f,\alpha}(\Phi^{c-}_{-j}(\chi)).\end{aligned}\tag{6.6}$$

6.2. Sufficient conditions for admissibility of measures with values in nearly holomorphic Siegel modular forms. In order to prove the admissibility property $\mathcal{D}^{c\pm}$ of we need to verify the following growth condition for $\mathcal{D}^{c}$:

$$\sup_{a\in Y}\left|\int_{a+(m)}(x-a_p)^r d\mathcal{D}^{c\pm}\right|_p = o\left(|M|_p^{r-2\mathrm{ord}_p\alpha_0(p)}\right),\tag{6.7}$$

for $r=0,1,\dots,k-m-1$, which will be implied by

$$\sup_{a\in Y}\left|\int_{a+(m)}(x-a_p)^r d\widetilde{\Phi}^{c\pm}_{\alpha}\right|_p = o\left(|M|_p^{r-2\mathrm{ord}_p\alpha_0(p)}\right),\tag{6.8}$$

Let us use a general result giving a sufficient condition for the admissibility of measures with values in nearly holomorphic Siegel modular forms (given in Theorem 3.4): with $\varkappa = 2$, $h=[2\mathrm{ord}_p\alpha_0(p)]+1$. Then we need to prove that

$$\left|U^{2v}\Big(\sum_{j'=0}^{j}\binom{j}{j'}(-a_p^0)^{j-j'}\Phi^{c+}_{1+j'}\big((a)_v\big)\Big)\right|_p \le Cp^{-vj}\tag{6.9}$$

and

$$\left|U^{2v}\Big(\sum_{j'=0}^{j}\binom{j}{j'}(-a_p^0)^{j-j'}\Phi^{c-}_{-j'}\big((a)_v\big)\Big)\right|_p \le Cp^{-vj}\tag{6.10}$$

for all $j=0,1,\dots,k-m-1$.

6.3. Fourier expansions of distributions with values in nearly holomorphic Siegel modular forms. Let us use that fact that

$$\begin{aligned}&\delta^{(r^+)}_{m/2+s}G^{+}_{m/2+s}(z,0,\bar\chi\psi\chi_{h_0}) = G^{+}_{k'}(z,s',\bar\chi\psi\chi_{h_0}),\\ &\delta^{(r^-)}_{m/2+1-s}G^{-}_{m/2+1-s}(z,s-1/2,\bar\chi\psi\chi_{h_0}) = G^{-}_{k'}(z,s',\bar\chi\psi\chi_{h_0}),\end{aligned}\tag{6.11}$$

where $k' = k - m/2 - \nu$, $s' = (s - k + m + \nu)/2$, $r^+ = r^- = (k - m - \nu - j - 1)/2 \in \mathbb{N}$ and if $j + 1 \equiv \delta \pmod 2$ (otherwise the series vanishes).

The Fourier expansions of these series are given by

$$\begin{aligned} G^+_{m/2+s}(z, 0, \bar\chi\psi\chi_{\xi_0}) &= \sum_{\xi\in C_m} b^+(\xi)e_m(\xi z), \\ G^-_{m/2+1-s}(z, s-1/2, \bar\chi\psi\chi_{\xi_0}) &= \sum_{\xi\in C_m} b^-(\xi)e_m(\xi z), \end{aligned} \tag{6.12}$$

where

$$\begin{aligned} b^+(\xi) &= 2^{-m\kappa}\det(\xi)^{m/2+s-\kappa}L^+_{N_1}(s, \omega\chi_\xi)M(\xi, \omega, m/2+s), \\ b^-(\xi) &= 2^{-m\kappa}L^-_{N_1}(s, \omega\chi_\xi)M(\xi, \omega, m/2+s), \end{aligned} \tag{6.13}$$

$\omega = \psi\chi_{\xi_0}\bar\chi$ is a Dirichlet character,

$$M(\xi, \omega, m/2+s) = \prod_{q\in P(\xi)} M_q(\xi, \omega(q)q^{-m/2-s}),$$

is a finite Euler product such that $M_q(\xi, t) \in \mathbb{Z}[t]$. The normalized Dirichlet L-functions are given by

$$\begin{aligned} L^+_{N_1}(s, \omega\chi_\xi) &= i^\delta\pi^{(1-2s)/2}\frac{\Gamma(s+\delta)/2)}{\Gamma((1-s+\delta)/2}L_{N_1}(s, \omega\chi_\xi), \\ L^-_{N_1}(s, \omega\chi_\xi) &= L_{N_1}(s, \omega\chi_\xi). \end{aligned} \tag{6.14}$$

Let us use these Fourier expansions in order to compute the following nearly holmorphic Fourier expansion

$$\begin{aligned} \Phi^\pm_s(\chi) &= \Phi_1(\chi)\Phi^\pm_{2,s}(\psi\chi_{\xi_0}\bar\chi) \\ \Phi_1(\chi)\delta^{(r^\pm)}\Psi^\pm_{2,s}(\omega) &= \sum_{\xi\in C_m} v^\pm(R, \xi, s, \chi)e_m(\xi z), \end{aligned}$$

$$\begin{aligned} \Phi^\pm_s(\chi)|U(M') &= \sum_{\xi\in C_m} v^\pm(M'R, M'\xi, s, \chi)e_m(\xi z) \\ &= \sum_{\xi\in C_m}\ \sum_{N\hat\xi_0[\xi_1]+\xi_2=M'\xi} d^\pm(s, M'R, \xi_1, \xi_2)e_m(\xi z) \end{aligned} \tag{6.15}$$

where $M' = p^{2v}$,

$$\begin{aligned} d^+(s, M'R, \xi_1, \xi_2) = (q_0N)^{-m(s-1+k-m-\nu)/2}&\det(\xi_0)^{(s-1+k-m-\nu)/2}\cdot \\ &\cdot\chi(\det(\xi_1))\det(\xi_1)^\nu\det(\xi_2)^{s-1/2}L^+_{N_1}(s, \omega\chi_{\xi_2})\cdot \\ &\cdot M(\xi_2, \omega, s+m/2)Q(M'R, \xi_2; r^+, \beta^+), \end{aligned} \tag{6.16}$$

under the condition $s \equiv \delta \pmod 2$ (otherwise $d^+(s, M'R\xi_1, \xi_2) = 0$). Recall the notation $\omega = \bar{\chi}\psi\chi_{\xi_0}$, $\beta^+ = (1-s-k+\nu+m)/2$ and $r^+ = (k-\nu-m-s)/2$. The coefficients $d^-(s, M'R, \xi_1, \xi_2)$ are given by

$$\begin{aligned} d^-(s, M'R, \xi_1, \xi_2) = (q_0 N)^{-m(s-1+k-m-\nu)/2} & \det(\xi_0)^{(s-1+k-m-\nu)/2} \cdot \\ & \cdot \chi_M(\det(\xi_1))\det(\xi_1)^\nu L^-_{N_1}(s, \omega\chi_{\xi_2}) \cdot \\ & \cdot M(\xi_2, \omega, s+m/2) Q(M'R, \xi_2; r^-, \beta^-), \end{aligned} \tag{6.17}$$

under the condition $s \not\equiv \delta \pmod 2$ (otherwise $d^-(s, \xi_1, \xi_2) = 0$. Recall also that $\beta^- = (s-k+\nu+m)/2$ and that $r^- = (k-\nu-m+s-1)/2$.

6.4. Fourier expansions of regularized distributions. Let us apply the above formulas for the normalized distributions:

$$\begin{aligned} \Phi^{c\pm}_s(\chi)|U(M') &= \sum_{\xi \in C_m} v^{c\pm}(M'R, M'\xi, s, \chi) e_m(\xi z) \\ &= \sum_{\xi \in C_m} \sum_{N\hat{\xi}_0[\xi_1] + \xi_2 = M'\xi} d^{c\pm}(s, M'R, \xi_1, \xi_2) e_m(\xi z) \end{aligned} \qquad (6.18),$$

where $M' = p^{2v}$,

$$\begin{aligned} d^{c+}(s, M'R, \xi_1, \xi_2) = (q_0 N)^{-m(s-1+k-m-\nu)/2} & \det(\xi_0)^{(s-1+k-m-\nu)/2} \cdot \\ & \cdot \chi(\det(\xi_1))\det(\xi_1)^\nu \det(\xi_2)^{s-1/2} Q(M'R, \xi_2; r^+, \beta^+) \cdot \\ & \cdot M(\xi_2, \bar{\chi}\psi\chi_{\xi_0}, s+m/2) L^+_{N_1}(s, \bar{\chi}\psi\chi_{\xi_0}\chi_{\xi_2}) \cdot \\ & \cdot \frac{C^s_{\bar{\chi}\psi}}{G(\bar{\chi}\psi)} (1 - (\bar{\chi}\psi)^2(c)c^{-2s}) \left(\prod_{q|N_0} \frac{(1 - (\chi\bar{\psi})_0(q)q^{s-1})}{(1 - (\bar{\chi}\psi)_0(q)q^{-s})} \right). \end{aligned} \tag{6.19}$$

The coefficients $d^{c-}(s, M'R, \xi_1, \xi_2)$ are given by

$$\begin{aligned} d^{c-}(s, M'R, \xi_1, \xi_2) = (q_0 N)^{-m(s-1+k-m-\nu)/2} & \det(\xi_0)^{(s-1+k-m-\nu)/2} \cdot \\ & \cdot \chi_M(\det(\xi_1))\det(\xi_1)^\nu (1 - (\bar{\chi}\psi)^2(c)c^{-2s}) L^-_{N_1}(s, \omega\chi_{\xi_2}) \cdot \\ & \cdot M(\xi_2, \omega, s+m/2) Q(M'R, \xi_2; r^-, \beta^-) \end{aligned} \tag{6.19}$$

under the condition $s \equiv \delta \pmod 2$ (otherwise $d^+(s, M'R\xi_1, \xi_2) = 0$). Recall the notation $\omega = \bar{\chi}\psi\chi_{\xi_0}$, $\beta^+ = (1-s-k+\nu+m)/2$ and $r^+ = (k-\nu-m-s)/2$, $\beta^- = (s-k+\nu+m)/2$ and that $r^- = (k-\nu-m+s-1)/2$.

6.5. Main congruences for the Fourier expansions of regularized distributions. Let us use the orthogonality relations for Dirichlet characters in order to prove (6.9) and (6.10) using the Fourier expansions (6.18).

$$\left|\sum_{j'=0}^{r}\binom{j}{j'}(-a_p^0)^{j-j'}\frac{1}{\varphi(M)}\sum_{\chi \bmod M}\chi^{-1}(a)v^{c+}(M'R, M'\xi, 1+j', \chi)\right|_p \le Cp^{-vj} \qquad (6.21)$$

and

$$\left|\sum_{j'=0}^{r}\binom{j}{j'}(-a_p^0)^{j-j'}\frac{1}{\varphi(M)}\sum_{\chi \bmod M}\chi^{-1}(a)v^{c-}(M'R, M'\xi, -j', \chi)\right|_p \le Cp^{-vj} \qquad (6.22)$$

for all $j = 0, 1, \ldots, k-m-1$, $M = p^v$, $M' = p^{2v}$.

6.6. Kummer congruences and Mazur's measure. *Non–Archimedean integration and admissible measures.* Let S be a finite set of primes containing p and all the prime divisors of N. Consider the $\mathbb{C}_p$–adic analytic Lie groups

$$\mathcal{X}_S = \mathrm{Hom}_{\mathrm{contin}}(\mathbb{Z}_S^\times, \mathbb{C}_p^\times), \qquad X = X_N = \mathrm{Hom}_{\mathrm{contin}}(Y, \mathbb{C}_p^\times)$$

where

$$\mathbb{Z}_S^\times = \varprojlim_Q (\mathbb{Z}/Q\mathbf{Z})^\times$$

The projection homomorphism $\mathbb{Z}_S^\times \to Y$ induces the analytic embedding $X \hookrightarrow \mathcal{X}_S$. Recall the notion of the h–admissible measures on $\mathbb{Z}_S^\times$ and properties of their Mellin transform. These Mellin transform are certain p–adic analytic functions on the $\mathbb{C}_p$–analytic Lie group $\mathcal{X}_S$. The canonical $\mathbb{C}_p$–analytic structure on $\mathcal{X}_S$ is obtained by shifts from the obvious $\mathbb{C}_p$–analytic structure on the subgroup

$$\mathrm{Hom}_{\mathrm{contin}}(\mathbf{Z}_p^\times, \mathbb{C}_p^\times) \subset \mathcal{X}_S.$$

We view the elements of finite order $\chi \in \mathcal{X}_S^{\mathrm{tors}}$ as Dirichlet characters whose conductor $C(\chi)$ contain only primes in S, by means of the decomposition

$$\chi : \mathbb{Z}_S^\times \to \overline{\mathbb{Q}}^\times \xrightarrow{i_\infty} \mathbb{C}^\times,$$

where i_∞ is the fixed embedding. The characters $\chi \in \mathcal{X}_S^{\rm tors}$ form a discrete subgroup $\mathcal{X}_S^{\rm tors} \subset \mathcal{X}_S$. We use also the following natural homomorphism

$$x_p : \mathbb{Z}_S^\times \to \mathbb{Z}_p^\times \to \mathbb{C}_p^\times, \quad x_p \in \mathcal{X}_S.$$

so that all integers $k \in \mathbb{Z}$ may be viewed as characters of the type $x_p^k : y \mapsto y^k$.

Recall that a p–adic measure on $\mathbb{Z}_S^\times$ may be viewed as a bounded $\mathbb{C}_p$–linear form μ on the space $\mathcal{C}(\mathbb{Z}_S^\times)$ of all continuous $\mathbb{C}_p$–valued functions

$$\varphi \to \mu(\varphi) = \int_{\mathbb{Z}_S^\times} d\mu \in \mathbb{C}_p, \varphi \in \mathcal{C}(\mathbb{Z}_S^\times),$$

which is uniquely determined by its restriction to the subspace $\mathcal{C}^1(\mathbb{Z}_S^\times)$ of locally constant functions. We denote by $\mu(a+(Q))$ the value of μ on the characteristic function of the set

$$a + (Q) = \{x \in \mathbb{Z}_S^\times \mid x \equiv a \bmod Q\} \subset \mathbb{Z}_S^\times.$$

The Mellin transform L_μ of μ is a bounded analytic function

$$L_\mu : \mathcal{X}_S \to \mathbb{C}_p, \quad L_\mu(\chi) = \int_{\mathbb{Z}_S^\times} d\mu \in \mathbb{C}_p, \chi \in \mathcal{X}_S,$$

on $\mathcal{X}_S$, which is uniquely determined by its values $L_\mu(\chi)$ for the characters $\chi \in \mathcal{X}_S^{\rm tors}$.

A more delicate notion of an h–admissible measure was introduced by Amice–Velu and Višik (see [Am-V], [V1]). Let $\mathcal{C}^h(\mathbb{Z}_S^\times)$ denote the space of $\mathbb{C}_p$–valued functions which can be locally represented by polynomials of degree less than a natural number h of the variable $x_p \in \mathcal{X}_S$ introduced above.

Definition. *A $\mathbb{C}_p$–linear form*

$$\mu : \mathcal{C}^h(\mathbb{Z}_S^\times) \to \mathbb{C}_p$$

is called h–admissible measure if for all $a \in \mathbb{Z}_S^\times$ and for all $r = 0, 1, \ldots, h-1$ the following growth condition is satisfied

$$| \sup_{a\in\mathbb{Z}_S^\times} \int_{a+(Q)} (x_p - a_p)^r d\mu \,|= o(| \, Q \, |_p^{r-h})$$

We know (essentially due to Amice–Velu and Višik) that each h–admissible measure can be uniquely extended to a linear form on the $\mathbb{C}_p$–space of all locally analytic functions so that one can associate to its Mellin transform

$$L_\mu : \mathcal{X}_S \to \mathbb{C}_p, \quad L_\mu(\chi) = \int_{\mathbb{Z}_S^\times} \chi d\mu \in \mathbb{C}_p, \quad \chi \in \mathcal{X}_S,$$

which is a $\mathbb{C}_p$–analytic function on $\mathcal{X}_S$ of the type $o(\log x_p^h)$. Moreover, the measure μ is uniquely determined by the special values of the type

$$L_\mu(\chi x_p^r) \quad (\chi \in \mathcal{X}_S^{\text{tors}}, r = 0, 1, \ldots, h-1).$$

Let ω mod A be a fixed primitive Dirichlet character such that $(A, M_0) = 1$ with $M_0 = \prod_{q\in S} q$. Put $\overline{S} = S \cup S(A)$, $\overline{M} = \prod_{q\in\overline{S}} q$. Then for any positive integer c with $(c, \overline{M}) = 1$, $c > 1$ there exist $\mathbb{C}_p$-measures $\mu^+(c,\omega), \mu^-(c,\omega)$ on $\mathbb{Z}_S^\times$ which are uniquely determined by the following conditions: for $s \in \mathbb{Z}, s > 0$

$$\begin{aligned} i_p^{-1}\left(\int_{\mathbb{Z}_S^\times} \chi\, x_p^s \, d\mu^+(c,\omega)\right) &= (1 - \overline{\chi}\omega(c)c^{-s})\frac{C_{\omega\overline{\chi}}}{G(\omega\overline{\chi})}\times \\ \times \prod_{q\in S\setminus S(\chi)} &\{(1-\chi\overline{\omega}(q)q^{s-1})/(1-\overline{\chi}\omega(q)q^{-s})\}L^+_{M_0}(s,\overline{\chi}\omega), \end{aligned} \tag{6.23}$$

and for $s \in \mathbb{Z}, s \le 0$

$$i_p^{-1}\left(\int_{\mathbb{Z}_S^\times} \chi\, x_p^s \, d\mu^-(c,\omega)\right) = (1 - \chi\overline{\omega}(c)c^{s-1})L^-_{M_0}(s,\overline{\chi},\omega), \tag{6.24}$$

where

$$L^+_{M_0}(s,\overline{\chi}\omega) = L_{\overline{M}}(s,\overline{\chi}\omega)\frac{2i^\delta\Gamma(s)\cos(\pi(s-\delta)/2)}{(2\pi)^s}, \tag{6.25}$$

$$L^+_{M_0}(s,\overline{\chi}\omega) = L_{\overline{M}}(s,\overline{\chi}\omega) \tag{6.26}$$

are the normalized Dirichlet L-functions with $\delta = 0, 1$, $(-1)^\delta = \overline{\chi}\omega(-1)$. The functions (6.23) and (6.24) satisfy the following functional equation

$$L_{M_0}(1-s,\chi\overline{\omega}) = \prod_{q\in S\setminus S(\chi)} \{(1-\chi\overline{\omega}(q)q^{s-1})/(1-\overline{\chi}\omega(q)q^{-s})\}L^+_{M_0}(s,\overline{\chi}\omega). \tag{6.27}$$

Recall that by the definition of the $\overline{S}$-adic Mazur measure μ^c on $\mathbb{Z}_{\overline{S}}^{\times}$ (6.23) and (6.24) are given by

$$\int_{\mathbb{Z}_S^{\times}} x\, d\mu^-(c,\omega) = \int_{\mathbb{Z}_{\overline{S}}^{\times}} x^{-1}\omega\, d\mu^c,$$

$$\int_{\mathbb{Z}_S^{\times}} d\mu^+(c,\omega) = \int_{\mathbb{Z}_{\overline{S}}^{\times}} x\, x_p^{-1}\omega^{-1}\, d\mu^c,$$

where $x \in X_S$, and $X_{\overline{S}}$ is regarded as a subgroup of $X_{\overline{S}}$.

6.7. Reduction of the Main congruence to congruences for partial sums. Recall from (2.4) that

$$\delta_k^{(r)} f(z) = \sum_{\xi \in B_m} c(\xi) Q(R,\xi) q^{\xi} \in \mathbb{Q}(f)[\![q^{B_m}]\!][R_{i,j}]_{i,j=1,\cdots,m},$$

where $\mathbb{Q}(f)$ is the subfield of $\mathbb{C}$ generated by the Fourier coefficients of f, L runs over all the multi-indices $0 \le l_1 \le \cdots \le l_t \le m$ as above,

$$\begin{aligned} Q(R,\xi) = Q(R,\xi;k,r) &= \sum_{t=0}^{r} \binom{r}{t} \det(\xi)^{r-t} \sum_{|L| \le mt-t} R_L(\kappa - k - r) Q_L(R,\xi), \\ Q_L(R,\xi) &= \operatorname{tr}\left({}^t\rho_{m-l_1}(R)\rho_{l_1}^{\star}(\xi)\right) \cdots \operatorname{tr}\left({}^t\rho_{m-l_t}(R)\rho_{l_t}^{\star}(\xi)\right). \end{aligned}$$

Hence combining all the above remarks we deduce that the coefficients

$$i_p(v^{c\pm}(M'R, M'\xi, s, \chi_M))$$

are linear combinations with coefficients independent of M, $M'$and s of the following expressions:

$$\begin{aligned} B^+ = \chi_M(\det(\xi_1))\det(\xi_1)^{\nu}\det(\xi_2)^{s-1/2}\det(\xi_2)^{r^+} \sum_{t=0}^{r^+} \binom{r^+}{t} \sum_{|L|\ge t} c_L(\beta^+)\cdot \\ \cdot Q_L(M'R,\xi_2)\left((q_0N)^{-m}\det(\xi_0)\right)^{r^+ + s - 1/2} \frac{\bar{\chi}\psi(\tau)}{\tau^{s-1/2}} \int_{\mathbb{Z}_S^{\times}} \chi x_p^s \mathrm{d}\mu^{c+}(\omega) \end{aligned} \tag{6.28}$$

$$\begin{aligned} B^- = \bar{\chi}_M(\det(\xi_1))\det(\xi_1)^{\nu}\det(\xi_2)^{r^-} \sum_{t=0}^{r^-} \binom{r^-}{t} \sum_{|L|\ge t} c_L(\beta^-)\cdot \\ \cdot Q_L(M'R,\xi_2)\left((q_0N)^{-m}\det(\xi_0)\right)^{r^-} \int_{\mathbb{Z}_S^{\times}} \bar{\chi} x_p^s \mathrm{d}\mu^{c-}(\omega), \end{aligned} \tag{6.29}$$

with $\omega = \psi\chi_{\xi_0}\chi_{\xi_2}$ and $\det(2\xi_2)\det(2\xi_0) = \tau a^2$ for $a, \tau \in \mathbb{Z}$ such that τ is square free. Hence $C_{\bar\chi\omega} = \tau C_{\bar\chi\psi}$ and $G(\bar\chi\omega) = \bar\chi\psi(\tau)\sqrt{\tau}G(\bar\chi\psi)$ and using $N\hat{\xi}_0[\xi_1] + \xi_2 = M'\xi$ we show the following congruence $\det(\xi_2)\det(\xi_0) \equiv (q_0N)^m\det(\xi_1)^2 \pmod{M'}$. Moreover m is even, and one deduces then that $\det(\xi_2)\det(\xi_0)$ is a square modulo q for all $q|N$, then we simply obtain (6.28), (6.29) from (6.23) and (6.24). Concider M' and M large enough such that $M^2|M'$ and $q_0^m|M$ and recall that $N \in \mathbb{Z}_S^\times$ hence on can find

$$\det(\xi_2)(q_0N)^{-m}\det(\xi_0) \equiv \det(\xi_1)^2 \pmod{M}.$$

Thus the coefficients $\chi_M(\det(\xi_1))$ do not vanish only if $\det(\xi_1) \in \mathbb{Z}_S^\times$. We see then that $\det(\xi_2)(q_0N)^{-m}\det(\xi_0)$ is a square in $\mathbb{Z}_S^\times$ and we put

$$\begin{aligned}\lambda &= \lambda^+ = \tau^{-1}\sqrt{\det(\xi_2)(q_0N)^{-m}\det(\xi_0)} \in \mathbb{Z}_S^\times \text{ in the case } +,\\ \lambda &= \lambda^- = \phantom{\tau^{-1}}\sqrt{\det(\xi_2)(q_0N)^{-m}\det(\xi_0)} \in \mathbb{Z}_S^\times \text{ in the case } -.\end{aligned}$$

It follows that the expressions B^+ and B^- become

$$\begin{aligned}B^+ &= \psi(\tau)\tau^{k-m-\nu-1/2}\det(\xi_1)^\nu\chi(\lambda)\lambda^{s-1+k-m-\nu}\sum_{t=0}^{[(k-m-1)/2]}\binom{r^+}{t}\cdot\\ &\cdot\sum_{|L|\geq t}c_L(\beta^+)Q_L(M'R,\xi_2)\int_{\mathbb{Z}_S^\times}\chi x_p^s d\mu^{c+}(\omega),\\ B^- &= \det(\xi_1)^\nu\bar\chi(\lambda)\lambda^{s-1+k-m-\nu}\sum_{t=0}^{[(k-m-1)/2]}\binom{r^-}{t}\sum_{|L|\geq t}c_L(\beta^-)\cdot\\ &\cdot Q_L(M'R,\xi_2)\int_{\mathbb{Z}_S^\times}\bar\chi x_p^s d\mu^{c-}(\omega).\end{aligned}\tag{6.30}$$

In this way $A^\pm$ is a linear combination with coefficients independent on M, M' and j of type

$$\begin{aligned}\underline{A}^+ =&\det(\xi_1)^\nu\lambda^{k-m-\nu-1-l}\sum_{t=0}^{[(k-m-1)/2]}\sum_{|L|\geq t}Q_L(M'R,\xi_2)\sum_{j=0}^{l}\binom{r_j^\pm}{t}\cdot\\ &\cdot c_L(\beta_j^\pm)lj(-a\lambda^{-1})^{l-j}\int_{\mathbb{Z}_S^\times}\frac{1}{\varphi(M)}\sum_{\chi \bmod M}\chi(xa^{-1})x^{j+1}d\mu^{c\pm}(\omega).\end{aligned}\tag{6.31}$$

Note that $c_L(\beta_j^+)$ is a polynomial of degree $|L|$ in β_j^+ hence of degree $|L|$ in j because $\beta_j^+ = (-j-k+\nu+m)/2$ and $\binom{r_j^+}{t}$ is a polynomial of degree t in r_j^+ thus of degree $t$in j in view of $r_j^+ = (k-m-\nu-j-1)/2$. One can therefore write

$$\binom{r_j^+}{t} c_L(\beta_j^+) = \sum_{n=0}^{t+|L|} \mu_n \frac{(j+n+1)!}{(j+1)!}.$$

Here the coefficients μ_n are certain fixed rational numbers.

Using again the orthogonality relations we see that the sum on j denoted by C^+ takes the form

$$C^+ = \int_{x \equiv a\lambda^{-1} \operatorname{mod} M} \sum_{n=0}^{t+|L|} \mu_n \underbrace{\sum_{j=0}^{l} \binom{l}{j} (-a\lambda^{-1})^{l-j} \frac{(j+n+1)!}{(j+1)!} x^{j+1}}_{x^{-n} \frac{\partial^n}{\partial x^n} \left(x^{n+1}(x-a\lambda^{-1})^l\right)} \mathrm{d}\mu^+(c,.)$$

Therefore $(x-a\lambda^{-1})^l \equiv 0 \pmod{M^l}$ giving

$$\begin{aligned} 2^{\nu_{m,k}} C^+ &\equiv 0 \pmod{M^{l-n}} \\ &\equiv 0 \pmod{M^{l-t-|L|}}. \end{aligned} \tag{6.32}$$

6.8. Proof of the Main congruence. Now the expression (6.31) transforms to

$$\underline{A}^+ = \det(\xi_1)^\nu \lambda^{k-m-\nu-1+l} \sum_{t=0}^{[(k-m-1)/2]} \sum_{|L| \geq t} Q_L(M'R, \xi_2) C^+, \tag{6.33}$$

where $Q_L(M'R,\xi_2)$ is a homogeneuos polynomial of degree $|L|$ in the variables $M'R_{ij}$ implying the congruence

$$\begin{aligned} Q_L(M'R,\xi_2) &\equiv 0 \pmod{M'^{|L|}} \\ &\equiv 0 \pmod{M^{2|L|}}. \end{aligned} \tag{6.34}$$

We deduce that

$$\underline{A}^+ = \mathrm{o}\left(|M|_p^{2|L|+l-t-|L|}\right). \tag{6.35}$$

On the other hand we know from the description of the polynomial

$$Q(R,\xi) = Q(R,\xi;k,r) = \sum_{t=0}^{r} \binom{r}{t} \det(\xi)^{r-t} \sum_{|L| \leq mt-t} R_L(\kappa-k-r) Q_L(R,\xi)),$$

$$Q_L(R,\xi) = \operatorname{tr}\left({}^t\rho_{m-l_1}(R)\rho_{l_1}^\star(\xi)\right) \cdots \operatorname{tr}\left({}^t\rho_{m-l_t}(R)\rho_{l_t}^\star(\xi)\right))$$

that $|L| \geq t$ so we obtain finally the congruence

$$\underline{A}^{+} = \mathrm{o}\left(|M|_p^l\right). \tag{6.36}$$

This shows that $\mathcal{D}^{c+}$ is an h-admissible measure by the sufficient condition (6.9) (implying (6.7)).

In the case $\mathcal{D}^{c-}$ the proof is similar but one needs to replace χ by $\bar{\chi}$. Thus A^{-} is also a linear combination with coefficients independent of M, M' and j of type

$$\begin{aligned} \underline{A}^{-} = \det(\xi_1)^{\nu}\lambda^{k-m-\nu-1} \sum_{t=0}^{[(k-m-1)/2]} \sum_{|L|\geq t} Q_L(M'R,\xi_2) \sum_{j=0}^{l} \binom{r_j^-}{t} \cdot \\ \cdot c_L(\beta_j^-)\binom{l}{j}(-a\lambda)^{l-j}\int_{\mathbb{Z}_{S(N_0)}^{\times}} \frac{1}{\varphi(M)} \sum_{\chi \bmod M} \bar{\chi}(a\lambda x)x^{-j}\mathrm{d}\mu^{-}(c). \end{aligned} \tag{6.37}$$

Let us use again the orthogonality for the Dirichlet characters and the fact that $\binom{r_j^-}{t}c_L(\beta_j^-)$ is a polynomial of degree $t+|L|$ en j. One represents this polynomial in the form

$$\binom{r_j^-}{t}c_L(\beta_j^-) = \sum_{n=0}^{t+|L|} \mu_n \frac{(j+1+n)!}{(j+1)!}. \tag{6.38}$$

One obtains the following expression for the sum C^{-} over j in the formula giving $\underline{A}^{-}$:

$$\begin{aligned} &C^{-} \\ &= \int_{x^{-1}\equiv a\lambda \bmod M} \sum_{n=0}^{t+|L|} \mu_n \underbrace{\sum_{j=0}^{l}\binom{l}{j}(-a\lambda)^{l-j}\frac{(j+1+n)!}{(j+1)!}x^{-j}}_{x_1^n \frac{\partial^n}{\partial x_1^n}\left(x_1^{n+1}(a\lambda - x_1)^l\right)} \mathrm{d}\mu^{-}(c,.), \end{aligned} \tag{6.39}$$

where $x_1 = x^{-1}$. Then we use the same argument as for $\underline{A}^{+}$ in (6.36) in order to prove that

$$\underline{A}^{-} = \mathrm{o}\left(|M|_p^l\right) = \mathrm{o}\left(|M|_p^{l-h}\right). \tag{6.40}$$

It follows from (6.40) that the linear form $\mathcal{D}^{c-}$ is an admissible measure.

Theorem 6.9. *Let $f \in \mathcal{S}_m^k(N,\psi)$ be a Siegel cusp eigenform of even degree m and of weight $k > 2m+2$ satisfying the nonvanishing condition (*) of Theorem 0.7. For a positive integer $c > 1$, $(c, Np) = 1$ there exist two $\mathbb{C}_p$-analytic functions $L^{c\pm} : X_S \to \mathbb{C}_p$ such that:*

(i) For all pairs (s,χ) with $s \in \mathbb{Z}$, $0 \le s \le k-m-2$ and a primitive non-trivial Dirichlet character $\chi \in X_S^{\rm tors}$ of conductor $C_\chi = p^v$ one has

$$\bullet L^{c+}(\chi x_p^s) = \frac{G_{C_\chi}(1_m,\chi)C_\chi^{m(2s+2k-m)/4}}{(iC_\chi)^{m^2/2}\alpha_0(C_\chi^2)}\frac{C_{\bar\chi\psi}^{s+1}}{G(\bar\chi\psi)}(1-(\bar\chi\psi)^2(c)c^{-2s-2})\cdot$$
$$\cdot\prod_{q|N}\frac{(1-(\chi\bar\psi)_0(q)q^s)}{(1-(\bar\chi\psi)_0(q)q^{-s-1})}\frac{\mathcal{D}^+(s+1,f,\bar\chi)}{\langle f,f\rangle_N}.$$
$$\bullet L^{c-}(\chi x_p^s) = \frac{G_{C_\chi}(1_m,\bar\chi)C_\chi^{m(2-2s+2k-m)/4}}{(iC_\chi)^{m^2/2}\alpha_0(C_\chi^2)}(1-(\bar\chi\bar\psi)^2(c)c^{-2s-2})\cdot$$
$$\cdot\frac{\mathcal{D}^-(-s,f,\chi)}{\langle f,f\rangle_N}.$$

These values are algebraic numbers $\bar{\mathbb{Q}}$ viewed as elements of the Tate field $\mathbb{C}_p$ (via the fixed embedding $i_p : \bar{\mathbb{Q}} \to \mathbb{C}_p$). On the other hand the $\mathbb{C}_p$-analytic functions vanish at the following points

$$L^{c+}(\chi x_p^s) = L^{c-}(\chi x_p^s) = 0 \quad \text{if} \quad s \equiv 0 \pmod 2.$$

(ii) If $\mathrm{ord}_p(\alpha_0(p)) = 0$ (i.e. f is p-ordinary), then the holomorphic functions L^{c+} et L^{c-} are bounded $\mathbb{C}_p$-analytic functions.

(iii) In the general case the functions L^{c+} and L^{c-} are of type $\mathrm{o}(\log(x_p^h))$ where $h = [2\mathrm{ord}_p(\alpha_0(p))]+1$ and they are Mellin transforms of certain h-admissible measuress.

(iv)If $h \le k-m-1$, then the functions $L^{c\pm}$ are uniquely determined by (i).

Proof of Theorem 6.9: The assertion (ii) (i.e. the case $\mathrm{ord}_p(\alpha_0(p)) = 0$) which was proved in [PaLNM], also follows easily from the Main congruence (6.9) and (6.10) taking into account the description of the canonical projection operator in Proposition 1.6: for $\alpha = \alpha_0(p)$,

$\Phi^\alpha = \pi_\alpha(\widetilde{\Phi}^{c\pm})$ let us develop the definition of $\widetilde{\Phi}^\alpha$ using the binomial formula:

$$\int_{(a)_v} (y_p - a_p)^j \, d\widetilde{\Phi}^\alpha =$$
$$\sum_{j'=0}^{j} \binom{j}{j'} (-a_p)^{j-j'} \Phi^\alpha_{j'}((a)_v) = \alpha^{-v\varkappa} \cdot$$
$$\alpha^{v\varkappa} \cdot U^{-v\varkappa} \left[\pi_{\alpha,1} U^{\varkappa v} \Big(\sum_{j'=0}^{j} \binom{j}{j'} (-a_p)^{j-j'} \Phi_{j'}((a)_v) \Big) \right]$$

First we note that all the operators

$$\alpha^{v\varkappa} \cdot U^{-v\varkappa} = \left(\alpha^{-1} U\right)^{-v\varkappa} = \left(I + \alpha^{-1} Z\right)^{-v\varkappa} = \sum_{t=0}^{n-1} \binom{-v\varkappa}{t} \left(\alpha^{-1} Z\right)^t$$

are uniformely bounded for $v \to \infty$ by a positive constant C_1 (where $U = \alpha I + Z$ and $Z^n = 0$ for $n = \mathrm{rk}_A \mathcal{M}_{r^*}(Np; A))$ because n does not depend on v and the binomial coefficients $\binom{-v\varkappa}{t}$ are all $\mathbb{Z}_p$-integers.

On the other hand by the Main congruences (6.9) and (6.10) with $\varkappa = 2$ (for the distributions $\Phi_j = \Phi^{c+}_{j+1}$ or $\Phi_j = \Phi^{c-}_{-j}$)

$$\left| U^{\varkappa v} \Big(\sum_{j'=0}^{j} \binom{j}{j'} (-a_p)^{j-j'} \Phi_{j'}((a)_v) \Big) \right|_{p, \mathcal{M}_{r^*}} \le C p^{-vj}$$

for all $j = 0, 1, \ldots, k - m - 1$. If we apply to this estimate the previous bounded operators $\pi_{\alpha,1} U^{\varkappa v}$ we get the following inequality

$$\left| \int_{(a)_v} (y_p - a_p)^j \, d\widetilde{\Phi}^\alpha \right|_{p, \mathcal{M}_{r^*}} \le C \cdot C_1 |\alpha^{-v\varkappa}|_p \cdot p^{-vj} = C C_1 p^{-vj}$$

because $|\alpha^{-v\varkappa}|_p = 1$. In particular, for $j = 0$ it implies that $\widetilde{\Phi}^\alpha = \pi_\alpha(\Phi_0)$ is a bounded measure, and for all $j = 0, 1, \ldots, k - m - 1$ we have moreover that

$$\int_{(a)_v} d\pi_\alpha(\Phi_j) = \int_{(a)_v} y_p^j \, d\pi_\alpha(\Phi_0) = \int_{(a)_v} y_p^j \, d\widetilde{\Phi}^\alpha$$

Remark. We see in this case that all distributions $\pi_\alpha(\Phi_j)$ are bounded but we do not claim that the distributions Φ_j are bounded themselves.

In the general case $h > k-m-1$ the integer j run over $\{0, \cdots, h-1\}$ and one can extend the values of our functions $L^{c\pm}$ by the equality $L^{c\pm}(\chi x_p^s) = 0$ (for all $\chi \in X_S^{\mathrm{tors}}$ of a conductor divisible by all the prime divisors of Np) for $j > k-m-1-\nu$ (but keeping the same values for $0 \le s \le k-m-1-\nu$). These are the factors $\binom{r^{\pm}}{t}$ which vanish in the above formulas (6.30) for the distributions $B^{\pm}$ and the verification of the h-admissibility goes without change in this situation. Also, one obtains again the h-admissible measures $\mathcal{D}^{c\pm}$ with $h = [2\mathrm{ord}_p(\alpha_0(p))]+1$. The functions $L^{c\pm}$ coincide therefore with the Mellin transforms of these h-admissible measures. Finally, if $h \le k-m-1$ then the conditions in (i) uniquely determine the analytic functions $L^{c\pm}$ of type $\mathrm{o}(\log(x_p^h))$ by their values following a general property of admissible measures [Am-Ve], [Vi]. In the case $h > k-m-1$, there exist many analytic functions $L^{c\pm}$ verifying the conditions in (i) which depend on a choice of analytic continuation (interpolation) for the values $L^{c\pm}(\chi x_p^j)$ if $j > k-m-1-\nu$ but one shows in the theorem 6.9 that there exists at least one such continuation (for example the one which was described in the proof of (iii)).

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