Ore extensions of Hopf algebras

A. N. Panov.

Abstract
In the paper we study Ore extensions in the framework of the theory of Hopf algebras. Our classificaton Theorem provides the method available for classification of all Hopf-Ore extensions of group algebras, $U(\mathfrak{g})$, $U_q(\mathfrak{g})$, quantum "$ax + b$".

1 Main Theorem

The Ore extension construction is the well known general construction available for arbitrary noncommutative rings. As a result we have got a ring which is sometimes called the skew polynomial extension of a given ring. In dozen papers were studed ideals in this extensions, their representations, structure of the center and so on. In recent papers [B],[BDG],[D],[N],[Ne] iterated Ore extensions were used for a construction of new examples of Hopf algebras (usually of finite dimension). The aim of this paper is to study Ore extension of Hopf algebras in general setting. In this section we present a criterion for an Ore extension of a given Hopf algebra to be also a Hopf algebra with the comultiplication (1.3). In the next section we use the above criterion for classification of Hopf-Ore extensions.

Let we have a commutative ring k, a k-algebra A, an endomorphism τ and a τ-derivation δ of A. That is

$$\delta(ab) = \delta(a)b + \tau(a)\delta(b) \tag{1.1}$$

An Ore extension $R = A[y; \tau, \delta]$ of an k-algebra A is the k-algebra R generated by indeterminate y and A with the relation

$$ya = \tau(a)y + \delta(a) \tag{1.2}$$

for all $a \in A$. Every element in $R = A[y; \tau, \delta]$ is uniquely expressed in the form $\sum c_i y^i$ [MC-R, 2.1].

Definition 1.0 Let A and $R = A[y; \tau, \delta]$ be Hopf algebras over k. We say that the extension $R = A[y; \tau, \delta]$ is a Hopf-Ore extension if $\Delta(y) = y \otimes r_1 + r_2 \otimes y$ holds for some $r_1, r_2 \in A$ and A is a Hopf subalgebra of R.

In what follows we use common notations for multiplication m, comultiplication Δ, counit ε and antipode S of A and R. We shall use the Sweedler's notation $\Delta y = a_1 \otimes a_2$ (implying the sum) for comultiplication. We shall write the property of coassociativity of comultiplication as follows $a_{11} \otimes a_{12} \otimes a_2 = a_1 \otimes a_{21} \otimes a_{22}$. We shall denote an element defined by the last equality by $a_1 \otimes a_2 \otimes a_3$.

Note that the elements r_1, r_2 in Definition 1.1 are group-like. Indeed, comparing $(1\otimes\Delta)\Delta y = y\otimes\Delta(r_1) + r_2\otimes y\otimes r_1 + r_2\otimes r_2\otimes y$ and $(\Delta\otimes 1)\Delta y = y\otimes r_1\otimes r_1 + r_2\otimes y\otimes r_1 + \Delta(r_2)\otimes y$, we get $\Delta r_i = r_i\otimes r_i$.
We change the generator y by $y' = yr_1^{-1}$ and get $\Delta(y') = y'\otimes 1 + r\otimes y'$. Preserving the previous notations we further require that y in a Hopf-Ore extension obeys

$$\Delta(y) = y\otimes 1 + r\otimes y \tag{1.3}$$

with group-like $r\in A$. As usual $\mathrm{Ad}_r(a) = raS(r) = rar^{-1}$.
Lemma 1.1. If $R = A[y;\tau,\delta]$ is a Hopf-Ore extension then

$$\varepsilon(y) = 0, \tag{1.4}$$

$$S(y) = -r^{-1}y \tag{1.5}$$

where $r^{-1} = S(r)$.
Proof.
1) Using the main property of counit $a = a_1\varepsilon(a_2)$ we obtain $y = \varepsilon(y) + \varepsilon(r)y = \varepsilon(y) + y$. This yields (1.4).
2) By the main property of antipode $\varepsilon(a) = a_1S(a_2)$, we get $0 = \varepsilon(y) = y + rS(y)$ and (1.5).□
Lemma 1.2. If $\Delta(b) = b\otimes 1$ then $b\in k\subset A$.
Proof. Using the main property of counit $a = a_1\varepsilon(a_2) = \varepsilon(a_1)a_2$, we have $b = \varepsilon(b)\in k$. □
Theorem 1.3. $R = A[y;\tau,\delta]$ is a Hopf-Ore extension if and only if
1) there exists a character $\chi : A\mapsto k$ such that

$$\tau(a) = \chi(a_1)(a_2) = \mathrm{Ad}_r(a_1)\chi(a_2) \tag{1.6}$$

for all $a\in A$ (that is τ is a winding automorphism of A);
2) for $\mathrm{Ad}_r, S, \tau : A\mapsto A$ there holds the relation

$$\mathrm{Ad}_r S = \tau S\tau; \tag{1.7}$$

3) the τ-derivation δ obeys

$$\Delta\delta(a) = \delta(a_1)\otimes a_2 + ra_1\otimes\delta(a_2). \tag{1.8}$$

Proof. The Proof is divided into three Steps. In Step 1 we shall show that the comultiplication Δ is extended from A to $R = A[y;\tau,\delta]$ via (1.3) if and only if (1.6) and (1.8) hold. In Step 2 we prove that R (with the above comultiplication) has a counit as an extension of counit of A via (1.4). In Step 3 we show that R (with the above comultiplicaton and counit) has an antipode S extending antipode $S|_A$ via (1.5) if and only if (1.7) holds.
Step 1. Comultiplication.
Suppose that the comultiplication $\Delta|_A$ can be extended to $R = A[y;\tau,\delta]$ via (1.3). Then the homomorphism Δ preserves (1.2). That is $\Delta y\Delta a = \Delta\tau(a)\Delta y + \Delta\delta(a)$. We have

$$\Delta y\Delta a = (y\otimes 1 + r\otimes y)a_1\otimes a_2 =$$

$$(\tau(a_1)y + \delta(a_1)) \otimes a_2 + ra_1 \otimes (\tau(a_2)y + \delta(a_2)) =$$
$$(\tau(a_1) \otimes a_2)(y \otimes 1) + (ra_1r^{-1} \otimes \tau(a_2))(r \otimes y) + \delta(a_1) \otimes a_2 + ra_1 \otimes \delta(a_2),$$
$$\Delta\tau(a)\Delta y + \Delta\delta(a) = \Delta\tau(a)(y \otimes 1 + r \otimes y) + \Delta\delta(a) =$$
$$\Delta\tau(a)y \otimes 1 + \Delta\tau(a)r \otimes y + \Delta\delta(a).$$

We see that the comultiplication of A extended to a comultiplication R via (1.3) if and only if the following conditions hold

$$\Delta\tau(a) = \tau(a_1) \otimes a_2, \qquad (1.9),$$

$$\Delta\tau(a) = \mathrm{Ad}_r(a_1) \otimes \tau(a_2), \qquad (1.10),$$
$$\Delta\delta(a) = \delta(a_1) \otimes a_2 + ra_1 \otimes \delta(a_2)$$

for all $a \in A$ and $\Delta a = a_1 \otimes a_2$. The last coincides with (1.8).

We are going to prove that (1.9) and (1.10) is equivalent to (1.6). Consider the element $\chi(a) := \tau(a_1)S(a_2) \in A$. By direct calculations,

$$\Delta\chi(a) = \Delta\tau(a_1)\Delta S(a_2) = (\tau(a_{11}) \otimes (a_{12})(S(a_{22}) \otimes S(a_{21})) =$$

$$\tau(a_{11})S(a_{22}) \otimes (a_{12})S(a_{21})) = \tau(a_1)S(a_4) \otimes (a_2S(a_3)) =$$
$$\tau(a_1)S(a_3) \otimes \varepsilon(a_2) = \tau(a_1\varepsilon(a_2))S(a_3) \otimes 1 = \tau(a_1)S(a_2) \otimes 1 = \chi(a) \otimes 1.$$

By Lemma 1.2, $\chi(a) \in k$. One can treat χ as a map $\chi : A \to k$. Since τ is an endomorphism, χ is a character:

$$\chi(ab) = \tau(a_1b_1)S(a_2b_2) = \tau(a_1)\tau(b_1)S(b_2)S(a_2) =$$

$$\tau(a_1)\chi(b)S(a_2) = \chi(a)\chi(b),$$
$$\chi(a+b) = \tau((a+b)_1)S((a+b)_2) = \tau(a_1)S(a_2) + \tau(b_1)S(b_2) = \chi(a) + \chi(b).$$

By χ, one can restore τ:

$$\chi(a_1)a_2 = \tau(a_1)S(a_2)a_3 = \tau(a_1)\varepsilon(a_2) = \tau(a_1\varepsilon(a_2)) = \tau(a).$$

This proves the left equality (1.6). We substitute it in (1.10) and get:

$$\Delta(\chi(a_1)a_2) = \mathrm{Ad}_r(a_1) \otimes \chi(a_2)a_3,$$

$$\chi(a_1)\Delta(a_2) = \mathrm{Ad}_r(a_1)\chi(a_2) \otimes a_3,$$
$$(\chi(a_1)a_2) \otimes a_3 = \mathrm{Ad}_r(a_1)\chi(a_2) \otimes a_3.$$

Hence
$$(\chi(a_1)a_2)a_3S(a_4) = \mathrm{Ad}_r(a_1)\chi(a_2)a_3S(a_4),$$
$$(\chi(a_1)a_2)\varepsilon(a_3) = \mathrm{Ad}_r(a_1)\chi(a_2)\varepsilon(a_3),$$
$$\chi(a_1)(a_2) = \mathrm{Ad}_r(a_1)\chi(a_2).$$

This proves the right equality (1.6). We have shown that (1.6) and (1.8) are necessary for an extension of comultiplication.

On the other hand, if (1.6) and (1.8) hold, then $\Delta\tau(a) = \chi(a_1)a_2 \otimes a_3 = \tau(a_1) \otimes a_2$ and $\Delta\tau(a) = \Delta(\mathrm{Ad}_r a_1 \chi(a_2)) = \mathrm{Ad}_{\Delta r}(a_1 \otimes a_2)\chi(a_3) = \mathrm{Ad}_r a_1 \otimes \mathrm{Ad}_r a_2 \chi(a_3) = \mathrm{Ad}_r a_1 \otimes \tau(a_2)$. This proves that (1.9) and (1.10) is true and the comuliplication $\Delta|_A$ extends to comultiplication of R via (1.3).

Step 2. Counit.

Let R admit the above comultiplicaion (i.e (1.6) and (1.8) are true). Let us prove an existence of counit extending $\varepsilon|_A$ and obeying (1.4) . This data defines a homomorphism ε of R if and only if ε preserves (1.2). That is $\varepsilon(y)\varepsilon(a) = \varepsilon(\tau(a))\varepsilon(y) + \varepsilon(\delta(a))$. Since $\varepsilon(y) = 0$, R admits the counit if and only if

$$\varepsilon(\delta(a)) = 0. \tag{1.11}$$

Hence, our goal in this Step is to prove (1.11). Acting by $m(1 \otimes \varepsilon)$ on (1.8), we get

$$m(\mathrm{id} \otimes \varepsilon)(\Delta\delta(a)) = m(\mathrm{id} \otimes \varepsilon)(\delta(a_1) \otimes a_2 + ra_1 \otimes \delta(a_2)),$$

$$\delta(a) = \delta(a_1)\varepsilon(a_2) + ra_1\varepsilon(\delta a_2)) = \delta(a) + ra_1\varepsilon(\delta a_2)).$$

It follows $a_1\varepsilon(\delta(a_2)) = 0$. Finally,

$$0 = S(a_1)a_2\varepsilon(\delta(a_3)) = \varepsilon(a_1)\varepsilon(\delta a_2) = \varepsilon\delta(\varepsilon(a_1)a_2) = \varepsilon(\delta(a)).$$

This proves (1.11) and an existence of counit.

Step 3. Antipode.

Let R be as in Step 2. Recall that S is an antihomomorphism of a Hopf algebra and $\Delta S(a) = S(a_2) \otimes S(a_1)$ [M,1.5.10]. Suppose that R has antipode S which is extended (as an antihomomorphism) from A to R via (1.3). Then S preserves (1.2). That is

$$S(a)S(y) = S(y)S\tau(a) + S\delta(a). \tag{1.12}$$

By Lemma 1.1,

$$-S(a)r^{-1}y = -r^{-1}yS\tau(a) + S\delta(a),$$

$$-S(a)r^{-1}y = -r^{-1}\tau(S\tau(a))y - r^{-1}\delta(S\tau(a)) + S\delta(a).$$

We get two necessary conditions

$$S(a)r^{-1} = r^{-1}\tau S\tau(a), \tag{1.13}$$

$$rS\delta(a) = \delta S\tau(a). \tag{1.14}$$

The first is equivalent to (1.7) (and proves it).

We shall show that (1.14) is a consequence of (1.1) and (1.8). By (1.6), $S\tau(a) = S(\chi(a_1)(a_2)) = \chi(a_1)S(a_2)$. After substitution in (1.14) one can present (1.14) in the equivalent form:

$$rS\delta(a) = \chi(a_1)\delta S(a_2). \tag{1.15}$$

Denote $L = rS\delta(a)$ and $M = \chi(a_1)\delta S(a_2)$. Acting by $m(\mathrm{id} \otimes S)$ on (1.8), we obtain

$$m(\mathrm{id} \otimes S)(\Delta\delta(a)) = m(\mathrm{id} \otimes S)(\delta(a_1) \otimes a_2 + ra_1 \otimes \delta(a_2)),$$

$$0 = \varepsilon\delta(a) = \delta(a_1)S(a_2) + ra_1S\delta(a_2),$$
$$a_1S\delta(a_2) = -r^{-1}\delta(a_1)S(a_2).$$

Then

$$L = rS\delta(a) = r\varepsilon(a_1)S\delta(a_2) = rS(a_1)a_2S\delta(a_3) =$$
$$-rS(a_1)r^{-1}\delta(a_2)S(a_3) = -\mathrm{Ad}_rS(a_1)\delta(a_2)S(a_3). \quad (1.16)$$

On the other hand, acting by δ on the left and right sides of $\varepsilon(a) = a_1Sa_2$, we get

$$0 = \delta(a_1)S(a_2) + \tau(a_1)\delta S(a_2),$$

$$M = \chi(a_1)\delta S(a_2) = \chi(a_1)\varepsilon(a_2)\delta S(a_3) = \chi(a_1)\tau(S(a_2)a_3)\delta S(a_4) =$$
$$= \chi(a_1)\tau S(a_2)\tau(a_3)\delta S(a_4) = -\chi(a_1)\tau S(a_2)\delta(a_3)S(a_4) =$$
$$-\chi(a_1)\mathrm{Ad}_rS(a_3)\chi(S(a_2))\delta(a_4)S(a_{(5)}) = -\mathrm{Ad}_rS(a_1)\delta(a_2)S(a_3). \quad (1.17)$$

Compairing (1.17) and (1.16), we get $L = M$. This proves (1.15).

It is proved that (1.7) is a necessary condition for an existence of antipode in R. The condition (1.7) is also a sufficient condition. If it holds, then (1.13) is true. As above the formula (1.14) is a corollary of properties of δ. This implies (1.12) an existence of antipode in R. $\square$

Corollary 1.4. If $R = A[y;\tau,\delta]$ is a Hopf-Ore extension and A is cocommutative a Hopf algebra, then r lies in the center of A.

Proof. Since $\tau(a) = \chi(a_1)(a_2) = \mathrm{Ad}_r(a_1)\chi(a_2) = \chi(a_1)\mathrm{Ad}_ra_2$, then

$$\chi(S(a_1))\chi(a_2)a_3 = \chi(S(a_1))\chi(a_2)\mathrm{Ad}_ra_3,$$

$$\chi(\varepsilon(a_1))a_2 = \chi(\varepsilon(a_1))\mathrm{Ad}_ra_2,$$
$$a = \mathrm{Ad}_r(a)$$

for all $a \in A$. $\square$

Remarks 1.5.

1) The left condition (1.6) states that $\tau(a) = \chi(a_1)a_2$. That is τ is a right χ-winding automorphism of R [M,1.6.5]. The automorphism $\tau'(a) = a_1\chi(a_2)$ is called a left χ-winding automorphism. The right condition (1.6) can be rewritten in an equivalent form $\tau = \mathrm{Ad}_r\tau'$.

2) Let us denote by $\mathcal{R}_r$ and $\mathcal{L}_r$ the right and left winding action of r in A^*. By definition, $\mathcal{R}_r\lambda(a) = \lambda(ra)$ and $\mathcal{L}_r\lambda(a) = \lambda(ar)$ for all $\lambda \in A^*$ and $a \in A$. One can prove that the right condition (1.6) has an equivalent form $\mathcal{R}_r = \mathrm{Ad}_\chi\mathcal{L}_r$.

Definition 1.6. 1) A map δ, obeying (1.8), will be called a r-coderivaton (notice that is δ is a r-coderivetion, then the map $\delta^* : A^* \to A^*$ is a $\mathcal{R}_r$-derivation);

2) we shall refer δ as $< \chi, r >$-derivation if it is a τ-derivation (see 1.1) and a r-coderivaton.

Notation 1.7. The Hopf-Ore extension $R = A[y;\tau,\delta]$ we shall denote $R = A(\chi, r, \delta)$ with $\chi : A \to k$ is a character, r is a grouplike element of A obeying (1.7) and δ is a $< \chi, r >$-derivation.

Definition 1.8. We say that δ is an inner $< \chi, r >$-derivation if there exists an

element $d \in A$ such that $\delta(a) = \tau(a)d - da$ and $\Delta(d) = d \otimes 1 + r \otimes d$.
Definition 1.9. We say that Hopf-Ore extensions R and R' of A and A' are isomorphic if there exists a Hopf isomorphism $\Phi : R \mapsto R'$ passing A to A'.
Proposition 1.10. 1) If δ is an inner $< \chi, r >$-derivation, then $A(\chi, r, \delta)$ is isomorphic to $A(\chi, r, 0)$.
2) Two Hopf-Ore extensions $R = A(\chi, r, \delta)$ and $R = A'(\chi', r', \delta')$ are isomorphic iff the exists a Hopf isomorphism $\Phi : A \mapsto A'$ such that $\chi' = \chi\Phi^{-1}$, $r' = \Phi(r)$, $\delta' = \Phi\delta\Phi^{-1} + \delta''$ where δ'' is an inner $< \chi', r' >$-derivation of R'.
Proof. Changing y by $y + d$.□
Proposition 1.11. The subHopfalgebra A in $R = A(\chi, r, \delta)$ is normal (see [M,3.4.1]) if and only if $\chi = \varepsilon$ and r lies in the center of A.
Proof. The left(resp.right) adjoint action on a Hopf algebra H is given by $(\mathrm{ad}_l h)(x) = h_1 x S(h_2)$ (resp.$(\mathrm{ad}_r h)(x) = S(h_1) x h_2$). By definition, a subHopfalgebra is normal if it is stable with respect to the left and the right adjoint action. The subHopfalgebra A in $R = A(\chi, r, \delta)$ is normal if $(\mathrm{ad}_l y)(a) \in A$ and $(\mathrm{ad}_r y)(a) \in A$ for any $a \in A$. That is $ya - rar^{-1}y \in A$ and $-r^{-1}ya + r^{-1}ay = r^{-1}(-ya + ay) \in A$. By (1.2), it follows $\tau(a) = rar^{-1}$ and $\tau(a) = a$ for any $a \in A$. This proves the claim. □

2 Classifying Hopf-Ore extensions

In this section we classify Hopf-Ore extensions for some classes of Hopf algebras.
Point 1. Cocommmutative Hopf algebras.
We present here a method of constructing $< \chi, r >$-derivations of cocommutative Hopf algebras.

Let A be a cocommutative Hopf algebra. Let χ be a A-charater and r be a group-like element in the center of A. Let α be a linear form $\alpha : A \to k$ obeying

$$\alpha(uv) = \alpha(u)\varepsilon(v) + \chi(a)\alpha(b). \tag{2.1}$$

Proposition 2.1.The map

$$\delta(a) := \alpha(a_1)(1 - r)a_2 \tag{2.2}$$

is a $< \chi, r >$-derivation. This provides an example $A(\chi, \tau, \delta)$ of a Hopf-Ore extenson of A.
Proof. The map δ is a τ-derivation:

$$\delta(ab) = \alpha((ab)_{(1)})(1 - r)(ab)_{(2)} = (\alpha(a_1)\varepsilon(b_1) + \chi(a_1)\alpha(b_1))(1 - r)a_2 b_2 =$$

$$\alpha(a_1)(1 - r)a_2\varepsilon(b_1)b_2 + \chi(a_1)a_2\alpha(b_1)(1 - r)b_2 = \delta(a)b + \tau(a)\delta(b).$$

The map δ is r-coderivation (see 1.8):

$$\Delta\delta(a) = \alpha(a_1)\Delta(1 - r)a_2 = \alpha(a_1)(1 \otimes 1 - r \otimes r)a_2 \otimes a_3 =$$

$$\alpha(a_1)(1 - r)a_2 \otimes a_3 + \alpha(a_1)ra_2 \otimes (1 - r)a_3 =$$

$$\alpha(a_1)(1 - r)a_2 \otimes a_3 + ra_1 \otimes \alpha(a_2)(1 - r)a_3 = \delta(a_1) \otimes a_2 + ra_1 \otimes \delta(a_2).$$

□
Remark. In the case $A = kG$ the linear form α obeying is determined by $\alpha(g)$,$g \in G$ and obeys $\alpha(gh) = \alpha(g) + \chi(g)\alpha(h)$. That is α is a 1-cocycle ($\alpha \in Z^1_\chi(G)$).
Point 2. Group algebras.
Propositon 2.2 Any Hopf-Ore extension of $A = kG$ has the form $A(\chi, r, \delta)$ where χ is a group character, r in the center of G and δ have the form (2.2) for some $\alpha \in Z^1_\chi(G)$.
Proof. By Theorem 1.3, all Hopf-Ore extesions have the form $A(\chi, r, \delta)$ where χ is a kG-character (i.e. χ is a G-character), r is a group-like element of kG (one can prove that all group-like elements of kG are elements of G), and δ is a $< \chi, r >$-derivation. We have proved that r lies in the center of G (Corollary 1.4). The condition (1.8) have the form

$$\Delta\delta(g) = \delta(g) \otimes g + rg \otimes \delta(g) \tag{2.3}$$

for all $g \in G$. The element $c := \delta(g)g^{-1}$ satisfies

$$\Delta(c) = c \otimes 1 + r \otimes c. \tag{2.4}$$

For any $\mu \in (kG)^*$ we have

$$\mu^2(c) = \mu \otimes \mu(\Delta c) = \mu(c)\mu(1) + \mu(r)\mu(c). \tag{2.5}$$

The element c is the sum $c = \sum \gamma(g)g$. Let g_0 be an element of G which is different from 1 and r. Take $\mu_0 \in (kG)^*$ in (2.5) such that $\mu_0(g_0) = 1$ and $\mu_0(g) = 0$ for all other elements of G. We put $\mu = \mu_0$ in (2.5) and get

$$\gamma^2(g_0) = \sum \gamma(g)\mu_0^2(g) = \mu_0^2(\sum \gamma(g)g) = \mu_0^2(c) = \mu_0(c)\mu_0(1) + \mu_0(r)\mu_0(c) = 0.$$

This yields $\gamma(g_0) = 0$ and $c = \gamma(1) + \gamma(r)r$. Substituting in (2.4), we obtain $c = \gamma(1 - r)$ for some $\gamma \in k$. Hence $\delta(g) = \alpha(g)(1 - r)g$. Since $\delta(g)$ is τ-derivation, then $\delta(gh) = \delta(g)h + \chi(g)\delta(h)$. After subsitution $\delta(g) = \alpha(g)(1 - r)g$ we obtain $(1 - r)(\alpha(gh) - \alpha(g) - \chi(g)\alpha(h))gh = 0$. Finally, $\alpha(g) \in Z\chi^1(G)$. □
Point 3. $A = U(\mathfrak{g})$ where $\mathfrak{g}$ is a Lie algebra over a field of zero characteristic.
By [D,2.7.2], one can prove that a group-like element in $U(\mathfrak{g})$ equals to 1 and any primitive element lies in $\mathfrak{g}$. We have $r = 1$. A $U(\mathfrak{g})$ character χ is determined by $\mathfrak{g}$-character, and $\tau(x) = x+\chi(x)$ for all $x \in \mathfrak{g}$. For $x \in \mathfrak{g}$ we also have $\Delta(x) = x\otimes 1+1\otimes x$ and (1.8) has the form $\Delta\delta(x) = \delta(x) \otimes 1 + 1 \otimes \delta(x)$. Therefore, $\delta(x) \in \mathfrak{g}$, $\delta : \mathfrak{g} \to \mathfrak{g}$ and any Hopf-Ore extension R of $U(\mathfrak{g})$ is generated by y and $U(\mathfrak{g})$ with the relation $yx = (x + \chi(x))y + \delta(x)$. In the case $\chi = 0$ the algebra R is universal enveloping algebra $U(\mathfrak{g}')$ of the Lie algebra $\mathfrak{g}' = ky + \mathfrak{g}$.
Point 4. Let A be $U_q(\mathfrak{g})$ in the Drilfeld-Jimbo sense, and q be an indeterminate. It as generated by $K_i, e_i, f_i, i = 1, \dots, n$ related as usual and $\Delta K_i = K_i \otimes K_i$, $\Delta e_i = e_i \otimes 1 + K_i \otimes e_i$ and $\Delta f_i = f_i \otimes K_i^{-1} + 1 \otimes e_i$.
Proposition 2.3 Any Hopf-Ore extension of $A = U(\mathfrak{g})$ is isomorphic to $A(1, 1, \delta)$ with $\delta(K_i) = 0$, $\delta(e_i) = \xi_i e_i$ and $\delta(f_i) = -\xi_i f_i$.
Proof Any character of A has the form $\chi(K_i) = \varepsilon_i = \pm 1$, $\chi(e_i) = \chi(f_i) = 0$. Then $\tau(K_i) = \varepsilon_i K_i$, $\tau(e_i) = \varepsilon_i e_i$, $\tau(f_i) = f_i$. The generator y in a Hopf-Ore extension of $U_q(\mathfrak{g})$ obeys (1.3) with $r = K_1^{\gamma_1} \cdots K_n^{\gamma_n} = K^\gamma$. For any simple root α_i we have

$$\mathrm{Ad}_r S(e_i) = \mathrm{Ad}_r(-K_i^{-1} e_i) = -K_i^{-1} q^{(\alpha_i, \gamma)} e_i.$$

On the other hand,

$$\tau S\tau(e_i) = \tau S(\varepsilon_i e_i) = -\tau(\varepsilon_i K_i^{-1} e_i) = -\varepsilon_i K_i^{-1} e_i.$$

By (1.7), $\gamma = 0$ and $\varepsilon_i = 1$. Hence, $r = 1$ and $\tau = \mathrm{id}$. We obtain, in particular, that y is primitive. By (1.8), $\Delta\delta(K) = \delta(K)\otimes K + K\otimes\delta(K)$. Whence $\delta(k)K^{-1}$ is a primitive element. By PBW Theorem, one can prove that $U_q(\mathfrak{g})$ has no primitive elements. This implies $\delta(K) = 0$. We get $ye_i = e_i y + \delta(e_i)$, $yf_i = f_i y + \delta(f_i)$, $yK_i = K_i y$. By (1.8), we have $\Delta\delta(e_i) = \delta(e_i)\otimes 1 + K_i\otimes\delta(e_i)$ and $\Delta\delta(f_i) = \delta(f_i)\otimes K_i^{-1} + 1\otimes\delta(f_i)$. It follows $\delta(e_i) = \xi_i e_i$, $\delta(f_i) = -\xi_i f_i$ for some numbers ξ_i.□

Point 5. Hopf-Ore extensions of quantum "$ax+b$".

Let q be an indeterminate. Quantum "$ax+b$" is generated over the ring $C = k[q, q^{-1}]$ by K, x with the relation $Kx = q^2xK$ and $\Delta(K) = K\otimes K$, $\Delta(x) = x\otimes 1 + K\otimes x$, $\varepsilon(K) = 1$, $\varepsilon(x) = 0$, $S(K) = K^{-1}$ and $S(x) = -K^{-1}x$.

Proposition 2.4. There are exist exactly four Hopf-Ore extensions of quantum "$ax+b$": $yK = Ky$, $yx = xy + \gamma x$ with $\gamma = 0, 1$, y is primitive and $Ky = q^{-2}yK$, $yx = xy + \gamma(K - K^{-1})$ with $\gamma = 0, 1$, $\Delta y = y\otimes K^{-1} + 1\otimes y$. Among them quantum $sl_q(2)$.

Proof. An character χ of A has the form $\chi(K) = \alpha = C^*$ and $\chi(x) = 0$. Then $\tau(K) = \chi(K)K = \alpha K$ and $\tau(x) = \chi(K)x = \alpha x$. The element r is group-like. Hence, $r = K^m$ for some $m\in\mathbb{Z}$ and

$$\Delta y = y\otimes 1 + K^m\otimes y \qquad (2.6).$$

The right condition (1.6) has the form:

$$\tau(K) = \mathrm{Ad}_{K^m}(K)\chi(K) = \alpha K,$$

$$\tau(x) = \mathrm{Ad}_{K^m}(x)\chi(1) = q^{2m}x.$$

It implies

$$\alpha = q^{2m}. \qquad (2.7)$$

and

$$\mathrm{Ad}_r S(x) = -q^{2m}K^{-1}x = \alpha\tau(-K^{-1}x) = \tau S\tau(x).$$

This proves that (1.7) holds.

The equation (1.8) has the form $\Delta\delta(K) = \delta(K)\otimes K + K^{m+1}\otimes\delta(K)$. The element $b := \delta(K)K^{-1}$ obeys

$$\Delta b = b\otimes 1 + K^m\otimes b. \qquad (2.8)$$

One can prove that any solution of (2.8) has the form: $b = \gamma x + \gamma'(1-K)$ (if $m = 1$) or $b = \gamma(1 - K^m)$. We shall show that changing the generator y one can consider that $\delta(K) = 0$.

1) Let $b = \gamma x + \gamma'(1-K)$ and $m = 1$. Then $\delta(K) = \gamma xK + \gamma'(1-K)K$ and $\alpha = q^2$. Consider the inner derivations $\delta_x(a) = \tau(a)x - xa$ and $\delta_{1-K}(a) = \tau(a)(1-K) - (1-K)a$ (see Definition 1.7). Then $\delta_x(K) = \tau(K)x - xK = xK(\alpha q^2 - 1) = xK(q^4 - 1)$ and $\delta_{1-K}(K) = (q^2-1)(1-K)K$. Changing y by $y' = y + \xi x + \xi'(1-K)$ with $\xi = \gamma(q^4-1)^{-1}$ and $\xi' = \gamma'(q^2-1)^{-1}$, we get the new generator y' obeying (2.6) with $\delta'(K) = 0$.

2) Let $b = \gamma(1-K^m)$. Then $\delta(K) = \gamma(1-K^m)K$. The element b satifies (2.6). Consider the inner derivation δ_b. Then $\delta_b(K) = \tau(K)b - bK = \gamma(\alpha-1)(1-K^m)K$. If $\alpha \neq 1$ we change y by $y' = y + \xi b$ with $\xi = (\alpha-1)^{-1}$ and get $\delta'(K) = 0$. If $\alpha = 1$, the equality (2.7) yields $m = 0$ and $\delta(K) = 0$.

We have proved that, changing y, one can consider that $\delta(K) = 0$. Hence $yK = \alpha Ky$. Our next goal is to determine $\delta(x)$. We act by δ on the relation $Kx = q^2xK$ and get $q^{2m}K\delta(x) = q^2\delta(x)K$ and

$$K\delta(x) = q^{2(1-m)}\delta(x)K. \tag{2.9}$$

The equality (1.8) yields:

$$\Delta\delta(x) = \delta(x)\otimes 1 + rK\otimes\delta(x) = \delta(x)\otimes 1 + K^{m+1}\otimes\delta(x).$$

Again two solutions

$$\delta(x) = \begin{cases} \gamma x + \gamma'(1-K) & \text{for} \quad m = 0, \\ \gamma(1-K^{m+1}) & \text{for} \quad m \neq 0. \end{cases} \tag{2.11}$$

By (2.9) and (2.11),

$$\delta(x) = \begin{cases} \gamma x & \text{for} \quad m = 0, \\ \gamma(1-K^2) & \text{for} \quad m = 1. \end{cases} \tag{2.12}$$

In the first case a Hopf-Ore extension R is generated by y and A with $yK = Ky$, $yx = x(y+\gamma)$ and y is primitive. One may consider that $\gamma = 0$ or $\gamma = 1$.

In the second case $\delta(x) = \gamma(1-K^2)$, $m = 1$ and $\alpha = q^2$. We obtain $yK = q^2Ky$, $yx = q^2xy + \gamma(1-K^2)$.

After changing $y' = yK^{-1}$ we get $\Delta y' = y'\otimes K^{-1} + 1\otimes y'$ and $y'x = xy' + \gamma q^{-2}(K^{-1}-K)$. We may assume that $-\gamma q^{-2}$ is equal to 0 or 1. $\square$

References

[B] M.Beatie, *An isomorphism theorem for Ore extension Hopf algebras*, Comm.Algebra, 2000, No.2, 569-584.

[BDG] M.Beatie, S.Dascalescu, L.Grunenfelder, *Constructing pointed Hopf algebras by Ore extensions*, Algebra,**225**, 2000, 743-770.

[D] L.Delvaux, *Pairing and Drinfeld-double of Ore-extensions*, Comm.Algebras,**29**, N7, 2001, 3167-3177.

[D] J.Dixmier, *Algebres enveloppantes*, Gauthier-Villars, Paris, 1974.

[MC-R] J.C.McConnel, J.C.Robson, *Noncommutative Noetherian rings*, Wileys-Interscience, New York, 1987.

[M] S.Montgomery, *Hopf algebras and their actions on rings*, Reg.Conf.Series in Math., V.82, 1993.

[N1] A.Nenciu, *Cleft extensions for a class of pointed Hopf algebras constructed by Ore extensions*, Comm.Algebra, **29**, N5, 2001, 1959-1981.

[N2] A.Nenciu, *Quasitriangular structures for a class of pointed Hopf algebras constructed by Ore extensions*, Comm.Algebra, **29**, N8, 2001, 3419-3432.