Geodesic equivalence and integrability.

Peter J. Topalov, Vladimir S. Matveev

Abstract

We suggest a construction that, given a trajectorial diffeomorphism between two Hamiltonian systems, produces integrals of them. As the main example we treat geodesic equivalence of metrics. We show that the existence of a non-trivially geodesically equivalent metric leads to Liouville integrability, and present explicit formulae for integrals.

1 Introduction

Integrals of a system are closely related to symmetries. A classical example is Noether's theorem: if a vector field X on a manifold M preserves a Lagrangian $L : \mathcal{T}M \to R$, then the function $I_X \stackrel{\text{def}}{=} \frac{\partial L}{\partial \dot{x}}(x, \dot{x})X(x)$ is a first integral of the corresponding Lagrangian system.

There are many generalizations of Noether's theorem, we recall the following two. In the paper [2] it was shown that the existence of a vector field on $\mathcal{T}^*M$ which commutes with a Hamiltonian vector field allows one to construct a (multi-valued) integral of the Hamiltonian system. In the paper [11] the result of [2] was generalized to tensor fields. It was shown that if a Hamiltonian flow preserves a tensor field on $\mathcal{T}^*M$, then there exists an (also multi-valued) integral of the Hamiltonian system.

In our paper we, following ideas of [11], present a construction which, given a diffeomorphism between two Hamiltonian systems that takes the trajectories and the isoenergy surfaces of the first Hamiltonian system to the trajectories and the isoenergy surfaces of the second one, produces n integrals of the first system, where n is the number of the degrees of freedom of the system.

The construction is applied to geodesically equivalent metrics. Let $g = (g_{ij})$ and $\bar{g} = (\bar{g}_{ij})$ be smooth metrics on the same manifold M^n.

Definition 1. *The metrics g and $\bar{g}$ are* geodesically equivalent*, if they have the same geodesics (considered as unparameterized curves).*

This is rather classical material. In 1869 Dini [3] formulated the problem of local classification of geodesically equivalent metrics, and solved it for dimension two. In 1896 Levi-Civita [4] got a local description of geodesically equivalent metrics on manifolds of arbitrary dimension. In the paper [6] a family of (non-trivial) examples of geodesically equivalent metrics on closed manifolds was constructed.

For geodesically equivalent metrics, a trajectorial diffeomorphism Φ is given by $\Phi(x,\xi) = (x, \frac{\|\xi\|_g}{\|\xi\|_{\bar g}}\xi)$. Here $(x,\xi) \in \mathcal{T}M^n$, x is a point of M^n and $\xi \in \mathcal{T}_x M^n$.

Theorem 1. *Let metrics g and $\bar g$ on M^n be geodesically equivalent. Denote by G the linear operator $g^{-1}\bar g = (g^{i\alpha}\bar g_{\alpha j})$. Consider the characteristic polynomial $det(G - \mu E) = c_0\mu^n + c_1\mu^{n-1} + ... + c^n$. The coefficients $c_1, .., c_n$ are smooth functions on the manifold M^n, and $c_0 \equiv (-1)^n$. Then the functions $I_k = \left(\frac{det(g)}{det(\bar g)}\right)^{\frac{k+2}{n+1}} \bar g(S_k\xi, \xi)$, $k = 0, ..., n-1$, where $S_k \stackrel{\text{def}}{=} \sum_{i=0}^{k} c_i G^{k-i}$, are integrals of the geodesic flow of the metric g and pairwise commute.*

Remark 1. *The integral $I_0 = \left(\frac{det(g)}{det(\bar g)}\right)^{\frac{2}{n+1}} \bar g(\xi,\xi)$ was obtained by Painlevé, see [4]. The integral I_{n-1} is the energy integral (multiplied by minus two).*

The integrals $I_1, I_2, ..., I_{n-2}$ seem to be new, although in each Levi-Civita chart the integrals are linear combinations of Levi-Civita integrals (see Section 3 for definitions). We touch on the connection between the integrals $I_0, ..., I_{n-1}$ and Levi-Civita integrals in Section 5.

Metrics $g, \bar g$ on M^n are *strictly non-proportional* at a point $x \in M^n$, if the characteristic polynomial $\frac{1}{det(g)} det(\bar g - tg))_{|x}$ has no multiple root.

Corollary 1. *Let M^n be a closed real-analytic manifold supplied with two real-analytic metrics $g, \bar g$ such that the metrics $g, \bar g$ are geodesically equivalent and strictly non-proportional at least at one point. Then the fundamental group $\pi_1(M^n)$ of the manifold M^n contains a commutative subgroup of finite index, and the dimension of the homology group $H_1(M^n; \mathbf{Q})$ is no greater than n.*

For dimension two the converse of Theorem 1 is also true, and the condition of Corollary 1 can be weakened.

Corollary 2. *Metrics g and $\bar g$ on a surface M^2 are geodesically equivalent, if and only if the function $\left(\frac{det(g)}{det(\bar g)}\right)^{\frac{2}{3}} \bar g(\xi,\xi)$ is an integral of the geodesic flow of the metric g.*

Corollary 3. *Let metrics $g, \bar g$ on a closed surface of negative Euler characteristic be geodesically equivalent. Then $g = C\bar g$, where C is a constant.*

Corollary 4. *Let metrics $g, \bar g$ on the torus T^2 be geodesically equivalent. If they are proportional at a point $x \in T^2$, then $g = C\bar g$, where C is a positive constant.*

Corollary 5. *Let metrics g, $\bar g$ on the sphere S^2 be geodesically equivalent. Then there are three possibilities.*

1. *The metrics are proportional at exactly two points*
2. *The metrics are proportional at exactly four points.*

3. *The metrics are completely proportional, i.e. $g = C\bar{g}$, where C is a positive constant.*

In the first case the metrics admit a Killing vector field.

Recall that a vector field on M^n is *Killing* (with respect to a metric), if the flow of the field preserves the metric.

Corollary 6. *Let metrics $g, \bar{g}$ on a surface M^2 be geodesically equivalent. If the metrics are proportional at each point of an open non-empty domain $U \subset M^2$, then $g = C\bar{g}$, where C is a positive constant.*

Corollary 7. *If metrics $g, \bar{g}$ on a surface M^n are geodesically equivalent, and if the metric g admits a non-trivial Killing vector field, then the metric $\bar{g}$ also admits a non-trivial Killing vector field.*

One of the most famouse integrable geodesic flow on closed surface is the geodesic flow of the metric on ellipsoid (see [7]). Consider the ellipsoid $\sum_{i=1}^{n} \frac{(x^i)^2}{a_i} = 1$, where $a_i > 0$, $i = 1, ..., n$.

Theorem 2. *The restriction of the metric $\sum_{i=1}^{n} (dx^i)^2$ to the ellipsoid $\sum_{i=1}^{n} \frac{(x^i)^2}{a_i} = 1$ is geodesically equivalent to the restriction of the metric*

$$\frac{1}{\sum_{i=1}^{n} \left(\frac{x^i}{a_i}\right)^2} \left(\sum_{i=1}^{n} \frac{(dx^i)^2}{a_i} \right)$$

to the ellipsoid.

The paper is organized as follows. In Section 2 we present the announced construction. Theorem 3 there gives an explicit formula for a one-parameter family of first integrals, if a trajectorial diffeomorphism between two Hamiltonian systems is given.

In Section 3, for use in Sections 4, 5, 7 we formulate Levi-Civita and Painlevé results about a local form of geodesically equivalent metrics.

In Section 4 we apply the construction to geodesically equivalent metrics, and prove that the functions $I_0, ..., I_{n-1}$ from Theorem 1 are integrals of the geodesic flow of the metric g.

In Section 5 we prove that the integrals $I_0, ..., I_{n-1}$ are in involution.

In Section 6 we prove Corollaries 1, 2, 3, 4, 5, 6, 7.

In Section 7 we prove Theorem 2.

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2 Trajectorial diffeomorphisms and integrals

Let v and $\bar{v}$ be Hamiltonian systems on symplectic manifolds (M,ω) and $(\bar{M},\bar{\omega})$ with Hamiltonians H and $\bar{H}$ respectively. Consider the isoenergy surfaces

$$Q \stackrel{\text{def}}{=} \{x \in M : H(x) = h\}, \quad \bar{Q} \stackrel{\text{def}}{=} \{x \in M : \bar{H}(x) = \bar{h}\},$$

where h and $\bar{h}$ are regular values of the functions $H, \bar{H}$ respectively. Let $U(Q) \subset M$ and $U(\bar{Q}) \subset \bar{M}$ be neighborhoods of the isoenergy surfaces Q and $\bar{Q}$.

Definition 2. *A diffeomorphism $\Phi : U(Q) \longrightarrow U(\bar{Q})$, $\Phi(Q) = \bar{Q}$, is said to be* trajectorial on Q, *if the restriction $\Phi|_Q$ takes the trajectories of the system v to the trajectories of the system $\bar{v}$.*

Denote the restriction $\Phi|_Q$ by ϕ. Since ϕ takes the trajectories of v to the trajectories of $\bar{v}$, it takes the vector field v to the vector field that is proportional to $\bar{v}$. Denote by $a_1 : Q \to R$ the coefficient of proportionality, i.e. $\phi_*(v) = a_1\bar{v}$. Since Φ takes Q to $\bar{Q}$, it takes the differential $\mathrm{d}H$ to a form that is proportional to $d\bar{H}$. Denote by $a_2 : Q \to R$ the coefficient of proportionality, i.e. $\phi_*\mathrm{d}H = a_2\mathrm{d}\bar{H}$. By a we denote the product a_1a_2. We denote the Pfaffian of a skew-symmetric matrix X by $\mathrm{Pf}(X)$.

Theorem 3. *Let a diffeomorphism $\Phi : U(Q) \to U(\bar{Q})$, $\Phi(Q) = \bar{Q}$, be trajectorial on Q. Then for each value of the parameter t the polynomial*

$$\mathcal{P}^{n-1}(t) \stackrel{\text{def}}{=} \frac{\mathrm{Pf}\,(\Phi^*\bar{\omega} - t\omega)}{\mathrm{Pf}\,(\omega)\,(t-a)} \tag{1}$$

is an integral of the system v on Q. In particular, all the coefficients of the polynomial $\mathcal{P}^{n-1}(t)$ are integrals.

Proof. Denote by σ, $\bar{\sigma}$ the restrictions of the forms $\omega, \bar{\omega}$ to Q, $\bar{Q}$ respectively. Consider the form $\phi^*\bar{\sigma}$ on Q.

Lemma 1 (Topalov, [11]). *The flow v preserves the form $\phi^*\bar{\sigma}$.*

Proof of Lemma 1. The Lie derivative L_v of the form $\phi^*\bar{\sigma}$ along the vector field v satisfies

$$L_v\phi^*\bar{\sigma} = \mathrm{d}\,[\imath_v\phi^*\bar{\sigma}] + \imath_v\mathrm{d}\,[\phi^*\bar{\sigma}]\,.$$

On the right side both terms vanish. More precisely, for an arbitrary vector $u \in \mathcal{T}_xQ$ at an arbitrary point $x \in Q$ we have

$$\begin{array}{rcll} \imath_v\phi^*\bar{\sigma}(u) & = & \bar{\sigma}(\phi_*(v), \phi_*(u)) & = \\ & = & \bar{\sigma}(a_1\bar{v}, \phi_*(u)) & = \\ & = & -a_1\mathrm{d}\bar{H}(\phi_*(u)) & = 0. \end{array}$$

Since the form $\bar{\omega}$ is closed, the form $\bar{\sigma}$ is also closed and $\mathrm{d}\,[\phi^*\bar{\sigma}] = \phi^*(\mathrm{d}\bar{\sigma}) = 0$, q. e. d.

It is obvious that the kernels of the forms σ and $\phi^*\bar{\sigma}$ coincide (in the space $\mathcal{T}_xQ$ at each point $x \in Q$) with the linear span of the vector v. Therefore these forms induce two non-degenerate tensor fields on the quotient bundle $\mathcal{T}Q/\langle v\rangle$. We shall denote the corresponding forms on $\mathcal{T}Q/\langle v\rangle$ also by the letters $\sigma, \bar{\sigma}$.

Lemma 2. *The characteristic polynomial of the operator $(\sigma)^{-1}(\phi^*\bar{\sigma})$ on $\mathcal{T}Q/\langle v\rangle$ is preserved by the flow v.*

Proof of Lemma 2. Since the flow v preserves the Hamiltonian H and the form ω, the flow v preserves the form σ. Since the flow v preserves both forms, it preserves the characteristic polynomial of the operator $(\sigma)^{-1}(\phi^*\bar{\sigma})$, q. e. d.

Since both forms are skew-symmetric, each root of the characteristic polynomial of the operator $(\sigma)^{-1}(\Phi^*\bar{\sigma})$ has an even multiplicity. Then the characteristic polynomial is the square of a polynomial $\delta^{n-1}(t)$ of degree $n-1$. Hence the polynomial $\delta^{n-1}(t)$ is also preserved by the flow v. It is obvious that

$$\delta^{n-1}(t) = (-1)^{n-1}\frac{\mathrm{Pf}\,(\phi^*\bar{\sigma} - t\sigma)}{\mathrm{Pf}\,(\sigma)}. \tag{2}$$

The last step of the proof is to verify that

$$(t-a)\delta^{n-1} = \frac{\mathrm{Pf}\,(\Phi^*\bar{\omega} - t\omega)}{\mathrm{Pf}\,(\omega)} \stackrel{\mathrm{def}}{=} \Delta^n.$$

Take an arbitrary point $x \in Q$. Consider the form $\Phi^*\bar{\omega} - a\omega$ on $\mathcal{T}_xM$. The form $\imath_v(\Phi^*\bar{\omega} - a\omega)$ equals zero. More precisely, for any vector $u \in \mathcal{T}_xM$ we have

$$\begin{aligned}
\imath_v(\Phi^*\bar{\omega} - a\omega) &= \bar{\omega}(\Phi_*(v), \Phi_*(u)) - a\omega(v,u) = \\
&= \bar{\omega}(a_1 v, \Phi_*(u)) - a\omega(v,u) = \\
&= -a_1 \mathrm{d}\bar{H}(\Phi_*(u)) + adH = \\
&= -a\mathrm{d}H + adH = 0.
\end{aligned}$$

There exists a vector $A \in \mathcal{T}_xM$ such that $\omega(A,v) \neq 0$ and the restriction of the form $\imath_A(\Phi^*\bar{\omega} - a\omega)$ to the space $\mathcal{T}_xM$ equals zero. More precisely, since the forms $\Phi^*\bar{\omega}$, ω are skew-symmetric, then the kernel $K_{\Phi^*\bar{\omega}-a\omega}$ of the form $\Phi^*\bar{\omega} - a\omega$ has an even dimension, and the kernel of the restriction of the form $\Phi^*\bar{\omega} - a\omega$ to $\mathcal{T}_xQ$ has an odd dimension. Thus the intersection $K_{\Phi^*\bar{\omega}-a\omega} \cap (\mathcal{T}_xM \setminus \mathcal{T}_xQ)$ is not empty. For each vector A from the intersection we obviously have $\omega(A,v) \neq 0$ and $\imath_A(\Phi^*\bar{\omega} - a\omega) = 0$. Without loss of generality we can assume $\omega(A,v) = 1$.

Consider a basis $(v, e_1, ..., e_{2n-2})$ for the space $\mathcal{T}_xQ$. The set $(A, v, e_1, ..., e_{2n-2})$ is a basis for the space $\mathcal{T}_xM$. In this basis we have

$$\begin{aligned}
\det(\Phi^*\bar{\omega} - t\omega) &= \det \left|\begin{array}{cc|c} 0 & a-t & (*) \\ -(a-t) & 0 & 0\cdots 0 \\ \hline -(*) & 0 & (\Phi^*\bar{\omega} - t\omega)_{\langle e_1,\ldots,e_{2n-2}\rangle} \end{array}\right| \\
&= (a-t)^2 \det((\Phi^*\bar{\omega} - t\omega)_{\langle e_1,\ldots,e_{2n-2}\rangle}) \\
&= (a-t)^2 \det(\phi^*\bar{\sigma} - t\sigma),
\end{aligned}$$

where $(\Phi^*\bar{\omega} - t\omega)_{\langle e_1,\ldots,e_{2n-2}\rangle}$ is the matrix of the form $\Phi^*\bar{\omega} - t\omega$ in the basis $(e_1, ...e_{2n-2})$. Finally, $\delta^{n-1} = \mathcal{P}^{n-1}$, q. e. d .

3 Levi-Civita theorem

Let g and $\bar{g}$ be smooth metrics on a manifold M^n. Denote by $\rho^1, ..., \rho^m$ $(1 \le m \le n)$ the common eigenvalues of the metrics g and $\bar{g}$. Suppose the functions $\rho^1, ..., \rho^m$ are differ at every point of an open domain $\mathcal{D} \subset M^n$. In the paper [4], Levi-Civita proved that for every point $P \in \mathcal{D}$ there is an open neighborhood $\mathcal{U}(P) \subset \mathcal{D}$ and a coordinate system $\bar{x} = (\bar{x}_1, ..., \bar{x}_m)$ (in $\mathcal{U}(P)$), where $\bar{x}_i = (x_i^1, ..., x_i^{k_i})$, $(1 \le i \le m)$, such that the quadratic forms of the metrics g and $\bar{g}$ have the following form:

$$\begin{aligned} g(\dot{\bar{x}}, \dot{\bar{x}}) &= \Pi_1(\bar{x}) A_1(\bar{x}_1, \dot{\bar{x}}_1) + \Pi_2(\bar{x}) A_2(\bar{x}_2, \dot{\bar{x}}_2) + \cdots + \\ &+ \Pi_m(\bar{x}) A_m(\bar{x}_m, \dot{\bar{x}}_m), \qquad (3) \\ \bar{g}(\dot{\bar{x}}, \dot{\bar{x}}) &= \rho^1 \Pi_1(\bar{x}) A_1(\bar{x}_1, \dot{\bar{x}}_1) + \rho^2 \Pi_2(\bar{x}) A_2(\bar{x}_2, \dot{\bar{x}}_2) + \cdots + \\ &+ \rho^m \Pi_m(\bar{x}) A_m(\bar{x}_m, \dot{\bar{x}}_m), \qquad (4) \end{aligned}$$

where $A_i(\bar{x}_i, \dot{\bar{x}}_i)$ are positive-definite quadratic forms in the velocities $\dot{\bar{x}}_i$ with coefficients depending on $\bar{x}_i$,

$$\Pi_i \stackrel{\text{def}}{=} (\phi_i - \phi_1) \cdots (\phi_i - \phi_{i-1})(\phi_{i+1} - \phi_i) \cdots (\phi_m - \phi_i) \qquad (5)$$

and $\phi_1, \phi_2, ..., \phi_m$, $0 < \phi_1 < \phi_2 < ... < \phi_m$, are smooth functions such that

$$\phi_i = \begin{cases} \phi_i(\bar{x}_i), & \text{if} \quad k_i = 1 \\ \text{constant}, & \text{else.} \end{cases}$$

It is easy to see that the functions ρ^i as functions of ϕ_i and the function ϕ_i as functions of ρ^i are given by

$$\begin{aligned} \rho^i &= \frac{1}{\phi_1 ... \phi_m} \frac{1}{\phi_i} \\ \phi_i &= \frac{1}{\rho_i} (\rho_1 \rho_2 ... \rho_m)^{\frac{1}{m+1}} \end{aligned}$$

Definition 3. *Let metrics g and $\bar{g}$ be given by formulae (3) and (4) in a coordinate chart $\mathcal{U}$. Then we say that the metrics g and $\bar{g}$ have* Levi-Civita local form (of type m)*, and the coordinate chart $\mathcal{U}$ is* Levi-Civita *coordinate chart (with respect to the metrics).*

Levi-Civita proved that the metrics g and $\bar{g}$ given by formulae (3) and (4) are geodesically equivalent. If we replace ϕ_i by $\phi_i + c$, $i = 1, ..., m$, where c is a (positive for simplicity) constant, in (3) and (4), we obtain the following one-parameter family of metrics, geodesically equivalent to g:

$$g_c(\dot{\bar{x}}, \dot{\bar{x}}) = \frac{1}{(\phi_1 + c) \cdots (\phi_m + c)} \left\{ \frac{1}{\phi_1 + c} \Pi_1 A_1 + \cdots + \frac{1}{\phi_m + c} \Pi_m A_m \right\}. \qquad (6)$$

The next theorem is essentially due to Painlevé, see [4].

Theorem 4. *If the metrics g and $\bar{g}$ are geodesically equivalent, then the function*

$$I_0 \stackrel{\text{def}}{=} \left(\frac{\det(g)}{\det(\bar{g})}\right)^{\frac{2}{n+1}} \bar{g}(\dot{x}, \dot{x}), \tag{7}$$

is an integral of the geodesic flow of the metric g.

Substituting g_c instead of $\bar{g}$ in (7), we obtain the following one-parameter family of integrals

$$\begin{aligned} I_c &\stackrel{\text{def}}{=} \left(\frac{\det(g)}{\det(g_c)}\right)^{\frac{2}{n+1}} g_c(\dot{x}, \dot{x}) = \\ &= C[(\phi_1 + c) \cdots (\phi_m + c)] \left\{ \frac{1}{\phi_1 + c} \Pi_1 A_1 + \cdots + \frac{1}{\phi_m + c} \Pi_m A_m \right\} \\ &= C\{L_1 c^{m-1} + L_2 c^{m-2} + \cdots + L_m\}, \end{aligned}$$

where

$$\begin{aligned} L_1 &= \Pi_1 A_1 + \cdots + \Pi_m A_m \ \text{— the twice energy integral,} \\ L_2 &= \sigma_1(\phi_2, ..., \phi_m)\Pi_1 A_1 + \cdots + \sigma_1(\phi_1, ..., \phi_{m-1})\Pi_m A_m, \\ L_3 &= \sigma_2(\phi_2, ..., \phi_m)\Pi_1 A_1 + \cdots + \sigma_2(\phi_1, ..., \phi_{m-1})\Pi_m A_m, \\ &\vdots \\ L_m &= (\phi_2 ... \phi_m)\Pi_1 A_1 + \cdots + (\phi_1 ... \phi_{m-1})\Pi_m A_m, \end{aligned}$$

σ_k denotes the elementary symmetric polynomial of degree k, and $C \stackrel{\text{def}}{=} \left[(\phi_1 + c)^{k_1 - 1} \cdots (\phi_m + c)^{k_m - 1}\right]^{\frac{2}{n+1}}$ is a constant. Therefore the functions L_k, $k = 1, ..., m$, are integrals of the geodesic flows of the metric g. We call these integrals *Levi-Civita integrals.*

From the results of [8] it follows that Levi-Civita integrals are in involution. More precisely, let $D = (d^i_j)$ be an $m \times m$ matrix. Suppose that for any i, j the element d^i_j depends only on the variables $\bar{x}_j$. Denote by Δ the determinant of the matrix D and by Δ^i_j the minor of the element d^i_j. In the paper [8] it was shown that, for arbitrary functions $A_i(\bar{x}_i, \dot{\bar{x}}_i)$, quadratic in velocities $\dot{\bar{x}}_i$, the Lagrangian system with Lagrangian

$$T_1 = \Delta \left(\frac{A_1(\bar{x}_1, \dot{\bar{x}}_1)}{\Delta^1_1} + \frac{A_2(\bar{x}_2, \dot{\bar{x}}_2)}{\Delta^1_2} + ... + \frac{A_m(\bar{x}_m, \dot{\bar{x}}_m)}{\Delta^1_m} \right)$$

admits $(m-1)$ integrals

$$T_i = \Delta \left(A_1(\bar{x}_1, \dot{\bar{x}}_1) \frac{\Delta^i_1}{(\Delta^1_1)^2} + A_2(\bar{x}_2, \dot{\bar{x}}_2) \frac{\Delta^i_2}{(\Delta^1_2)^2} + ... + A_m(\bar{x}_m, \dot{\bar{x}}_m) \frac{\Delta^i_m}{(\Delta^1_m)^2} \right),$$

where $i = 2, ..., m$, and if we identify the tangent and cotangent bundles the Lagrangian T_1 and consider the standart symplectic form on the cotangent bundle, then the integrals are in involution.

If we take $d^i_j = (\phi_j)^{m-i}$, then Δ and Δ^i_j are given by

$$\Delta^i_j = (-1)^{m-1}\sigma^{i-1}(\phi_1, \phi_2, ..., \phi_{j-1}, \phi_{j+1}, ..., \phi_m) \prod_{\alpha>\beta\geq 1, \alpha\neq j, \beta\neq j} (\phi_\alpha - \phi_\beta),$$

$$\Delta = (-1)^m \prod_{\alpha>\beta\geq 1} (\phi_\alpha - \phi_\beta).$$

Therefore,

$$\frac{\Delta\Delta^i_j}{(\Delta^1_j)^2} = \sigma^{i-1}(\phi_1, \phi_2, ..., \phi_{j-1}, \phi_{j+1}, ..., \phi_m)\Pi_j,$$

so $T_i = -L_i$ and thus the integrals L_i are in involution, q. e. d.

4 Geodesic equivalence and corresponding integrals

Let metrics g and $\bar{g}$ on a manifold M (of dimension n) be geodesically equivalent. By definition, put

$$U^r_g M \stackrel{\text{def}}{=} \{(x, \xi) \in \mathcal{T}M : \quad ||\xi||_g = r\},$$

where $x \in M$, $\xi \in \mathcal{T}_x M$ and $||\xi||_g \stackrel{\text{def}}{=} \sqrt{g(\xi,\xi)} = \sqrt{g_{ij}\xi^i\xi^j}$ is the norm of the vector ξ in the metric g.

By the geodesic flow of the metric g we mean the Lagrangian system of differential equations $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$ on $\mathcal{T}M$ with Lagrangian $L \stackrel{\text{def}}{=} \frac{1}{2}g_{ij}\dot{x}^i\dot{x}^j$. Because of the Legendre transformation, the geodesic flow could be considered as a Hamiltonian system on $\mathcal{T}M$ (as a symplectic form we take $\omega_g \stackrel{\text{def}}{=} \mathrm{d}[g_{ij}\xi^j dx^i]$) with the Hamiltonian $H_g \stackrel{\text{def}}{=} \frac{1}{2}g_{ij}\xi^i\xi^j$.

Since the metrics $g, \bar{g}$ are geodesically equivalent, the mapping $\Phi : \mathcal{T}M \to \mathcal{T}M$, $\Phi(x,\xi) = \left(x, \xi\frac{||\xi||_g}{||\xi||_{\bar{g}}}\right)$, takes the trajectories of the geodesic flow of the metric g to the trajectories of the geodesic flow of the metric $\bar{g}$. This mapping is a diffeomorphism (for $r \neq 0$), takes $U^r_g M$ to $U^r_{\bar{g}} M$ and is trajectorial on $U^r_g M$. Obviously the surfaces U^r_g, $U^r_{\bar{g}}$ are regular isoenergy surfaces $\{H_g = \frac{r}{2}\}$, $\{H_{\bar{g}} = \frac{r}{2}\}$.

By Theorem 3, in order to obtain a family of first integrals we have to find the polynomial $\Delta^n(t)$ and divide it by $(t-a)$. In our case $H_g = H_{\bar{g}} \circ \Phi$. Therefore the function a from Theorem 3 equals to $\frac{||\xi||_{\bar{g}}}{||\xi||_g}$.
In coordinates we have

$$\omega_g = \mathrm{d}[g_{ij}\xi^j dx^i]$$

and

$$\omega_{\bar{g}} = \mathrm{d}[\bar{g}_{ij}\xi^j dx^i].$$

Therefore,

$$\begin{aligned}
\Phi^*\omega_{\bar g} &= \mathrm{d}\left[\frac{||\xi||_g}{||\xi||_{\bar g}}\bar g_{ij}\xi^j dx^i\right] = \\
&= \frac{\partial}{\partial x^k}\left[\frac{||\xi||_g}{||\xi||_{\bar g}}\bar g_{ij}\xi^j\right] dx^k\wedge dx^i - \\
&- \frac{\partial}{\partial \xi^k}\left[\frac{||\xi||_g}{||\xi||_{\bar g}}\bar g_{ij}\xi^j\right] dx^i\wedge d\xi^k .
\end{aligned}$$

It is easy to see that at a point $\xi\in\mathcal{T}_xM$ the quantities

$$A_{ik} \stackrel{\rm def}{=} -\frac{\partial}{\partial \xi^k}\left[\frac{||\xi||_g}{||\xi||_{\bar g}}\bar g_{ij}\xi^j\right]$$

form an element of $\mathcal{T}_xM\otimes\mathcal{T}_xM$. Without loss of generality we can assume that in the space $\mathcal{T}_xM$ the metrics g and $\bar g$ are given in principal axes. Then

$$\begin{aligned}
A_{ij} &\stackrel{\rm def}{=} -\rho^i(x)\frac{\partial}{\partial\xi^j}\left(\xi^i\frac{\sqrt{{\xi^1}^2+...+{\xi^n}^2}}{\sqrt{\rho^1{\xi^1}^2+...+\rho^n{\xi^n}^2}}\right) = \\
&= \rho^i\delta^i_j\frac{||\xi||_g}{||\xi||_{\bar g}} - \rho^i\xi^i\left(\frac{\frac{||\xi||_{\bar g}}{||\xi||_g}-\rho^j\frac{||\xi||_g}{||\xi||_{\bar g}}}{||\xi||^2_{\bar g}}\xi^j\right) = \\
&= \mathrm{diag}(\mu_1,...,\mu_n) - A\otimes B.
\end{aligned}$$

Here $\rho^i, i=1,...,n$ are common eigenvalues (here we allow ρ^i to be equal to ρ^j for some i,j) of the metrics g and $\bar g$, $\mu_i \stackrel{\rm def}{=} -\rho^i\frac{||\xi||_g}{||\xi||_{\bar g}}$, $A_i \stackrel{\rm def}{=} \rho^i\xi^i$ and

$$B_i \stackrel{\rm def}{=} \frac{\frac{||\xi||_{\bar g}}{||\xi||_g}-\rho^i\frac{||\xi||_g}{||\xi||_{\bar g}}}{||\xi||^2_{\bar g}}\xi^i .$$

We have

$$\begin{aligned}
\det(\Phi^*\omega_{\bar g}-t\omega_g) &= \det\left|\begin{array}{c|c} (*) & (A_{ij}+t\delta_{ij}) \\ \hline -(A_{ij}+t\delta_{ij}) & 0\end{array}\right| \\
&= \det(A_{ij}+t\delta_{ij})^2 .
\end{aligned}$$

Therefore,

$$\Delta^n(t) = \det\left(\mathrm{diag}(t+\mu_1,...,t+\mu_n) - a\otimes b\right). \tag{8}$$

Lemma 3. *The following relation holds:*

$$\begin{aligned}
\Delta^n(t) &= (t+\mu_1)\cdots(t+\mu_n) - (a_1b_1)(t+\mu_2)\cdots(t+\mu_n) - \dots \\
&- (t+\mu_1)\cdots(t+\mu_{n-1})(a_nb_n).
\end{aligned} \tag{9}$$

The lemma follows from induction considerations.

To divide the polynomial by $(t-a)$ we shall use the Horner scheme. Suppose that $\Delta^n(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0$ and $\delta^{n-1}(t) = t^{n-1} + b_{n-2}t^{n-2} + \cdots + b_0$. Then we have

$$b_{n-1} = a_n = 1, \tag{10}$$

$$b_{n-2} = a_{n-1} + a, \tag{11}$$

$$\cdots$$

$$b_k = a_{k+1} + ab_{k+1}, \tag{12}$$

$$\cdots$$

$$0 = a_0 + ab_0. \tag{13}$$

It follows from lemma 3 that

$$\begin{aligned} a_0 &= (\mu_1 ... \mu_n) - (A_1B_1)(\mu_2...\mu_n) - \cdots - (\mu_1...\mu_{n-1})A_nB_n = \\ &= (-1)^n \left(\frac{||\xi||_g}{||\xi||_{\bar g}}\right)^n (\rho^1 \cdots \rho^n). \end{aligned}$$

Combining with (13) we get

$$b_0 = -\frac{a_0}{a} = (-1)^{n+1} \left(\frac{||\xi||_g}{||\xi||_{\bar g}}\right)^{n+1} (\rho^1 \cdots \rho^n).$$

Since $\frac{1}{2}g_{ij}\xi^i\xi^j$ is an integral of the geodesic flow of the metric g, the function

$$I_0 \stackrel{\text{def}}{=} (\rho^1 \cdots \rho^n)^{-\frac{2}{n+1}} \bar g(\xi, \xi) \tag{14}$$

is also an integral of the geodesic flow of the metric g. Using Lemma 3 we have

$$\begin{aligned} a_{n-1} &= (\mu_1 + ... + \mu_n) - (A_1B_1 + ... + A_nB_n) = \\ &= \frac{||\xi||_g}{||\xi||^3_{\bar g}} \Big\{ ({\rho^1}^2 {\xi^1}^2 + ... + {\rho^n}^2 {\xi^n}^2) - \\ &- (\rho^1 + ... + \rho^n)(\rho^1 {\xi^1}^2 + ... + \rho^n {\xi^n}^2) \Big\} - \frac{||\xi||_{\bar g}}{||\xi||_g}. \end{aligned}$$

Using (11) we get

$$\begin{aligned} b_{n-2} &= a_{n-2} + a = \\ &= \frac{||\xi||_g}{||\xi||^3_{\bar g}} \Big\{ ({\rho^1}^2 {\xi^1}^2 + ... + {\rho^n}^2 {\xi^n}^2) - \\ &- (\rho^1 + ... + \rho^n)(\rho^1 {\xi^1}^2 + ... + \rho^n {\xi^n}^2)\} \end{aligned}$$

Therefore, the function

$$\begin{aligned} I_1 &\stackrel{\text{def}}{=} (\rho^1 \cdots \rho^n)^{-\frac{3}{n+1}} \Big\{ ({\rho^1}^2 {\xi^1}^2 + ... + {\rho^n}^2 {\xi^n}^2) - \\ &- (\rho^1 + ... + \rho^n)(\rho^1 {\xi^1}^2 + ... + \rho^n {\xi^n}^2) \Big\} \end{aligned}$$

is an integral. (It is easy to see that $\frac{||\xi||^2_g}{||\xi||^2_{\bar g}} = (\rho^1\cdots\rho^n)^{-\frac{2}{n+1}}\frac{||\xi||^2_g}{I_0}$.)

Arguing as above, we see that the functions

$$\begin{aligned} I_k &\stackrel{\rm def}{=} (\rho^1\cdots\rho^n)^{-\frac{k+2}{n+1}}\Big\{({\rho^1}^{k+1}{\xi^1}^2+...+{\rho^n}^{k+1}{\xi^n}^2)- \\ &- (\rho^1+...+\rho^n)({\rho^1}^k{\xi^1}^2+...+{\rho^n}^k{\xi^n}^2)+\cdots \\ &+ (-1)^k\sigma_k(\rho^1,...,\rho^n)(\rho^1{\xi^1}^2+...+\rho^n{\xi^n}^2)\Big\}, \end{aligned}$$

are integrals of the geodesic flow of the metric g, where by σ_k we denote the elementary symmetric polynomial of degree k. It is obvious that $(-1)^k\sigma_k = c_k$ from Theorem 1, and therefore $I_k = \left(\frac{\det(g)}{\det(\bar g)}\right)^{\frac{k+2}{n+1}}\bar g(S_k\xi,\xi)$. Thus I_k, $k = 0,...,n-1$, are integrals of the geodesic flow of the metric g, q. e. d.

5 Liouville integrability

The last step of the proof of Theorem 1 is to verify that the integrals $I_0,...,I_{n-1}$ are in involution. We proceed along the following plan. First we show that it is sufficient to prove the involutivity in each Levi-Civita chart. Then we prove that in each Levi-Civita chart the integrals $I_0,...,I_{n-1}$ are linear combinations of Levi-Civita integrals, and therefore commute.

Let $g,\bar g$ be metrics on M. A point $x\in M$ is called *stable*, if in a neighborhood of x the number of different eigenvalues of the metrics $g,\bar g$ does not depend of a point.

Denote by $\mathcal{M}$ the set of stable points of M. The set $\mathcal{M}$ is an open subset of M. Obviously

$$\mathcal{M} = \bigsqcup_{1\le q\le n}\mathcal{M}^q, \tag{15}$$

where $\mathcal{M}^q$ denotes the set of stable points whose number of distinct common eigenvalues equals q. Points $x\in M\setminus\mathcal{M}$ are called *points of bifurcation.*

Lemma 4. *The set $\mathcal{M}$ is everywhere dense in M.*

Proof of Lemma 4. Denote by $N(x)$ the number of distinct common eigenvalues of the metrics $g,\hat g$ at a point x. Recall that the common eigenvalues of the metrics $g,\hat g$ at a point $x\in M$ are roots of the characteristic polynomial $P_x(t) = \det(G-tE)_{|x}$, where $G = \left(g^{i\alpha}\bar g_{\alpha j}\right)$. In particular, all roots of $P_x(t)$ are real.

Let us prove that, for a sufficiently small neighborhood of an arbitrary point $x\in M$, for any y from the neighborhood the number $N(x)$ is no less than $N(y)$. Take a small $\epsilon>0$ and an arbitrary root ρ of $P_x(t)$. Let us prove that for a sufficiently small neighborhood $U(x)\subset M$, for any $y\in U(x)$ there is a root ρ_y, $\rho-\epsilon<\rho_y<\rho+\epsilon$, of the polynomial $P_y(t)$. If ϵ is small, then for a sufficiently small neighborhood $U(x)$ of the point x, for any $y\in U(x)$ the numbers $\rho+\epsilon$ and $\rho-\epsilon$ are not roots of $P_y(t)$. Consider the circle $S_\epsilon \stackrel{\rm def}{=} \{z\in C: \quad |z-\rho|=\epsilon\}$ on the complex plane C. Clearly the number of roots (with multiplicities) of the polynomial P_y inside the circle is equal to

$$\frac{1}{2\pi i}\int_{S_\epsilon}\frac{P_y'(z)}{P_y(z)}dz.$$

Since for any $y \in U(x)$ there are no roots of P_y on the circle S_ϵ, then the function

$$\frac{1}{2\pi i}\int_{S_\epsilon}\frac{P_y'(z)}{P_y(z)}dz$$

continuously depends on $y \in U(x)$, and therefore is a constant. Clearly it is positive. Thus for any $y \in U(x)$ there is at least one root of P_y that lies between $\rho + \epsilon$ and $\rho - \epsilon$. Then for any y from a sufficiently small neighborhood of x we have $N(y) \geq N(x)$.

Now let us prove the lemma. Evidently the set $\mathcal{M}$ is an open subset of M. Then it is sufficient to prove that for any open subset $U \subset M$ there is a stable point $x \in U$. Suppose otherwise, i.e. let all the points of U be points of bifurcation. Take a point $y \in M$ with maximal value of the function N on it. We have that in a neighborhood $U(y)$ of the point y the function N is constant and equals $N(y)$. Then the point y is a stable point, and we get a contradiction, q. e. d.

Now let the metrics $g, \bar g$ be geodesically equivalent. Since the set of points of bifurcation is nowhere dense, it is sufficient to prove the involutivity in each Levi-Civita chart. Let the metrics g and $\bar g$ be given by

$$\begin{aligned} g(\dot{\bar x},\dot{\bar x}) &= \Pi_1(\bar x)A_1(\bar x_1,\dot{\bar x}_1) + \Pi_2(\bar x)A_2(\bar x_2,\dot{\bar x}_2) + \cdots + \\ &+ \Pi_m(\bar x)A_m(\bar x_m,\dot{\bar x}_m), \qquad (16)\\ \bar g(\dot{\bar x},\dot{\bar x}) &= \rho^1\Pi_1(\bar x)A_1(\bar x_1,\dot{\bar x}_1) + \rho^2\Pi_2(\bar x)A_2(\bar x_2,\dot{\bar x}_2) + \cdots \\ &+ \rho^m\Pi_m(\bar x)A_m(\bar x_m,\dot{\bar x}_m). \qquad (17)\end{aligned}$$

We show that the integrals I_k are linear combinations of the Levi-Civita integrals. We have

$$\bar G = \mathrm{diag}(\underbrace{\rho_1,...,\rho_1}_{k_1},...,\underbrace{\rho_m,...,\rho_m}_{k_m}), \qquad (18)$$

where $\rho_k = \frac{1}{(\phi_1\ldots\phi_m)}\frac{1}{\phi_k}$. It is easy to check that

$$S_k = (-1)^k\mathrm{diag}(\underbrace{\sigma_k^1,...,\sigma_k^1}_{k_1},...,\underbrace{\sigma_k^m,...,\sigma_k^m}_{k_m}), \qquad (19)$$

where

$$\sigma_k^l \stackrel{\mathrm{def}}{=} \sigma_k(\underbrace{\rho_1,...,\rho_1}_{k_1},...,\underbrace{\rho_l,...,\rho_l}_{k_l-1},...,\underbrace{\rho_m,...,\rho_m}_{k_m}). \qquad (20)$$

We have

$$\sigma_k^1 = \frac{1}{(\phi_1...\phi_m)^k}\sigma_k\Big(\underbrace{\frac{1}{\phi_1},...,\frac{1}{\phi_1}}_{k_1-1},...,\underbrace{\frac{1}{\phi_m},...,\frac{1}{\phi_m}}_{k_m}\Big) = \tag{21}$$

$$= \frac{1}{(\phi_1...\phi_m)^k}\sum_{|\alpha|=k}\binom{k_1-1}{\alpha_1}\binom{k_2}{\alpha_2}\cdots\binom{k_m}{\alpha_m}\frac{1}{\phi_1^{\alpha_1}}\frac{1}{\phi_2^{\alpha_2}}\cdots\frac{1}{\phi_m^{\alpha_m}}, \tag{22}$$

$$\tag{23}$$

where $|\alpha| \stackrel{\text{def}}{=} \alpha_1 + \cdots + \alpha_m$ and $\alpha_i \geq 0$. Substituting $\binom{k_l-1}{\alpha_l} + \binom{k_l-1}{\alpha_l-1}$ for $\binom{k_l}{\alpha_l}$ (we assume that $\binom{k}{0} = 1$, $\binom{k}{-1} = 0$, $k \geq 0$) for $2 \leq l \leq m$ we obtain

$$\sigma_k^1 = \frac{1}{(\phi_1...\phi_m)^k}\Bigg(B_k + B_{k-1}\sigma_1\Big(\frac{1}{\phi_2},...,\frac{1}{\phi_m}\Big) + \cdots +$$
$$+ B_{k-m+1}\sigma_{m-1}\Big(\frac{1}{\phi_2},...,\frac{1}{\phi_m}\Big)\Bigg),$$

where

$$B_k \stackrel{\text{def}}{=} \sum_{|\alpha|=k}\binom{k_1-1}{\alpha_1}\cdots\binom{k_m-1}{\alpha_m}\frac{1}{\phi_1^{\alpha_1}}\cdots\frac{1}{\phi_m^{\alpha_m}}. \tag{24}$$

Note that

$$\left(\frac{\det(g)}{\det(\bar g)}\right)^{\frac{k+2}{n+1}} = C_k(\phi_1...\phi_m)^{k+2}, \tag{25}$$

where $C_k = [\phi_1^{k_1-1}...\phi_m^{k_m-1}]^{\frac{k+2}{n+1}}$. Therefore,

$$I_k \stackrel{\text{def}}{=} \left(\frac{\det(g)}{\det(\bar g)}\right)^{\frac{k+2}{n+1}}\bar g(S_k\dot x, \dot x) =$$
$$= (-1)^k C_k(\phi_1...\phi_m)^{k+2}\left\{\rho^1\sigma_k^1\Pi_1 A_1 + \cdots + \rho^m\sigma_k^m\Pi_m A_m\right\} =$$
$$= (-1)^k C_k(\phi_1...\phi_m)^{k+2}\Bigg\{\frac{1}{\phi_1...\phi_m}\frac{1}{\phi_1}\Bigg\{\frac{1}{(\phi_1...\phi_m)^k}(B_k +$$
$$+ \cdots + B_{k-m+1}\sigma_{m-1}\Big(\frac{1}{\phi_2},...,\frac{1}{\phi_m}\Big)\Big)\Bigg\}\Pi_1 A_1 + \cdots\Bigg\} =$$
$$= (-1)^k C_k\left\{B_k L_m + B_{k-1}L_{m-1} + \cdots + B_{k-m+1}L_1\right\}, \tag{26}$$

where L_i are Levi-Civita integrals.

Finally, since the integrals $I_0, ..., I_{n-1}$ are linear combinations of Levi-Civita integrals with constant coefficients, and since Levi-Civita integrals commute, then the integrals $I_0, ..., I_{n-1}$ also commute, q. e. d.

Remark 2. *Let m be the number of distinct common eigenvalues of geodesically equivalent metrics $g, \bar g$ at a point x. Then in a neighborhood U of the point x the number of functionally independent almost everywhere Levi-Civita integrals is no less than m. Therefore the dimension of the space generated by the differentials $(dI_0, dI_1, ..., dI_{n-1})$ no less than m in almost all points of TU.*

6 Topological obstructions

Corollary 1 follows immediately from the following theorem. Recall that a group G is *almost commutative*, if there exists a commutative subgroup $P \subset G$ of finite index.

Theorem 5 (Taimanov, [10]). *If a real-analytic closed manifold M^n with a real-analytic metric satisfies at least one of the conditions:*

a) $\pi_1(M^n)$ is not almost commutative

b) $dim H_1(M^n; \mathbf{Q}) > dim M^n$,

then the geodesic flow on M^n is not analytically integrable.

If metrics $g, \bar{g}$ are real-analytic and geodesically equivalent, then the integrals $I_0, ..., I_{n-1}$ are also real-analytic. If the metrics are strictly non-proportional at least at one point of M^n, then the integrals are functionally independent almost everywhere in a neighborhood of that point. Since the integrals are real-analytic, then they are functionally independent almost everywhere and we can apply Theorem 5, q. e. d.

Proof of Corollary 2. Let metrics $g, \bar{g}$ on a surface M^2 be geodesically equivalent. Using Theorem 1 we have that the function $I_0 = \left(\frac{\det(g)}{\det(\bar{g})}\right)^{\frac{2}{n+1}} \bar{g}(\xi, \xi)$ is an integral of the geodesic flow of the metric g. In one direction Corollary 2 is proved. In other direction the statement of Corollary 2 can be verified by direct calculation, and it was done in [12].

Proof of Corollaries 3, 4, 5, 6. Let g be a metric on a surface M^2. The following lemma is essentially due to [1], see also [5]. For simplicity assume that the surface M^2 is oriented, otherwise finitely cover the surface by an oriented one. Consider the complex structure on M^2 corresponding to the metric g. Let z be a complex coordinate in a open domain $U \subset M^2$. Consider the complex momentum p. We shall denote by $\bar{z}$ and $\bar{p}$ the complex conjugation of z and p respectively. In complex variables the Hamiltonian $H : \mathcal{T}^*M^2 \to R$ of the geodesic flow of the metric g reads $\frac{p\bar{p}}{\lambda(z)}$, where λ is a real-valued function. Suppose that the real-valued function

$$F = A(z)p^2 + B(z)p\bar{p} + \bar{A}(z)\bar{p}^2$$

is an integral of the geodesic flow of the metric g.

Lemma 5. *The form $\frac{1}{A(z)}dzdz$ is meromorphic.*

Remark 3. *If the Hamiltonian and the integral are proportional at each point of M^2, i.e. if $F \equiv \alpha(z)H$, where $\alpha : M^2 \to R$, then by definition put $\frac{1}{A(z)}dzdz$ equal zero.*

Proof of Lemma 5. Since F is an integral of the Hamiltonian system with the Hamiltonian H, the Poisson bracket $\{H, F\}$ equals zero. We have

$$\{H, F\} = H_p F_z - H_z F_p + H_{\bar{p}} F_{\bar{z}} - H_{\bar{z}} F_{\bar{p}} = 0 \tag{27}$$

On the right side of (27) each term is a polynomial of third degree in momenta. Then the bracket is also a polynomial of third degree in momenta. In order for a polynomial to equal zero, all coefficients must be zero, in particular the coefficient of p^3. Thus $\frac{A_{\bar{z}}}{\lambda}$ equals zero, and A is holomorphic. Then $\frac{1}{A(z)}$ is meromorphic, q. e. d.

Let $g, \bar{g}$ be geodesically equivalent metrics on a closed surface M^2 of Euler characteristic $\chi(M^2)$. Then the function $I_0 = \left(\frac{\det(g)}{\det(\bar{g})}\right)^{\frac{2}{n+1}} \bar{g}(\xi, \xi)$ is an integral of the geodesic flow of the metric g, and is quadratic in momenta (if we identify with the help of the metric g the tangent and cotangent bundles of M^2). Consider the form $\frac{1}{A(z)} dzdz$ corresponding to the integral I_0. Suppose that the form is not identical zero. For a meromorphic 2-form on a closed Riemann surface, the number of poles P minus the number of zeros Z is equal to twice the Euler characteristic. It is easy to see that the form $\frac{1}{A(z)} dzdz$ has no zeros (otherwise the metric $\bar{g}$ has singularities). Then $P = 2\chi(M^2)$, and the Euler characteristic $\chi(M^2)$ can not be negative, q. e. d. Now assume the metrics are proportional at each point of an open subset $U \subset M^2$. Since the form is meromorphic, it must be zero. Thus $\bar{g} = \alpha(z) g$, where α is a function on M^2. Let us show that the function α is constant. Actually, $I_0 = 2\left(\frac{1}{\alpha}\right)^{\frac{1}{3}} H$ (here we identify $\mathcal{T}^*M$ and $\mathcal{T}M$ with the help of the metric g). We have

$$\{H, I_0\} = \{H, 2\left(\frac{1}{\alpha}\right)^{\frac{1}{3}} H\} = \{H, H\} 2 \left(\frac{1}{\alpha}\right)^{\frac{1}{3}} + 2H\{\left(\frac{1}{\alpha}\right)^{\frac{1}{3}}, H\}.$$

Since $\{H, H\}$ equals zero, we have that $\{\left(\frac{1}{\alpha}\right)^{\frac{1}{3}}, H\}$ equals zero and the function α is constant, q. e. d.

Remark 4. *For non-orientable surfaces the sign of the Euler characteristic coincides with the sign of the Euler characteristic of the oriented covering. Therefore Corollary 3 is true also for non-orientable surfaces.*

It is easy to see that the form $\frac{1}{A(z)} dzdz$ has poles precisely at points, where the metrics are proportional. If the surface M^2 is the torus, then $\chi(M^2) = 0$ and either the metrics $g, \bar{g}$ are proportional at every point, or there are no points of proportionality of the metrics, q. e. d.

The following lemma is essentially due to Kolokol'tzov [5]. It completes the proof of Corollary 5.

Lemma 6. *On the sphere S^2 there are the following three possibilities for the form $\frac{1}{A(z)} dzdz$.*

1. *The form $\frac{1}{A(z)} dzdz$ is identical zero.*

2. *The form* $\frac{1}{A(z)}dzdz$ *has exactly two zeros (both zeros are of multiplicity two).*

3. *The form* $\frac{1}{A(z)}dzdz$ *has exactly four zeros.*

In the second case the metric g *admits a non-trivial Killing vector field.*

Proof of Corollary 7. Because of Noether's theorem, if a metric admits a (non-trivial) Killing vector field, then the geodesic flow of the metric admits a (non-trivial) integral, linear in velocities, and vice versa.

Suppose the function

$$F_1 = \sum_{i=1}^{n} a_i(x)\xi^i$$

is constant on the trajectories of the geodesic flow of the metric $\bar g$. Then the function

$$\Phi^* F_1 = \frac{||\xi||_g}{||\xi||_{\bar g}} \sum_{i=1}^{n} a_i(x)\xi^i$$

is constant on the trajectories of the geodesic flow of the metric g. Since the function $I_0 = \left(\frac{\det(g)}{\det(\bar g)}^{\frac{2}{n+1}}\right)\bar g(\xi,\xi)$ is an integral of the geodesic flow of the metric g, and since the function $||\xi||_g = \sqrt{g(\xi,\xi)}$ is also an integral of the geodesic flow of the metric g, then the function

$$\frac{\sqrt{G(\xi,\xi)}}{\sqrt{I_0}}\Phi^* F_1 = \left(\frac{\det(g)}{\det(\bar g)}\right)^{\frac{1}{n+1}} \sum_{i=1}^{n} a_i(x)\xi^i,$$

linear in velocities, is also an integral of the geodesic flow of the metric g, q. e. d.

7 Geodesically equivalent metrics on the ellipsoid.

Proof of Theorem 2. We show that in the elliptic coordinate system the restriction of the metrics

$$ds^2 \stackrel{\text{def}}{=} \sum_{i=1}^{n}(dx^i)^2 \text{ and } dr^2 \stackrel{\text{def}}{=} \frac{1}{\sum_{i=1}^{n}\left(\frac{x^i}{a_i}\right)^2}\left(\sum_{i=1}^{n}\frac{(dx^i)^2}{a_i}\right)$$

to the ellipsoid $\sum_{i=1}^{n}\frac{(x^i)^2}{a_i} = 1$ have Levi-Civita local form, and therefore are geodesically equivalent. More precisely, consider elliptic coordinates $\nu^1, ..., \nu^n$. Without loss of generality we can assume that $a^1 < a^2 < ... < a^n$. Then the relation between the elliptic coordinates $\bar\nu$ and the Cartesian coordinates $\bar x$ is given by

$$x^i = \sqrt{\frac{\prod_{j=1}^{n}(a^i - \nu^j)}{\prod_{j=1, j\neq i}^{n}(a^i - a^j)}}. \tag{28}$$

Recall that the elliptic coordinates are non-degenerate almost everywhere, and the set

$$\{\nu^1 = 0, a_1 < \nu^2 < a_2, a_2 < \nu^3 < a_3, ..., a_{n-1} < \nu^n < a^n\}$$

is the part of the ellipsoid, lieing in the quadrant $\{x^1 > 0, x^2 > 0, ..., x^n > 0\}$. Since for any i the symmetry $x^i \to -x^i$ takes the ellipsoid to the ellipsoid and preserves the metrics ds^2 and dr^2, it is sufficent to check the statement of the theorem only in the quadrant $\{x^1 > 0, x^2 > 0, ..., x^n > 0\}$.

In the elliptic coordinates the restriction of the metric ds^2 to the ellipsoid has the following form

$$\sum_{i=1}^{n} \Pi_i A_i (d\nu^i)^2, \tag{29}$$

where $\Pi_i \stackrel{\text{def}}{=} \prod_{j=1, j\neq i}^{n} (\nu^i - \nu^j)$, and $A_i \stackrel{\text{def}}{=} \frac{\nu^i}{\prod_{j=1}^{n}(a^i - \nu^j)}$. The restriction of the metric dr^2 to the ellipsoid is

$$(a^1 a^2 ... a^n) \sum_{i=1}^{n} \rho^i \Pi_i A_i (d\nu^i)^2, \tag{30}$$

where $\rho^i \stackrel{\text{def}}{=} \frac{1}{\nu^i(\nu^1\nu^2...\nu^n)}$. We see that the metrics ds^2, dr^2 have Levi-Civita local form, and therefore are geodesically equivalent, q. e. d.

References

[1] G. D. Birkhoff, *Dynamical Systems,* A.M.S. Colloq. Publ. **9**, Amer. Math. Soc., New York, 1927.

[2] S. V. Bolotin and V. V. Kozlov, *Symmetry fields of geodesic flows*, Russian J. Math. Phys., **3**(1995), 279–295.

[3] U. Dini, *Sopra un problema che si presenta nella theoria generale delle rappresetazioni geografice di una superficie su un'altra*, Ann. di Math., ser.2, **3**(1869), 269–293.

[4] T. Levi-Civita, *Sulle trasformazioni delle equazioni dinamiche* , Ann. di Mat., serie 2^a, **24**(1896), 255–300.

[5] V. N. Kolokol'tzov, *Geodesic flows on two-dimensional manifolds with an additional first integral that is polynomial whit respect to velocities,* Math. USSR-Izv. **21**(1983), no. 2, 291–306.

[6] J. Mikesh, *On the existence of n-dimentional compact Riemannian spaces that admit nontrivial global projective transformations "in the large",* Soviet Math. Docl, **39**(1989), no. 2, 315–317.

[7] J. Moser, *Various aspects of integrable Hamiltonian systems,* Dynamical systems (Bressanone, 1978), 137–195, Liguori, Naples, 1980.

[8] P. Painlevé, *Sur les intégrale quadratiques des équations de la Dynamique,* Compt.Rend., **124**(1897), 221–224.

[9] A. M. Perelomov, *Integrable Systems of Classical Mechanics and Lie Algebras*, Birkhäuser Verlag, Basel, 1990.

[10] I. A. Taimanov, *Topological obstructions to the integrability of geodesic flow on nonsimply connected manifold*, Math.USSR-Izv., **30**(1988), no. 2, pp.403–409.

[11] P. J. Topalov, *Tensor invariants of natural mechanical systems on compact surfaces, and the corresponding integrals*, Sb. Math., **188**(1997), no. 1–2, 307–326.

[12] E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies,* Cambridge University Press, Cambridge, 1937.