Estimates for the Bennequin number of Legendrian links from state models for knot polynomials

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MPI 96-64

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1. Introduction

The standard contact structure in 3-space is given by the contact 1-form $\lambda = dz - y\,dx$; a Legendrian knot is a knot everywhere tangent to the contact distribution Ker λ. The vertical vector field ∂_z is transverse to the contact distribution; it gives every Legendrian knot a natural framing. Given a Legendrian knot K, shift it slightly along the field ∂_z to obtain a new knot K'. The linking number $\beta(K)$ of K and K' is called the Bennequin number of the Legendrian knot K. Everything, said so far, extends verbatum from knots to links.

In the seminal paper [Be] D. Bennequin proved that for every Legendrian knot in the standard contact space its Bennequin number is less than twice its genus. In particular, the Bennequin number of a topologically unknotted Legendrian knot is always negative. This remarkable inequality is specific to the standard contact structure and distiguishes it from other, so called, overtwisted ones (see [El 1,2]).

One may notice that Bennequin's inequality has two shortcomings. First, the genus of a knot, in general, is not computable from knot diagram. Secondly, the genus is insensitive to mirroring, and Bennequin's inequality gives the same estimate for a knot and its mirror image. This makes Bennequin's inequality far from being optimal. For example, the genus of the trefoil knot is 1, so Bennequin's inequality gives $\beta \leq 1$ for its every Legendrian realization. However the maximal Bennequin number of Legendrian right- and left-handed trefoils equals 1 and -6 (as follows from Theorem 2.1 below; see [F-T] and also [Ka]).

It was observed in [F-T] that the Bennequin number of a Legendrian link in the standard contact space has an upper bound in terms of its 2-variable Homfly and Kauffman polynomials, namely, β is bounded above by the least degree in the framing variable of the Homfly polynomial, and of the Kauffman polynomial, reduced modulo 2 (it is interesting that the Kauffman polynomial seems to be slightly better in this game than Homfly: the former gives the exact estimate -6 for the trefoil while the latter gives -5). The new inequalities are free from the above mentioned shortcomings: knot polynomials are easy to compute, and they are very sensitive to mirroring.

These Bennequin number estimates were extended in [C-G] in two directions: the reduced mod 2 Kauffman polynomial replaced by the usual one with integer coefficients, and the standard contact space replaced by the space of 1-jets of functions on the circle

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(accordingly, the above coordinate x becomes the cyclic coordinate x mod 1). The standard contact space, being the space of 1-jets of functions on the line, has a contact embedding to $J^1\mathbf{S}^1$. The space $J^1\mathbf{S}^1$ is topologically the solid torus; it is contactomorphic to the space of cooriented contact elements in the plane with its canonical contact structure.

The proofs of the inequalities for the Bennequin number proceed (implicitly in [F-T] and explicitly in [C-G]) by induction in the number of double points of a link diagram, using the skein relations for the knot polynomials. On the other hand, Yang-Baxter state models are available for knot polynomials ([Tu 1,2]). In this note we deduce the Bennequin number estimates directly from the state models.

We mainly consider the standard contact 3-space, briefly indicating the necessary changes in the solid torus case in the last section of the paper.

2. Setting the scene: Legendrian link diagrams

Consider the two projections of the contact (x, y, z)–space: on the (x, y)–plane and on the (x, z)–plane.

The (x, y)–projection of a Legendrian knot is an immersed curve; since $dz = y\,dx$ along a Legendrian curve, this immesed curve bounds zero area. Likewise the (x, y)–projection of a Legendrian knot may have a kink shown in Fig. 1 on the left but not the opposite kink shown on the right.

Figure 1: possible and impossible kinks

The natural framing of Legendrian links is the blackboard framing in this projection, and the Bennequin number equals the writhe, i.e., the algebraic sum of double points:

Bennequin number = #  - # 

The (x, z)–projection of a Legendrian curve is called its front. A front does not have vertical tangents; generically, its only singularities are transverse double points and semicubical cusps. Since $y = dz/dx$ along a Legendrian curve, the missing y coordinate is the slope of the front. Therefore the front of a Legendrian link is free from selftangencies, and, at a double point, the branch with a greater slope is higher along the y axis.

The Bennequin number of a Legendrian link is expressed in terms of the double points and cusps of its front:

Bennequin number = #  + #  - #  - #  - *1/2 # of cusps*

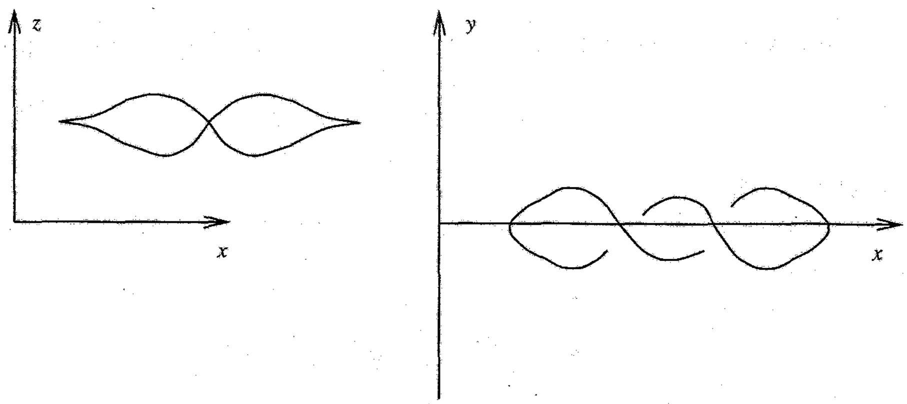


Figure 2: two projections of a Legendrian (un)knot

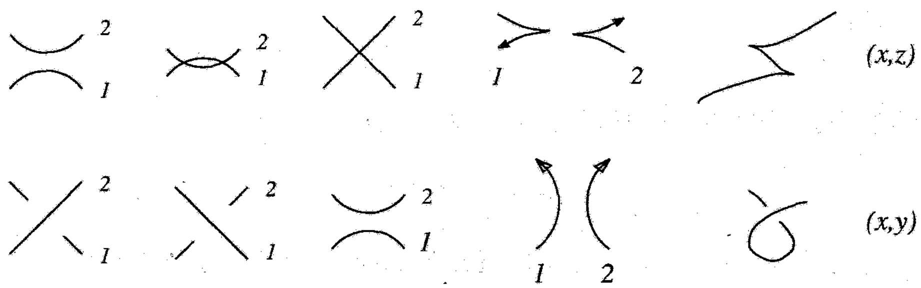


Figure 3: correspondence between two projections

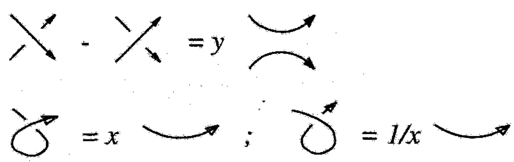
For example, $\beta = -2$ for the front in Fig. 2.

Figure 3 shows the correspondence between the (x,z)– and (x,y)–projections.

Two Legendrian links are Legendrian isotopic if and only if their fronts are related by a sequence of the Legendrian versions of Reidemeister moves shown in Fig. 4 (see [Sw]).

We consider the following versions of the Homfly and Kauffman polynomials (slightly different from the ones in [F-T]), described in terms of the (x,y)–projection.

The framed Homfly polynomial $F_L(x,y)$ is a Laurent polynomial in x, y * depending on a link L which satisfies the following skein relations (here and in further skein relations we omit the symbol for the polynomial; it is understood that F takes equal values on the right and the left hand sides):



In addition,

$$F_{L_1 \cup L_2} = F_{L_1} \; F_{L_2}$$

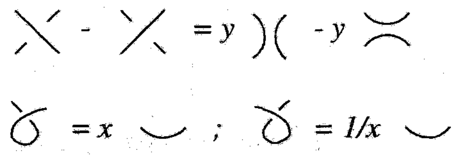
* surely these variables have nothing to do with the coordinates in 3-space.

where $L_1 \cup L_2$ is the disjoint union of the links L_1 and L_2. The Homfly polynomial is

$$\bar{F}(x,y) = x^w \; F(x,y),$$

where w is the writhe.

Likewise, the framed Kauffman polynomial $K_L(x,y)$ for nonoriented links satisfies the skein relations:



In addition,

$$K_{L_1 \cup L_2} = K_{L_1} \; K_{L_2}.$$

The Kauffman polynomial for oriented links is

$$\bar{K}(x,y) = x^w \; K(x,y).$$

The polynomials $\bar{F}$ and $\bar{K}$ are topological isotopy invariants of links.

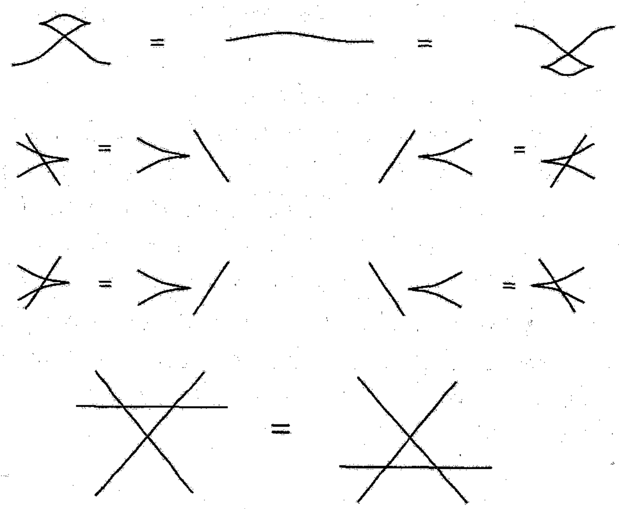
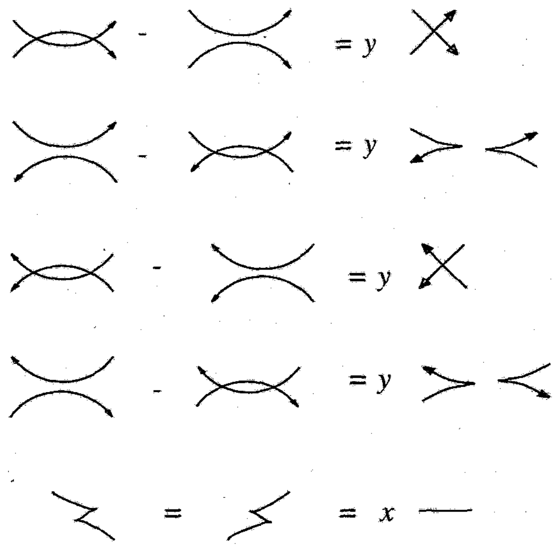
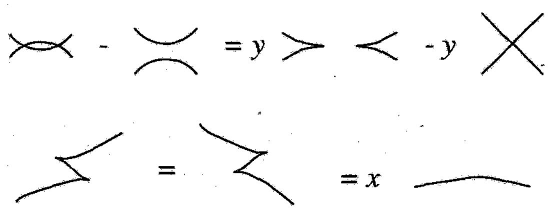


Figure 4: Legendrian Reidemeister moves

Following [C-G] one expresses the skein relations for the framed polynomials of Legendrian links in terms of their fronts. In view of Fig. 3, the Homfly polynomial F satisfies the following equations which will be referred to as front skein relations:



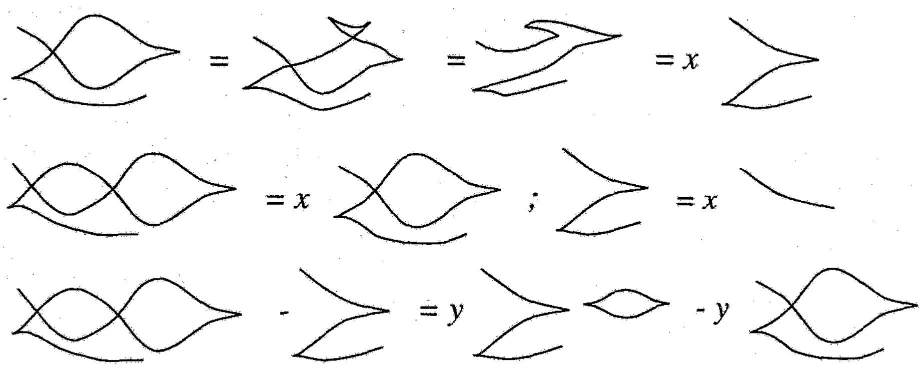
The front skein relations for the Kauffman polynomial K are as follows:



In addition, F and K are invariant under the Legendrian Reidemeister moves and

$$F_{L_1 \cup L_2} = F_{L_1}\ F_{L_2}, \quad K_{L_1 \cup L_2} = K_{L_1}\ K_{L_2}.$$

As an example, the next equalities for the Kauffman polynomial follow from its Legendrian isotopy invariance and the skein relations:



It follows that K takes the value $x + (x^2 - 1)/y$ on the simplest front, the "flying saucer". The value of F on this front is $(1 - x^2)/y$ (as the reader will easily check).

The inequalities for the Bennequin number from [F-T] and [C-G] are as follows.

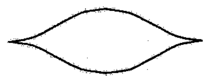


Figure 5: flying saucer

Theorem 2.1. *The Bennequin number of a Legendrian link L in the standard contact space does not exceed the minimum of the two numbers: the least degree in x of the Homfly polynomial $\bar{F}_L$, and that of the Kauffman polynomial $\bar{K}_L$. Equivalently, the framed polynomials F_L and K_L do not contain negative powers of the variable x.*

The equivalence of the two statements follows from the fact that the Bennequin number is the writhe in the (x, y)–projection.

Remarks. 1. It follows from Theorem 2.1 that there exists the 1-variable Legendrian link polynomials obtained from $F(x, y)$ and $K(x, y)$ by setting $x = 0$. This does not seem to have a counterpart for topological links.

2. Both polynomials F and K take equal values on fronts, symmetric with respect to the x axis. The corresponding contactomorphism of 3-space

$$T : (x, y, z) \to (x, -y, -z)$$

is topologically but not contactly isotopic to identity (T changes the sign of the contact 1-form). No nontrivial invariants * are known, at least to the author, which can distinguish between Legendrian links L and $T(L)$.

3. Uniqueness of the polynomials F and K

In this section we show that the front skein relations determine the Homfly and Kauffman polynomials unambiguously. This result is proved in [C-G] in quite a different way.

Theorem 3.1. *The front skein relations along with the Legendrian Reidemeister moves invariance uniquely determine the Laurent polynomials F and K on all fronts of Legendrian links.*

Proof. Consider the Homfly polynomial, the case of the Kauffman one being completely analogous. Our argument is an adaptation of the standard proof of the fact that skein relations uniquely determine knot polynomials (the existence is quite a different, and harder, matter!)

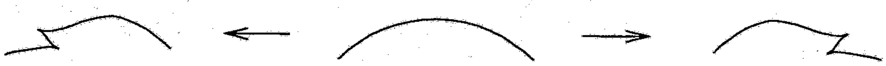


Figure 6: incerting a zigzag into a front

* except the Maslov number which is mentioned below

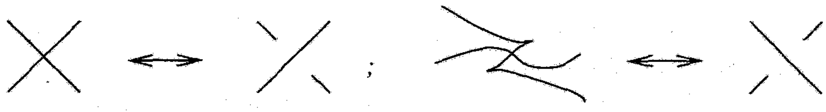
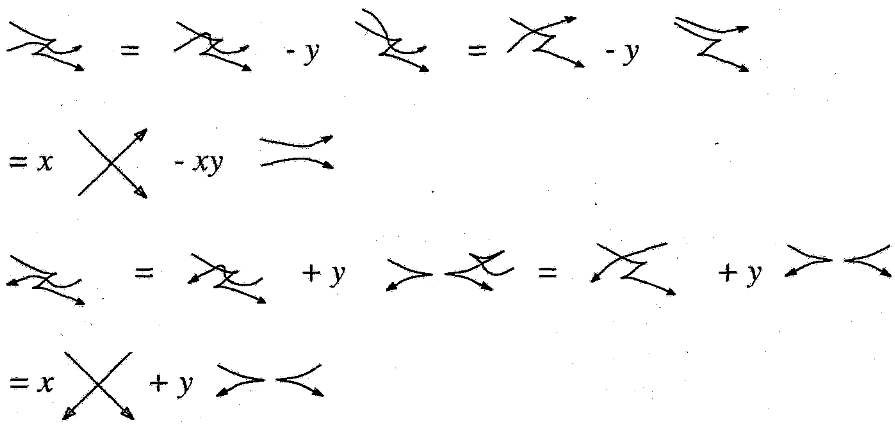


Figure 7: front versions of crossing changes

Let F satisfy the Homfly front skein relations. A double point free front is a disjoint union of "flying saucers" with a number of "zigzags" incerted. Therefore, as was above mentioned, the value of F on such a front is determined by the skein relations.

Given a front L with N double points consider it as a link diagram (whose every double point is of the type shown in Fig. 7 on the left). One may trade some overcrossings for undercrossings to obtain a link diagram of a topologically trivial link. The front versions of the crossing change is shown in Fig. 7; note that the two front fragments are not Legendrian equivalent.

The front skein relations and the Legendrian isotopy invariance imply:



and the other two similar formulas with other orientations of the branches.

Thus, modulo the values of F on fronts with fewer than N double points, the computation of $F(L)$ reduces to that of $F(L_0)$ where L_0 is a front of a topologically trivial Legendrian link. That is, L_0 is topologically isotopic to a Legendrian link with a double point free front.

Next we make use of the following lemma from [F-T] (see also [El 1]):

Lemma 3.2. *If two Legendrian links are topologically isotopic then they become Legendrian isotopic after incerting a sufficient number of zigzags in their fronts.*

Incerting a zigzag into a front amounts to multiplying F by x. Thus the value of $F(L_0)$ is uniquely determined. The proof of the theorem is completed by induction in the number of double points N.

4. State models for polynomials F and K.

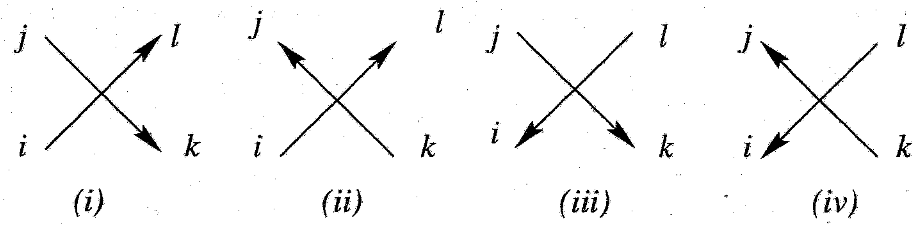
We modify the state models for the Homfly and Kauffman polynomials from [Tu 1,2]. These models come from the solutions of the quantum Yang-Baxter equation, associated with the classical Lie algebras of series A and D, respectively.

Consider a generic front L as a graph whose vertices are the double points and cusps of L. Given a finite set (of colors) C, a state of the graph is an assignment of an element of C to each edge. To each vertex a weight corresponds depending on the colors of the edges incident to this vertex. The total weight of a state is the product of the weights of all vertices, and the state sum is the sum of total weights over all colorings.

We specify the set of colors C and the weights below. C will depend on a positive integer n, and the weights also on a variable q. Thus the state sum will be a function of q and n. The state sums for the Homfly and Kauffman polynomials are denoted by $S_F(q,n)$ and $S_K(q,n)$, respectively.

1). *Homfly polynomial.*

The set of colors $C = \{1, 2, ..., n\}$. Set: $y = q - q^{-1}$, $x = q^n$. There are four types of double points:



and the corresponding weights are as follows.

(i)
if $i = j = k = l$ then $w = -q^{-1}$;
if $j = k \neq i = l$ then $w = 1$;
if $i = k < j = l$ then $w = y$.

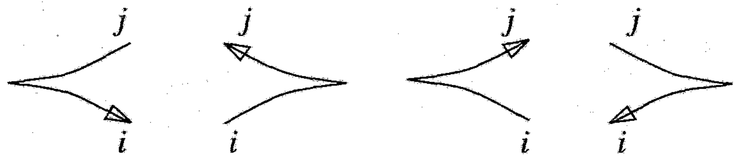
(ii)
if $i = j = k = l$ then $w = -q$;
if $j = k \neq i = l$ then $w = 1$;
if $i = j < k = l$ then $w = -yq^{k-i}$.

(iii)
if $i = j = k = l$ then $w = -q$;
if $j = k \neq i = l$ then $w = 1$;
if $i = j > k = l$ then $w = -yq^{i-k}$.

(iv)
if $i = j = k = l$ then $w = -q^{-1}$;
if $j = k \neq i = l$ then $w = 1$;
if $i = k > j = l$ then $w = y$.

In all other cases the double points weights are equal to zero.

There are four types of cusps



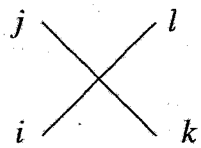
and the weights vanish unless $i = j$; if $i = j$ they equal, respectively,

$$-q^{n+0.5-i}, \quad q^{n+0.5-i}, \quad q^{i-0.5}, \quad -q^{i-0.5}.$$

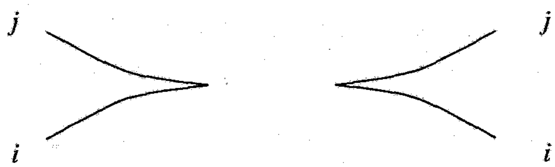
2). *Kauffman polynomial.*

The set of colors $C = \{-(2n-1), -(2n-3), ..., -3, -1, 1, 3, ..., (2n-1)\}$. Set: $y = q - q^{-1}$, $x = q^{2n-1}$. For $i \in C$ denote by $\bar{i}$ the number $i+1$ if $i < 0$ and $i - 1$ if $i > 0$.

There is only one kind of double point and the weights are:



if $i = j = k = l$ then $w = q^{-1}$;
if $i = l = -j = -k$ then $w = q$;
if $i = l, j = k$ and $i \neq \pm j$ then $w = 1$;
if $i = k < j = l$ then $w = -y$;
if $i = -j, k = -l$ and $i < l$ then $w = yq^{(\bar{l}-\bar{i})/2}$.
In all other cases the double points weights are equal to zero.



The cusp weights vanish unless $j = -i$; if $j = -i$ then, for both types of cusps,

$$w = q^{n-(\bar{i}+1)/2}.$$

With this choice of weights the state sums enjoy the following property.

Theorem 4.1. *$S_F(q,n)$ and $S_K(q,n)$ are invariant under the Legendrian Reidemeister moves and satisfy the front skein relations (with the above specified x and y).*

We omit the proof which is essentially computational and repeats the argument in [Tu 1,2]; the above weights are slight modifications of the ones from [Tu 1,2].

5. Proof of Theorem 2.1

We are ready to prove that the Homfly and Kauffman polynomials $F(x,y)$ and $K(x,y)$ are genuine polynomials in the variable x. In a nutshell, the state sums do not contain too great negative powers of q because each weight contributes at most q^{-1}. On the other hand, a negative power of x would contribute a great negative exponent of q for n great enough.

Proof of Theorem 2.1. Consider $F(x,y)$, the case of $K(x,y)$ being completely analogous.

The state sum $S_F(q,n)$ is a Legendrian isotopy invariant and satisfies the front skein relations with $x = q^n$ and $y = q - q^{-1}$. It follows from Theorem 3.1 that for every front

$$S_F(q,n) = F(q^n, q - q^{-1}).$$

Fix a front L; let $F(x,y)$ and $S_F(q,n)$ be the corresponding Homfly polynomial and the state sum. Notice that the only negative power of q which appears in the weights of each vertex is q^{-1}. Let v be the number of vertices of L. It follows that the exponent of each monomial q^i in S_F satisfies the inequality $i \geq -v$.

Let m be the least degree of F in y, and let $u = -\min\{m, 0\} \geq 0$. Set:

$$F_1(x,y) = y^u\ F(x,y);$$

this is a genuine polynomial in y. The exponent of each monomial q^i in $F_1(q^n, q - q^{-1})$ satisfies the inequality $i \geq -(u+v)$.

Arguing by contradiction, assume that $F(x,y)$ contains negative powers of x. Then

$$F_1(x,y) = \sum_{i=k}^{l} a_i(y) x^i; \quad k < 0.$$

Let d be the top degree of $F_1(x,y)$ in y, and $e = \deg a_k(y)$. The term $a_k(y)x^k$ contributes the monomials q^{kn+j} to $F_1(q^n, q - q^{-1})$ with $j \leq e$, and the coefficient of q^{kn+e} in $a_k(y)x^k$ does not vanish.

On the other hand, the exponent of each monomial q^j in the terms $a_i(y)x^i$ with $i > k$ satisfies the inequality $j \geq n(k+1) - d$. Therefore, for sufficiently great n (namely, $n > e+d$) the monomial q^{kn+e} does not cancel in $F_1(q^n, q - q^{-1})$. If, in addition, $n > e + u + v$ then $kn + e < -(u+v)$, the latter number being the least posible exponent of the variable q in $F_1(q^n, q - q^{-1})$. This is a desired contradiction.

Remark. The Maslov number μ of an oriented front is half the difference between the numbers of its descending and ascending cusps; μ is a Legendrian isotopy invariant. It is proved in [F-T] and [C-G] that for every front the number $\beta + |\mu|$ is also bounded above by the least degree in x of the corresponding Homfly polynomial $F(x,y)$. It is easy to incorporate μ into the state model (multiplying the cusp weights by $q^{\pm n/2}$). However the inequality for $\beta + |\mu|$ does not seem to follow the same way, as the one for β, from the state model.

6. The space $J^1\mathbf{S}^1$

We briefly indicate the modifications of the previous arguments needed in this case. The Homfly and Kauffman polynomials for links in the solid torus were constructed by V. Turaev in [Tu 3].

Fronts lie on the cylinder $\mathbf{S}^1 \times \mathbf{R}^1$ rather than in the plane. Each irreducible component of a front contributes an integer, the degree of its projection to $\mathbf{S}^1$, in the oriented Homfly case, and a nonnegative integer, the absolute value of the degree of its projection to $\mathbf{S}^1$, in the nonoriented Kauffman case. The degree of a front is the sum of these numbers over all components.

Accordingly, the Homfly and Kauffman polynomials depend on extra variables z_i with i a nonzero integer in the former and a positive integer in the latter cases. The polynomials $F(x, y, z_i)$ and $K(x, y, z_i)$ satisfy the same front skein relations (involving x and y), and they take the values z_i on the simplest fronts of degrees i shown in Fig. 8 (oriented in the Homfly and not oriented in the Kauffman cases).

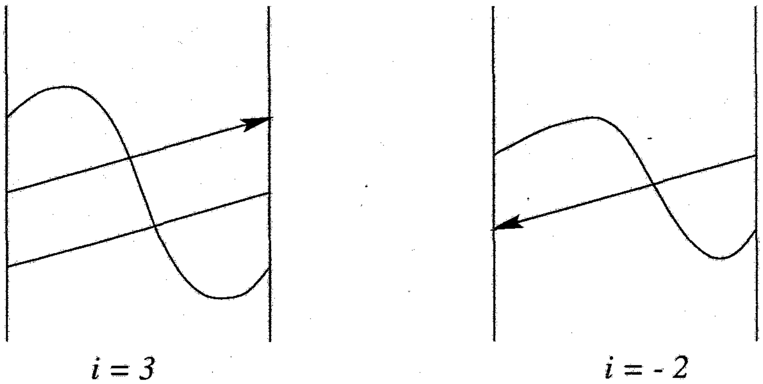


Figure 8: simple fronts of degree i

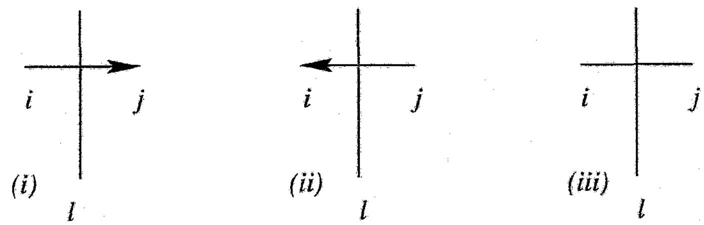
The Bennequin number of a front is given by the same local formula as before, and the polynomials

$$\bar{F}(x, y, z_i) = x^{\beta} F(x, y, z_i), \quad \bar{K}(x, y, z_i) = x^{\beta} K(x, y, z_i)$$

are isotopy invariants of links in the solid torus.

The state models are modified as follows. To incorporate the new variables one chooses a vertical line l on the cylinder (say, $x = 0$). A generic front intersects l off its double points and cusps. These intersections are considered new vertices.

Let $t_1, ..., t_n$ be new commuting variables, also commuting with q.



The weights assigned to the new vertices vanish unless $i = j$; if $i = j$ they are:

$$(i)w = t_i; (ii)w = t_i^{-1}; (iii)w = t_i^{sngi}$$

(cases (i) and (ii) are those of the Homfly and (iii) of the Kauffman polynomials).

The state sums become Laurent polynomials in $t_1, ..., t_n, q$ and do not change under the moves in Fig. 9. The variables x and y are related to q and n as before, and z_i equals the state sum, corresponding to the front of index i in Fig. 8.

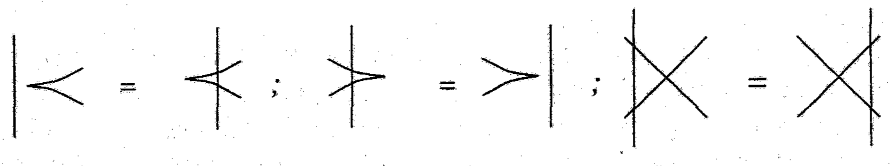


Figure 9: relative position of a front and the vertical line l

After these preparations the previous arguments apply to show that $F(x, y, z_i)$ and $K(x, y, z_i)$ are genuine polynomials in x for every front. This gives an upper bound for the Bennequin number of a Legendrian link in $J^1\mathbf{S}^1$ within a topological isotopy class.

Acknowledgements. I am grateful to S. Chmutov and V. Goryunov for numerous stimulating discussions. It is a pleasure to acknowledge the hospitality of the Max-Planck-Institut fur Mathematik in Bonn. The research was supported in part by NSF grant DMS-9402732.

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