MIRROR SYMMETRY FOR ABELIAN VARIETIES

V. GOLYSHEV, V. LUNTS, AND D. ORLOV

0. Introduction.

0.1. We define the relation of mirror symmetry on the class of pairs (complex abelian variety A + an element of the complexified ample cone of A) and study its properties. More precisely, let A be a complex abelian variety, $C_A^a \subset NS_A(\mathbb{R})$ – the ample cone of A and put

$$C_A^{\pm} := NS_A(\mathbb{R}) \pm iC_A^a,$$

$$C_A := C_A^+ \sqcup C_A^- \subset NS_A(\mathbb{C}).$$

If $\omega_A \in C_A$ we call (A, ω_A) an *algebraic pair.* In this work we define the notion of mirror symmetry for algebraic pairs . The definition is given below in 0.4.1. Our notion of mirror symmetry is defined purely in the language of algebraic geometry and algebraic groups. We explain in 0.9 below that our construction is compatible with the pictures of mirror symmetry of Kontsevich [Ko] (see also [PZ]) and Strominger, Yau, Zaslow [SYZ] (see also [Mo],[Gro1],[Gro2]).

0.2. Let us discuss some properties of this notion.

1. Not every algebraic pair has a mirror symmetric one (9.5.1), but for "general" abelian varieties A and any $\omega_A \in C_A$ the pair (A, ω_A) does have a symmetric one (9.6.3). Also for any abelian variety A there exists $\omega_A \in C_A$ such that the algebraic pair (A, ω_A) has a mirror symmetric one (9.6.1).

2. Suppose two algebraic pairs (B, ω_B) and (C, ω_C) are both mirror symmetric to the same algebraic pair (A, ω_A) then the derived categories of coherent sheaves on B and C are equivalent (as triangulated categories) (9.2.6). Thus in particular for any algebraic pair (A, ω_A) there are only finitely many isomorphism classes of abelian varieties which are mirror symmetric to (A, ω_A) (9.2.3).

Conversely, if some algebraic pairs (B, ω_B) and (A, ω_A) are mirror symmetric and an abelian variety C is such that the derived categories of coherent sheaves on C and B are equivalent, then there is ω_C such that the pairs (C, ω_C) and (A, ω_A) are mirror symmetric too.

3. Suppose that an algebraic pair (B, ω_B) is mirror symmetric to the algebraic pair (A, ω_A) . Starting with (A, ω_A) we cannot construct B , but we can construct the product $B \times \widehat{B}$, where $\widehat{B}$ is the dual abelian variety.

Given an abelian variety A consider its total cohomology $H^*(A, \mathbb{Q})$. This space has a "horizontal" and "vertical" structures which are discussed below in 0.3. More precisely, $H^*(A, \mathbb{Q})$ is acted upon by two reductive algebraic $\mathbb{Q}$ –groups – the "horizontal" and the "vertical", – and these groups commute. The "horizontal" group is the Hodge group or the special Mumford-Tate group (its action preserves each cohomology space $H^k(A, \mathbb{Q})$). The vertical group is defined below in 0.3.

4. Suppose that a mirror symmetry between algebraic pairs (A, ω_A) and (B, ω_B) is given. Then there exists a natural isomorphism

$$\beta : H^*(A, \mathbb{Z}) \xrightarrow{\sim} H^*(B, \mathbb{Z})$$

such that $\beta_{\mathbb{Q}}$ "interchanges" the horizontal and vertical structures on $H^*(A, \mathbb{Q})$ and $H^*(B, \mathbb{Q})$ respectively (9.3.3). There exists a canonical choice (up to $\pm$) of such a β . This isomorphism will either preserve or switch the parity of the cohomology groups depending on the parity of the dimension of A and B. For example, if A and B are elliptic curves then β induces isomorphisms

$$\beta : H^1(A, \mathbb{Z}) \xrightarrow{\sim} H^0(B, \mathbb{Z}) \oplus H^2(B, \mathbb{Z}),$$

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$$\beta : H^0(A, \mathbb{Z}) \oplus H^2(A, \mathbb{Z}) \xrightarrow{\sim} H^1(B, \mathbb{Z}).$$

The mirror symmetry works "best" for abelian varieties A which are obtained by a version of the **G**–construction of Gerritzen (section 10).

5. If the abelian variety A is obtained by the **G**– construction, then for any $\omega_A \in C_A$ the algebraic pair (A, ω_A) has a mirror symmetric pair (B, ω_B) , such that B is also obtained by the **G**–construction. Moreover, in this case we can describe the isomorphism β explicitly. Namely, β is given by an element in $H^*(A \times B, \mathbb{Z})$ which is the Chern character of some natural line bundle on a certain real subtorus of $A \times B$ of dimension $3n$ (where $n = dimA = dimB$).

0.3. Let us describe the two $\mathbb{Q}$ –algebraic groups attached to any abelian variety A , which act on the total cohomology $H^*(A, \mathbb{Q})$ and commute with each other.

0.3.1. The first group is the classical *Hodge group* or the *special Mumford-Tate group.* Let us recall its definition.

Put $\Gamma = \Gamma_A = H_1(A, \mathbb{Z})$, $V = \Gamma_{\mathbb{R}} = H_1(A, \mathbb{R})$. The complex structure on A induces the operator of complex structure J_A on V . Consider the homomorphism of algebraic $\mathbb{R}$ -groups

$$h_A : \boldsymbol{S}^1 \longrightarrow \mathrm{GL}(V), \qquad h_A(e^{i\theta}) = \cos(\theta) \cdot Id + \sin(\theta) \cdot J_A,$$

so that $h_A(e^{i\pi/2}) = J_A$. Then $Hdg_{A,\mathbb{Q}}$ is defined as the smallest $\mathbb{Q}$ –algebraic subgroup G of $\mathrm{GL}(H_1(A, \mathbb{Q}))$ such that $h_A(\boldsymbol{S}^1) \subset G(\mathbb{R})$. By functoriality $Hdg_{A,\mathbb{Q}}$ acts on $H^1(A, \mathbb{Q})$ and on the total cohomology $H^*(A, \mathbb{Q}) = \Lambda^{\cdot} H^1(A, \mathbb{Q})$. We call this group "horizontal" because it preserves each subspace $H^k(A, \mathbb{Q})$.

0.3.2. Perhaps the main point of our work is the consideration of the second ("vertical") algebraic group which we view as the mirror image of the Hodge group. This second group is the Zariski closure $\overline{Spin(A)}$ in $\mathrm{GL}(H^*(A, \mathbb{Q}))$ of a certain discrete subgroup $Spin(A) \subset \mathrm{GL}(H^*(A, \mathbb{Q}))$.

Let $D^b(A)$ be the bounded derived category of coherent sheaves on A . Consider its group $Auteq(D^b(A))$ of exact autoequivalences. We construct in 4.3 a natural representation

$$\rho_A : Auteq(D^b(A)) \longrightarrow \mathrm{GL}(H^*(A, \mathbb{Z}))$$

and denote its image by $Spin(A)$. The elements of $Spin(A)$ act by algebraic correspondences, hence commute with the Hodge group and preserve the Hodge verticals $\oplus_{p-q=fixed} H^{p,q}(A, \mathbb{C})$. Hence the same is true for the algebraic group $\overline{Spin(A)}$. Let us discuss some properties of the groups $Spin(A)$ and $\overline{Spin(A)}$.

0.3.3. Let $\widehat{A}$ be the dual abelian variety and $\Gamma_{\widehat{A}}$ be its first homology lattice that can be identified with $\Gamma_A^* := \mathrm{Hom}(\Gamma_A, \mathbb{Z})$. Consider the lattice

$$\Lambda := \Gamma_A \oplus \Gamma_{\widehat{A}}$$

with the canonical bilinear symmetric form

$$Q((a, b), (c, d)) = b(c) + d(a).$$

Let $\mathrm{SO}(\Lambda, Q)$ be the corresponding special orthogonal group. Let $Spin(\Lambda, Q)$ be the corresponding spinorial group. We have the canonical exact sequence

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow Spin(\Lambda, Q) \longrightarrow \mathrm{SO}(\Lambda, Q).$$

Note that the group $Spin(\Lambda, Q)$ acts naturally on the total cohomology group $\Lambda^{\cdot}\Gamma_A^* = H^*(A, \mathbb{Z})$.

For any abelian variety A , there are two representations of $Auteq(D^b(A))$. One, in $H^*(A, \mathbb{Z})$, is the representation ρ_A considered above, in (0.3.2). The existence of the other, in Λ is implied by the explicit description of $Auteq(D^b(A))$ given in [24]. In fact, a commutative diagram exists:

$$\begin{array}{ccc} Spin(A) & \hookrightarrow & Spin(\Lambda, Q) \\ \downarrow & & \downarrow \\ \mathrm{U}(A) & \hookrightarrow & \mathrm{SO}(\Lambda, Q) \end{array}$$

where the horizontal arrows are compatible with the actions on $H^*(A, \mathbb{Z})$ and Λ respectively. The group $\mathrm{U}(A)$ was first introduced independently by S.Mukai [18] and by A.Polishchuk [27], so we call it the Mukai-Polishchuk group.

Let $\overline{\mathrm{U}(A)} \subset \mathrm{GL}(\Lambda_{\mathbb{Q}})$ be the $\mathbb{Q}-$ algebraic subgroup which is the Zariski closure of $\mathrm{U}(A)$ in $\mathrm{GL}(\Lambda_{\mathbb{Q}})$. Thus the groups $\overline{Spin(A)}$ and $\overline{\mathrm{U}(A)}$ are isogeneous.

0.3.4. The algebraic groups $\overline{Spin(A)}$ and $\overline{\mathrm{U}(A)}$ are semisimple (7.2.1).

0.3.5. Consider the set $C_A = C_A^+ \sqcup C_A^- \subset NS_A(\mathbb{C})$ as in 0.1. Both C_A^+ and C_A^- can be considered as Siegel domains of the first kind: the Lie group $\overline{\mathrm{U}(A)}(\mathbb{R})$ acts naturally on C_A preserving C_A^+ and C_A^-. Moreover, C_A^+ and C_A^- are single $\overline{\mathrm{U}(A)}(\mathbb{R})$ –orbits and the stabilizer K_ω of a point $\omega \in C_A$ is a maximal compact subgroup of the semisimple Lie group $\overline{\mathrm{U}(A)}(\mathbb{R})$. Further, for each $\omega \in C_A$ there is a natural choice of an element $I_\omega \in K_\omega$ (14) such that I_ω defines a complex structure on the space $\Lambda_{\mathbb{R}}$ (0.3.3).

0.4. Now we are ready to give the main definition of mirror symmetry for algebraic pairs.

Let (A, ω_A) be an algebraic pair, $\Lambda_A = \Gamma_A \oplus \Gamma_{\widehat{A}}$ with the symmetric form Q_A as in 0.3.3. Note that the Hodge group $Hdg_{A,\mathbb{Q}} = Hdg_{A\times\widehat{A},\mathbb{Q}}$ is naturally a subgroup of $\mathrm{SO}(\Lambda_{A,\mathbb{Q}}, Q_{A,\mathbb{Q}})$. Moreover, it commutes with $\overline{\mathrm{U}(A)}$. Thus we obtain two commuting complex structures on $\Lambda_{A,\mathbb{R}}$: $J_{A\times\widehat{A}} \in Hdg_{A,\mathbb{Q}}(\mathbb{R}) = Hdg_{A\times\widehat{A},\mathbb{Q}}(\mathbb{R})$ and $I_{\omega_A} \in \overline{\mathrm{U}(A)}(\mathbb{R})$.

0.4.1 Definition. *We call algebraic pairs* (A, ω_A) *and* (B, ω_B) mirror symmetric *if there is given an isomorphism of lattices*

$$\alpha : \Lambda_A \xrightarrow{\sim} \Lambda_B,$$

which identifies the forms Q_A *and* Q_B *and satisfies the following conditions:*

$$\alpha_{\mathbb{R}} \cdot J_{A\times\widehat{A}} = I_{\omega_B} \cdot \alpha_{\mathbb{R}},$$
$$\alpha_{\mathbb{R}} \cdot I_{\omega_A} = J_{B\times\widehat{B}} \cdot \alpha_{\mathbb{R}}.$$

0.4.2. Note that if we identify Λ_A and Λ_B by means of α then

$$Hdg_{A,\mathbb{Q}} \subseteq \overline{\mathrm{U}(B)}, \qquad Hdg_{B,\mathbb{Q}} \subseteq \overline{\mathrm{U}(A)}.$$

0.5. Actually we work in a more general context. Namely, we consider *weak pairs* (A, ω_A) , where A is a *complex torus* and $\omega_A = \eta_1 + i\eta_2 \in NS_A(\mathbb{C})$ is such that η_2 is nondegenerate. (Thus an algebraic pair is in particular a weak pair.) And we define the notion of mirror symmetry for weak pairs. However we prove that if two weak pairs are mirror symmetric and one of them is algebraic then so is the other.

0.6. For an abelian variety A the algebraic group $\overline{Spin(A)}$ or rather its Lie algebra has a different description. It turns out to be isomorphic (as a Lie subalgebra of $\mathfrak{gl}(H^*(A,\mathbb{Q}))$) to the *Neron-Severi* Lie algebra $\mathfrak{g}_{NS}(A)$ defined in [LL].

0.6.1. Let us recall the definition of $\mathfrak{g}_{NS}(X)$ for a smooth complex projective variety X . If $\kappa \in H^{1,1}(X) \cap H^2(X,\mathbb{Q})$ is an ample class, then cupping with it defines an operator e_κ in the total cohomology $H^*(X) = H^*(X,\mathbb{C})$ of degree 2 and the hard Lefschetz theorem asserts that for $s = 0,1,...,n$, e_κ^s maps $H^{n-s}(X)$ isomorphically onto $H^{n+s}(X)$. As is well known, this is equivalent to the existence of a (unique) operator f_κ on $H^*(X)$ of degree -2 such that the commutator $[e_\kappa, f_\kappa]$ is the operator h which on $H^k(X)$ is multiplication by $k-n$. The elements e_κ, f_κ, h make up a Lie subalgebra $\mathfrak{g}_\kappa$ of $\mathfrak{gl}(H^*(X))$ isomorphic to $sl(2)$. Define the *Neron-Severi* Lie algebra $\mathfrak{g}_{NS}(X)$ as the Lie subalgebra of $\mathfrak{gl}(H^*(X))$ generated by $\mathfrak{g}_\kappa$'s with κ an ample class. This Lie subalgebra is defined over $\mathbb{Q}$ and is evenly graded by the adjoint action by the semisimple element h. The Lie algebra $\mathfrak{g}_{NS}(X)$ is semisimple.

0.6.2. Note that the equality

$$Lie\overline{Spin(A)} \quad = \quad \mathfrak{g}_{NS}(A) \qquad (*)$$

implies one of the standard conjectures of Grothendieck for A . Namely, the algebraicity of the operator f_κ (0.5.1) that has been proved for abelian varieties by Kleiman [12].

0.6.3. *Question.* Let X be a smooth complex projective variety. Assume that the canonical sheaf K_X is trivial. Does the analogue of (*) hold for X ?

We believe that the answer is "yes". Note that if K_X or K_X^{-1} is ample then the analogue of (*) cannot hold. Indeed, by a theorem in [BO] the group $Auteq(D^b(X))$ is then generated by $Aut(X)$, the shift operator [1], and by operators of tensoring with a line bundle on X . Thus (if $Aut(X)$ is trivial) $Auteq(D^b(X))$ is abelian, whereas the Lie algebra $\mathfrak{g}_{NS}(X)$ is semisimple.

0.7. Our picture of the mirror symmetry provides an explanation of the phenomenon in Hodge theory, which was noticed long ago (see for example p.68 in [Gr]). Namely, consider a variation of Hodge structures which degenerates along a divisor. Then the logarithm of the monodromy operator has properties which are similar to the properties of a Lefschetz operator on the cohomology. But in our construction this logarithm of the monodromy is transformed by the mirror symmetry to a Lefschetz operator indeed.

0.8. A similar picture exists for hyperkahler manifolds and will be described in our next paper.

0.9. Our notion of mirror symmetry for abelian varieties is compatible with other (conjectural) pictures of mirror symmetry. We will explain the compatibility with the ideas of Kontsevich [Ko] on one hand and with the T –duality picture of Strominger, Yau and Zaslow [SYZ] on the other hand.

0.9.1. In [Ko] Kontsevich proposes the following thesis. If a symplectic manifold (V,ω) is mirror symmetric to an algebraic variety W , then the Fukaya category $F(V,\omega)$ of (V,ω) is equivalent to the derived category $D^b(W)$ of coherent sheaves on W . This has been proved for elliptic curves in [PZ].

Assume that the algebraic pairs (A,ω_A) and (B,ω_B) are mirror symmetric (0.4.1). Let $\omega_A = \eta_1 + i\eta_2 \in C_A$. Then η_2 is a Kahler form on A and in particular (A,η_2) is a symplectic manifold. (The class η_1 plays the role of a B-field). The Hodge group $Hgd_{A,\mathbb{Q}}$ acts on $H_1(A)$ preserving the form η_2 . Hence its arithmetic subgroup

$$Hdg_A(\mathbb{Z}) := \{g \in Hdg_{A,\mathbb{Q}} |\ \ g(\Lambda_A) \subset \Lambda_A\}$$

acts by symplectomorphisms of the symplectic manifold (A,η_2) . Thus

$$Hdg_A(\mathbb{Z}) \subset AuteqF(A,\eta_2).$$

But our definition of mirror symmetry implies the natural inclusion of groups

$$Hdg_A(\mathbb{Z}) \subset \mathrm{U}(B)$$

(0.3.3, 0.4.2), where the group $\mathrm{U}(B)$ captures the essential part of $AuteqD^b(B)$. Thus we do not establish the equivalence of categories $F(A,\eta_2)$ and $D^b(B)$ as suggested by Kontsevich, but rather compare their groups of autoequivalences.

0.9.2. Let us show the compatibility with the mirror symmetry proposed in [SYZ] and more specifically in [Mo] (see also [Gro1],[Gro2]). Let algebraic pairs (A,ω_A) and (B,ω_B) be mirror symmetric. Assume that the pair (A,ω_A) is obtained by the **G**–construction (10.2.1) (this is so for "most" pairs). Then there exists a real subtorus $T \subset A \times B$ of dimension $3n$ ($n = dim_{\mathbb{C}} A = dim_{\mathbb{C}} B$) with the following property: The fibres of the projection of T onto B (resp. A) are special Lagrangian subtori of A (resp. B) of dimension n (10.3.2, 10.7). The existence of such a correspondence is part of the "geometric mirror symmetry" of Morrison (Definition 4 in [Mo]). Moreover, there is a natural complex line bundle L on T such that its Chern class $c_1(L)$ considered as an element in $H^*(A\times B) \simeq Hom(H^*(A),H^*(B))$ is the isomorphism β mentioned in 0.2.

0.10. Let us briefly discuss the contents of each section.

Section 1 contains some background material on abelian varieties and complex tori. In section 2 we recall the Hodge group of a complex torus and its basic properties. Section 3 contains a review of the Clifford algebra (of a certain quadratic form which we define there) and its spinorial representation. This Clifford algebra plays the key role in the definition of the isomorphism β discussed in 0.2.

Section 4 is devoted to the bounded derived category $D^b(X)$ of coherent sheaves on a smooth projective variety X . We define the group $Auteq(D^b(X))$ of autoequivalences of $D^b(X)$ and construct a canonical representation

$$\rho_X : Auteq(D^b(X)) \longrightarrow \mathrm{GL}(H^*(X,\mathbb{Q})).$$

We then show that if $X = A$ is an abelian variety then $Im\rho_A \subset \mathrm{GL}(H^*(A,\mathbb{Z}))$. The Mukai-Polishchuk group is introduced and then the main effort is applied to prove the commutativity of the diagram in 0.3.3 (see Proposition 4.3.7).

In section 5 we discuss various $\mathbb{Q}$ –algebraic subgroups of $\mathrm{GL}(H_1(A\times \widehat{A},\mathbb{Q}))$, which enter the picture of mirror symmetry. In particular we recall the algebraic group $\mathrm{U}_{A,\mathbb{Q}}$ which was introduced by A. Polishchuk and present his result on the group of $\mathbb{R}$ –points of $\mathrm{U}_{A,\mathbb{Q}}$. This group $\mathrm{U}_{A,\mathbb{Q}}$ is closely related to the algebraic groups $\overline{Spin(A)}$ and $\overline{\mathrm{U}(A)}$. Namely, the semisimple group $\overline{\mathrm{U}(A)}$ is a subgroup of the reductive group $\mathrm{U}_{A,\mathbb{Q}}$ which consists (up to isogeny) of all noncompact factors of $\mathrm{U}_{A,\mathbb{Q}}$ (7.2.1).

In section 6 we recall the Neron-Severi Lie algebra $\mathfrak{g}_{NS}(X)$ of a smooth projective variety X and describe $\mathfrak{g}_{NS}(A)$ for an abelian variety A following [LL]. Then in section 7 we establish the relationship between this Lie algebra and the group $Auteq(D^b(A))$.

In section 8 we define a natural action of the Lie group $\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$ on the Siegel domain C_A^+ (and C_A^-). We then show that this action is transitive and the stabilizer of a point is a maximal compact subgroup of $\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$. To each $\omega \in C_A$ we associate a natural element $I_\omega \in \mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$, which is the main ingredient in the definition of mirror symmetry.

Finally, in section 9 we define the notion of mirror symmetry for complex tori and abelian varieties and study its properties.

Section 10 contains a variant of the **G**–construction and its relations with the mirror symmetry.

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1. Some preliminaries on complex tori and abelian varieties.

1.1. Let $\Gamma \cong \mathbb{Z}^{2n}$ be a lattice, $V = \Gamma \otimes \mathbb{R} \cong \mathbb{R}^{2n}$ and $J \in \mathrm{End}(V)$, s.t. $J^2 = -1$. That is J is a complex structure on V . This way we obtain an n -dimensional complex torus

$$A = (V/\Gamma, J).$$

Note the canonical isomorphisms

$$\Gamma = H_1(A, \mathbb{Z}), \quad V = H_1(A, \mathbb{R}).$$

Sometimes we will add the subscript "A" to the symbols Γ, V, J.

Given another complex torus $B = (V_B/\Gamma_B, J_B)$, the group $\mathrm{Hom}(A, B)$ consists of homomorphisms $f : \Gamma_A \to \Gamma_B$ such that

$$J_B \cdot f_{\mathbb{R}} = f_{\mathbb{R}} \cdot J_A : V_A \longrightarrow V_B.$$

Thus the abelian group $\mathrm{Hom}(A, B)$ can be considered as a subgroup of $\mathrm{Hom}(\Gamma_A, \Gamma_B)$.

1.2. One has the dual torus $\widehat{A}$ defined as follows. Put $\Gamma^* = \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{Z})$, $V^* = \Gamma^* \otimes \mathbb{R} = \mathrm{Hom}(V, \mathbb{R})$ and $\widehat{J} : V^* \xrightarrow{\sim} V^*$, s.t. $(\widehat{J}w)(v) = w(-Jv)$ for $v \in V$, $w \in V^*$. Then by definition

$$\widehat{A} = (V^*/\Gamma^*, \widehat{J}).$$

1.3. Denote by Pic_A the Picard group of A . Let $\mathrm{Pic}^0_A \subset \mathrm{Pic}_A$ be the subgroup of line bundles with the trivial Chern class. It has a natural structure of a complex torus. Moreover, there exists a natural isomorphism of complex tori

$$\widehat{A} \cong \mathrm{Pic}^0_A.$$

Every line bundle L on A defines a morphism $\varphi_L : A \to \widehat{A}$ by the formula

$$\varphi_L(a) = T_a^* L \otimes L^{-1}.$$

(Here $T_a : A \to A$ is the translation by a .) We have $\varphi_L = 0$ iff $L \in \mathrm{Pic}^0_A$ and φ_L is an isogeny if L is ample. Thus the correspondence $L \mapsto \varphi_L$ identifies the Neron-Severi group $NS_A := \mathrm{Pic}_A/\mathrm{Pic}^0_A$ as a subgroup in $\mathrm{Hom}(A, \widehat{A})$. Also NS_A is naturally a subgroup of $H^2(A, \mathbb{Z})$: to a line bundle L there corresponds its first Chern class that can be considered as a skew-symmetric bilinear form on Γ . Put $c_1(L) = c$. Then the morphism ϕ_L is given by the map

$$V_A \to V_{\widehat{A}}, \quad v \mapsto c(v, \cdot).$$

We will identify NS_A either as a subgroup of $\mathrm{Hom}(A, \widehat{A})$ or $\mathrm{Hom}(\Gamma_A, \Gamma_{\widehat{A}})$ or as a set of (integral) skew-symmetric forms c on Γ_A such that the extension $c_{\mathbb{R}}$ on V_A is J -invariant. Put $NS_A(\mathbb{R}) = NS_A \otimes \mathbb{R}$. We will denote by $NS_A(\mathbb{R})^0 \subset NS_A(\mathbb{R})$ the subset consisting of nondegenerate forms.

1.4. There is a unique line bundle P_A on the product $A \times \widehat{A}$ such that

i) for any point $\alpha \in \widehat{A}$ its restriction P_α on $A \times \{\alpha\}$ represents an element of Pic^0_A, corresponding to α with respect to the fixed isomorphism $\mathrm{Pic}^0_A = \widehat{A}$;

ii) the restriction $P|_{\{0\}\times \widehat{A}}$ is trivial.

Such P_A is called Poincare line bundle.

Given a morphism of complex tori $f : A \to B$, the dual morphism $\widehat{f} : \widehat{B} \to \widehat{A}$, is defined. Pointwise, for $\alpha \in \widehat{A}$ and $\beta \in \widehat{B}$ one has $\widehat{f}(\beta) = \alpha$ if and only if the line bundle f^*P_β on A coincides with P_α .

The double dual torus $\widehat{\widehat{A}}$ is naturally identified with A by means of the Poincare line bundle on $A \times \widehat{A}$ and $\widehat{A} \times \widehat{\widehat{A}}$. In other words,there exists a unique isomorphism $\kappa_A : A \tilde{\to} \widehat{\widehat{A}}$ such that under the isomorphism $1 \times \kappa_A : \widehat{A} \times A \xrightarrow{\sim} \widehat{A} \times \widehat{\widehat{A}}$ the Poincare line bundle pulls back to the Poincare line bundle: $(1 \times \kappa_A)^* P_{\widehat{A}} \cong P_A$.

Thus $\widehat{\ }$ is an antiinvolution on the category of complex tori, i.e. a contravariant functor whose square is isomorphic to the identity: $\kappa : Id \tilde{\to} \widehat{\widehat{\ }}$. It is known that $\widehat{\varphi}_L = \varphi_L$ for any $L \in \mathrm{Pic}_A$ (see next remarks).

1.5 Remark. Let A be a complex torus. Consider the Poincare line bundles P_A and $P_{\widehat{A}}$ on $A \times \widehat{A}$ and $\widehat{A} \times \widehat{\widehat{A}}$ respectively. Choose a basis $l_1, ..., l_{2n}$ of Γ_A and the dual basis $l_1^*, ..., l_{2n}^*$ of $\Gamma_{\widehat{A}} = \Gamma_A^*$ (1.2). Let $l_1^{**}, ..., l_{2n}^{**}$ be the basis of $\Gamma_{\widehat{\widehat{A}}} = \Gamma_{\widehat{A}}^*$ dual to $l_1^*, ..., l_{2n}^*$. The isomorphism $\kappa_A : A \xrightarrow{\sim} \widehat{\widehat{A}}$ induces an identification of Γ_A and $\Gamma_{\widehat{\widehat{A}}}$ that takes l_i to $-l_i^{**}$ (!), due to the fact that the forms $c_1(P_A)$ and $c_1(P_{\widehat{A}})$ are skew-symmetric.

1.6 Remark. Let $f : A \to B$ be a morphism of complex tori and $\widehat{f} : \widehat{A} \to \widehat{B}$ be the dual morphism Choose basises $l_1, ..., l_{2n}$, $m_1, ..., m_{2n}$ for Γ_A , Γ_B and the dual basises $l_1^*, ..., l_{2n}^*$, $m_1^*, ..., m_{2n}^*$ for $\Gamma_{\widehat{A}}$, $\Gamma_{\widehat{B}}$. Denote by F and $\widehat{F}$ the matrices of linear maps $f : \Gamma_A \to \Gamma_B$ and $\widehat{f} : \Gamma_{\widehat{B}} \to \Gamma_{\widehat{A}}$ in these basises. The matrices F and $\widehat{F}$ are transposes of each other, i.e. $\widehat{F} = F^t$.

Now, let $f : A \to \widehat{A}$. Using the isomorphism κ , we will consider $\widehat{f}$ as morphism form A to $\widehat{A}$ too. If, as above, F and $\widehat{F}$ are the matrices of linear maps $f, \widehat{f} : \Gamma_A \to \Gamma_{\widehat{A}}$, then they are skew-transposes of each other, i.e. $\widehat{F} = -F^t$. This immediately follows from the previous remark.

Thus, in particular, $\widehat{\varphi}_L = \varphi_L$ for any $L \in \mathrm{Pic}_A$.

1.7. A complex torus $A = (V/\Gamma, J)$ is algebraic, i.e. an abelian variety, iff there exists $c \in NS_A$ such that the symmetric bilinear form $c_{\mathbb{R}}(J\cdot, \cdot)$ on V is positive definite.

For the rest of this section 1 we assume that A is an abelian variety.

The endomorphism ring $\mathrm{End}(A)$ is a finitely generated $\mathbb{Z}$ –module, and we denote , as usual, $\mathrm{End}^0(A) := \mathrm{End}(A) \otimes \mathbb{Q}$. By the Poincare reducibility theorem the variety A is isogeneous to a product of simple abelian varieties. So $\mathrm{End}^0(A)$ is a semisimple finite $\mathbb{Q}$ –algebra. Put $\Gamma_{A,\mathbb{Q}} = \Gamma_A \otimes \mathbb{Q}$.

Let $L \in \mathrm{Pic}_A$ be ample. Then the induced map

$$\varphi_L : \Gamma_{A,\mathbb{Q}} \to \Gamma_{\widehat{A},\mathbb{Q}}$$

is an isomorphism. The map

$$' : \mathrm{End}^0(A) \to \mathrm{End}^0(A), \quad a \mapsto \varphi_L^{-1} \cdot \widehat{a} \cdot \varphi_L$$

is a positive (anti-)involution of $\mathrm{End}^0(A)$ called the Rosati involution.

1.8. Assume that A is simple, so that $\mathrm{End}^0(A)$ is a division algebra. Finite division $\mathbb{Q}$ –algebras with positive involutions were classified by Albert. Below is his classification which consists of four cases.

Let us introduce some notation. Put $F = \mathrm{End}^0(A)$, K - the center of F . Fix a Rosati involution $'$ of F . The involution that it induces on K is independent of $'$: for any embedding of K in $\mathbb{C}$ it is given by complex conjugation. The fixed subfield $K_0 = K'$ is totally real. Put $[F : K] = d^2, [K_0 : \mathbb{Q}] = e_0$.

I. The case of totally real multiplication.

Here $F = K = K_0$.

II. The case of totally indefinite quaternion multiplication.
Here $K = K_0$ and F is a K_0 -form of $\mathrm{End}(2)$: there is an $\mathbb{R}$ –algebra isomorphism $F\otimes_{\mathbb{Q}}\mathbb{R} \cong \mathrm{End}(2,\mathbb{R})^{e_0}$ such that the involution corresponds to the transpose in every summand.

III. The case of totally definite quaternion multiplication.
Here also $K = K_0$ and F is a K_0 –form of the quaternion algebra $\mathbb{H}$ over K_0 : there is an $\mathbb{R}$ –algebra isomorphism $F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{H}^{e_0}$ such that the involution corresponds to the quaternion conjugation in every summand.

IV. The case of totally complex multiplication.
The field K has no real embedding (so is a totally imaginary quadratic extension of K_0) and F is a K –form of $\mathrm{End}(d)$: there is an $\mathbb{R}$ –algebra isomorphism $F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathrm{End}(d,\mathbb{C})^{e_0}$ such that the involution corresponds to the conjugate transpose in every summand.

2. The Hodge group $Hdg_{A,\mathbb{Q}}$.

2.1. Let $A = (V/\Gamma, J)$ be a complex torus. Consider the homomorphism of algebraic $\mathbb{R}$ -groups

$$h_A : \boldsymbol{S}^1 \longrightarrow \mathrm{GL}(V), \qquad h_A(e^{i\theta}) = \cos(\theta)\cdot Id + \sin(\theta)\cdot J,$$

so that $h_A(e^{i\pi/2}) = J$. This defines a representation of $\boldsymbol{S}^1$ on the exterior algebra $\Lambda^{\cdot} H_1(A,\mathbb{R})$, hence on the total (co-)homology of A . Decomposing the representation of $\boldsymbol{S}^1$ on the k-th cohomology space with respect to the characters of $\boldsymbol{S}^1$

$$\boldsymbol{S}^1 \longrightarrow \mathbb{C}^*, \quad e^{i\theta} \mapsto e^{i(q-p)\theta}, \quad p+q=k,$$

we get the usual Hodge decomposition

$$H^k(A,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(A,\mathbb{C})$$

2.2 Definition. *The* Hodge group (or special Mumford-Tate group) $Hdg_{A,\mathbb{Q}}$ *of* A *is defined as a minimal algebraic* $\mathbb{Q}$ *-subgroup* G *of* $\mathrm{GL}(H_1(A,\mathbb{Q}))$ *such that* $h_A(\boldsymbol{S}^1) \subset G(\mathbb{R})$.

The group $Hdg_{A,\mathbb{Q}}$ acts naturally on the homology $H_k(A,\mathbb{Q})$ and the cohomology $H^k(A,\mathbb{Q})$.

It is clear that $Hdg_{A,\mathbb{Q}}$ acts trivially on the Hodge spaces $H^{p,p}(A,\mathbb{Q}) := H^{p,p}(A,\mathbb{C}) \cap H^{2p}(A,\mathbb{Q})$.

Consider the dual torus $\widehat{A}$. We have canonical identifications

$$Hdg_{A,\mathbb{Q}} = Hdg_{\widehat{A},\mathbb{Q}} = Hdg_{A\times\widehat{A},\mathbb{Q}}.$$

The representation of $Hdg_{A,\mathbb{Q}}$ on $\Gamma_{\widehat{A},\mathbb{Q}}$ is the contragradient of its representation on $\Gamma_{A,\mathbb{Q}}$. Depending upon a context, we will view $Hdg_{A,\mathbb{Q}}$ as a subgroup of $\mathrm{GL}(\Gamma_{A,\mathbb{Q}})$ or $\mathrm{GL}(\Gamma_{\widehat{A},\mathbb{Q}})$ or $\mathrm{GL}(\Gamma_{A,\mathbb{Q}} \oplus \Gamma_{\widehat{A},\mathbb{Q}})$.

2.3. The following facts about $Hdg_{A,\mathbb{Q}}$ are known (see [20], [4]).

2.3.1 Theorem. *Assume that* A *is an abelian variety.*

a) $Hdg_{A,\mathbb{Q}}$ *is a connected reductive algebraic* $\mathbb{Q}$ *-group.*
b) $h_A(-1)$ *is in the center of* $Hdg_{A,\mathbb{Q}}$.
c) The involution $Ad(h_A(i))$ *on* $Hdg_{A,\mathbb{Q}}(\mathbb{R})^0$ *is a Cartan involution, i.e. its fixed subgroup is a maximal compact subgroup* K *of* $Hdg_{A,\mathbb{Q}}(\mathbb{R})^0$.
d) The symmetric space $Hdg_{A,\mathbb{Q}}(\mathbb{R})^0/K$ *is of hermitian type.*
e) The reductive group $Hdg_{A,\mathbb{Q}}$ *has no simple factors of exceptional type.*

3. The Clifford algebra and the spinor representation.

3.1. Fix a lattice $\Gamma \cong \mathbb{Z}^{2n}$ for some $n \geq 1$. (In the future applications Γ will be Γ_A for a complex torus A). Put $\Lambda = \Gamma \oplus \Gamma^*$ with the canonical symmetric bilinear form $Q : \Lambda \times \Lambda \longrightarrow \mathbb{Z}$ defined by

$$Q((a_1,b_1),(a_2,b_2)) = b_1(a_2) + b_2(a_1).$$

This form is even and unimodular. Moreover, $Q \simeq U^{2n}$, where $U = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Consider the Clifford algebra $Cl(\Lambda, Q)$ defined as follows

$$Cl(\Lambda, Q) := T(\Lambda)/ < x \otimes x = \frac{1}{2}Q(x,x) >,$$

where $T(\Lambda)$ is the tensor algebra of the lattice Λ .

3.2. Choose a basis $l_1, ..., l_{2n}$ of Γ and the dual basis $x_1, ..., x_{2n}$ of Γ^* . Let $I \subset Cl(\Lambda, Q)$ be the left ideal generated by the product $\mathbf{l} := l_1 \cdots l_{2n}$. The left multiplication in $Cl(\Lambda, Q)$ defines a homomorphism of algebras

$$\sigma : Cl(\Lambda, Q) \longrightarrow \mathrm{End}_{\mathbb{Z}}(I).$$

3.2.1 Proposition.

a) $Cl(\Lambda, Q)$ *is a free abelian group of rank* 2^{4n} *with the basis consisting of monomials*

$$x_{j_1} \cdots x_{j_l} l_{i_1} \cdots l_{i_k}, \quad 1 \le k, s \le 2n, \quad i_1 < \cdots < i_k, j_1 < \cdots < j_s.$$

In particular, Λ *is naturally a subgroup of* $Cl(\Lambda, Q)$ *.*

b) $Cl(\Lambda, Q)$ *is a* $\mathbb{Z}_2$ *-graded algebra,* $Cl(\Lambda, Q) = Cl^+(\Lambda, Q) \oplus Cl^-(\Lambda, Q)$, *where* $Cl^+(\Lambda, Q)$ *(resp.* $Cl^-(\Lambda, Q)$ *) is the span of monomials with even (resp. odd) number of elements.* $Cl^+(\Lambda, Q)$ *is a subalgebra of* $Cl(\Lambda, Q)$ *.*

c) The ideal I *is independent of the choice of the basis* $l_1, ..., l_{2n}$ *of* $H_1(A, \mathbb{Z})$.

d) The ideal I *has a* $\mathbb{Z}$ *-basis consisting of the monomials*

$$x_{j_1} \cdots x_{j_s} l_1 \cdots l_{2n}, \quad 1 \le s \le 2n, \quad j_1 < \cdots < j_s.$$

Hence $I \cong \Lambda^{\cdot}\Gamma^*$ *as abelian groups.*

e) The homomorphism $\sigma : Cl(\Lambda, Q) \longrightarrow \mathrm{End}_{\mathbb{Z}}(I)$ *is an isomorphism, hence* $Cl(\Lambda, Q) \cong \mathrm{M}_{2^{2n}}(\mathbb{Z})$.

Proof. Assertions a), b) are standard and easy
c) is obvious since l_i , l_j anticommute for $i \neq j$.
d) follows from a).
Let us prove e). Since $Cl(\Lambda, \mathbb{Q})$ and $\mathrm{End}_Z(I)$ are both free Z -modules of rank 2^{4n} it suffices to prove that σ is a surjection. For a subset $p = \{i_1, ..., i_k\} \subset [1, 2n]$, $i_1 \le ... \le i_k$, denote by $\mathbf{l}_p$ (resp. $\mathbf{x}_p$) the ordered monomial $l_{i_1} \cdots l_{i_k}$ (resp. $x_{i_1} \cdots x_{i_k}$). Let $p' = [1, 2n] - p$ be the complementary set. One immediately checks that for subsets $p, h, m \subset [1, 2n]$

$$\mathbf{x}_h \mathbf{l} \mathbf{x}_{p'}(\mathbf{x}_m \mathbf{l}) = \left\{ \begin{array}{cc} \pm \mathbf{x}_h \mathbf{l} & \text{if} \quad p = m \\ 0 & \text{otherwise} \end{array} \right\},$$

where $\mathbf{l} = l_1 \cdots l_{2n}$. This implies the surjectivity of σ in view of d) above (see also [1]). □

3.3 Corollary. *Let* $\Lambda = M_1 \oplus M_2$ *be a decomposition such that* M_1, M_2 *are (maximal)* Q *-isotropic sublattices. Let* $m_1, ..., m_{2n}$ *be a basis of* M_1 , $\mathbf{m} = m_1 \cdots m_{2n}$ *and* $I' = Cl(\Lambda, Q)\mathbf{m}$ *be the corresponding left ideal. There exists a unique (up to* ± 1 *) isomorphism of left* $Cl(\Lambda, Q)$ *- modules* I *and* I' *.*

Proof. The form Q identifies M_2 with the dual M_1^* of M_1 . Moreover, the form Q' on $\Lambda = M_1 \oplus M_1^*$ defined by

$$Q'((m_1, n_1), (m_2, n_2)) = n_1(m_2) + n_2(m_1)$$

coincides with Q , hence gives rise to the same Clifford algebra. Therefore we may apply the previous proposition (parts d),e)) to the ideal I' instead of I . Since $\mathrm{End}_Z(I) = Cl(\Lambda, Q) = \mathrm{End}_Z(I')$ there exists a unique (up to a sign) isomorphism $I \cong I'$ of left $Cl(\Lambda, Q)$ -modules. This proves the corollary.

3.4. Let us recall the spinorial group of the Clifford algebra.

There exists a unique involution $'$ on $Cl(\Lambda, Q)$ that is the identity on Λ . In terms of the monomial basis as in Proposition 3.2.1a) it is defined by rule

$$(x_{j_1} \cdots x_{j_s} l_{i_1} \cdots l_{i_k})' = l_{i_k} \cdots l_{i_l} x_{j_s} \cdots x_{j_1}$$

3.4.1 Definition. The spinorial group *$Spin(\Lambda, Q)$ is the multiplicative subgroup of such elements z in $Cl^+(\Lambda, Q)$ that*

1. $z\Lambda z^{-1} = \Lambda$,
2. $N(z) := zz' = 1$.

3.4.2 Proposition. *For $z \in Spin(\Lambda,Q)$ denote by r_z the conjugation by z restricted to Λ . Then $r_z \in SO(\Lambda,Q)$. The kernel of the map r = { 1,-1 }. In other words, there is an exact sequence*

$$0 \to Z/2\mathbb{Z} \to Spin(\Lambda,Q) \to SO(\Lambda,Q).$$

Proof. See [PR].

3.5. Let us extend the scalars from $\mathbb{Z}$ to $\mathbb{Q}$. That is we consider $\Gamma_{\mathbb{Q}}$, $\Lambda_{\mathbb{Q}}$, $Q_{\mathbb{Q}}$, $\mathrm{SO}(\Lambda_{\mathbb{Q}},Q_{\mathbb{Q}})$, etc. The conditions 1, 2 in the Definition 3.4.1 above define a $\mathbb{Q}$ -algebraic group $Spin_{\mathbb{Q}}(\Lambda_{\mathbb{Q}},Q_{\mathbb{Q}})$. The map r as in Proposition 3.4.2 induces an exact sequence of $\mathbb{Q}$ -algebraic groups ([PR])

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow Spin_{\mathbb{Q}}(\Lambda_Q,Q_{\mathbb{Q}}) \longrightarrow \mathrm{SO}(\Lambda_{\mathbb{Q}},Q_{\mathbb{Q}}) \longrightarrow 1.$$

3.5.1. Since $Spin_{\mathbb{Q}}(\Lambda_{\mathbb{Q}},Q_{\mathbb{Q}})$ is a subgroup of the multiplicative group $Cl^{+}(\Lambda_{\mathbb{Q}},Q_{\mathbb{Q}})^*$, it acts on the space $I_{\mathbb{Q}}$ (3.2). This is called the *spinorial* representation of the group $Spin_{\mathbb{Q}}(\Lambda_{\mathbb{Q}},Q_{\mathbb{Q}})$. It is a direct sum of two (nonequivalent) irreducible semispinorial representations $I_{\mathbb{Q}} = I_{\mathbb{Q}}^{ev} \oplus I_{\mathbb{Q}}^{odd}$, where $I_{\mathbb{Q}}^{ev}$ (resp. $I_{\mathbb{Q}}^{odd}$) is the direct sum of monomials $x_{j_1}\cdots x_{j_s} l_1 \cdots l_{2n}$ with s even (resp. odd) (3.2.1 d)). We have a similar $Spin_{\mathbb{Q}}(\Lambda_{\mathbb{Q}},Q_{\mathbb{Q}})$ – decomposition $I'_{\mathbb{Q}} = I_{\mathbb{Q}}^{'ev} \oplus I_{\mathbb{Q}}^{'odd}$ (3.3). The isomorphism of $Cl(\Lambda,Q)$ -modules $\beta : I \tilde{\to} I'$ induces an isomorphism of $Spin_{\mathbb{Q}}(\Lambda_{\mathbb{Q}},Q_{\mathbb{Q}})$ –modules $\beta_{\mathbb{Q}} : I_{\mathbb{Q}} \tilde{\to} I'_{\mathbb{Q}}$. Thus

$$\beta_{\mathbb{Q}} : I_{\mathbb{Q}}^{ev} \tilde{\to} I_{\mathbb{Q}}^{'ev}, \quad I_{\mathbb{Q}}^{ev} \tilde{\to} I_{\mathbb{Q}}^{'odd}$$

or

$$\beta_{\mathbb{Q}} : I_{\mathbb{Q}}^{ev} \tilde{\to} I_{\mathbb{Q}}^{'odd}, \quad I_{\mathbb{Q}}^{ev} \tilde{\to} I_{\mathbb{Q}}^{'ev}.$$

3.5.2 Corollary. *The isomorphism of $Cl(\Lambda,Q)$ - modules $\beta : I \xrightarrow{\sim} I'$ (3.3) satisfies*

$$\beta(I^{ev}) = I^{'ev}, \quad \beta(I^{odd}) = I^{'odd}$$

or

$$\beta(I^{ev}) = I^{'odd}, \quad \beta(I^{odd}) = I^{'ev}$$

3.5.3 Definition. *We call β even (resp. odd) if $\beta(I^{ev}) = I^{'ev}$ (resp. $\beta(I^{ev} = I^{'odd})$).*

3.5.4 Proposition. *The isomorphism β is even (resp. odd) if $dim_{\mathbb{Q}}(\Gamma_{\mathbb{Q}} \cap M_{1\mathbb{Q}})$ is even (resp. odd).*

Proof. Let $d = dim_{\mathbb{Q}}(\Gamma_{\mathbb{Q}} \cap M_{1\mathbb{Q}})$. Consider the obvious $\mathbb{Q}$ -versions of 3.2.1 and 3.3. We may assume that $l_1 = m_1, ..., l_d = m_d$. Put $\mathbf{m}' := m_{d+1}\cdots m_{2n}$. Then the right multiplication $R_{\mathbf{m}'}$ by $\mathbf{m}'$ induces an isomorphism $R_{\mathbf{m}'} : I_{\mathbb{Q}} \xrightarrow{\sim} I'_{\mathbb{Q}}$ of left $Cl(\Lambda_{\mathbb{Q}},Q_{\mathbb{Q}})$ –modules. Hence $R_{\mathbf{m}'}$ is a scalar multiple of $\beta_{\mathbb{Q}}$. But $R_{\mathbf{m}'}(\mathbf{l}) = l_{d+1}\cdots l_{2n}\mathbf{m}'$. Hence the parity of β is equal to the parity of d . This proves the proposition. □

3.6 Remark. In the notations of 3.2 the choice of a basis $x_1, ...x_{2n}$ of Γ induces an isomorphism of abelian groups

$$I \cong \Lambda^{\cdot}\Gamma^*$$

(see Prop. 3.2.1 d)). A different choice of a basis of Γ may change this isomorphism by -1 . Thus we get a canonical left $Cl(\Lambda,Q)$ -module structure on $\Lambda^{\cdot}\Gamma^*$. Similarly in the notation of Corollary 3.3 the group $\Lambda^{\cdot}M_2$ is canonically a left $Cl(\Lambda,Q)$ -module. Moreover, this corollary asserts that there exists a unique (up to ± 1) isomorphism of $Cl(\Lambda,Q)$ -modules

$$\Lambda^{\cdot}\Gamma^* \cong \Lambda^{\cdot}M_2.$$

3.6.1. More generally, we can introduce the following equivalence relation $\sim$ on the set of maximal $Q_{\mathbb{Q}}$ -isotropic subspaces of $\Lambda_{\mathbb{Q}}$. Namely, let $L \subset \Lambda_{\mathbb{Q}}$ be a maximal isotropic subspace. Then the $\mathbb{Q}$ -versions of 3.2.1 and 3.6 provide a canonical $Cl(\Lambda_{\mathbb{Q}},Q_{\mathbb{Q}})$ -module structure on the exterior algebra $\Lambda^{\cdot}L^*$ of the dual space L^* . Given two maximal isotropic subspaces L_1 , L_2 we get a unique (up to a scalar) isomorphism $\beta_{\mathbb{Q}} : \Lambda^{\cdot}L_1^* \xrightarrow{\sim} \Lambda^{\cdot}L_2^*$ of $Cl(\Lambda_{\mathbb{Q}},Q_{\mathbb{Q}})$ -modules (by the $\mathbb{Q}$ -version of 3.3). We say that $L_1 \sim L_2$ if $dim(L_1 \cap L_2)$ is even. Then by 3.5.4 $L_1 \sim L_2$ iff $\beta_{\mathbb{Q}}(\Lambda^{ev}L_1^*) = \Lambda^{ev}L_2^*$. Thus $\sim$ is indeed an equivalence relation, which divides the set of maximal isotropic subspaces into two equivalence classes.

3.7. We will be interested in the following situation. Let A be a complex torus. Put $\Gamma = \Gamma_A$, $\Gamma^* = \Gamma_{\widehat{A}}$, $\Lambda_A = \Gamma_A \oplus \Gamma_{\widehat{A}}$ with the symmetric bilinear form Q_A on Λ_A as in 3.1. Then by Remark 3.4 above $\Lambda^{\cdot}\Gamma_{\widehat{A}}$ is naturally a $Cl(\Lambda_A, Q_A)$ -module. Note the canonical isomorphisms $\Gamma_{\widehat{A}} = H_1(\widehat{A}, \mathbb{Z}) = H^1(A, \mathbb{Z})$. Thus the total cohomology of A $\Lambda^{\cdot}H^1(A, \mathbb{Z}) = H^*(A, \mathbb{Z})$ is naturally a $Cl(\Lambda_A, Q_A)$ -module.

Assume that B is another complex torus and there exists an isomorphism

$$\alpha : \Lambda_A \xrightarrow{\sim} \Lambda_B$$

which identifies the forms Q_A and Q_B. Let us identify

$$\Lambda_A = \Lambda = \Lambda_B, \quad Q_A = Q = Q_B$$

$$Cl(\Lambda_A, Q_A) = Cl(\Lambda, Q) = Cl(\Lambda_B, Q_B)$$

by means of α. Then by Remark 3.6 there exists a unique (up to ± 1) isomorphism of $Cl(\Lambda, Q)$ –modules

$$\beta : H^*(A, \mathbb{Z}) \xrightarrow{\sim} H^*(B, \mathbb{Z}).$$

By Corollary 3.5.2 this isomorphism either preserves even and odd cohomology groups or interchanges them. By Proposition 3.5.4 the parity of β (3.5.3) is equal to the parity of the dimension $\dim_{\mathbb{Q}}(\Gamma_{A,\mathbb{Q}} \cap \Gamma_{B,\mathbb{Q}})$.

4. Derived categories and their groups of autoequivalences.

4.1 Categories of coherent sheaves and functors between them.

4.1.1. Let X be an algebraic variety over an arbitrary field k. By $\mathcal{O}_X$ denote the structure sheaf on X. Let $coh(X)$ (resp. $Qcoh(X)$) be the category of (quasi)–coherent sheaves on X. Recall that a quasicoherent (coherent) sheaf is a sheaf of $\mathcal{O}_X$ -modules, which locally on X has a (finite) presentation by free $\mathcal{O}_X$ -modules.

It is more convenient to work with derived categories instead of abelian categories. Let us denote by $D^b(X)$ the bounded derived category of $coh(X)$.

Any derived category $\mathcal{D}$ has a structure of a triangulated category. This means that there are fixed

a) a translation functor $[1] : \mathcal{D} \longrightarrow \mathcal{D}$ that is an additive autoequivalence,

b) a class of distinguished (or exact) triangles:

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

that satisfies a certain set of axioms (see [29]).

An additive functor $F : \mathcal{D} \longrightarrow \mathcal{D}'$ between two triangulated categories is called exact if it commutes with the translation functors, and it takes every distinguished triangle in $\mathcal{D}$ to a distinguished triangle in $\mathcal{D}'$.

A natural example of an exact functor is related to a map of algebraic varieties. Every such map $f : X \to Y$ induces the functor of direct image $f_* : Qcoh(X) \to Qcoh(Y)$. Since a category $Qcoh(X)$ has enough injectives there exists the right derived functor $\mathbf{R}f_* : D^b(Qcoh(X)) \longrightarrow D^b(Qcoh(Y))$. This gives us an example of exact functor between derived categories. If the map f is proper then for any i the functor $\mathbf{R}^i f_*$ takes a coherent sheaf to a coherent one. It is known that $D^b(coh(X))$ is equivalent to the full subcategory $D^b(Qcoh(X))_{coh} \subset D^b(Qcoh(X))$ objects of which have coherent cohomologies. Hence any proper map f induces the exact functor $\mathbf{R}f_* : D^b(coh(X)) \longrightarrow D^b(coh(Y))$. Further, the functor f_* has a left adjoint functor $f^* : coh(Y) \longrightarrow coh(X)$. If the map f has finite Tor–dimension, then there exists a left derived functor $\mathbf{L}f^* : D^b(coh(Y)) \longrightarrow D^b(coh(X))$, which is left adjoint for $\mathbf{R}f_*$. This is another example of an exact functor. (see [9], [2] Ex.I,II,III).

4.1.2. From now on we will assume that all varieties are smooth and projective and will denote $D^b(coh(X))$ by $D^b(X)$. In this case every map $f : X \to Y$ induces the functors $\mathbf{R}f_* : D^b(X) \longrightarrow D^b(Y)$ and $\mathbf{L}f^* : D^b(Y) \longrightarrow D^b(X)$. Also each complex $\mathcal{F} \in D^b(X)$ induces an exact functor $\overset{\mathbf{L}}{\otimes} \mathcal{F} : D^b(X) \longrightarrow D^b(X)$. Using these functors one can introduce a larger class of exact functors.

Let X and Y be smooth projective varieties of dimension n and m respectively. Consider the projections

$$X \xleftarrow{p} X\times Y \xrightarrow{q} Y$$

Every object $\mathcal{E}\in D^b(X\times Y)$ defines an exact functor $\Phi_{\mathcal{E}}: D^b(X)\longrightarrow D^b(Y)$ by the following formula:

$$\Phi_{\mathcal{E}}(\cdot) := \mathbf{R}q_*(\mathcal{E}\overset{\mathbf{L}}{\otimes} p^*(\cdot)). \tag{1}$$

Obviously, the same object defines another functor $\Psi_{\mathcal{E}}: D^b(Y)\longrightarrow D^b(X)$ by the similar formula

$$\Psi_{\mathcal{E}}(\cdot) := \mathbf{R}p_*(\mathcal{E}\overset{\mathbf{L}}{\otimes} q^*(\cdot)).$$

The functor $\Phi_{\mathcal{E}}$ has left and right adjoint functors $\Phi^*_{\mathcal{E}}$ and $\Phi^!_{\mathcal{E}}$ respectively, defined by the rule:

$$\begin{array}{ccc} \Phi^*_{\mathcal{E}} & \cong & \Psi_{\mathcal{E}^\vee\otimes q^*K_Y[m]} \\ \Phi^!_{\mathcal{E}} & \cong & \Psi_{\mathcal{E}^\vee\otimes p^*K_X[n]} \end{array} \tag{2}$$

Here K_X and K_Y are the canonical sheaves on X and Y respectively, and $\mathcal{E}^\vee$ is notation for $\mathbf{R}\underline{\mathcal{H}om}(\mathcal{E},\mathcal{O}_{X\times Y})$.

4.1.2.1 Example. Let $f: X\to Y$ be a morphism of varieties. Denote by $\Gamma(f)$ a subvariety of $X\times Y$ that is the graph of f . It is clear that the functor $\mathbf{R}f_*: D^b(X)\longrightarrow D^b(Y)$ is isomorphic to $\Phi_{\mathcal{O}_{\Gamma(f)}}$, where $\mathcal{O}_{\Gamma(f)}$ is the structure sheaf of the graph $\Gamma(f)$. In the same way the functor $\mathbf{L}f^*: D^b(Y)\longrightarrow D^b(X)$ is isomorphic to $\Psi_{\mathcal{O}_{\Gamma(f)}}$.

4.1.2.2 Example. Let $\delta:\Delta\hookrightarrow X\times X$ be the embedding of the diagonal. First, since the Δ is the graph of the identity morphism the identity functor from the derived category $D^b(X)$ to itself is represented by the structure sheaf $\mathcal{O}_\Delta$. Moreover, for any object $\mathcal{F}\in D^b(X)$ the functor $\overset{\mathbf{L}}{\otimes}\mathcal{F}: D^b(X)\longrightarrow D^b(X)$ is isomorphic to $\Phi_{\delta_*\mathcal{F}}$

4.1.2.3. Thus there is a reasonably large class of exact functors between derived categories of smooth projective varieties that consists of functors having the form $\Phi_{\mathcal{E}}$ for some complex $\mathcal{E}$ on the product. This class is closed under composition of functors. Actually, let X,Y,Z be three (smooth projective) varieties and let

$$\Phi_I: D^b(X)\longrightarrow D^b(Y),\qquad \Phi_J: D^b(Y)\longrightarrow D^b(Z)$$

be two functors, where I and J are objects of $D^b(X\times Y)$ and $D^b(Y\times Z)$ respectively. Denote by p_{XY}, p_{YZ}, p_{XZ} the projections of $X\times Y\times Z$ on the corresponding pair of factors.

4.1.2.4 Lemma.[17] *In above notations, the composition $\Phi_J\circ\Phi_I$ is isomorphic to Φ_K , where $K\in D^b(X\times Z)$ is given by the formula:*

$$K\cong \mathbf{R}p_{XZ*}(p^*_{YZ}(J)\otimes p^*_{XY}(I))$$

Presumably, the class of exact functors described above embraces all exact functors between bounded derived categories of coherent sheaves on smooth projective varieties. We do not know if it is true or not. However it is definitely true for exact equivalences.

4.1.2.5 Theorem.([23]) *Let X and Y be smooth projective varieties. Suppose $F: D^b(X)\xrightarrow{\sim} D^b(Y)$ is an exact equivalence. Then there exists a unique (up to an isomorphism) object $\mathcal{E}\in D^b(X\times Y)$ such that the functors F and $\Phi_{\mathcal{E}}$ are isomorphic.*

4.1.2.6 Definition. *The group of isomorphism classes of exact equivalences $F: D^b(X)\xrightarrow{\sim} D^b(X)$ is called the* group of autoequivalences *of $D^b(X)$ and is denoted by $Auteq(D^b(X))$.*

4.1.3. Assume for simplicity that $k=\mathbb{C}$. Analogously to derived categories and functors between them one can consider cohomologies and maps between them. The latter is much simpler.

For every element $\xi\in H^*(X\times Y,\mathbb{Q})$ let us define linear maps

$$v_\xi: H^*(X,\mathbb{Q})\longrightarrow H^*(Y,\mathbb{Q}),\qquad w_\xi: H^*(Y,\mathbb{Q})\longrightarrow H^*(X,\mathbb{Q})$$

by the formula :

$$v_\xi(\cdot) = q_*(\xi\cup p^*(\cdot)),\qquad w_\xi(\cdot) = p_*(\xi\cup q^*(\cdot)). \tag{3}$$

It follows from the Künneth formula that any linear map between cohomologies have this form for some ξ .

The composition formula for cohomological correspondence is well-known and is similar to the composition of functors. Let as above X, Y, Z be three varieties and let ξ and η be elements of $H^*(X\times Y)$ and $H^*(Y \times Z)$ respectively.

4.1.3.1 Lemma. *The composition* $v_\eta \circ v_\xi$ *is isomorphic to* v_ζ , *where* $\zeta \in H^*(X \times Z)$ *is given by the formula:*

$$\zeta = p_{XZ*}(p_{YZ}^*(\eta) \cup p_{XY}^*(\xi)).$$

4.1.4. There is a natural correspondence that attaches to a functor $\Phi_{\mathcal{E}} : D^b(X) \longrightarrow D^b(Y)$ a linear map of vector spaces $\phi_{\mathcal{E}} : H^*(X,\mathbb{Q}) \longrightarrow H^*(Y,\mathbb{Q})$. To describe it note that each exact functor $F : D^b(X) \longrightarrow D^b(Y)$ induces a homomorphism $\overline{F} : K(X) \longrightarrow K(Y)$, where $K(X)$ and $K(Y)$ are the Grothendieck groups of the categories $D^b(X)$ and $D^b(Y)$. On the other side, there is a map

$$ch : K(X) \longrightarrow H^*(X,\mathbb{Q})$$

that is called the Chern character. The map ch is a ring homomorphism, if to recall that $K(X)$ and $H^*(X,\mathbb{Q})$ are endowed with multiplications induced by $\otimes$ –product and $\cup$ –product respectively.

4.1.4.1 Definition. *For every object* $\mathcal{E} \in D^b(X \times Y)$ *put*

$$\begin{array}{l}\phi_{\mathcal{E}}(\cdot) := v_{ch(\mathcal{E})\cup p^*(td_X)}(\cdot) = q_*(ch(\mathcal{E}) \cup p^*(td_X) \cup p^*(\cdot)),\\ \psi_{\mathcal{E}}(\cdot) := w_{ch(\mathcal{E})\cup q^*(td_Y)}(\cdot) = p_*(ch(\mathcal{E}) \cup q^*(td_Y) \cup q^*(\cdot)),\end{array} \tag{4}$$

where td_X *and* td_Y *are the Todd classes of* X *and* Y .

4.1.4.2 Lemma. *For given* $\mathcal{E} \in D^b(X \times Y)$ *the following diagram*

$$\begin{array}{ccc} K(X) & \xrightarrow{\overline{\Phi}_{\mathcal{E}}} & K(Y) \\ {\scriptstyle ch}\downarrow & & \downarrow{\scriptstyle ch} \\ H^*(X,\mathbb{Q}) & \xrightarrow{\phi_{\mathcal{E}}} & H^*(Y,\mathbb{Q}) \end{array}$$

is commutative.

4.1.4.3 Lemma. *Assume that a functor* $\Phi_K : D^b(X) \longrightarrow D^b(Z)$ *is isomorphic to* $\Phi_J \circ \Phi_I$ *for some*

$$\Phi_I : D^b(X) \longrightarrow D^b(Y), \qquad \Phi_J : D^b(Y) \longrightarrow D^b(Z).$$

Then $\phi_K = \phi_J \circ \phi_I$. *Consequently, the correspondence* $\Phi_{\mathcal{E}} \mapsto \phi_{\mathcal{E}}$ *induces a representation*

$$\rho_X : Auteq(D^b(X)) \longrightarrow \mathrm{GL}(H^*(X,\mathbb{Q})).$$

Both lemmas immediately follows from the Riemann-Roch-Grothendieck theorem.

4.1.4.4 Example. Let $f : X \to Y$ be a morphism. The functors $\mathbf{L}f^* \cong \Psi_{\mathcal{O}_{\Gamma(f)}}$ and $\mathbf{R}f_* \cong \Phi_{\mathcal{O}_{\Gamma(f)}}$ give the maps $\psi_{\mathcal{O}_{\Gamma(f)}}$ and $\phi_{\mathcal{O}_{\Gamma(f)}}$ between cohomologies. Also the morphism f induces the map: $f^* : H^i(Y,\mathbb{Q}) \longrightarrow H^i(X,\mathbb{Q})$. It can be checked that

$$\psi_{\mathcal{O}_{\Gamma(f)}} = f^*.$$

On the other side, there is a map of homologies: $f_* : H_i(X,\mathbb{Q}) \longrightarrow H_i(Y,\mathbb{Q})$. Using the Poincare isomorphism $H^{2n-i}(X,\mathbb{Q}) \cong H_i(X,\mathbb{Q})$, the map f_* can be considered as a map between cohomologies: $f_* : H^{2n-i}(X,\mathbb{Q}) \longrightarrow H^{2m-i}(Y,\mathbb{Q})$. In this case $\phi_{\mathcal{O}_{\Gamma(f)}}$ does not coincide with f_* , but they are connected by the following relation:

$$\phi_{\mathcal{O}_{\Gamma(f)}}(x) \cup td_Y = f_*(x \cup td_X)$$

In particular, if $td_X = td_Y = 1$, then these maps coincide.

4.1.4.5 Example. Let as in Example 4.1.2.2 $\delta : \Delta \hookrightarrow X \times X$ be an embedding of the diagonal. For any $\mathcal{F} \in D^b(X)$ a functor $\overset{\mathbf{L}}{\otimes} \mathcal{F}$ is isomorphic to $\Phi_{\delta_*\mathcal{F}}$. Since ch is a ring homomorphism Lemma 4.1.4.2 implies that

$$\phi_{\delta_*\mathcal{F}}(\cdot) = (\cdot) \cup ch(\mathcal{F}).$$

4.1.5. For any variety X the group $Auteq(D^b(X))$ contains a subgroup $G(X) = (\mathbb{Z} \times \mathrm{Pic}(X)) \rtimes Aut(X)$ that is the semi-direct product of the normal subroup $(\mathbb{Z} \times \mathrm{Pic}(X))$ and $Aut(X)$ naturally acting on the former. Under the inclusion $G(X) \subset Auteq(D^b(X))$ the generator of $\mathbb{Z}$ goes to the translation functor $[1]$, a line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ maps to the functor $\otimes\mathcal{L} = \Phi_{\delta_*\mathcal{L}}$ and an automorphism $f : X \to X$ induces the autoequivalence $\mathbf{R}f_* \cong \Phi_{\mathcal{O}_{\Gamma(f)}}$.

4.1.5.1 Lemma. *In the above notation* $\rho_X([1]) = -Id_{H^*(X,\mathbb{Q})}$ *and* $\rho_X(\otimes\mathcal{L}) = \cup ch(\mathcal{L})$.

The proof follows from the Example 4.1.4.5 above and the equality $ch(E[1]) = -ch(E)$.

4.1.6. Now let us show that any equivalence $\Phi_{\mathcal{E}} : D^b(X) \xrightarrow{\sim} D^b(Y)$ defines a functor between derived categories of coherent sheaves on $X \times X$ and $Y \times Y$, which will be called adjoined. Consider two functors

$$\Phi_{\mathcal{E}} : D^b(X) \longrightarrow D^b(Y), \qquad \Psi_{\mathcal{F}} : D^b(Y) \longrightarrow D^b(X),$$

where $\mathcal{E}$ and $\mathcal{F}$ are objects of $D^b(X \times Y)$. By $\mathcal{F} \boxtimes \mathcal{E}$ denote the object $p_{13}^*(\mathcal{F}) \otimes p_{24}^*(\mathcal{E})$ from $D^b(X \times X \times Y \times Y)$, which is called the external tensor product of $\mathcal{E}$ and $\mathcal{F}$. It defines the functor

$$\Phi_{\mathcal{F}\boxtimes\mathcal{E}} : D^b(X \times X) \longrightarrow D^b(Y \times Y).$$

Let us take an object $\mathcal{G} \in D^b(X \times X)$ and denote $\Phi_{\mathcal{F}\boxtimes\mathcal{E}}(\mathcal{G}) \in D^b(Y \times Y)$ by $\mathcal{H}$ for short. These two objects define the functors

$$\Phi_{\mathcal{G}} : D^b(X) \longrightarrow D^b(X), \qquad \Phi_{\mathcal{H}} : D^b(Y) \longrightarrow D^b(Y).$$

4.1.6.1 Lemma. *In above notation, there is an isomorphism of functors* $\Phi_{\mathcal{H}} \cong \Phi_{\mathcal{E}}\Phi_{\mathcal{G}}\Psi_{\mathcal{F}}$.

This follows from Lemma 4.1.2.4. □

4.1.6.2 Lemma. *If* $\Phi_{\mathcal{E}}$ *and* $\Psi_{\mathcal{F}}$ *are equivalences, then the functor* $\Phi_{\mathcal{F}\boxtimes\mathcal{E}}$ *is an equivalence too.*

The proof is straightforward (see for example [24]).

4.1.6.3 Definition. *Assume that* $\Phi_{\mathcal{E}} : D^b(A) \longrightarrow D^b(B)$ *is an equivalence and* $\Psi_{\mathcal{F}} \cong \Phi_{\mathcal{E}}^{-1}$. *In this case the functor* $\Phi_{\mathcal{F}\boxtimes\mathcal{E}}$ *and the map* $\phi_{\mathcal{F}\boxtimes\mathcal{E}}$ *will be denoted by* $Ad_{\mathcal{E}}$ *and by* $ad_{\mathcal{E}}$ *respectively.*

Thus any exact equivalence $\Phi_{\mathcal{E}} : D^b(X) \xrightarrow{\sim} D^b(Y)$ defines an exact equivalence $Ad_{\mathcal{E}} : D^b(X \times X) \xrightarrow{\sim} D^b(Y \times Y)$ such that, by Lemma 4.1.6.1, there is an isomorphism:

$$\Phi_{Ad_{\mathcal{E}}(\mathcal{G})} \cong \Phi_{\mathcal{E}}\Phi_{\mathcal{G}}\Phi_{\mathcal{E}}^{-1}. \tag{5}$$

4.2 Category of coherent sheaves on an abelian variety.

4.2.1. Let A be an abelian variety of dimension n over $\mathbb{C}$. Denote by $m : A \times A \to A$ the multiplication morphism. For any point $a \in A$ by T_a denote the translation automorphism $m(\cdot, a) : A \to A$.

Let $\widehat{A}$ be dual abelian variety (1.2). It is canonically isomorphic to $\mathrm{Pic}^0(A)$ (1.3). It is well-known that there is a unique line bundle P on the product $A \times \widehat{A}$ such that for any point $\alpha \in \widehat{A}$ the restriction P_α on $A \times \{\alpha\}$ represents an element of $\mathrm{Pic}^0 A$, corresponding to α and, in addition, the restriction $P\big|_{\{0\}\times\widehat{A}}$ should be trivial. Such P is called Poincare line bundle (1.4). As in 1.4, 1.5 we identify A with $\widehat{\widehat{A}}$ using the Poincare line bundles on $A \times \widehat{A}$ and $\widehat{A} \times \widehat{\widehat{A}}$.

4.2.2. The Poincare line bundle gives an example of an exact equivalence between derived categories of coherent sheaves of two non-isomorphic varieties. Let us consider the projections

$$A \xleftarrow{\ p\ } A \times \widehat{A} \xrightarrow{\ q\ } \widehat{A}$$

and the functor $\Phi_P : D^b(A) \longrightarrow D^b(\widehat{A})$, defined as in 4.1.2, i.e. $\Phi_P(\cdot) = \mathbf{R}q_*(P \otimes p^*(\cdot))$.

The following proposition was proved by Mukai.

4.2.2.1 Proposition.([17]) *Let* P *be an Poincare line bundle on* $A \times \widehat{A}$. *Then the functor* $\Phi_P : D^b(A) \longrightarrow D^b(\widehat{A})$ *is an exact equivalence, and there is an isomorphism of functors:*

$$\Psi_P \circ \Phi_P \cong (-1_A)^*[n],$$

where (-1_A) *is the inverse map on the group* A.

4.2.3. Let $x_1,...,x_{2n}$ be a basis of $H^1(A,\mathbb{Z})$. Let $l_1,...,l_{2n}$ be the dual basis of $H^1(\widehat{A},\mathbb{Z})$. It is clear that $p^*(x_1),...,p^*(x_{2n}),q^*(l_1),...,q^*(l_{2n})$ is a basis of $H^1(A\times\widehat{A},\mathbb{Z})$.

4.2.3.1 Lemma.([19]) *The first Chern class of the Poincare line bundle P on $A\times\widehat{A}$ satisfies the equality:*

$$c_1(P)=\sum_{i=1}^{2n}p^*(x_i)\cup q^*(l_i). \tag{6}$$

Now consider the map $\phi_P: H^*(A,\mathbb{Q})\longrightarrow H^*(\widehat{A},\mathbb{Q})$, given by formula (4).

4.2.3.2 Lemma. *For any k the map ϕ_P sends k-th cohomology to $(2n-k)$-th cohomology and induces isomorphisms*

$$H^k(A,\mathbb{Z})\xrightarrow{\sim}H^{2n-k}(\widehat{A},\mathbb{Z}),$$

under which a monomial $x_{i_1}\cup\cdots\cup x_{i_k}$ goes to $(-1)^{\varepsilon}l_{j_1}\cup\cdots\cup l_{j_{2n-k}}$, where $(j_1<\cdots<j_{2n-k})$ is the complement of $(i_1<\cdots<i_k)$ in the set $(1,...,2n)$, and $\varepsilon=\sum_t j_t$.

Proof. Take $\alpha\in H^k(A,\mathbb{Z})$. Since $ch(P)=\exp(c_1(P))$ and $c_1(P)$ has the form (6), we obtain

$$\phi_P(\alpha)=q_*(ch(P)\cup p^*(\alpha))=\frac{1}{(2n-k)!}q_*(c_1(P)^{2n-k}\cup p^*(\alpha))$$

This implies that ϕ_P sends k-th cohomology to $(2n-k)$-th. Moreover, it is easy to see that

$$\frac{1}{s!}c_1(P)^s=\sum_{j_1<\cdots<j_s}p^*(x_{j_s}\cup\cdots\cup x_{j_1})\cup q^*(l_{j_1}\cup\cdots\cup l_{j_s})$$

Substituting this expression in previous formula, we get

$$\begin{aligned}\phi_P(x_{i_1}\cup\cdots\cup x_{i_k}) &= q_*p^*(x_{i_1}\cup\cdots\cup x_{i_k}\cup x_{j_{2n-k}}\cup\cdots\cup x_{j_1})\cup l_{j_1}\cup\cdots\cup l_{j_{2n-k}}=\\ &(-1)^{\varepsilon}q_*p^*(x_1\cup\cdots\cup x_{2n})\cup l_{j_1}\cup\cdots\cup l_{j_{2n-k}}=(-1)^{\varepsilon}l_{j_1}\cup\cdots\cup l_{j_{2n-k}}\end{aligned}$$

where $(j_1<\cdots<j_{2n-k})$ is the complement of $(i_1<\cdots<i_k)$, and $\varepsilon=\sum_t j_t$. The monomials $x_{i_1}\cup\cdots\cup x_{i_k}$ and $l_{j_1}\cup\cdots\cup l_{j_{2n-k}}$ form basises of $H^k(A,\mathbb{Z})$ and $H^{2n-k}(\widehat{A},\mathbb{Z})$ respectively. Hence the map ϕ_P gives an isomorphism between these lattices. □

4.2.4. For any point $(a,\alpha)\in A\times\widehat{A}$ one can introduces a functor from $D^b(A)$ to itself, defined by the rule:

$$\Phi_{(a,\alpha)}(\cdot):=T_a^*(\cdot)\otimes P_\alpha. \tag{7}$$

A functor $\Phi_{(a,\alpha)}$ is represented by a sheaf $S_{(a,\alpha)}=\mathcal{O}_{\Gamma_a}\otimes p_2^*(P_\alpha)$ on $A\times A$, where Γ_a is the graph of the automorphism $T_a: A\longrightarrow A$. It is clear that the functor $\Phi_{(a,\alpha)}$ is an equivalence. Since an automorphism T_a acts identically on the cohomology group and $ch(P_\alpha)=1$ the functors $\Phi_{(a,\alpha)}$ belongs to kernel of the homomorphism ρ (4.1, 4.3).

The set of the functors $\Phi_{(a,\alpha)}$, parametrized by $A\times\widehat{A}$, can be unified in one functor from $D^b(A\times\widehat{A})$ to $D^b(A\times A)$ that takes a skyscraper $\mathcal{O}_{(a,\alpha)}$ to $S_{(a,\alpha)}$. (Note that this condition does not define the functor by uniquely, because it can be changed by tenzoring with any line bundle pulling back from $A\times\widehat{A}$.) Let us define such a functor $\Phi_{S_A}: D^b(A\times\widehat{A})\longrightarrow D^b(A\times A)$ by the representing object

$$S_A=(m\cdot p_{13},p_4)^*\mathcal{O}_\Delta\otimes p_{23}^*P_A$$

on the product $(A\times\widehat{A})\times(A\times A)$. Here $(m\cdot p_{13},p_4)$ is a morphism onto $A\times A$ that takes (a_1,α,a_3,a_4) to $(m(a_1,a_3),a_4)$. A direct check shows that Φ_{S_A} takes the skyscraper sheaf $\mathcal{O}_{(a,\alpha)}$ to $S_{(a,\alpha)}$.

The functor Φ_{S_A} can be consider as a composition of two functors. Denote by $\mathcal{R}_A=p_{14}^*\mathcal{O}_\Delta\otimes p_{23}^*P\in D^b((A\times\widehat{A})\times(A\times A))$. By $\mu_A: A\times A\longrightarrow A\times A$ denote a morphism that takes (a_1,a_2) to $(a_1,m(a_1,a_2))$. Consider two functors

$$\Phi_{\mathcal{R}_A}: D^b(A\times\widehat{A})\longrightarrow D^b(A\times A),\qquad \mathbf{R}\mu_{A*}: D^b(A\times A)\longrightarrow D^b(A\times A).$$

4.2.4.1 Lemma. *The functor Φ_{S_A} is isomorphic to the composition $\mathbf{R}\mu_{A*}\circ\Phi_{\mathcal{R}_A}$.*

The proof is omitted. Also this statement can be used as another definition of the functor Φ_{S_A}.

4.2.4.2 Lemma. *The functor* Φ_{S_A} *is an equivalence.*

PROOF. By Lemma 4.1.6.2 the functor $\Phi_{\mathcal{R}_A}$ is an equivalence. Since μ_A is an automorphism of $A \times A$ the functor $\mathbf{R}\mu_{A*}$ is an equivalence too. This implies that Φ_{S_A} is also an equivalence.

4.2.5. Let A and B be two abelian varieties. Fix an equivalence $\Phi_{\mathcal{E}} : D^b(A) \xrightarrow{\sim} D^b(B)$. Consider the functor $\Phi_{S_B}^{-1} Ad_{\mathcal{E}} \Phi_{S_A}$ from $D^b(A \times \widehat{A})$ to $D^b(B \times \widehat{B})$ (4.1, 6.3). By $\mathcal{J}(\mathcal{E})$ denote an object that represents this functor. Thus there is the commutative diagram:

$$\begin{array}{ccc} D^b(A \times \widehat{A}) & \xrightarrow{\Phi_{S_A}} & D^b(A \times A) \\ {\scriptstyle \Phi_{\mathcal{J}(\mathcal{E})}}\downarrow & & \downarrow{\scriptstyle Ad_{\mathcal{E}}} \\ D^b(B \times \widehat{B}) & \xrightarrow{\Phi_{S_B}} & D^b(B \times B) \end{array} \tag{8}$$

Let us describe the object $\mathcal{J}(\mathcal{E})$.

4.2.5.1 Theorem.([24]) *There exists a group isomorphism* $f_{\mathcal{E}} : A \times \widehat{A} \longrightarrow B \times \widehat{B}$ *and a line bundle* $L_{\mathcal{E}}$ *on* $A \times \widehat{A}$ *such that the object* $\mathcal{J}(\mathcal{E})$ *is isomorphic to* $i_*(L_{\mathcal{E}})$ *, where* i *is the embedding of* $A \times \widehat{A}$ *in* $(A \times \widehat{A}) \times (B \times \widehat{B})$ *as the graph* $\Gamma(f_{\mathcal{E}})$ *of the isomorphism* $f_{\mathcal{E}}$ *.*

In particular, it immediately follows from the theorem that if two abelian varieties A and B have equivalent derived categories of coherent sheaves, then the varieties $A \times \widehat{A}$ and $B \times \widehat{B}$ are isomorphic.

4.2.5.2 Corollary. *An isomorphism* $f_{\mathcal{E}}$ *takes a point* $(a, \alpha) \in A \times \widehat{A}$ *to a point* $(b, \beta) \in B \times \widehat{B}$ *iff the equivalences*

$$\Phi_{(a,\alpha)} : D^b(A) \xrightarrow{\sim} D^b(A), \qquad \Phi_{(b,\beta)} : D^b(B) \xrightarrow{\sim} D^b(B),$$

defined by formula (7), are connected by the relation

$$\Phi_{(b,\beta)} \cong \Phi_{\mathcal{E}} \Phi_{(a,\alpha)} \Phi_{\mathcal{E}}^{-1}.$$

PROOF. By the theorem, $\Phi_{\mathcal{J}(\mathcal{E})}$ takes a skyscraper $\mathcal{O}_{(a,\alpha)}$ to $\mathcal{O}_{(b,\beta)}$, where $(b, \beta) = f_{\mathcal{E}}(a, \alpha)$. By construction of the functor Φ_{S_A} it takes a skyscraper $\mathcal{O}_{(a,\alpha)}$ to $S_{(a,\alpha)}$, where the object $S_{(a,\alpha)}$ represents the functor $\Phi_{(a,\alpha)}$. Thus diagram (8) implies that the morphism $f_{\mathcal{E}}$ takes (a, α) to (b, β) iff $S_{(b,\beta)} \cong Ad_{\mathcal{E}}(S_{(a,\alpha)})$. Now the formula (5) implies that $\Phi_{(b,\beta)} \cong \Phi_{\mathcal{E}} \Phi_{(a,\alpha)} \Phi_{\mathcal{E}}^{-1}$. □

4.2.5.3 Proposition.([24]) *Let* $B \cong A$ *, the correspondence* $\Phi_{\mathcal{E}} \mapsto f_{\mathcal{E}}$ *induces a group homomorphism*

$$\gamma_A : Autoeq(D^b(A)) \longrightarrow Aut(A \times \widehat{A}).$$

PROOF. Let $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ be objects of $D^b(A \times A)$ such that $\Phi_{\mathcal{E}_i}$ are equivalences and $\Phi_{\mathcal{E}_3} \cong \Phi_{\mathcal{E}_2} \circ \Phi_{\mathcal{E}_1}$. This implies that $Ad_{\mathcal{E}_3} \cong Ad_{\mathcal{E}_2} \circ Ad_{\mathcal{E}_1}$. Further we have the sequence of isomorphisms:

$$\Phi_{\mathcal{J}(\mathcal{E}_2)} \circ \Phi_{\mathcal{J}(\mathcal{E}_1)} \cong \Phi_{S_A}^{-1} Ad_{\mathcal{E}_2} \Phi_{S_A} \Phi_{S_A}^{-1} Ad_{\mathcal{E}_1} \Phi_{S_A} \cong \Phi_{S_A}^{-1} Ad_{\mathcal{E}_3} \Phi_{S_A} \cong \Phi_{\mathcal{J}(\mathcal{E}_3)}$$

By theorem 4.2.5.1 the object $\mathcal{J}(\mathcal{E})$ is a line bundle over the graph of some automorphism $f_{\mathcal{E}}$. Thus we obtain that $f_{\mathcal{E}_3} = f_{\mathcal{E}_2} \cdot f_{\mathcal{E}_1}$. □

4.2.5.4 Proposition.([24]) *The kernel of the homomorphism* γ_A *coincides with* $\mathbb{Z} \times A \times \widehat{A}$ *, consisting of all equivalences* $\Phi_{(a,\alpha)}[i] = T_a^*(\cdot) \otimes P_\alpha[i]$ *.*

4.3 The Mukai-Polishchuk group $\mathrm{U}(A)$ and the spinorial group $Spin(A)$.

In his recent preprint S.Mukai [18] was interested in the group of autoequivalences $Auteq(D^b(A))$ of the bounded derived category $D^b(A)$ of coherent sheaves on an abelian variety A and he defined a certain related discrete group (which he called $U(A \times \widehat{A})$). Independently A.Polishchuk considered the same group, which he called $\mathrm{SL}_2(A)$, and proved that $\mathrm{SL}_2(A)$ acts naturally on $D^b(A)$ "up to the shift functor"(see [Po2]). Let us recall the definition of this group for complex tori. We call this group $\mathrm{U}(A)$.

4.3.1 Definition. *Let* A *be a complex torus. Put*

$$\mathrm{U}(A) := \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} \mathrm{End}(A) & \mathrm{Hom}(\widehat{A}, A) \\ \mathrm{Hom}(A, \widehat{A}) & \mathrm{End}(\widehat{A}) \end{pmatrix} \middle| \ g^{-1} = \begin{pmatrix} \widehat{d} & -\widehat{b} \\ -\widehat{c} & \widehat{a} \end{pmatrix} \right\}$$

and call it the Mukai-Polishchuk group.

We may consider $\mathrm{U}(A)$ as a subgroup of $\mathrm{GL}(\Gamma_A \oplus \Gamma_{\widehat{A}})$. Consider the canonical symmetric bilinear form Q on $\Lambda := \Gamma_A \oplus \Gamma_{\widehat{A}}$ as in 3.1 above. Here is another description of the group $\mathrm{U}(A)$.

4.3.2 Proposition. *There are equalities*

$$\mathrm{U}(A) = O(\Lambda, Q) \cap Aut(A \times \widehat{A}) = SO(\Lambda, Q) \cap Aut(A \times \widehat{A})$$

of subgroups of $\mathrm{GL}(\Lambda)$.

Proof. The second equality follows from the fact that elements in $Aut(A \times \widehat{A})$ preserve the complex structure on $V_A \oplus V_{\widehat{A}}$, hence have a positive determinant.

To prove the first equality it suffices to show that the group

$$T := \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(\Lambda) \middle| \ g^{-1} = \begin{pmatrix} \widehat{d} & -\widehat{b} \\ -\widehat{c} & \widehat{a} \end{pmatrix} \right\}$$

coincides with the orthogonal group $O(\Lambda, Q)$. Choose a basis $l_1, ..., l_{2n}$ of Γ_A and the dual basis $x_1, ..., x_{2n}$ of $\Gamma_{\widehat{A}}$. Consider the basis $l_1, ..., l_{2n}, x_1, ..., x_{2n}$ of the lattice Λ . Then the group $O(\Lambda, Q)$ consists of matrices

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

such that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha^t & \gamma^t \\ \beta^t & \delta^t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The equality $T = O(\Lambda, Q)$ now follows from Remark 1.6. □

4.3.3 Proposition. ([Or2]) *The image of the homomorphism* $\gamma_A : Autoeq(D^b(A)) \longrightarrow Aut(A \times \widehat{A})$ *coincides with the Mukai-Polishchuk group* $\mathrm{U}(A)$. *Moreover there is an exact sequence of groups*

$$0 \longrightarrow \mathbb{Z} \times A \times \widehat{A} \longrightarrow Auteq(D^b(A)) \longrightarrow \mathrm{U}(A) \longrightarrow 1.$$

This was essentially conjectured by Polishchuk in [Po2].

By Lemma 4.1.4.3 the correspondence $\Phi_{\mathcal{E}} \mapsto \phi_{\mathcal{E}}$ induces a homomorphism

$$\rho_A : Auteq(D^b(A)) \longrightarrow \mathrm{GL}(H^*(A, \mathbb{Q})).$$

As a matter of fact, this representation preserves the integral cohomology lattice.

4.3.4 Proposition. *For any equivalence* $\Phi_{\mathcal{E}} : D^b(A) \xrightarrow{\sim} D^b(B)$ *the linear map* $\phi_{\mathcal{E}}$ *preserves the integral cohomology, i.e.*

$$\phi_{\mathcal{E}}(H^*(A, \mathbb{Z})) = H^*(B, \mathbb{Z}).$$

Proof. By definition of $\phi_{\mathcal{E}}$, one needs to show that $ch(\mathcal{E})$ belongs to integral cohomology $H^*(A \times B, \mathbb{Z})$. This is equivalent to checking that $ch(\mathcal{E}^\vee \boxtimes \mathcal{E})$ belongs to the integral cohomology $H^*((A \times A) \times (B \times B), \mathbb{Z})$. This object represents the functor $Ad_{\mathcal{E}}$. Since $Ad_{\mathcal{E}} \cong \Phi_{S_B} \Phi_{\mathcal{J}(\mathcal{E})} \Phi_{S_A}^{-1}$ it is sufficient to show that $ch(S_A)$, $ch(S_B^\vee)$ and $ch(\mathcal{J}(\mathcal{E}))$ are integral. But , by construction, all these objects are line bundles on abelian subvarieties. Thus, the following lemma implies the proposition. □

4.3.4.1 Lemma. *For any line bundle* L *on an abelian variety* A *the Chern character* $ch(L)$ *is integral, i.e belongs to* $H^*(A, \mathbb{Z})$.

Proof. Denote by $c_1(L)$ the first Chern class of L . The Chern character $ch(L)$ is equal to $exp(c_1(L))$. Thus we must show that $\frac{1}{k!} c_1(L)^k$ belong to integral cohomology for any k . If $x_1, ..., x_{2n}$ be a basis of $H^1(A, \mathbb{Z})$, then the first Chern class as an element of $H^2(A, \mathbb{Z})$ can be written as:

$$c_1(L) = \sum_{i<j} d_{ij} \cdot (x_i \cup x_j), \qquad d_{ij} \in \mathbb{Z}$$

Taking k -power, we obtain

$$c_1(L)^k = k! \sum_{\substack{(i_1<j_1);\dots;(i_k<j_k)\\(i_1,j_1,\dots,i_k,j_k)\subset(1,\dots,2n)}} d_{i_1j_1}\cdots d_{i_kj_k}(x_{i_1}\cup x_{j_1}\cup\cdots\cup x_{i_k}\cup x_{j_k})$$

That proves this lemma and the proposition. □

This way, the correspondence $\Phi_{\mathcal{E}}\mapsto\phi_{\mathcal{E}}$ gives us the homomorphism

$$\rho_A : Auteq(D^b(A)) \longrightarrow \mathrm{GL}(H^*(A,\mathbb{Z})),$$

denoted by the same letter as in Lemma 4.1.4.3.

4.3.5 Definition. *The subgroup of* $\mathrm{GL}(H^*(A,\mathbb{Z}))$ *that is the image of* ρ_A *will be called the* spinor group of A *and will be denoted by* $Spin(A)$ *.*

4.3.6. To justify this definition consider the Clifford algebra $Cl(\Lambda,Q)$ (3.1) and the homomorphism of algebras

$$cor_A : Cl(\Lambda,Q) \longrightarrow \mathrm{End}(H^*(A,\mathbb{Z})) \cong H^*(A\times A,\mathbb{Z})$$

defined on the $\Lambda\subset Cl(\Lambda,Q)$ by rule

$$cor_A((l,x))(\cdot) = l(\cdot) + x\cup(\cdot),$$

where $(l,x)\in\Gamma_A\oplus\Gamma_A^*=\Lambda$ and $l(\cdot)$ is the convolution with l , i.e. for a monomial $\alpha = x_1\cup\ldots\cup x_k$ it is defined as

$$l(\alpha) := \sum(-1)^{i-1}l(x_i)\cdot x_1\cup\ldots\cup x_{i-1}\cup x_{i+1}\cup\ldots\cup x_k.$$

By Proposition 3.2.1e) the homomorphism cor_A is an isomorphism. Clearly it sends $Cl^+(\Lambda,Q)$ onto $H^{ev}(A\times A,\mathbb{Z})$. The homomorphism cor_A induces the action of the group $Spin(\Lambda,Q)$ (3.4.1) on cohomology lattice $H^*(A,\mathbb{Z})$. Henceforth, we will consider $Spin(\Lambda,Q)$ as a subgroup of $\mathrm{GL}(H^*(A,\mathbb{Z}))$ with respect to cor_A .

There is the standard involution $'$ on $Cl(\Lambda)$. It is defined as a unique ring involution on $Cl(\Lambda)$ that is the identity on Λ . The isomorphism cor_A induces an involution on $\mathrm{End}(H^*(A,\mathbb{Z}))$, which will be denoted the same symbol $'$. It is not hard to check that for any $\xi\in H^d(A\times A,\mathbb{Z})$ there is the equality

$$v_\xi{}' = (-1)^{n+\frac{d(d-1)}{2}}w_\xi,$$

where v_ξ and w_ξ defined in 4.1.3 and $n=\dim A$. In particular, this implies that

$$\phi_{\mathcal{E}}{}' = \psi_{\mathcal{E}^\vee[n]}. \tag{9}$$

for any object $\mathcal{E}\in D^b(A\times A)$.

Thus there are two correspondences $\Phi_{\mathcal{E}}\mapsto f_{\mathcal{E}}$ and $\Phi_{\mathcal{E}}\mapsto\phi_{\mathcal{E}}$, which gives two group homomorphisms

$$\gamma_A : Auteq(D^b(A)) \longrightarrow \mathrm{U}(A)\subset SO(\Lambda,Q), \qquad \rho_A : Auteq(D^b(A)) \longrightarrow Spin(A)\subset \mathrm{GL}(H^*(A,\mathbb{Z}))$$

such that $\gamma_A(\Phi_{\mathcal{E}}) = f_{\mathcal{E}}$ and $\rho_A(\Phi_{\mathcal{E}}) = \phi_{\mathcal{E}}$. The following proposition relates these two homomorphisms.

4.3.7 Proposition.

a) The group $Spin(A)$ *is contained in* $Spin(\Lambda,Q)\subset\mathrm{GL}(H^*(A,\mathbb{Z}))$ *.*

b) Let $\pi : Spin(\Lambda,Q)\longrightarrow SO(\Lambda,Q)$ *be the canonical map. Then* $\pi\cdot\rho_A=\gamma_A$ *and* $\pi^{-1}(\mathrm{U}(A)) = Spin(A)$ *.*

Proof. Let $\Phi_{\mathcal{E}} : D^b(A)\xrightarrow{\sim} D^b(A)$ be an autoequivalence. The following diagram is a cohomological analog of diagram (8) with $B=A$:

$$\begin{array}{ccc} H^*(A\times\widehat{A},\mathbb{Z}) & \xrightarrow{\phi_{S_A}} & H^*(A\times A,\mathbb{Z}) \\ {\scriptstyle\phi_{\mathcal{J}(\mathcal{E})}}\downarrow & & \downarrow{\scriptstyle ad_{\mathcal{E}}} \\ H^*(A\times\widehat{A},\mathbb{Z}) & \xrightarrow{\phi_{S_A}} & H^*(A\times A,Z) \end{array} \tag{10}$$

By Theorem 2.4.5.1 the sheaf $\mathcal{J}(\mathcal{E})$ is a line bundle over the subvariety $\Gamma(f_{\mathcal{E}})$ that is the graph of the isomorphism $f_{\mathcal{E}}$. This implies that $\phi_{\mathcal{J}(\mathcal{E})}$ sends $H^{4n-1}(A\times\widehat{A},\mathbb{Z})$ to itself. Moreover, the restriction of $\phi_{\mathcal{J}(\mathcal{E})}$ on the $(4n-1)-th$ cohomology coincides with

$$f_{\mathcal{E}}: H_1(A\times\widehat{A},\mathbb{Z})\longrightarrow H_1(A\times\widehat{A},\mathbb{Z})$$

under the Poincare isomorphism $D: H_1(A\times\widehat{A},\mathbb{Z})=\Lambda\xrightarrow{\sim}H^{4n-1}(A\times\widehat{A},\mathbb{Z})$. Fix this identification of $H^{4n-1}(A\times\widehat{A},\mathbb{Z})$ with Λ.

Let us assume that the restriction of ϕ_{S_A} on $H^{4n-1}(A\times\widehat{A},\mathbb{Z})$ coincides with the restriction of cor_A on Λ . Under this assumption, it follows from the commutativity of the diagram (10) that $ad_{\mathcal{E}}$ takes $cor_A(\Lambda)$ to itself. Hence $\phi_{\mathcal{E}}cor(\Lambda)\phi_{\mathcal{E}}^{-1}=cor(\Lambda)$. Further, since the inverse of an equivalence $\Phi_{\mathcal{E}}$ is isomorphic to $\Psi_{\mathcal{E}^{\vee}[n]}$ we get from (9) that $\phi_{\mathcal{E}}{}'=\phi_{\mathcal{E}}^{-1}$. Therefore $N(\phi_{\mathcal{E}})=\phi_{\mathcal{E}}\cdot\phi_{\mathcal{E}}{}'=id$. By Definition 3.4.1 $\phi_{\mathcal{E}}$ belongs to $Spin(\Lambda,Q)$ and a) is proved.

Moreover, we have that action of $f_{\mathcal{E}}$ on Λ coincides with the action of $ad_{\mathcal{E}}$ on $cor_A(\Lambda)$. This implies that $\gamma_A=\pi\rho_A$. Finally, since $Ker\pi=\mathbb{Z}/2\mathbb{Z}$ and $\rho([1])=-Id$ the full inverse image $\pi^{-1}(\mathrm{U}(A))$ not only contains but also coincides with $Spin(A)$.

Thus to complete the proof of the proposition it is sufficient to prove the following lemma.

4.3.7.1 Lemma. *The restriction of the map* $\phi_{S_A}D$ *on* $H_1(A\times\widehat{A},\mathbb{Z})=\Lambda$ *coincides with the restriction of the map* cor_A *on* Λ *.*

PROOF. As above let $x_1,...,x_{2n}$ and $l_1,...,l_{2n}$ be the dual bases of Γ_A^* and Γ_A respectively. For any $\alpha\in H_i(A\times\widehat{A},\mathbb{Z})$ and $\beta\in H^i(A\times\widehat{A},\mathbb{Z})$ the following equality holds

$$\langle\beta,\alpha\rangle=\langle\beta\cup D(\alpha),[A\times\widehat{A}]\rangle,$$

where $\langle,\rangle$ is the canonical pairing of cohomology with homology and $[A\times\widehat{A}]\in H_{4n}(A\times\widehat{A},\mathbb{Z})$ is the fundamental class. It is easy to check that

$$\begin{array}{l}D(x_i)=(-1)^{i-1}p^*(x_1\cup\cdots\cup x_{2n})\cup q^*(l_1\cup\cdots\cup l_{i-1}\cup l_{i+1}\cup\cdots\cup l_{2n}),\\ D(l_i)=(-1)^{i-1}p^*(x_1\cup\cdots\cup x_{i-1}\cup x_{i+1}\cup\cdots\cup x_{2n})\cup q^*(l_1\cup\cdots\cup l_{2n}).\end{array}$$

Denote $\mu_A=\mu$ (4.2.4). By Lemma 4.2.4.1 the functor Φ_{S_A} is the composition $\mathbf{R}\mu_*\Phi_{\mathcal{R}_A}$. Hence $\phi_{S_A}=\mu_*\phi_{\mathcal{R}_A}$. By construction of $\Phi_{\mathcal{R}_A}$, the map $\phi_{\mathcal{R}_A}$ takes $q^*(l)\cup p^*(x)$ to $p_1^*(\psi_P(l))\cup p_2^*(x)$. Combining Lemma 4.2.3.2 and Proposition 4.2.2.1 we get

$$\phi_{\mathcal{R}_A}D(x_i)=p_1^*(x_i)\cup p_2^*(x_1\cup\cdots\cup x_{2n})\tag{11}$$

$$\phi_{\mathcal{R}_A}D(l_i)=(-1)^{i-1}p_2^*(x_1\cup\cdots\cup x_{i-1}\cup x_{i+1}\cup\cdots\cup x_{2n})\tag{12}$$

Further, to compute $\mu_*(\alpha)$ it is sufficient to note that μ_* is a ring homomorphism because μ is an isomorphism and, consequently, μ_* is the inverse of μ^* . Besides, since $\mu(a_1,a_2)=(a_1,m(a_1,a_2))$ we have

$$\mu^*(p_1^*(\alpha))=p_1^*(\alpha),\qquad\mu^*(p_2^*(\alpha))=m^*(\alpha)$$

for any $\alpha\in H^*(A,\mathbb{Z})$. Moreover, if $\alpha=x\in H^1(A,\mathbb{Z})$ then

$$\mu^*(p_2^*(x))=m^*(x)=p_1^*(x)+p_2^*(x)$$

Therefore, for any $x\in H^1(A,\mathbb{Z})$ there are equalities

$$\mu_*(p_1^*(x))=p_1^*(x)\quad\text{and}\quad\mu_*(p_2^*(x))=p_2^*(x)-p_1^*(x).$$

To find an element $\phi_{S_A}(\lambda)$ for some $\lambda\in H^{4n-1}(A\times\widehat{A},\mathbb{Z})$ it is sufficient to compute the map that is defined by this element, i.e.

$$\phi_{S_A}(\lambda)(\cdot):=p_{2*}(\phi_{S_A}(\lambda)\cup p_1^*(\cdot)).$$

Substituting (11) in the last expression, we get

$$\begin{array}{l}\phi_{S_A}D(x_i)(\alpha)=p_{2*}(\mu_*(p_1^*(x_i)\cup p_2^*(x_1\cup\cdots\cup x_{2n}))\cup p_1^*(\alpha))=m_*(p_1^*(x_i\cup\alpha)\cup p_2^*(x_1\cup\cdots\cup x_{2n}))\\ =m_*(m^*(x_i\cup\alpha)\cup p_2^*(x_1\cup\cdots\cup x_{2n}))=x_i\cup\alpha\cup m_*(p_2^*(x_1\cup\cdots\cup x_{2n}))=x_i\cup\alpha\end{array}$$

Hence $\phi_{S_A}D(x)$ coincides with $cor_A(x)$.

Let us compute the action of $\phi_{S_A} D(l_i)$ on the cohomology. We have

$$\phi_{S_A} D(l_i)(x_{s_1} \cup \cdots \cup x_{s_k}) = p_{2*}(\mu_*((-1)^{i-1} p_2^*(\bigcup_{\substack{j=1,\\ j\neq i}}^{2n} x_j)) \cup p_1^*(x_{s_1} \cup \cdots \cup x_{s_k})) =$$
$$(-1)^i p_{2*}(\bigcup_{\substack{j=1,\\ j\neq i}}^{2n} (p_1^*(x_j) - p_2^*(x_j)) \cup p_1^*(x_{s_1} \cup \cdots \cup x_{s_k}))$$

If the set $(s_1, ..., s_k)$ does not contain i, then this expression equals 0. Suppose that $s_1 = i$. In this case the last expression can be simplified

$$\phi_{S_A} D(l_i)(x_i \cup x_{s_2} \cup \cdots \cup x_{s_k}) = p_{2*}(p_1^*(x_1 \cup \cdots \cup x_{2n}) \cup p_2^*(x_{s_2} \cup \cdots \cup x_{s_k})) = x_{s_2} \cup \cdots \cup x_{s_k}$$

Thus, $\phi_{S_A} D(l_i)$ coincides with $cor_A(l_i)$, because

$$cor_A(l_i)(x_{s_1} \cup \cdots \cup x_{s_k}) = 0 \quad \text{if} \quad i \notin (s_1, ..., s_k),$$
$$cor_A(l_i)(x_i \cup x_{s_2} \cup \cdots \cup x_{s_k}) = x_{s_2} \cup \cdots \cup x_{s_k}.$$

This concludes the proof of the lemma and, consequently, completes the proof of the proposition. □

4.3.8 Corollary. *$Ker\rho_A$ coincides with $2\mathbb{Z} \times A \times \widehat{A}$, consisting of all equivalences $\Phi_{(a,\alpha)}[2i] = T_a^*(\cdot) \otimes P_\alpha[2i]$.*

Proof. It is clear that $\Phi_{(a,\alpha)}[2i]$ are contained in $Ker\rho_A$. On the other side, it follows from Proposition 4.3.7 that $Ker\rho_A$ is a subgroup of $Ker\gamma_A \cong \mathbb{Z} \times A \times \widehat{A}$. Note that by Lemma 4.1.5.1 $\rho_A([1]) = -Id$. Corollary is proved. □

4.3.9 Corollary. *There is the exact sequence of groups*

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow Spin(A) \longrightarrow \mathrm{U}(A) \longrightarrow 1.$$

The last theorem of this chapter gives description of abelian varieties that have equivalent derived categories of coherent sheaves.

4.3.10 Theorem.([Or2]) *Let B and C be abelian varieties. Then the derived categories $D^b(B)$ and $D^b(C)$ are equivalent if and only if there exists an isomorphism*

$$\gamma : B \times \widehat{B} \xrightarrow{\sim} C \times \widehat{C}$$

which identifies the forms Q_B and Q_C on $\Gamma_B \oplus \Gamma_{\widehat{B}}$ and $\Gamma_C \oplus \Gamma_{\widehat{C}}$..

The "if" direction in the above theorem was first proved by A. Polishchuk in [Po1].

5. The algebraic group $\mathrm{U}_{A,\mathbb{Q}}$.

5.1. Fix a complex torus A. Put $\Lambda := \Gamma_A \oplus \Gamma_{\widehat{A}}$ and consider the group $\mathrm{GL}(\Lambda_{\mathbb{Q}})$. We have the following $\mathbb{Q}$-algebraic subgroups of $\mathrm{GL}(\Lambda_{\mathbb{Q}})$.

1) $Hdg_{A,\mathbb{Q}} = Hdg_{A\times\widehat{A},\mathbb{Q}}$.
2) $Aut_{\mathbb{Q}}(A \times \widehat{A})$, which is defined as the group of invertible elements in $\mathrm{End}^0(A \times \widehat{A})$.
3) $O(\Lambda_{\mathbb{Q}}, Q_{\mathbb{Q}})$, $SO(\Lambda_{\mathbb{Q}}, Q_{\mathbb{Q}})$, where $Q_{\mathbb{Q}}$ is the extension to $\Lambda_{\mathbb{Q}}$ of the canonical symmetric bilinear form on Λ (3.1).
4) $\mathrm{U}_{A,\mathbb{Q}}$, which is defined as follows

$$\mathrm{U}_{A,\mathbb{Q}} = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Aut_{\mathbb{Q}}(A \times \widehat{A}) \;\middle|\; g^{-1} = \begin{pmatrix} \widehat{d} & -\widehat{b} \\ -\widehat{c} & \widehat{a} \end{pmatrix} \right\}$$

Thus the discrete group $\mathrm{U}(A)$ defined in 4.3.1 is the arithmetic subgroup of $\mathrm{U}_{A,\mathbb{Q}}$ consisting of elements which preserve the lattice Λ.

The following proposition summarizes the interrelations of these subgroups.

5.2 Proposition.

1) $Hdg_{A,\mathbb{Q}} \subset \mathrm{SO}(\Lambda_{\mathbb{Q}}, Q_{\mathbb{Q}})$.
2) $Aut_{\mathbb{Q}}(A \times \widehat{A})$ *is the centralizer of* $Hdg_{A,\mathbb{Q}}$ *in* $\mathrm{GL}(\Lambda_{\mathbb{Q}})$.
3) $\mathrm{U}_{A,\mathbb{Q}} = Aut_{\mathbb{Q}}(A \times \widehat{A}) \cap \mathrm{SO}(\Lambda_{\mathbb{Q}}, Q_{\mathbb{Q}})$.

4) $\mathrm{U}_{A,\mathbb{Q}}$ *is the centralizer of* $Hdg_{A,\mathbb{Q}}$ *in* $\mathrm{SO}(\Lambda_{\mathbb{Q}}, Q_{\mathbb{Q}})$.

PROOF. 1) The operator J_A of complex structure on $\Lambda_{\mathbb{R}}$ has determinant 1 and preserves the form Q. Thus $h_A(\boldsymbol{S}^1)$ lies in the group of $\mathbb{R}$-points of $\mathrm{SO}(\Lambda_{\mathbb{Q}}, Q_{\mathbb{Q}})$. Since $Hdg_{A,\mathbb{Q}}$ is the $\mathbb{Q}$-closure of $h_A(\boldsymbol{S}^1)$, we have $Hdg_{A,\mathbb{Q}} \subset \mathrm{SO}(\Lambda_{\mathbb{Q}}, Q_{\mathbb{Q}})$.

2) By definition $\mathrm{End}^0(A \times \widehat{A})$ is the centralizer of $Hdg_{A,\mathbb{Q}}$ in $\mathrm{End}(\Lambda_{\mathbb{Q}})$. Thus $Aut_{\mathbb{Q}}(A \times \widehat{A})$ is the centralizer of $Hdg_{A,\mathbb{Q}}$ in $\mathrm{GL}(\Lambda_{\mathbb{Q}})$.

3) The proof is the same as that of proposition 4.3.2 above.

4) This follows from 1), 2), 3). □

5.3. Let us study the $\mathbb{Q}$-algebraic group $\mathrm{U}_{A,\mathbb{Q}}$ in case A is an abelian variety.

Clearly, the group $U_{A,\mathbb{Q}}$ depends only on the isogeny class of A. If $A = A_1^{m_1} \times ... \times A_k^{m_k}$, where A_i are pairwise nonisogeneous simple abelian varieties, then

$$\mathrm{U}_{A,\mathbb{Q}} = \prod_i \mathrm{U}_{A_i^{m_i},\mathbb{Q}}.$$

Below we describe the $\mathbb{Q}$-algebraic group $\mathrm{U}_{A^m,\mathbb{Q}}$ and its group of $\mathbb{R}$-points $\mathrm{U}_{A^m,\mathbb{Q}}(\mathbb{R})$ for a simple abelian variety A.

5.3.1. Let us use the notations of section 1.8. In particular denote $F = \mathrm{End}^0(A)$. Let $\varphi \in \mathrm{Hom}(A, \widehat{A})$ be a polarisation and $' : F \to F$ be the corresponding Rosati involution. We identify naturally $M_m(F) = \mathrm{End}^0(A^m)$, which makes $\Gamma_{A^m,\mathbb{Q}}$ a left $M_m(F)$ module. Consider the diagonal polarisation $\sigma = (\varphi, ...\varphi)$ of A^m. Then the corresponding Rosati involution on $M_m(F)$ is

$$M \mapsto {}^tM'.$$

Consider the following homomorphism of algebras

$$\tau : \mathrm{End}^0(A \times \widehat{A}) \longrightarrow \mathrm{End}(\Gamma_{A^m,\mathbb{Q}} \oplus \Gamma_{A^m,\mathbb{Q}})$$

$$\tau : \begin{pmatrix} B & C \\ D & E \end{pmatrix} \mapsto \begin{pmatrix} B & C\sigma \\ \sigma^{-1}D & \sigma^{-1}E\sigma \end{pmatrix}.$$

Obviously, the image of τ is contained in $M_{2m}(F)$. We have

$$\tau(\mathrm{U}_{A^m,\mathbb{Q}}) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2m, F) \,\middle|\, g^{-1} = \begin{pmatrix} {}^t\widehat{d} & -{}^t\widehat{b} \\ -{}^t\widehat{c} & {}^t\widehat{a} \end{pmatrix} \right\}$$

Thus $\tau(\mathrm{U}_{A^m,\mathbb{Q}})$ is the group of isometries of the skew-hermitian form on $F^m \oplus F^m$

$$\vartheta((z_1, ..., z_m, z_{-1}, ..., z_{-m}), (w_1, ..., w_m, w_{-1}, ..., w_{-m})) = \sum_{k=1}^m z_k w'_{-k} - z_k w'_k.$$

5.3.2. Denote by $U^*_{2m,F} \subset \mathrm{GL}(2m, F)$ the K_0-algebraic group of isometries of the skew-hermitian form ϑ. Then the $\mathbb{Q}$-algebraic group $\mathrm{U}_{A^m,\mathbb{Q}}$ is obtained from $U^*_{2m,F}$ by restriction of scalars from K_0 to $\mathbb{Q}$. Fix an embedding $K_0 \hookrightarrow \mathbb{R}$. The corresponding group of real points $U^*_{2m,F}(\mathbb{R})$ was computed by A. Polishchuk ([Po2]). His result according to the four cases of Albert's classification is the following

I. $Sp_{2m}(\mathbb{R})$,
II. $Sp_{4m}(\mathbb{R})$,
III. $U^*_{2m}(\mathbb{H})$ – the group of automorphisms of $\mathbb{H}^{2m}$ preserving the standard skew-hermitian form,
IV. $U(md, md)$.

5.3.3 Corollary. *The reductive Lie group* $U^*_{2m,F}(\mathbb{R})$ *is semisimple unless we are in case IV. This group has no compact factors unless we are in case IV (then it is isogeneous to the product* $\boldsymbol{S}^1 \times SU(md, md)$ *) or in case III and* $m = 1$ *(then it is isogeneous to the product* $SU(2) \times \mathrm{SL}(2, \mathbb{R})$ *).*

5.3.4. We have

$$\mathrm{U}_{A^m,\mathbb{Q}}(\mathbb{R}) \cong \prod_{e_0} U^*_{2m,F}(\mathbb{R}),$$

hence the above provides the corresponding description of the group $\mathrm{U}_{A^m,\mathbb{Q}}(\mathbb{R})$.

Corollary. *The reductive group* $\mathrm{U}_{A^m,\mathbb{Q}}(\mathbb{R})$ *is semisimple unless we are in case IV. This group has no compact factors unless we are in case IV (then the compact part is isogeneous to the product of* e_0 *copies of* $\boldsymbol{S}^1$ *) or in case III and* $m = 1$ *(then the compact part is isogeneous to the product of* e_0 *copies of* $SU(2)$ *).*

5.3.5 Corollary. *For any abelian variety* A *the Lie group* $\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$ *is connected.*

6. The Neron-Severi Lie algebra $\mathfrak{g}_{NS}(X)$.

6.1. Let X be a smooth projective variety of dimension n . Let us recall the definition of the Neron-Severi Lie algebra $\mathfrak{g}_{NS}(X)$ from [LL]. If $\kappa \in H^{1,1}(X) \cap H^2(X,\mathbb{Q})$ is an ample class, then cupping with it defines an operator e_κ in the total cohomology $H^*(X) = H^*(X,\mathbb{C})$ of degree 2 and the hard Lefschetz theorem asserts that for $s = 0, 1, ..., n$, e_κ^s maps $H^{n-s}(X)$ isomorphically onto $H^{n+s}(X)$. As is well known, this is equivalent to the existence of a (unique) operator f_κ on $H^*(X)$ of degree -2 such that the commutator $[e_\kappa, f_\kappa]$ is the operator h which on $H^k(X)$ is multiplication by $k-n$. The elements e_κ, f_κ, h make up a Lie subalgebra $\mathfrak{g}_\kappa$ of $\mathfrak{gl}(H^*(X))$ isomorphic to $sl(2)$. Define the *Neron-Severi* Lie algebra $\mathfrak{g}_{NS}(X)$ as the Lie subalgebra of $\mathfrak{gl}(H^*(X))$ generated by $\mathfrak{g}_\kappa$'s with κ an ample class. This Lie subalgebra is defined over $\mathbb{Q}$ and is evenly graded by the adjoint action by the semisimple element h .

In the above definition we could replace an ample class κ by a Kahler class or by any element $\kappa \in H^2(X)$, such that e_κ satisfies the conclusion of the hard Lefschetz theorem. The resulting Lie subalgebras of $\mathfrak{gl}(H^*(X))$ are called the *Kahler* Lie algebra and the *total* Lie algebra respectively and denoted by $\mathfrak{g}_K(X)$ and $\mathfrak{g}_{tot}(X)$ respectively. Obviously we have the inclusions $\mathfrak{g}_{NS}(X) \subset \mathfrak{g}_K(X) \subset \mathfrak{g}_{tot}(X)$.

The above Lie algebras preserve (infinitesimally) the following form on the cohomology $H^*(X)$.

$$\chi(\alpha,\beta) := (-1)^q \int_X \alpha \cup \beta,$$

where α is homogeneous of degree $n+2q$ or $n+2q+1$. The main fact about these Lie algebras is that they are semisimple ([LL],p.369).

6.2. In case X is a complex torus (resp. an abelian variety) the Lie algebras $\mathfrak{g}_{tot}(X)$, $\mathfrak{g}_K(X)$ (resp. $\mathfrak{g}_{NS}(X)$) were computed explicitly in [LL], p.381. Let us recall the result.

6.2.1. Let $X = (V_X/\Gamma_X, J_X)$ be a complex torus. Put $\Lambda = \Gamma_X \oplus \Gamma_{\widehat{X}}$ with the canonical symmetric bilinear form Q (3.1). There exists a natural isomorphism of Lie algebras

$$\mathfrak{g}_{tot}(X) \cong \mathfrak{so}(\Lambda_\mathbb{Q}, Q_\mathbb{Q}),$$

such that the semisimple element $h \in \mathfrak{g}_{tot}(X)$ corresponds to the element $-1_{\Gamma_X} \oplus 1_{\Gamma_{\widehat{X}}} \in \mathfrak{so}(\Lambda_\mathbb{Q}, Q_\mathbb{Q})$. The natural representation of $\mathfrak{g}_{tot}$ on $H^*(X,\mathbb{Q})$ under the above isomorphism is the spinorial representation of $\mathfrak{so}(\Lambda_\mathbb{Q}, Q_\mathbb{Q})$ (which is the direct sum of two semi-spinorial ones corresponding to H^{ev} and H^{odd} respectively).

Let $J = (J_X, J_{\widehat{X}})$ be the complex structure on $V_X \oplus V_{\widehat{X}}$. Then the form $Q_\mathbb{R}$ is J -invariant, hence defines a hermitian form on $\Lambda_\mathbb{R}$. Let $\mathfrak{su}(\Lambda_\mathbb{R})$ denote the Lie algebra of the corresponding special unitary Lie group. Then the isomorphism mentioned above $\mathfrak{g}_{tot}(X) \cong \mathfrak{so}(\Lambda_\mathbb{Q}, Q_\mathbb{Q})$ identifies the Lie subalgebras $\mathfrak{g}_K(X;\mathbb{R}) \cong \mathfrak{su}(\Lambda_\mathbb{R})$.

6.2.2. Finally, let X be an abelian variety. The Lie algebra $\mathfrak{g}_{NS}(X)$ depends only on the isogeny type of X . If

$$X = X_1^{m_1} \times ... \times X_k^{m_k},$$

then

$$\mathfrak{g}_{NS}(X) = \mathfrak{g}_{NS}(X_1^{m_1}) \times ... \times \mathfrak{g}_{NS}(X_k^{m_k}).$$

We therefore assume that X is a power A^m of a simple abelian variety A . Let us stick to notations of section 1.8. In particular, $F = \mathrm{End}^0(A)$. Choose a polarization $\varphi \in \mathrm{Hom}(A, \widehat{A})$ of A . Denote by $'$ the corresponding Rosati involution on F . Consider the diagonal polarisation $\sigma = (\varphi, ..., \varphi)$ of A^m . If we naturally identify $M_m(F) = \mathrm{End}^0(A^m)$ then the Rosati involution defined by σ is $M \mapsto {}^tM'$.

Define a K_0 –Lie subalgebra of $\mathrm{End}(2m,F)$ by

$$\mathfrak{slu}(2m,F,') = \{\begin{pmatrix} B & C \\ D & -{}^tB' \end{pmatrix} | B,C,D \in \mathrm{End}(m,F); \quad C = {}^tC', \quad D = {}^tD'\}.$$

This is the Lie algebra of infinitesimal isometries of the skew-hermitian form ϑ on $F^m \oplus F^m$, which was defined in 5.3.1 above. It is a reductive K_0 –Lie algebra whose center is the space of scalars $\lambda \in K$ with $\lambda' = -\lambda$. So $\mathfrak{slu}(2m,F,')$ is semisimple unless we are in the case of totally complex multiplication (case IV). We grade this Lie algebra by means of the semisimple element

$$u_m := \begin{pmatrix} -1_m & 0 \\ 0 & 1_m \end{pmatrix} \in \mathfrak{slu}(2m,F,'),$$

so that B , C and D parametrize the summands of degree 0 , -2 and 2 respectively. Let $\mathfrak{g}(2m,F,')$ denote the K_0 –Lie subalgebra of $\mathfrak{slu}(2m,F,')$ generated by summands of degree 2 and -2; let $\mathfrak{u}(m,F,')$ denote the union of $\mathrm{GL}(m,F)$ –conjugacy classes in $\mathrm{End}(m,F)$ made up of anti-invariants with respect to the involution $M \mapsto {}^tM'$ and identify $\mathfrak{u}(m,F,')$ with the subspace of $\mathfrak{slu}(2m,F,')$ in an obvious way. We have the following result.

6.2.2.1 Lemma. ([LL], Lemma 3.9) *We have*

$$\mathfrak{slu}(2m,F,') = \mathfrak{g}(2m,F,') \times \mathfrak{u}(m,F,').$$

The summand $\mathfrak{u}(m,F,')$ is trivial except in the following cases

1) $m=1$ and F is totally definite quaternion (case III): then $\mathfrak{g}(2,F,') \cong sl(2,K_0)$ and $\mathfrak{u}(1,F,')$ can be identified with pure quaternians in F (i.e. the $'$ -antiinvariants in F) or

2) K is totally complex (case IV): then $\mathfrak{g}(2m,F,')$ consists of the matrices for which B has its K –trace in K_0 , whereas $\mathfrak{u}(m,F.')$ can be identified with the purely imaginary scalars in K (i.e. the $\lambda \in K$ with $\bar{\lambda} = -\lambda$).

6.2.2.2 Remark. It was noted in [LL] that in the exceptional cases the connected Lie subgroup of $\mathrm{GL}(m,F\otimes_{\mathbb{Q}}\mathbb{R})$ with the Lie algebra $\mathfrak{u}(m,F,')\otimes_{\mathbb{Q}}\mathbb{R}$ is a product of e_0 copies of $\mathrm{U}(1)$, resp $\mathrm{SU}(2)$, and hence is compact.

In view of 6.2.1 above we may consider $\mathfrak{g}_{NS}(X)$ as a Lie subalgebra of $\mathrm{End}(\Lambda_{\mathbb{Q}}) = \mathrm{End}(\Gamma_{X,\mathbb{Q}} \oplus \Gamma_{\hat{X},\mathbb{Q}})$. Recall the homomorphism of algebras τ as in 5.3.1 above.

$$\tau : \mathrm{End}^0(\Gamma_{X,\mathbb{Q}} \oplus \Gamma_{\hat{X},\mathbb{Q}}) \longrightarrow \mathrm{End}(\Gamma_{X,\mathbb{Q}} \oplus \Gamma_{X,\mathbb{Q}})$$

$$\tau : \begin{pmatrix} B & C \\ D & E \end{pmatrix} \mapsto \begin{pmatrix} B & C\sigma \\ \sigma^{-1}D & \sigma^{-1}E\sigma \end{pmatrix}.$$

Note that $\Gamma_{X,\mathbb{Q}} \oplus \Gamma_{X,\mathbb{Q}}$ is naturally a $\mathrm{End}(2m,F)$ - module.

6.2.2.3 Proposition. ([LL], Prop.3.10) *The above homomorphism τ identifies the Lie algebras $\mathfrak{g}_{NS}(X)$ and $\mathfrak{g}(2m,F.')$. In particular, we get an isomorphism*

$$\mathfrak{slu}(2m,F,') \cong \mathfrak{g}_{NS}(X) \times \mathfrak{u}(m,F,').$$

7. Relation between the group $Auteq(D^b(A))$ and the Neron-Severi Lie algebra $\mathfrak{g}_{NS}(A)$ for an abelian variety A .

7.1. Let A be an abelian variety. We know that the group $Auteq(D^b(A))$ acts on the total cohomology $H^*(A,\mathbb{Z})$ of A and we denoted its image in $\mathrm{GL}(H^*(A,\mathbb{Z}))$ by $Spin(A)$ (4.3.5). Put $\Lambda = \Gamma_A \oplus \Gamma_{\hat{A}}$ and let Q be the canonical symmetric bilinear form on Λ (3.1). Recall the exact sequence of group homomorphisms

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow Spin(A) \longrightarrow \mathrm{U}(A) \longrightarrow 1$$

(4.3.9) and the commutative diagram (4.3.7)

$$\begin{array}{ccc} Spin(A) & \hookrightarrow & Spin(\Lambda,Q) \\ \downarrow & & \downarrow \\ \mathrm{U}(A) & \hookrightarrow & \mathrm{SO}(\Lambda,Q) \end{array},$$

where horizontal arrows are compatible with the actions on $H^*(A,\mathbb{Z})$ and Λ respectively.

7.1.1 Definition. *Let* $\overline{Spin(A)}$ *(resp.* $\overline{\mathrm{U}(A)}$ *) be the* Zariski closure *of* $Spin(A)$ *(resp.* $\mathrm{U}(A)$ *) in* $\mathrm{GL}(H^*(A,\mathbb{Q}))$ *(resp* $\mathrm{GL}(\Lambda_{\mathbb{Q}})$ *).*

These are $\mathbb{Q}$ -algebraic groups. These groups are isogeneous, hence have isomorphic Lie algebras.

7.1.2 Remark. Since the group $Spin(A)$ acts on $H^*(A,\mathbb{Q})$ by algebraic correspondences its action commutes with $Hdg_{A,\mathbb{Q}}$. Hence the same is true for $\overline{Spin(A)}$.

On the other hand we know that $\mathfrak{g}_{NS}(A)$ can be identified as a Lie subalgebra of $\mathfrak{so}(\Lambda_{\mathbb{Q}},Q_{\mathbb{Q}})$ so that the representation of $\mathfrak{g}_{NS}(A)$ in $H^*(A,\mathbb{Q})$ is the restriction of the spinorial representation of $\mathfrak{so}(\Lambda_{\mathbb{Q}},Q_{\mathbb{Q}})$ (6.2.1).

7.2 Theorem.

a) The Lie algebra of $\overline{Spin(A)}$ *is* $\mathfrak{g}_{NS}(A)$ *considered as a Lie subalgebra of* $\mathfrak{gl}(H^*(A,\mathbb{Q}))$.

b) The Lie algebra of $\overline{\mathrm{U}(A)}$ *is* $\mathfrak{g}_{NS}(A)$ *considered as a Lie subalgebra of* $\mathfrak{gl}(\Lambda_{\mathbb{Q}})$.

Proof. The discussion in 7.1 above implies that a) and b) are equivalent. Let us prove b).

Both $\overline{\mathrm{U}(A)}$ and $\mathfrak{g}_{NS}(A)$ depend only on the isogeny type of A . If A is a product of pairwise nonisogeneous abelian varieties then both $\overline{\mathrm{U}(A)}$ and $\mathfrak{g}_{NS}(A)$ are products of the corresponding factors. Thus it suffices to prove that $\mathfrak{g}_{NS}(A^m)$ is the Lie algebra of the algebraic group $\overline{\mathrm{U}(A^m)}$ for a simple abelian variety A .

By definition $\mathrm{U}(A^m)$ is an arithmetic subgroup of the reductive $\mathbb{Q}$ -algebraic group $\mathrm{U}_{A^m,\mathbb{Q}}$ (5.1). By Corollary 5.3.4 all abelian factors of $\mathrm{U}_{A^m,\mathbb{Q}}$ are compact. Thus by the density theorem ([PR]) the $\mathbb{Q}$ -algebraic group $\overline{\mathrm{U}(A^m)}$ consists (up to isogeny) of all noncompact factors of $\mathrm{U}_{A^m,\mathbb{Q}}$. Let $\mathfrak{g}(A^m)$ be the Lie algebra of $\mathrm{U}_{A^m,\mathbb{Q}}$. As follows from the results in 5.3.1, 6.2.2, 6.2.2.3 that it is a product of Lie algebras

$$\mathfrak{g}(A^m)=\mathfrak{g}_{NS}(A^m)\times\mathfrak{u}_{A^m},$$

where $\mathfrak{u}_{A^m}=\tau^{-1}(\mathfrak{u}(m,F,'))$ in the notation of 5.3.1, 6.2.2 above. Moreover, it follows from the results in 5.3.4, 6.2.2.1, 6.2.2.2 above that the $\mathbb{Q}$ -algebraic subgroup of $\mathrm{U}_{A^m,\mathbb{Q}}$ corresponding to $\mathfrak{u}_{A^m}$ consists (up to isogeny) of all compact factors of $\mathrm{U}_{A^m,\mathbb{Q}}$. Thus $\mathfrak{g}_{NS}(A^m)$ is the Lie algebra of the group $\overline{\mathrm{U}(A^m)}$. This proves the theorem. □

7.2.1 Corollary.

a) The algebraic groups $\overline{Spin(A)}$ *and* $\overline{\mathrm{U}(A)}$ *are semisimple.*

b) The group $\overline{\mathrm{U}(A)}$ *consists (up to isogeny) of all noncompact factors of* $\mathrm{U}_{A,\mathbb{Q}}$.

Proof. a) Indeed, this follows from the above theorem, since the Lie algebra $\mathfrak{g}_{NS}(A)$ is semisimple. b) This was obtained in the proof of the above theorem. □

7.3. Let A be an abelian variety. Consider $\mathfrak{g}_{NS}(A;\mathbb{R})$ as a Lie subalgebra of $\mathfrak{gl}(\Lambda_{\mathbb{R}})$. Let $\varphi\in NS_A(\mathbb{R})^0\subset\mathrm{Hom}(V_A,V_{\widehat{A}})$. Then $\varphi,\varphi^{-1}\in\mathfrak{g}_{NS}(A;\mathbb{R})$ and

$$[\varphi,\varphi^{-1}]=h=\begin{pmatrix}-1_A & 0\\ 0 & 1_{\widehat{A}}\end{pmatrix}.$$

The elements φ,φ^{-1},h make up a Lie subalgebra $\mathfrak{g}_\varphi$ of $\mathfrak{g}_{NS}(A;\mathbb{R})$ isomorphic to $sl(2)$. By definition the Lie algebra $\mathfrak{g}_{NS}(A;\mathbb{R})$ is generated by these subalgebras.

Consider the group of $\mathbb{R}$ -points $\overline{\mathrm{U}(A)}(\mathbb{R})$ of the $\mathbb{Q}$ -algebraic group $\overline{\mathrm{U}(A)}$. It is a semisimple Lie subgroup of the reductive Lie group $\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$ (5.3.4) which consists (up to isogeny) of all noncompact factors of $\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$. By the above theorem the Lie algebra of $\overline{\mathrm{U}(A)}(\mathbb{R})$ is $\mathfrak{g}_{NS}(A;\mathbb{R})$. For $\varphi\in NS_A(\mathbb{R})^0$ denote by $G_\varphi\subset\overline{\mathrm{U}(A)}(\mathbb{R})$ the connected Lie subgroup corresponding to the Lie subalgebra $\mathfrak{g}_\varphi$. We have $G_\varphi\cong\mathrm{SL}(2;\mathbb{R})$.

8. Action of $\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$ on a Siegel domain.

8.1. Let A be a complex torus. Let us define a rational (i.e. not defined everywhere) action of the group $\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$ (5.1) on the complex space $NS_A(\mathbb{C})=NS_A\otimes\mathbb{C}\subset\mathrm{Hom}(A,\widehat{A})\otimes\mathbb{C}$. It is given by the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \omega := (c+d\omega)(a+b\omega)^{-1}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{U}_{A,\mathbb{Q}}(\mathbb{R}), \quad \omega \in NS_A(\mathbb{C}) \subset \mathrm{Hom}(A, \widehat{A}) \otimes \mathbb{C}. \tag{13}$$

Here multiplication is understood as composition of maps.

In case A is an abelian variety $NS_A(\mathbb{C})$ contains Siegel domains of the first kind on which this action is well-defined. Namely, let $C_A^a \subset NS_A(\mathbb{R})$ be the ample cone of A, which is defined as $\mathbb{R}^+$ -linear combinations of ample classes in NS_A. It is an open subset in $NS_A(\mathbb{R})$. Put

$$C_A^{\pm} := NS_A(\mathbb{R}) \pm iC_A^a \subset NS_A(\mathbb{C}),$$

$$C_A := C_A^+ \amalg C_A^-.$$

Thus $C_A \neq \emptyset$ if and only if the complex torus A is an abelian variety.

8.2 Theorem. *Let A be an abelian variety.*

a) The action of $\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$ on C_A is well defined.

b) This action on C_A^+ (resp. C_A^-) is transitive.

c) The stabilizer of a point in C_A is a maximal compact subgroup of the Lie group $\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$.

PROOF. This theorem is proved in Appendix.

8.3 Remark. Let A be an abelian variety. The action of $\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$ on C_A restricts to an action of the Lie subgroup $\overline{\mathrm{U}(A)}(\mathbb{R})$. Since this subgroup consists (up to isogeny) of all noncompact factors of $\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$ its action of C_A^+ (resp. C_A^-) is also transitive. Thus C_A^+ (resp. C_A^-) is a hermitian symmetric space for the semisimple Lie group $\overline{\mathrm{U}(A)}(\mathbb{R})$.

8.4. Let A be a complex torus. As above, denote by $NS_A(\mathbb{R})^0 \subset NS_A(\mathbb{R})$ the subset consisting of nondegenerate forms. Assume that $NS_A(\mathbb{R})^0 \neq \emptyset$. Put

$$NS_A(\mathbb{C})^0 := \{\varphi_1 + i\varphi_2 \in NS_A(\mathbb{C}) \mid \varphi_2 \in NS_A(\mathbb{R})^0\}$$

Thus, $NS_A(\mathbb{C})^0$ is a nonempty Zariski open subset in $NS_A(\mathbb{C})$. (Note that if A is an abelian variety then C_A^+ (resp. C_A^-) is a connected component of $NS_A(\mathbb{C})^0$) .

Given $\omega = \varphi_1 + i\varphi_2 \in NS_A(\mathbb{C})^0$ consider the following element

$$I_\omega := \begin{pmatrix} \varphi_2^{-1}\varphi_1 & -\varphi_2^{-1} \\ \varphi_2 + \varphi_1\varphi_2^{-1}\varphi_1 & -\varphi_1\varphi_2^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \varphi_1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\varphi_2^{-1} \\ \varphi_2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\varphi_1 & 1 \end{pmatrix} \in \mathrm{U}_{A,\mathbb{Q}}(\mathbb{R}) \tag{14}$$

and the morphism of algebraic $\mathbb{R}$ -groups

$$\mu_\omega : \boldsymbol{S}^1 \longrightarrow \mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$$

given by the formula $\mu_\omega(e^{i\theta}) = \cos\theta \cdot Id + \sin\theta \cdot I_\omega$, so that $\mu_\omega(e^{i\frac{\pi}{2}}) = I_\omega$.

The correspondence $\omega \mapsto I_\omega$ is injective.

8.4.1 Properties of I_ω and μ_ω .

1. $I_{\overline{\omega}} = -I_\omega$, hence $\mu_{\bar{\omega}} = \mu_\omega \cdot (-Id_{\boldsymbol{S}^1})$.
   This is obvious.
2. $I_\omega^2 = -Id$.
   This is a direct computation.

   By $K_\omega \subset \mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$ denote the stabilizer of ω with respect to the action (13), and $Z(K_\omega)$ denote its center. One checks directly that

$$K_\omega = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{U}_{A,\mathbb{Q}}(\mathbb{R}) \;\middle|\; \begin{matrix} \varphi_2 a + \varphi_1 b \varphi_2 + \varphi_2 b \varphi_1 = d\varphi_2; \\ \varphi_1 a + \varphi_1 b \varphi_1 - \varphi_2 b \varphi_2 = c + d\varphi_1 \end{matrix} \right\}$$

3. $\mu_\omega(\boldsymbol{S}^1) \subseteq Z(K_\omega)$.
   This is also a direct computation.
4. $Z_{\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})}(I_\omega) = Z_{\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})}(\mu_\omega(\boldsymbol{S}^1)) = K_\omega$.
   The first equality is obvious and the second is a direct computation.

   Let $T_\omega NS_A(\mathbb{C})$ be the tangent space to $NS_A(\mathbb{C})$ at the point ω . Each element $g \in K_\omega$ defines a linear operator on $T_\omega NS_A(\mathbb{C})$.

5. $\mu_\omega(e^{i\frac{\pi}{4}})$ defines the complex structure (i.e. multiplication by i) on $T_\omega NS_A(\mathbb{C})$.
Indeed, given $\gamma + i\delta \in NS_A(\mathbb{C})$ it suffices to check that

$$\lim_{\epsilon\in\mathbb{R},\epsilon\to 0}\frac{1}{\epsilon}\left[\begin{pmatrix} 1+\varphi_2^{-1}\varphi_1 & -\varphi_2 \\ \varphi_2+\varphi_1\varphi_2^{-1}\varphi_1 & 1-\varphi_1\varphi_2^{-1}\end{pmatrix}(\varphi_1+i\varphi_2+\epsilon(\gamma+i\delta))-(\varphi_1+i\varphi_2)\right]=i(\gamma+i\delta)$$

This is a straightforward computation, which we omit.

6. If A is an abelian variety and $\omega \in C_A$, then Ad_{I_ω} is a Cartan involution in $\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$, i.e. $\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})^{Ad_{I_\omega}}$ is a maximal compact subgroup K_ω of $\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$.
Indeed, this follows from properties 2, 3, 4 above and from Theorem 8.2 c).

7. If A is an abelian variety, then $\mu_\omega(\boldsymbol{S}^1) \subset \overline{\mathrm{U}(A)}(\mathbb{R})$ (7.3).
Let us prove this. If $\omega' = i\varphi_2$, then $\mu_\omega = g_{\varphi_1}\mu_{\omega'}g_{\varphi_1}^{-1}$ where $g_{\varphi_1} = \begin{pmatrix} 1 & 0 \\ \varphi_1 & 1\end{pmatrix} \in G_{\varphi_1}$ (7.3).
So we may assume that $\omega = i\varphi$ and hence

$$\mu_\omega(e^{i\theta}) = \cos\theta\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} + \sin\theta\begin{pmatrix} 0 & -\varphi^{-1} \\ \varphi & 0\end{pmatrix}.$$

But then $\mu_\varphi(\boldsymbol{S}^1) \subset G_\varphi$. (7.3). □

9. Mirror symmetry.

Here we introduce the notion of mirror symmetry for abelian varieties and complex tori.

9.1 Definition. *An* algebraic pair *(resp. a* weak pair*)* (A, ω_A) *consists of an abelian variety (resp. a complex torus)* A *and an element* $\omega_A \in C_A$ *(8.1) (resp.* $\omega_A \in NS_A(\mathbb{C})^0$ *(8.4)).*

Thus a algebraic pair is a special case of a weak pair.

Let (A, ω_A) be a weak pair. Recall the canonical symmetric bilinear form Q_A (3.1) on the lattice $\Gamma_A \oplus \Gamma_{\widehat{A}}$. The group $Hdg_{A,\mathbb{Q}}(\mathbb{R}) \times \mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$ acts on the space $V_A \oplus V_{\widehat{A}}$, preserving the form $Q_{A,\mathbb{R}}$. In fact $\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$ is the centralizer of $Hdg_{A,\mathbb{Q}}(\mathbb{R})$ in $\mathrm{SO}(V_A \oplus V_{\widehat{A}}, Q_{A,\mathbb{R}})$ (Prop. 5.2.4). We have the elements

$$J_{A\times\widehat{A}} \in Hdg_{A,\mathbb{Q}}(\mathbb{R}) = Hdg_{A\times\widehat{A},\mathbb{Q}}(\mathbb{R}), \qquad I_{\omega_A} \in \mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$$

(see 2.1, 8.4) which commute and define two complex structures on $V_A \oplus V_{\widehat{A}}$.

9.2 Definition. *We say that algebraic pairs (resp. weak pairs)* (A, ω_A), (B, ω_B) *are* mirror symmetric *if there is an isomorphism*

$$\alpha : \Gamma_A \oplus \Gamma_{\widehat{A}} \xrightarrow{\sim} \Gamma_B \oplus \Gamma_{\widehat{B}}$$

which identifies bilinear forms Q_A *and* Q_B *and satisfies the following conditions*

$$\begin{aligned} \alpha_{\mathbb{R}} \cdot J_{A\times\widehat{A}} &= I_{\omega_B} \cdot \alpha_{\mathbb{R}}, \\ \alpha_{\mathbb{R}} \cdot I_{\omega_A} &= J_{B\times\widehat{B}} \cdot \alpha_{\mathbb{R}}. \end{aligned}$$

9.2.1 Remark. Let the weak pairs (A, ω_A) and (B, ω_B) be mirror symmetric. We will identify the lattices $\Gamma_A \oplus \Gamma_{\widehat{A}}$ and $\Gamma_B \oplus \Gamma_{\widehat{B}}$ by means of α . That is we have

$$\Gamma_A \oplus \Gamma_{\widehat{A}} = \Lambda = \Gamma_B \oplus \Gamma_{\widehat{B}}$$

with the bilinear form $Q_A = Q = Q_B$.

Then the $\mathbb{Q}$ -algebraic groups $Hdg_{A,\mathbb{Q}} \times \mathrm{U}_{A,\mathbb{Q}}$ and $Hdg_{B,\mathbb{Q}} \times \mathrm{U}_{B,\mathbb{Q}}$ are subgroups of $\mathrm{SO}(\Lambda_{\mathbb{Q}}, Q_{\mathbb{Q}})$. By the definition of the Hodge group we have the following group inclusions

$$Hdg_{A,\mathbb{Q}} \subseteq \mathrm{U}_{B,\mathbb{Q}}, \qquad Hdg_{B,\mathbb{Q}} \subseteq \mathrm{U}_{A,\mathbb{Q}}.$$

If A and B are abelian varieties, then stronger inclusions hold (see 8.4.1 (7))

$$Hdg_{A,\mathbb{Q}} \subseteq \overline{\mathrm{U}(B)}, \qquad Hdg_{B,\mathbb{Q}} \subseteq \overline{\mathrm{U}(A)}.$$

9.2.2 Remark. Let (A,ω_A) be a weak pair. Assume that weak pairs (B,ω_B) and (C,ω_C) are mirror symmetric to (A,ω_A). If B is an abelian variety, then so is C. Indeed, composing the α 's for B and C we obtain an isomorphism of complex tori

$$B\times\widehat{B}\cong C\times\widehat{C}.$$

Thus C is an abelian variety (as a subtorus of an abelian variety).

9.2.3 Remark. Let (A,ω_A) be a weak pair. The collection of isomorphism classes of abelian varieties B for which there exists $\omega_B\in NS_B(\mathbb{C})^0$ such that the pairs (A,ω_A) and (B,ω_B) are mirror symmetric, is finite. Indeed, fix one such B. Then for any other such C there exists an isomorphism of abelian varieties

$$B\times\widehat{B}\cong C\times\widehat{C},$$

hence C is isomorphic to an abelian subvariety of $B\times\widehat{B}$. The finiteness follows from the following theorem of Lenstra, Oort and Zarhin.

Theorem.([LOZ]) *Let A be an abelian variety. There are finitely many isomorphism classes of abelian varieties which admit an embedding as an abelian subvariety in A.*

9.2.4 Proposition. *Let (A,ω_A) and (B,ω_B) be two mirror symmetric weak pairs. Then the set of α 's that establish a mirror symmetry between the pairs is a torsor over the stabilizer of ω_A in $\mathrm{U}(A)$. In particular, if the pair (A,ω_A) is algebraic then this set is finite.*

Proof. Let α_1, α_2 be two isomorphisms that establish a mirror symmetry of (A,ω_A) and (B,ω_B). Then $\gamma:=\alpha_2^{-1}\alpha_1$ is an automorphism of the torus $A\times\widehat{A}$ which preserves the form Q_A. Thus by Proposition 4.3.2 $\gamma\in\mathrm{U}(A)$. Also $\gamma I_{\omega_A}\gamma^{-1}=I_{\omega_A}$. Therefore by property 4 in 8.4.1 $\gamma\omega_A=\omega_A$. This proves the first assertion. If $\omega_A\in C_A$ then its stabilizer in $\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$ is compact (Theorem 8.2 c). This implies the second assertion since the group $\mathrm{U}(A)$ is discrete. □

9.2.5 Proposition. *Two weak pairs (B,ω_1) and (B,ω_2) are both mirror symmetric to the same weak pair (A,ω_A) if and only if there exists $g\in\mathrm{U}(B)$ such that $g\omega_1=\omega_2$.*

Proof. $\Rightarrow$ Similarly to the previous proof we find $g\in\mathrm{U}(B)$ such that $gI_{\omega_1}g^{-1}=I_{\omega_2}$. This means that $g\omega_1=\omega_2$.

$\Leftarrow$ If α establishes a mirror symmetry of (A,ω_A) and (B,ω_1), then $g\alpha$ induces a mirror symmetry of (A,ω_A) and (B,ω_2). □

9.2.6 Proposition. *Let A, B, C be abelian varieties.*

a) If some algebraic pairs (B,ω_B), (C,ω_C) are both mirror symmetric to an algebraic pair (A,ω_A), then the derived categories $D^b(B)$ and $D^b(C)$ are equivalent.

b) If some algebraic pairs (B,ω_B) and (A,ω_A) are mirror symmetric and the derived categories $D^b(B)$ and $D^b(C)$ are equivalent, then there exists $\omega_C\in C_C$ such that the pairs (C,ω_C) and (A,ω_A) are mirror symmetric.

Proof. This follows immediately from the Theorem 4.3.10 and Proposition 9.4.3 below. □

9.3. Assume that weak pairs (A,ω_A), (B,ω_B) are mirror symmetric. Use the identification of 9.2.1. Consider the Clifford algebra $Cl(\Lambda,Q)$ (3.1). The total cohomology groups $H^*(A,\mathbb{Z})$ and $H^*(B,\mathbb{Z})$ are naturally $Cl(\Lambda,Q)$ –modules (3.7). There exists a unique (up to ±1) isomorphism of these $Cl(\Lambda,Q)$ –modules

$$\beta:H^*(A,\mathbb{Z})\xrightarrow{\sim}H^*(B,\mathbb{Z}).$$

This isomorphism either preserves the even and odd cohomology groups or interchanges them (3.7). In the first case we called β *even* and in the second – *odd.*

9.3.1. Assume that the weak pair (A,ω_A) is mirror symmetric to yet another weak pair (C,ω_C) by an isomorphism

$$\alpha':\Gamma_A\oplus\Gamma_{\widehat{A}}\xrightarrow{\sim}\Gamma_C\oplus\Gamma_{\widehat{C}}.$$

Identify $\Gamma_C\oplus\Gamma_{\widehat{C}}$ with $\Gamma_B\oplus\Gamma_{\widehat{B}}$ by means of α and α'. We claim that $dim(\Gamma_{B,\mathbb{Q}}\cap\Gamma_{C,\mathbb{Q}})$ is even and hence $\Gamma_{B,\mathbb{Q}}\sim\Gamma_{C,\mathbb{Q}}$ in the sense of 3.6.1. Indeed, both V_B and V_C are complex subspaces of $\Lambda_{\mathbb{R}}$ with respect to the complex structure I_{ω_A} (8.4). Hence $dim_{\mathbb{R}}(V_B\cap V_C)$ is even.

9.3.2 Proposition. *Let weak pairs (A,ω_A) and (B,ω_B) be mirror symmetric. Let $n=dimA=dimB$. Then the parity of β is equal to the parity of n.*

PROOF. Recall that $\Lambda^{\cdot}V_A^* = H^*(A,\mathbb{R})$ and $\Lambda^{\cdot}V_B^* = H^*(B,\mathbb{R})$ are both spinorial representations of the Lie algebra $\mathfrak{so}(\Lambda_{\mathbb{R}}, Q_{\mathbb{R}})$, and $\beta_{\mathbb{R}} : \Lambda^{\cdot}V_A^* \xrightarrow{\sim} \Lambda^{\cdot}V_B^*$ is an $\mathfrak{so}(\Lambda_{\mathbb{R}}, Q_{\mathbb{R}})$ -morphism. Thus the parity of $\beta_{\mathbb{R}}$ will not change if we replace V_B by a subspace gV_B for some $g \in \mathrm{SO}(\Lambda_{\mathbb{R}}, Q_{\mathbb{R}})$. Let $\omega_A = \varphi_1 + i\varphi_2$, where $\varphi_1 \in NS_A(\mathbb{R})$, $\varphi_2 \in NS_A(\mathbb{R})^0$. Choose $g = g_{\varphi_1}^{-1} \in G_{\varphi_1} \subset \mathrm{SO}(\Lambda_{\mathbb{R}}, Q_{\mathbb{R}})$ as in 8.4.1 (7). Then gV_B is a subspace of $\Lambda_{\mathbb{R}}$ which is preserved by the complex structure

$$I_{i\varphi_2} = \begin{pmatrix} o & -\varphi_2^{-1} \\ \varphi_2 & 0 \end{pmatrix}.$$

It suffices to find a maximal isotropic subspace $V \subset \Lambda_{\mathbb{R}}$ which is preserved by $I_{i\varphi_2}$ (hence $V \sim gV_B$) and such that $dim_{\mathbb{R}}(V \cap V_A) = n$. Let us construct such a subspace V .

Consider φ_2 as a skew-symmetric form on V_A . There exists a basis $e_1, ..., e_n, e_{-1}, ..., e_{-n}$ of V_A in which

$$\varphi_2 = \begin{pmatrix} 0 & -\Delta \\ \Delta & 0 \end{pmatrix}, \quad \Delta = \begin{pmatrix} \delta_1 & & & \\ & \delta_2 & & \\ & & \dots & \\ & & & \delta_n \end{pmatrix}.$$

Let $e_1^*, ..., e_n^*, e_{-1}^*, ..., e_{-n}^*$ be the dual basis of $V_{\widehat{A}}$. Then $V := < e_1, ..., e_n, e_{-1}^*, ..., e_{-n}^* >$ is clearly maximal isotropic subspace of Λ and $I_{i\varphi_2}$ -invariant. This proves the proposition. □

9.3.3. Let A be an abelian variety. Recall that the total cohomology $H^*(A,\mathbb{Q})$ is a representation space of the $\mathbb{Q}$ –algebraic group $Hdg_{A,\mathbb{Q}} \times \overline{Spin(A)}$ (7.1.2).

Proposition. *Let algebraic pairs* (A, ω_A) *and* $(B\omega_B)$ *be mirror symmetric. As in 9.3 above consider the canonical (up to* $\pm$ *) isomorphism of* $Cl(\Lambda, Q)$ *–modules*

$$\beta : H^*(A,\mathbb{Z}) \xrightarrow{\sim} H^*(B,\mathbb{Z}).$$

Then $\beta_{\mathbb{Q}}$ *induces the inclusions of algebraic groups*

$$Hdg_{A,\mathbb{Q}} \subseteq \overline{Spin(B)}, \qquad Hdg_{B,\mathbb{Q}} \subseteq \overline{Spin(A)}.$$

PROOF. Let us identify $H^*(A,\mathbb{Z})$ and $H^*(B,\mathbb{Z})$ by means of β. It suffices to prove one inclusion $Hdg_{A,\mathbb{Q}} \subset \overline{Spin(B)}$.

We know that the action of $Hdg_{A,\mathbb{Q}}$ on $H^*(A,\mathbb{Q}) = \Lambda^{\cdot}\Gamma_{\widehat{A},\mathbb{Q}}$ is induced from its action on $\Gamma_{\widehat{A},\mathbb{Q}}$ by functoriality. It suffices to show that this action agrees with the spinorial representation of $Spin(\Lambda_Q, Q_{\mathbb{Q}})$ when $Hdg_{A,\mathbb{Q}}$ is considered as a subgroup of $\mathrm{SO}(\Lambda_Q, Q_{\mathbb{Q}})$. This is done by identifying the weights of the Lie algebra $\mathfrak{sl}(\Gamma_{A,\mathbb{Q}}) \subset \mathfrak{so}(\Lambda_{\mathbb{Q}}, Q_{\mathbb{Q}})$ in the spinorial representation and the natural representation on $\Lambda^{\cdot}\Gamma_{\widehat{A},\mathbb{Q}}$. □

9.4. Let (A,ω) be a weak pair. Let $I \in U_{A,\mathbb{Q}}(\mathbb{R})$, $I^2 = -1$. Then I and $J_{A\times\widehat{A}} \in Hdg_{A,\mathbb{Q}}(\mathbb{R})$ are two complex structures on the space

$$\Lambda_{\mathbb{R}} = V_A \oplus V_{\widehat{A}}.$$

They commute and preserve the bilinear form $Q_{A,\mathbb{R}}$. Denote by c the product $J_{A\times\widehat{A}}I$. The operator c preserves the bilinear form $Q_{A,\mathbb{R}}$ and $c^2 = 1$. Hence, the bilinear form $Q_{A,\mathbb{R}}(c(\cdot),\cdot)$ is symmetric. Denote by $E_{A,I}$ the corresponding quadratic form. In case $I = I_\omega$ for some $\omega \in NS_A(\mathbb{C})^0$ we denote $E_{A,I_\omega} = E_{A,\omega}$.

9.4.1 Lemma. *Let* A *be a complex torus. Suppose* $I \in \mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$, $I^2 = -1$ *and* I *has the form* $\left(\begin{smallmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{smallmatrix}\right)$. *The following conditions are equivalent:*

1) $I = I_\omega$ *for some* $\omega \in NS_A(\mathbb{C})^0$.
2) $I_{12} : V_{\widehat{A}} \longrightarrow V_A$ *is invertible.*
3) *The restriction of bilinear form* $Q_{A,\mathbb{R}}(c\cdot,\cdot)$ *on* $V_{\widehat{A}}$ *is non-degenerate.*

PROOF. $1) \Rightarrow 2)$ This immediately follows from the formula (14) for I_ω in 8.4.

$2) \Leftrightarrow 3)$ Let $x_1, x_2 \in V_{\widehat{A}}$. We have

$$Q_{A,\mathbb{R}}(c(0,x_1),(0,x_2)) = x_2(J_A I_{12}(x_1)).$$

Thus, the restriction of $Q_{A,\mathbb{R}}(c(\cdot),\cdot)$ to $V_{\widehat{A}}$ is non-degenerate iff I_{12} is invertible.

2) $\Rightarrow$ 1) Let I_{12} is invertible. Since $I^2=-1$ and $I\in \mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$ we have equalities

$$\widehat{I_{12}}=I_{12},\quad \widehat{I_{21}}=I_{21},\quad \widehat{I_{11}}=-I_{22}$$

Put $\varphi_2=-I_{12}^{-1}$ and $\varphi_1=I_{22}I_{12}^{-1}$. Take $\omega=\varphi_1+i\varphi_2$. It is easy to see that $I_\omega=I$. $\Box$

9.4.2 Lemma. *Let A be a complex torus and let $I\in \mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$, $I^2=-1$. Then I coincides with I_ω for some $\omega\in C_{\hat A}^{+}$ (resp. $C_{\hat A}^{-}$) and, consequently, A is an abelian variety iff the quadratic form $E_{A,I}$ is positive (resp. negative) definite.*

PROOF. $\Rightarrow$ Suppose $I=I_\omega$ with $\omega=\varphi_1+i\varphi_2\in NS_A(\mathbb{C})^0$. It is not hard to check that $I_\omega=g_{\varphi_1}I_{i\varphi_2}g_{\varphi_1}^{-1}$, where $g_{\varphi_1}=\left(\begin{smallmatrix}1&0\\ \varphi_1&1\end{smallmatrix}\right)$. Therefore, we have an equality

$$E_{A,I}(\lambda)=Q_{A,\mathbb{R}}(c\lambda,\lambda)=Q_{A,\mathbb{R}}(I_{i\varphi_2}J_A(\lambda^{'}),\lambda^{'}),$$

where $\lambda^{'}=g_{\varphi_1}^{-1}(\lambda)$.

Taking $\lambda'=(l,x)$ with $l\in V_A$ and $x\in V_{\hat A}$, we obtain

$$E_{A,I}(\lambda)=Q_{A,\mathbb{R}}((-\varphi_2^{-1}(J_{\hat A}x),\varphi_2(J_Al)),(l,x))=\varphi_2(J_Al,l)-x(\varphi_2^{-1}(J_{\hat A}x))=\varphi_2(J_Al,l)-x(J_A\varphi_2^{-1}(x)).$$

Denoting $\varphi_2^{-1}(x)$ by m in the last expression, we get

$$E_{A,I}(\lambda)=-\varphi_2(m,J_Am)+\varphi_2(J_Al,l)=\varphi_2(J_Am,m)+\varphi_2(J_Al,l)$$

This shows that the quadratic form $E_{A,J}$ is positive (resp. negative) definite on $\Lambda_{A,\mathbb{R}}$ iff the form $\varphi_2(J_A\cdot,\cdot)$ is positive (resp. negative) definite on V_A , which is equivalent to $\omega\in C_{\hat A}^{+}$ (resp. $\omega\in C_{\hat A}^{-}$).

$\Leftarrow$ Assume that the form $E_{A,I}$ is definite. Let $I=\left(\begin{smallmatrix}I_{11}&I_{12}\\ I_{21}&I_{22}\end{smallmatrix}\right)$. If I_{12} is degenerate , then there is $0\neq x\in V_{\hat A}$ such that $I_{12}(x)=0$. Hence

$$E_{A,I}((0,x))=Q_{A,\mathbb{R}}((J_AI_{12}(x),J_{\hat A}I_{22}(x)),(0,x))=Q_{A,\mathbb{R}}((0,x'),(0,x))=0$$

This contradicts the definitness of $E_{A,I}$. By Lemma 9.4.1, as I_{12} is invertible, $I=I_\omega$ for some $\omega\in NS_A(\mathbb{C})^0$. Then the previous argument shows that ω belongs to C_A . Lemma is proved. $\Box$

9.4.3 Proposition. *Let weak pairs (A,ω_A), (B,ω_B) be mirror symmetric. Suppose that the weak pair (A,ω_A) is in fact an algebraic pair. Then (B,ω_B) is an algebraic pair too.*

PROOF. Since (A,ω_A) is an algebraic pair we have $\omega_A\in C_A$. Hence, by Lemma 9.4.2, the quadratic form E_{A,ω_A} (9.4) is definite. Since the weak pairs (A,ω_A) and (B,ω_B) are mirror symmetric the quadratic forms E_{A,ω_A} and E_{B,ω_B} coincide under an identification α . Therefore, the quadratic form E_{B,ω_B} is definite. Lemma 9.4.2 implies that $\omega_B\in C_B$. Thus B is an abelian variety and the weak pair (B,ω_B) is an algebraic pair. $\Box$

9.4.4 Lemma. *Let (A,ω_A) be an algebraic pair and B be a complex torus. Let $\alpha:\Gamma_A\oplus\Gamma_{\hat A}\stackrel{\sim}{\longrightarrow}\Gamma_B\oplus\Gamma_{\hat B}$ be an isomorphism which identifies the forms Q_A and Q_B and $\alpha_{\mathbb{R}}\cdot I_{\omega_A}=J_{B\times\hat B}\cdot\alpha_{\mathbb{R}}$. Then there exists $\omega_B\in C_B$ (in particular B is an abelian variety) such that $\alpha_{\mathbb{R}}\cdot J_{A\times\hat A}=I_{\omega_B}\cdot\alpha_{\mathbb{R}}$. That is α establishes the mirror symmetry of algebraic pairs (A,ω_A), (B,ω_B) .*

PROOF. Since (A,ω_A) is an algebraic pair the form E_{A,ω_A} on $V_A\oplus V_{\hat A}$ is definite (9.4.2). Identify the spaces $V_A\oplus V_{\hat A}$ and $V_B\oplus V_{\hat B}$ with the forms $Q_{A,\mathbb{R}}$ and $Q_{B,\mathbb{R}}$ by means of $\alpha_{\mathbb{R}}$. Thus

$$I:=J_{A\times\hat A}\in Hdg_{A,\mathbb{Q}}(\mathbb{R})$$

and

$$I_{\omega_A}=J_{B\times\hat B}\in Hdg_{B,\mathbb{Q}}(\mathbb{R}).$$

Since the $\mathbb{Q}$ -closures of $J_{A\times\hat A}$ and I_{ω_A} commute (5.2,4)) we find that $I\in \mathrm{U}_{B,\mathbb{Q}}(\mathbb{R})$ (5.2,4)). Thus we may apply Lemma 9.4.2 to I and B . Note that $\alpha_{\mathbb{R}}$ identifies the quadratic forms E_{A,ω_A} and $E_{B,I}$. Since the first one is definite it follows from Lemma 9.4.2 that $I=I_\omega$ for some $\omega=\omega_B\in C_B$. This proves the lemma. $\Box$

9.4.5 Lemma. *Let $\Lambda=\Gamma\oplus\Gamma^*$ be a lattice with canonical symmetric bilinear form Q (3.1). Let there be given an isotropic decomposition $\Lambda=\Lambda_1\oplus\Lambda_2$. Suppose $I\in \mathrm{GL}(\Lambda_{\mathbb{R}})$ is a complex structure on $\Lambda_{\mathbb{R}}$, i.e. $I^2=-1$, that satisfies the following assumptions*

$$a)\quad I(\Lambda_{i,\mathbb{R}})=\Lambda_{i,\mathbb{R}},\quad i=1,2$$
$$b)\quad I\in O(\Lambda_{\mathbb{R}},Q_{\mathbb{R}})$$

Then the complex tori

$$B_1 := (\Lambda_{1,\mathbb{R}}/\Lambda_1, I_1), \quad B_2 := (\Lambda_{2,\mathbb{R}}/\Lambda_2, I_2),$$

where I_i *are the restriction of* I *on* $\Lambda_{i,\mathbb{R}}$ *, are dual to each other, i.e.* $B_2 \cong \widehat{B_1}$ *.*

PROOF. The form Q induces the isomorphism $t : \Lambda_2 \longrightarrow \Lambda_1^*$ that takes $\lambda_2 \in \Lambda_2$ to linear functional $Q(\lambda_2, \cdot)$. It remains to show that for any $v_1 \in \Lambda_{1,\mathbb{R}},\ v_2 \in \Lambda_{2,\mathbb{R}}$ the following equality holds

$$t(I_2(v_2))(v_1) = t(v_2)(-I_1(v_1))$$

(3.1). This is straightforward using the inclusion $J \in O(\Lambda_{\mathbb{R}}, Q_{\mathbb{R}})$.

$$t(I_2(v_2))(v_1) = Q_{\mathbb{R}}(I_2(v_2), v_1) = Q_{\mathbb{R}}(I_2^2(v_2), I_1(v_1)) = Q_{\mathbb{R}}(-v_2, I_1(v_1)) = t(v_2)(-I_1(v_1)).$$

Lemma is proved. □

9.4.6 Summary. Let (A, ω_A) be an algebraic pair. The last two lemmas give us a "method" for construction of a mirror symmetric pair. Namely, put $\Lambda = \Gamma_A \oplus \Gamma_{\widehat{A}}$. It suffices to find a Q_A –isotropic decomposition $\Lambda = \Lambda_1 \oplus \Lambda_2$ such that $\Lambda_{1\mathbb{R}}$, $\Lambda_{2\mathbb{R}}$ are preserved by I_{ω_A} . Then by Lemma 9.4.5 the tori $B := (\Lambda_{1\mathbb{R}}/\Lambda_1, I_{\omega_A})$ and $\widehat{B} := (\Lambda_{2\mathbb{R}}/\Lambda_2, I_{\omega_A})$ are dual. Moreover, the natural identification

$$\alpha = id : \Gamma_A \oplus \Gamma_{\widehat{A}} \xrightarrow{\sim} \Gamma_B \oplus \Gamma_{\widehat{B}}$$

($\Gamma_B = \Lambda_1$, $\Gamma_{\widehat{B}} = \Lambda_2$) identifies the forms Q_A and Q_B . Thus by Lemma 9.4.4 there exists $\omega_B \in C_B$ such that the pairs (A, ω_A) , (B, ω_B) are mirror symmetric.

9.5. It is not true that for every pair (A, ω_A) there exists a mirror symmetric pair (B, ω_B) . The problem may occur if the group $\mathrm{U}_{A,\mathbb{Q}}$ is too big and $\omega_A \in C_A$ is chosen too general.

9.5.1 Counterexample. Let E be an elliptic curve with complex multiplication. We have $e_0 = 1, d = 1$ (1.8) and $\mathrm{U}_{E\times E,\mathbb{Q}}(\mathbb{R}) = \mathrm{U}(2,2)$ (5.3.2). Therefore this group acts irreducibly on $V_{E\times E} \oplus V_{\widehat{E\times E}}$. Assume that for all $\omega \in C_{E\times E}$ there exists a mirror symmetric pair (B, ω') , i.e. there exists α as in 9.2 with the corresponding properties. Let us use α to identify

$$\Gamma_{E\times E} \oplus \Gamma_{\widehat{E\times E}} = \Gamma_B \oplus \Gamma_{\widehat{B}}.$$

The Hodge group $Hdg_{B,\mathbb{Q}}$ is the $\mathbb{Q}$ –closure of the compact torus $\mu_\omega(\boldsymbol{S}^1) \subset Aut(V_B \oplus V_{\widehat{B}})$. For a general ω this $\mathbb{Q}$ –closure is the group $\mathrm{SU}(2,2) \subset \mathrm{U}_{E\times E,\mathbb{Q}}(\mathbb{R})$. But the group $Hdg_{B,\mathbb{Q}}(\mathbb{R})$ acts faithfully on $V_B \oplus V_{\widehat{B}}$ and preserves each summand. This is a contradiction.

9.6. However, each torus A with a nonempty NS_A^0 has an $\omega_A \in NS_A(\mathbb{C})^0$ such that the pair (A, ω_A) has a mirror symmetric pair. In particular for every abelian variety A there exists $\omega_A \in NS_A(\mathbb{C})^0$ (even $\omega_A \in C_A$) such that the pair (A, ω_A) has a mirror symmetric one. This follows from the next proposition.

9.6.1 Proposition. *Let* A *be a complex torus of dimension* n *. Let* $\varphi \in NS_A^0$ *, i.e.* $\varphi \in \mathrm{Hom}(A, \widehat{A})$ *is an isogeny. Let* $\tau = a + ib \in \mathbb{C}$ *,* $b \neq 0$ *. Consider the element* $\omega_A := \tau\varphi \in NS_A(\mathbb{C})^0$ *and the weak pair* (A, ω_A) *. Then there exist isogeneous elliptic curves* $E_1, ..., E_n$ *and an element* $\omega_E \in NS_E(\mathbb{C})^0$ *, where* $E = E_1 \times ... \times E_n$ *, such that the weak pair* (E, ω_E) *is mirror symmetric to the weak pair* (A, ω_A) *.*

PROOF. There exists a basis $e_1, ..., e_n, e_{-1}, ..., e_{-n}$ of Γ_A in which the bilinear form φ has a matrix

$$\begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix}, \quad \text{where} \quad \Delta = \begin{pmatrix} \delta_1 & & 0 \\ & ... & \\ 0 & & \delta_n \end{pmatrix}, \quad \delta_i \in \mathbb{Z}, \quad \delta_1|\delta_2|\delta_3...$$

Let $e_1^*, ...e_n^*, e_{-1}^*, ..., e_{-n}^*$ be the dual basis of $\Gamma_{\widehat{A}}$. Then the map $\varphi : \Gamma_A \longrightarrow \Gamma_{\widehat{A}}$ is

$$\varphi : e_i \mapsto \delta_i e_{-i}^*, \quad e_{-i} \mapsto -\delta_i e_i^*.$$

Put

$$\Gamma_i := \mathbb{Z}e_i \oplus \mathbb{Z}e_{-i}^*, \quad \Gamma_i^* := \mathbb{Z}e_{-i} \oplus \mathbb{Z}e_i^*, \quad i = 1, ..., n.$$

Clearly the subgroups $\Lambda_1 := \oplus\Gamma_i$, $\Lambda_2 := \oplus\Gamma_i^*$ are isotropic. We have

$$I_{\omega_A} = \begin{pmatrix} b^{-1}a & -b^{-1}\varphi^{-1} \\ (b + ab^{-1}a)\varphi & -ab^{-1} \end{pmatrix}.$$

This operator preserves each subspace $\Gamma_{i\mathbb{R}}$, $\Gamma^*_{i\mathbb{R}}$. In particular it preserves $\Lambda_{1,\mathbb{R}}$, $\Lambda_{2,\mathbb{R}}$. Thus by Lemma 9.4.5 the tori $(\Lambda_{1,\mathbb{R}}/\Lambda_1, I_{\omega_A})$, $(\Lambda_{2,\mathbb{R}}/\Lambda_2, I_{\omega_A})$ are dual. Moreover, it is clear that the torus $(\Lambda_{1,\mathbb{R}}/\Lambda_1, I_{\omega_A})$ is a product of elliptic curves $E_i := (\Gamma_{i\mathbb{R}}/\Gamma_i, I_\omega)$, $i = 1, ..., n$.

For each i the map

$$\Gamma_1 \longrightarrow \Gamma_i, \quad e_1 \mapsto e_i, \quad e^*_{-1} \mapsto \delta_1^{-1}\delta_i e^*_{-i}$$

commutes with I_{ω_A} and so is an isogeny of elliptic curves E_1 and E_i.

If the weak pair (A, ω_A) were an algebraic pair, then by Lemma 9.4.4 there would exist an element $\omega_E \in C_E$ such that the pairs (A, ω_A) and (E, ω_E) are mirror symmetric. Thus we are done in the algebraic case. In general there may not exist the desired $\omega_E \in NS_E(\mathbb{C})^0$, so it may be necessary to choose a different decomposition $\Lambda = \Lambda_1 \oplus \Lambda_2$.

By Lemma 9.4.1 in order for ω_E to exist the symmetric bilinear form $Q_{\mathbb{R}}(I_{\omega_A} J_{A\times\widehat{A}}(\cdot), \cdot)$ must be nondegenerate on $\Lambda_{2,\mathbb{R}}$. This is equivalent to the statement that $J_{A\times\widehat{A}}\Lambda_{2,\mathbb{R}} \cap \Lambda_{2,\mathbb{R}} = 0$. Put $W := \oplus \mathbb{R}e_{-i}$. Then since $J_{A\times\widehat{A}}$ preserves the form $Q_{\mathbb{R}}$ the last equality is equivalent to

$$J_{A\times\widehat{A}} W \cap W = 0. \qquad (*)$$

We are free to apply elements of the symplectic group $\mathrm{Sp}(\Gamma_A, \varphi; \mathbb{Z})$ to the sublattice $\oplus \mathbb{Z}e_{-i}$ to achieve the transversality condition (*) above. Since this discrete group is Zariski dense in the corresponding group of $\mathbb{R}$ –points $\mathrm{Sp}(V_A, \varphi; \mathbb{R})$ and since $\mathrm{Sp}(V_A, \varphi; \mathbb{R})$ acts transitively on the collection of maximal φ –isotropic subspaces of V_A it suffices to prove the following lemma

9.6.2 Lemma. *Let $V \cong \mathbb{R}^{2n}$ be a real vector space with a nondegenerate symplectic form φ. Let $J \in \mathrm{End}(V)$ be a complex structure on V, which preserves φ. Then there exists a maximal φ –isotropic subspace $W \subset V$ such that*

$$JW \cap W = 0.$$

PROOF. Note that the form $\varphi(J\cdot, \cdot)$ on V is symmetric and nondegenerate.

We will choose elements $x_1, ..., x_n \in V$ such that $W := \oplus \mathbb{R}x_i$ has the desired properties by induction on i. Choose x_1 such that $\varphi(Jx_1, x_1) \neq 0$. Let $V =< x_1, Jx_1 > \oplus V'$ be a φ -orthogonal decomposition. Note that V' is J –invariant. Hence we may replace V by V' and choose $x_2 \in V'$ s.t. $\varphi(Jx_2, x_2) \neq 0$. And so on. Cleary the resulting space $W = \oplus \mathbb{R}x_i$ is maximal φ -isotropic and $JW \cap W = 0$. This proves the lemma and the Proposition 9.6.1. □

9.6.3 Corollary. *Let A be an abelian variety with $NS(A) \cong \mathbb{Z}$. Then $C_A = NS_A(\mathbb{C})^0$ and for any $\omega_A \in C_A$ the algebraic pair (A, ω_A) has a mirror symmetric one (E, ω_E) which is like in Proposition 9.6.1 above.*

PROOF. The first assertion is obvious. The second is a direct consequence of Proposition 9.6.1. □

10. The G–construction and its application to mirror symmetry.

10.1. Let A be an abelian variety. The ring $R = \mathrm{End}(A)$ is finite over $\mathbb{Z}$ such that $D := \mathrm{End}^0(A) = R \otimes_{\mathbb{Z}} \mathbb{Q}$ is semisimple with a positive definite anti-involution $* : D \to D^{op}$. The lattice $\Gamma = H_1(A, \mathbb{Z})$ carries a natural structure of a faithful left R -module.

For a given D with an anti-involution $*$ the construction by Gerritzen [5] gives an abelian variety X with $\mathrm{End}^0(X) \cong D$. Furthermore, it was proved in [22] that any finite dimensional algebra over $\mathbb{Q}$ can be realized as $\mathrm{End}^0(T)$ for some complex torus T.

Now we are going to describe these constructions with some corrections and modifications related to the specifics of our situation being that we work with R and not with D.

10.1.1. Let R be an order in $\mathbb{Q}$ -algebra D of finite dimension over $\mathbb{Q}$, i.e. $D = R \otimes_{\mathbb{Z}} \mathbb{Q}$. Denote by Γ a left R -module that satisfies the following conditions:

1. Γ is a lattice of even dimension, i.e. $\Gamma = \mathbb{Z}^{2n}$ as abelian group.
2. Γ is a faithful module, i.e. the homomorphism $R \longrightarrow \mathrm{End}_{\mathbb{Z}}(\Gamma)$ is an embedding.
3. The order $O(\Gamma) := \{ a \in D \mid a(\Gamma) \subset \Gamma \}$ coincides with R.
4. Γ is a direct sum of R-modules Γ_1 and Γ_2 such that there exists an isomorphism $e : M_1 \xrightarrow{\sim} M_2$ of D–modules $M_i = \Gamma_i \otimes_{\mathbb{Z}} \mathbb{Q}$, $(i = 1, 2)$. (15)

Let us fix an isomorphism e and, henceforth, we will often identify D –modules M_1 and M_2 with respect to e and will use notation M .

Denote by C the algebra $\mathrm{End}_D(M)$. The algebras D and C are two subalgebras of $\mathrm{End}_{\mathbb{Q}}(M) \cong M(n,\mathbb{Q})$. Moreover C is a centralizer of D in $M(n,\mathbb{Q})$. On the other side, it is true that the centralizer of C contains D but does not necessarily coincide with D .

Let J be an element of $M(2, C\otimes\mathbb{R}) = (\mathrm{End}_D M^{\oplus 2})\otimes_{\mathbb{Q}}\mathbb{R}$ such that $J^2 = -Id$. Any such J defines a complex structure on the real space $\Gamma_{\mathbb{R}} = (M_1\oplus M_2)_{\mathbb{R}}$ and, as consequence, defines a complex torus

$$A_J := (\Gamma_{\mathbb{R}}/\Gamma, J). \tag{16}$$

10.1.2 Lemma. *For any complex torus A_J , defined above, the endomorphism ring $\mathrm{End}(A_J)$ contains R as subring.*

PROOF. It is clear, because $\mathrm{End}(A_J)$ coincides with maximal subring of $\mathrm{End}_{\mathbb{Z}}(\Gamma)$ that commutes with J . And each $r\in R\subset D$ commutes with any element from $M(2, C\otimes\mathbb{R})$. $\Box$

10.1.3. The set of operators $J\in M(2, C\otimes\mathbb{R})$ with $J^2=-Id$ is not empty. Actually, any element $q = q_1 + iq_2 \in C\otimes_{\mathbb{Q}}\mathbb{C}$ with nondegenerate q_2 defines the operator

$$J(q) = \begin{pmatrix} q_2^{-1}q_1 & -q_2^{-1} \\ q_2 + q_1q_2^{-1}q_1 & -q_1q_2^{-1}\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ q_1 & 1\end{pmatrix}\begin{pmatrix} 0 & -q_2^{-1} \\ q_2 & 0\end{pmatrix}\begin{pmatrix} 1 & 0 \\ -q_1 & 1\end{pmatrix} \tag{17}$$

It is easy to check that $J(q)^2 = -Id$.

10.1.4 Proposition. *Suppose that the centralizer of C in $M(n,\mathbb{Q})$ coincides with D . Then there exists $q\in C\otimes\mathbb{C}$ such that $\mathrm{End}(A_{J(q)}) = R$.*

PROOF. Let $\{e_1 = 1, ..., e_t\}$ be a basis of $\mathbb{Q}$ -vector space C . Set $q_1 = 0$ and $q_2 = \sum\limits_{i=1}^{t} r_ie_i$, where $r_i\in\mathbb{R}$.

Let us describe the algebra $\mathrm{End}^0(A_{J(q)})$. It consists of elements of $\mathrm{End}_{\mathbb{Q}}(M^{\oplus 2})$ that commute with J . Each such element is a 2×2 matrix that satisfies the condition

$$\begin{pmatrix} 0 & q_2^{-1} \\ -q_2 & 0\end{pmatrix}\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} 0 & -q_2^{-1} \\ q_2 & 0\end{pmatrix} = \begin{pmatrix} q_2^{-1}dq_2 & -q_2^{-1}cq_2^{-1} \\ -q_2bq_2 & q_2aq_2^{-1}\end{pmatrix} = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$$

This implies that $c = -q_2bq_2$ and $q_2a = dq_2$.

Let us take q_2 such that $(r_1, ..., r_t)$ are algebraic independent over $\mathbb{Q}$. For such q_2 we obtain $c = 0 = b$ and $ae_i = e_id$ for all i . As $e_1 = 1$ we have $a = d$ and a commutes with any element of C . It follows that a belongs to centralizer of C which coincides with D . Thus for general q_2 the algebra $\mathrm{End}^0(A(q))$ is isomorphic to D .

Now, the algebra $\mathrm{End}(A_{J(q)})$ is an order in $\mathrm{End}^0(A_{J(q)}) = D$ that takes the lattice Γ to itself. By condition 3) of (15) it coincides with R . $\Box$

10.1.5 Remark. Note that if $\Gamma_i\cong R$ as left R -modules for $i = 1,2$ then the lattice $\Gamma = \Gamma_1\oplus\Gamma_2$ satisfies the condition (15). In this case $C\cong D^{op}$ and, in addition, the centralizer of C coincides with D . This way by Proposition 10.1.4 for any R there exists a complex torus with R as the endomorphism ring.

10.2. Let Γ be a lattice as in (15). Suppose that, in addition, Γ satisfies the following extra condition:

5. There is a rational symmetric form $g : M\times M\to\mathbb{Q}$ such that the algebra D is invariant with respect to an anti-involution $\sim$ on $\mathrm{End}_{\mathbb{Q}}(M)$, defined by the rule $\widetilde{\alpha} = g^{-1}\alpha^t g.$ (18)

Consider a rational skew-symmetric form s on $\Gamma_{\mathbb{Q}}$ given as

$$s\langle(m_1, m_2), (n_1, n_2)\rangle = g(m_2, e(n_1)) - g(e(m_1), n_2) \tag{19}$$

Both subspaces $M_1 = \Gamma_1\otimes\mathbb{Q}$ and $M_2 = \Gamma_2\otimes\mathbb{Q}$ are isotropic with respect to s . Since C is a centralizer of D in $\mathrm{End}_{\mathbb{Q}}(M)$ it is invariant with respect to $\sim$ too. Denote by $S(D)$ and $S(C)$ the subspaces of symmetric elements $\widetilde{\alpha} = \alpha$ of D and C respectively.

Take an element $q = q_1 + iq_2 \in S(C)\otimes\mathbb{C}$ with nondegenerate q_2 and consider the complex structure $J(q)$ defined by the formula (17). Since q is symmetric element a computation shows that the form $s_{\mathbb{R}}$ is $J(q)$ -invariant. A some multiple of s is integral. Hence it determines a line bundle on the complex torus $A_{J(q)} = (\Gamma_{\mathbb{R}}/\Gamma, J(q))$. Further, any element $d \in S(D)$ defines the rational form

$$s\langle (dm_1, dm_2), (n_1, n_2)\rangle$$

that is skew-symmetric. This correspondence gives the embedding $i_s : S(D) \hookrightarrow NS_A(\mathbb{Q})$.

10.2.1 Definition. *We say that a weak pair* $(A_{J(q)}, \omega)$ *is obtained by* **G***–construction if the complex torus* $A_{J(q)}$ *is* $(\Gamma_{\mathbb{R}}/\Gamma, J(q))$ *, where* Γ *satisfies contions (15) and (18),* $q \in S(C) \otimes \mathbb{C}$ *, and the form* ω *belongs to* $i_s(S(D)) \otimes \mathbb{C}$ *.*

Now we show that under some assumption we can guarantee the existence of such a form s and moreover the complex torus $A_{J(q)}$ would be an abelian variety.

10.2.2 Lemma. *Let the algebra* D *be semi-simple with a positive definite anti-involution* $* : D \to D^{op}$ *. Then for any* D *-module* M *there exists a positive definite symmetric form* $g : M \times M \longrightarrow \mathbb{Q}$ *such that the anti-involution* $\sim$ *on* $\mathrm{End}_{\mathbb{Q}}(M)$ *, defined by the rule* $\widetilde{a} = g^{-1} a^t g$ *, under the restriction on* D *coincides with* $*$ *.*

Proof. Actually, since D is semi-simple each module is projective. Hence M can be considered as submodule of left free module $D^{\oplus k}$ for some natural number k . Let us define a bilinear form $u : D^{\oplus k} \times D^{\oplus k} \longrightarrow \mathbb{Q}$ by formula:

$$u(x,y) := \sum_{i=1}^{k} [Tr(x_i^* y_i) + Tr(y_i^* x_i)] \quad \text{where} \quad x = (x_1, ..., x_k), y = (y_1, ..., y_k) \in D^{\oplus k}$$

The bilinear form u is symmetric and positive definite. Denote by g the restriction of u on M . It is a positive definite symmetric bilinear form on M . For any endomorphism $a \in \mathrm{End}_{\mathbb{Q}}(M)$ there exists an adjoint with respect to g endomorphism $\widetilde{\alpha}$ given by rule

$$g(m, a(n)) = g(\widetilde{a}(m), n), \quad \text{i.e.} \quad \widetilde{a} = g^{-1} a^t g$$

It is clear that $\sim$ is anti-involution on $\mathrm{End}_{\mathbb{Q}}(W)$ and $\widetilde{d} = d^*$ for any $d \in D$. $\square$

10.2.3 Proposition. *Let a lattice* Γ *be as in (15) and the algebra* D *be a semi-simple with a positive definite anti-involution* $*$ *. Then there exists a neighbourhood* $W \subset S(C) \otimes \mathbb{C}$ *of the point* i *such that for any* $q \in W$ *the complex torus* $A_{J(q)}$ *is an abelian variety.*

If, in addition, the algebra C *is symmetrically generated (i.e. is generated by the subspace* $S(C)$ *). Then there exists* $q \in W$ *such that* $\mathrm{End}(A_{J(q)}) = R$ *.*

Proof. By Lemma 1.1.2 there is a positive definite symmetric form g on M . consider the skew-symmetric form s defined as in (19). The form $s_{\mathbb{R}}$ on $M_{\mathbb{R}}^{\oplus 2}$ is invariant under the action of $J(q)$, and, consequently, a some multiple of s gives a line bundle on $A_{J(q)}$. If this line bundle is ample, then $A_{J(q)}$ is an abelian variety. The ampleness is equivalent to positive definiteness of the symmetric form $s_{\mathbb{R}}\langle J(q)(\cdot), \cdot\rangle$. For $q_1 = 0$ and $q_2 = 1$ we have

$$s_{\mathbb{R}}\langle J(q)(m_1, m_2), (n_1, n_2)\rangle = s_{\mathbb{R}}\langle (-m_2, m_1), (n_1, n_2)\rangle = g(m_1, n_1) + g(m_2, n_2)$$

Hence the positive definiteness of g implies the positive definiteness of the form $s_{\mathbb{R}}\langle J(q)(\cdot), \cdot\rangle$ for $q = i$ and, consequently, for any q from some neighbourhood W of i .

Further, since D is a semi-simple algebra the cenralizer of C coincides with D . By assumption C is generated by $S(C)$ and , consequently, D is the centralizer of $S(C)$. Now the existence of q for which $\mathrm{End} A_{J(q)} = R$ is proved the same way as in Proposition 10.1.4. $\square$

10.2.4 Remark. Note that if the endomorphism ring $\mathrm{End} A_{J(q)}$ coincides with R then the embedding $i_s : S(D) \hookrightarrow NS_{A_J}(\mathbb{Q})$ is an isomorphism.

10.3. Now we are going to give an alternative description of weak pairs which are obtained by the **G**–construction.

10.3.1 Definition. *We say that a weak pair* (A, ω) *is* well-becoming*, if there is a decomposition* $\Gamma = H_1(A, \mathbb{Z}) = \Gamma_1 \oplus \Gamma_2$ *such that the following conditions hold:*

1. *The subspaces* $\Gamma_{k\mathbb{R}} (k = 1, 2)$ *are isotropic with respect to* $\omega = \varphi_1 + i\varphi_2$ *, i.e.* $\omega |_{\Gamma_{k\mathbb{R}}} \equiv 0$ *.*
2. *Maps* $J_{21} : \Gamma_{1\mathbb{R}} \to \Gamma_{2\mathbb{R}}$ *and* $J_{12} : \Gamma_{2\mathbb{R}} \to \Gamma_{1\mathbb{R}}$ *, which are the components of the complex structure* $J = \left(\begin{smallmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{smallmatrix} \right)$ *on* A *, are invertible.*

10.3.2 Remark. The second condition of the previous definition is equivalent to saying that the restrictions of holomorphic n -form to the tori $\Gamma_{k\mathbb{R}}/\Gamma_k$ are non–zero. Moreover, these tori are special Lagragian submanifolds of A .

The following technical results are needed in the sequel.

10.3.3 Definition. *Let M be a vector space over $\mathbb{Q}$ and let $r_1,\dots,r_k$ be elements of real space $M_{\mathbb{R}}$. The minimal vector subspace $W\subset M$ sucht that $r_1,\dots,r_k\in W_{\mathbb{R}}$ is called $\mathbb{Q}$ –envelope of the set $\{r_1,\dots,r_k\}$.*

10.3.4 Lemma. *Let (A,ω) be a weak pair. Suppose there is a decomposition $\Gamma=H_1(A,\mathbb{Z})=\Gamma_1\oplus\Gamma_2$ such that the subspaces $\Gamma_{k\mathbb{R}}$ are ω –isotropic and the map $J_{12}:\Gamma_{2\mathbb{R}}\longrightarrow\Gamma_{1\mathbb{R}}$ is invertible. Then the pair (A,ω) is well-becoming.*

PROOF. Consider the subspace $U\in\mathrm{Hom}_{\mathbb{Q}}(\Gamma_{1\mathbb{Q}},\Gamma_{2\mathbb{Q}})$ consisting of such maps that satisfy the condition $s'\langle l,e(m)\rangle=s'\langle m,e(l)\rangle$ with $l,m\in\Gamma_{1\mathbb{Q}}$ for any $s'\in NS_A\otimes\mathbb{Q}$. One can see that J_{12}^{-1} belongs to $U_{\mathbb{R}}$. Hence, there is invertible $e\in U$. Fix one such e .

Take a sublattice $\Gamma_1'=(\gamma_1,Ne(\gamma_1))$ for sufficient large N , where $\gamma_1\in\Gamma_1$. Consider the decomposition $\Gamma=\Gamma_1'\oplus\Gamma_2$. It easy to check that $\Gamma_{1\mathbb{R}}'$ is isotropic with respect to ω . In addition, maps $J_{21}':\Gamma_{1\mathbb{R}}'\to\Gamma_{2\mathbb{R}}$ and $J_{12}':\Gamma_{2\mathbb{R}}\to\Gamma_{1\mathbb{R}}'$ will be invertible. Hence the pair $(A_{J(q)},\omega)$ is well-becoming. □

10.3.5 Proposition. *The weak pair (A,ω) is well-becoming if and only if it can be obtained by the* **G**–*construction.*

PROOF. $\Leftarrow$ Suppose that a weak pair $(A_{J(q)},\omega)$ is obtained by the **G**–construction. The lattice $H_1(A_{J(q)},\mathbb{Z})=\Gamma=\Gamma_1\oplus\Gamma_2$ satisfies the conditions (15). Moreover, there is a skew-symmetric form s with isotropic sublattices Γ_i for $i=1,2$. This imples that $\Gamma_{i\mathbb{Q}}$ are isotropic with respect to any element from $i_s(S(D))$. As ω belongs to $i_s(S(D))\otimes\mathbb{C}$, the real spaces $\Gamma_{i\mathbb{R}}$ are isotropic with respect to ω . Further, it follows from the formula (17) for $J(q)$ that the map J_{21} is invertible. Hence, by Lemma 10.3.4 the pair $(A_{J(q)},\omega)$ is well-becoming.

$\Rightarrow$ Set $\omega=\varphi_1+\varphi_2$. Denote by W the $\mathbb{Q}$ -envelope of $\{\varphi_1,\varphi_2\}$. Let $D\subset\mathrm{End}(\Gamma_{\mathbb{Q}})$ be a $\mathbb{Q}$ -algebra that is generated by all elements of the form $s_2^{-1}s_1$ with $s_1,s_2\in W$. Any element $s_2^{-1}s_1$ sends the space $\Gamma_{i\mathbb{Q}}$ to itself, because $\Gamma_{i\mathbb{Q}}$ are maximal isotropic with respect to any non-degenerate $s_2\in W$. Therefore the rational spaces $\Gamma_{i\mathbb{Q}}$ are modules over D . Take some integral element $s\in W\bigcap NS_A^0$. Define an anti-involution $\sim$ on D by rule $\widetilde{d}=s^{-1}d^ts$. It is clear that an element $s^{-1}s'$ belongs to the subspace of symmetric elements $S(D)$ for any s' . As in previous lemma let us consider the subspace $U\in\mathrm{Hom}_{\mathbb{Q}}(\Gamma_{1\mathbb{Q}},\Gamma_{2\mathbb{Q}})$ consisting of such maps that satisfy the condition $s'\langle l,e(m)\rangle=s'\langle l,e(m)\rangle$ with $l,m\in\Gamma_{1\mathbb{Q}}$ for any $s'\in NS_A\otimes\mathbb{Q}$. Since J_{12}^{-1} belongs to $U_{\mathbb{R}}$, there is an invertible $e\in U$. It follows from definition of D that e gives an isomorphism between D -modules $\Gamma_{1\mathbb{Q}}$ and $\Gamma_{2\mathbb{Q}}$.

Let R be a maximal order in D that takes Γ to itself. It follows from above that $\Gamma=\Gamma_1\oplus\Gamma_2$ satisfies the conditions (15). By construction, Γ_1 and Γ_2 are isotropic by means of s . Identifying $\Gamma_{1\mathbb{Q}}$ and $\Gamma_{2\mathbb{Q}}$ with respect to e , we can consider J_A as element of $M(2,C\otimes\mathbb{R})$, where C is a centralizer of D . In addition, J_A preserves the form s . Let it has a form $\left(\begin{smallmatrix}J_{11}&J_{12}\\J_{21}&J_{22}\end{smallmatrix}\right)$ with respect to decomposition $\Gamma=\Gamma_1\oplus\Gamma_2$. As in the proof of Lemma 9.4.1, we have

$$\widetilde{J_{21}}=J_{21},\quad \widetilde{J_{12}}=J_{12},\quad \widetilde{J_{11}}=-J_{22}.$$

Put $q_2=-e^{-1}J_{12}^{-1}$ and $q_1=e^{-1}J_{22}J_{12}^{-1}$. It is easy to see that $q=q_1+iq_2$ belongs to $S(C)\otimes\mathbb{C}$ and $J_A=J(q)$, where $J(q)$ is defined by the formula (17).

Finally, any $\omega'\in W_{\mathbb{C}}$ belongs to $i_s(S(D))\otimes\mathbb{C}\subseteq NS_A(\mathbb{C})$ because $W\subseteq i_s(S(D))$. Thus (A,ω') is obtained by **G**–consruction. Proposition is proved. □

We claim that any well-becoming pair has a mirror symmetric pair. First, let us give a construction.

10.4. Construction. Let (A_J,ω) be a well-becoming pair. As above by Λ denote $H_1(A_J\times\widehat{A_J},\mathbb{Z})$. There is a decomposition

$$\Lambda=\Gamma\oplus\Gamma^*=\Gamma_1\oplus\Gamma_2\oplus\Gamma_1^*\oplus\Gamma_2^*.$$

The form ω defines an element $I_\omega\in SO_{\mathbb{Q}}(Q,\mathbb{R})$ by formula (14), where Q is canonical symmetric bilinear form on Λ as in (3.1). By definition the subspaces $\Gamma_{k,\mathbb{R}}$ are isotropic with respect to a ω . This implies that $(\Gamma_1^*\oplus\Gamma_2)_{\mathbb{R}}$ and $(\Gamma_1\oplus\Gamma_2^*)_{\mathbb{R}}$ are the I_ω –invariant subspaces. By I and I' denote the restriction of the linear operator I_ω on the subspaces $(\Gamma_1^*\oplus\Gamma_2)_{\mathbb{R}}$ and $(\Gamma_1\oplus\Gamma_2^*)_{\mathbb{R}}$ respectively. Denote the sublattice $(\Gamma_1^*\oplus\Gamma_2)$ by Σ . Define a complex torus B_I by the rule $B_I=(\Sigma_{\mathbb{R}}/\Sigma, I)$.

10.4.1 Theorem. *Any well-becoming pair (A_J,ω) has a mirror symmetric pair (B_I,θ) , where the complex torus B_I is constructed above. Moreover, the pair (B_I,θ) is well-becoming too.*

PROOF. First, take $\Sigma^*=\Gamma_1\oplus\Gamma_2^*$ and consider the complex torus

$$C_{I'}=(\Sigma^*_{\mathbb{R}}/\Sigma^*, I')$$

By Lemma 9.4.5, the torus $C_{I'}$ is isomorphic to $\widehat{B_I}$. Therefore, there is a canonical identification of lattices $H_1(B_I\times\widehat{B_I},\mathbb{Z})$ and Λ . Moreover, by construction, the element $J_{B\times\widehat{B}}$ coincides with I_ω . Hence, to define mirror symmetric pair we have to find $\theta\in NS_{B_I}(\mathbb{C})^0$ such that $I_\theta=J_{A\times\widehat{A}}$.

Denote by $\left(\begin{smallmatrix}\alpha&\beta\\ \gamma&\delta\end{smallmatrix}\right)$ a matrix of $J_{A\times\widehat{A}}$ with respect to the decomposition $\Lambda=\Sigma\oplus\Sigma^*$. Since $J_A=\left(\begin{smallmatrix}J_{11}&J_{12}\\ J_{21}&J_{22}\end{smallmatrix}\right)$ the map β sends $(a,b)\in\Sigma^*_{\mathbb{R}}=(\Gamma_1\oplus\Gamma_2^*)_{\mathbb{R}}$ to $(-J_{21}^*(b),J_{21}(a))\in\Sigma_{\mathbb{R}}=(\Gamma_1^*\oplus\Gamma_2^*)_{\mathbb{R}}$. As J_{21} is invertible β is invertible too. Hence, by Lemma 9.4.1, there is $\theta=\psi_1+i\psi_2\in NS_{B_I}(\mathbb{C})^0$ such that $I_\theta=J_{A\times\widehat{A}}$. Thus, there is a mirror symmetric pair (B_I,θ) .

Further, since $\psi_2=-\beta^{-1}$ and $\psi_1=\delta\beta^{-1}$ we have that $\psi_k(\Gamma_{2\mathbb{R}})=\Gamma_{1\mathbb{R}}$ and $\psi_k(\Gamma_{1\mathbb{R}}^*)=\Gamma_{2\mathbb{R}}^*$, for $k=1,2$. Hence the subspaces $\Gamma_{1\mathbb{R}},\Gamma_{2\mathbb{R}}^*$ are isotropic with respect to θ . In addition, the map $I_{21}:\Gamma_{1\mathbb{R}}^*\to\Gamma_{2\mathbb{R}}$ coincides with the restriction on $\Gamma_{1\mathbb{R}}$ of the map $-\varphi_2^{-1}$. Hence I_{21} is invertible and the pair (B_I,θ) is well-becoming by Lemma 10.11. This completes the proof. □

10.4.2 Corollary. *Let $A_{J(q)}=(\Gamma_{\mathbb{R}}/\Gamma,J(q))$ be a complex torus obtained by* **G***–constriction with $R=\mathrm{End}A_{J(q)}$. Then for any $\omega=\varphi_1+i\varphi_2\in NS_{A_{J(q)}}(\mathbb{C})$ with non-degenerate φ_2 the pair $(A_{J(q)},\omega)$ has a mirror symmetric pair.*

PROOF. Since $\mathrm{End}A_{J(q)}=R$ the map $i_s:S(D)\longrightarrow NS_{A_{J(q)}}(\mathbb{Q})$ is an isomorphism. Hence, by Lemma 10.3.5, any weak pair $(A_{J(q)},\omega)$ is well-becoming. By Theorem 10.4.1, it has a mirror symmetric pair (B_I,θ) . □

10.4.3 Corollary. *For any abelian variety A there exists an element $\omega\in C_A$ such that the pair (A,ω) has a mirror symmetric pair.*

PROOF. By Theorem 10.4.1, it is sufficient to show that there is ω such that the pair (A,ω) is well-becoming. Since A is an abelian variety there is a non-degenerate skew-symmetric form s on $\Gamma=H_1(A,\mathbb{Z})$ such that the form $s\langle J(\cdot),\cdot\rangle$ is positive definite. There exists isotropic with respect to s decomposition $\Gamma=\Gamma_1\oplus\Gamma_2$. Since the form $s\langle J(\cdot),\cdot\rangle$ is positive definite the linear maps J_{12} and J_{21} are invertible. Thus, the pair (A,ω) with $\omega=cs$ is well-becoming for any $c\in\mathbb{C}^*$. Therefore, it has a mirror symmetric pair. □

Notice that this corollary is a particular case of (9.6.1).

10.5 Theorem. *Let A be an abelian variety of dimension n and $\omega=\varphi_1+i\varphi_2\in C_A$. Let $W\subset NS_A(\mathbb{Q})$ be an $\mathbb{Q}$ –envelope of $\{\varphi_1,\varphi_2\}$. Suppose that algebraic pair $(A,t\omega)$ has a mirror symmetric pair for any $t\in\mathbb{R}^*$. Then for any $\omega'\in C_A\bigcap W_{\mathbb{C}}$ the pair (A,ω') is well-becoming.*

PROOF. As above, consider operators

$$I_{t\omega}:=\begin{pmatrix}\varphi_2^{-1}\varphi_1 & -t^{-1}\varphi_2^{-1}\\ t(\varphi_2+\varphi_1\varphi_2^{-1}\varphi_1) & -\varphi_1\varphi_2^{-1}\end{pmatrix}$$

that act on the real space $\Lambda_{\mathbb{R}}=(\Gamma\oplus\Gamma^*)_{\mathbb{R}}$. By assumption, any pair $(A,t\omega)$ has a mirror symmetric one. Therefore there is a decomposition $\Lambda=\Sigma_{1t}\oplus\Sigma_{2t}$ for any $t\in\mathbb{R}^*$ such that both Σ_{kt} and are Q –isotropic and $\Sigma_{kt\mathbb{R}}$ are $I_{t\omega}$ –invariant ($k=1,2$). Since the set of all sublattices in a lattice is countable there is a decomposition $\Lambda=\Sigma_1\oplus\Sigma_2$ such that $\Sigma_{k\mathbb{R}}$ are $I_{t\omega}$ –invariant for infinite set of t . Hence these subspaces are invariant with respect to the operators

$$\gamma_1=\begin{pmatrix}0&\varphi_2^{-1}\\0&0\end{pmatrix},\qquad \gamma_2=\begin{pmatrix}\varphi_2^{-1}\varphi_1&0\\0&-\varphi_1\varphi_2^{-1}\end{pmatrix},\qquad \gamma_3=\begin{pmatrix}0&0\\ \varphi_2+\varphi_1\varphi_2^{-1}\varphi_1&0\end{pmatrix}.$$

It follows from Lemmas 9.4.1 and 9.4.2 that the map $\psi=\varphi_2+\varphi_1\varphi_2^{-1}\varphi_1$ from $\Gamma_{\mathbb{R}}$ to $\Gamma^*_{\mathbb{R}}$ is invertible.

Denote by Π and Ξ the projections of sublattice Σ_1 on Γ and Γ^* respectively. It is clear that $\Sigma_1\subseteq\Pi\oplus\Xi$. Since Σ_1 is invariant under the action of γ_1 and γ_3 we have $\gamma_1\gamma_3(\Sigma_{1\mathbb{R}})\subset\Sigma_{1\mathbb{R}}$. One can see that $\gamma_1\gamma_3(\Sigma_{1\mathbb{R}})=(\psi\varphi_2^{-1}(\Pi_{\mathbb{R}}),0)$ This imples that $\psi\varphi_2^{-1}(\Pi_{\mathbb{R}})\subseteq\Pi_{\mathbb{R}}$. As ψ is invertible, we have that $\psi\varphi_2^{-1}(\Pi_{\mathbb{R}}) = \Pi_{\mathbb{R}}$ and $(\Pi_{\mathbb{R}}, 0) \subset \Sigma_{1\mathbb{R}}$. By the same argument, $(0, \Xi_{\mathbb{R}}) \subset \Sigma_{1\mathbb{R}}$. Hence $\Sigma_{1\mathbb{R}} = \Pi_{\mathbb{R}} \oplus \Xi_{\mathbb{R}}$ and, consequently, Σ_1 is a sublattice of finite index in $\Pi \oplus \Xi$. But, as the sublattice Σ_1 is a direct summand of Λ it coincides with $\Pi \oplus \Xi$. Moreover, it is easy to see that $\dim \Pi = \dim \Xi = n$, because there are inclusions $\psi(\Pi_{\mathbb{R}}) \subseteq \Xi_{\mathbb{R}}$ and $\varphi^{-1}(\Xi_{\mathbb{R}}) \subseteq \Pi_{\mathbb{R}}$ which are actually equalities.

Futher, take $x, y \in \Pi_{\mathbb{R}}$. There is $l \in \Xi_{\mathbb{R}}$ such that $x = \varphi_2^{-1}(l)$. Since Σ_1 is isotropic with respect to the form Q we obtain that $\varphi_2(x, y) = l(y) = 0$. Similarly, we have $\varphi_1(x, y) = \varphi_1\varphi_2^{-1}(l)(y) = 0$. Hence $\Pi_{\mathbb{R}}$ is isotropic with respect to φ_2 and φ_1 . This yields that the lattice Π is isotropic with respect to any element from W that is the $\mathbb{Q}$ –envelope of $\{\varphi_1, \varphi_2\}$.

Now put $\Gamma_1 := \Pi$. Produce a sublattice $\Gamma_2 \subset \Gamma$ from Σ_2 in the same way as Γ_1 from Σ_1 . It is clear that $\Gamma = \Gamma_1 \oplus \Gamma_2$. By construction, Γ_1 and Γ_2 are isotropic with respect to any element of W . Let $s \in W \bigcap C_A$ be a form that corresponding to an ample line bundle on A . Hence the operator J_A preserves $s_{\mathbb{R}}$ and the symmetric form $s\langle J(\cdot), \cdot\rangle$ is positive definite. It can be checked as in Lemma 9.4.2 that J_{12}, J_{21} are invertible. Thus (A, ω') is well-becoming. $\square$

10.6. Let (A, ω) and (B, θ) be two mirror symmetric well–becoming pairs, as in Theorem 10.4.1. Then $H_1(A, \mathbb{Z}) = \Gamma = \Gamma_1 \oplus \Gamma_2$ and $H_1(B, \mathbb{Z}) = \Sigma = \Gamma_1^* \oplus \Gamma_2$. According to (3.7), both cohomology lattices $H^*(A, \mathbb{Z})$ and $H^*(B, \mathbb{Z})$ carry the structure of $Cl(\Lambda, Q)$ –modules.

Recall that by Proposition 9.3.3 there exists a unique (up to ± 1) isomorphism of $Cl(\Lambda, Q)$ –modules $\beta : H^*(A, \mathbb{Z}) \xrightarrow{\sim} H^*(B, \mathbb{Z})$. It is clear that any such isomorphism can be represented as

$$v_\xi(\cdot) = p_{2*}(\xi \cup p_1^*(\cdot)) \tag{20}$$

for some class $\xi \in H^*(A \times B, \mathbb{Z})$.

We want to show that this class ξ is in fact the Chern character of some complex line bundle on a real subtorus of the product $A \times B.$ As we shall see, this torus is of real dimension $3n,$ where $n = \dim_{\mathbb{C}} A = \dim_{\mathbb{C}} B.$

10.6.1. Let us fix bases $\langle l_1, \dots, l_n\rangle$ of Γ_1 and $\langle l_{n+1}, \dots, l_{2n}\rangle$ of Γ_2. Let the dual bases of Γ_1^* and Γ_2^* be $\langle x_1, \dots, x_n\rangle$ and $\langle x_{n+1}, \dots, x_{2n}\rangle,$ respectively.

Let, as in (3.2), I_A and I_B be the following left ideals of $Cl(\Lambda, Q)$:

$$I_A = Cl(\Lambda, Q) \cdot l_1 \cdots l_{2n}, \qquad I_B = Cl(\Lambda, Q) \cdot l_{n+1} \cdots l_{2n} x_1 \cdots x_n.$$

Since $H^1(A, \mathbb{Z}) \cong \Gamma^*,$ we identify $H^*(A, \mathbb{Z})$ with the ideal I_A as $Cl(\Lambda, Q)$ –modules by the following rule: $x_{i_1} \cup \dots \cup x_{i_k} \in H^k(A, \mathbb{Z})$ goes to $x_{i_1} \cdots x_{i_k} \cdot l_1 \cdots l_{2n} \in I_A$. In the same manner, since $H^1(B, \mathbb{Z}) \cong \Sigma^* = \Gamma_1 \oplus \Gamma_2^*,$ we identify $H^*(B, \mathbb{Z})$ with the ideal I_B as $Cl(\Lambda, Q)$ –modules.

Both I_A and I_B are isomorphic irreducible $Cl(\Lambda, Q)$ –modules. Therefore, there is a unique (up to ± 1) isomorphism between them. It is given by the right multiplication $I_A \cdot x_1 \cdots x_n = I_B$.

All the above gives us a formula for cohomology map β in terms of the bases chosen above:

$$x_S \cup x_R \mapsto (-1)^\epsilon x_R \cup l_{\overline{S}}, \tag{21}$$

with

$$\epsilon = |S| \cdot |R| + \sum_{i \in S} (i - 1), \tag{22}$$

Here $S \subset \{1, \dots, n\},$ $R \subset \{n+1, \dots, 2n\}$, $\overline{S}$ is the complement subset, and $x_S = x_{i_1} \cup \dots \cup x_{i_{|S|}}$, where $i_1 < i_2 < \dots < i_{|S|}$ are the elements of $S,$ etc.

10.7. Now that we have an explicit formula for the map β in the chosen bases, our next step is to compute the class $\xi \in H^*(A \times B, \mathbb{Z}),$ that represents β in the form (20). In fact, we will define a real subtorus in $A \times B$ with a complex line bundle on it, and then show that the direct image of the Chern character of this line bundle is the desired $\xi.$ First, take this real torus to be $T := \Pi_{\mathbb{R}}/\Pi,$ where $\Pi = \Gamma_1 \oplus \Gamma_2 \oplus \Gamma_1^*.$ Clearly, since Π is a sublattice of Λ one can consider T as a subtorus both of $A \times \widehat{A}$ and $B \times \widehat{B},$ because the first homology lattices of these two are identified with $\Lambda.$

10.7.1 Lemma. *The restrictions* $P_A|_T$ *and* $P_B^{-1}|_T$ *are isomorphic as complex line bundles on* T .

Proof. It follows from the well-known fact that the complex line bundle is determined by its first Chern class (see for example [10] or [11]). Using Lemma 4.2.3.1, it is easy to check that the first Chern classes of these complex line bundles coincide. $\square$

10.7.2. Put $L := P_A|_T$. Lemma 4.2.3.1 gives us the formula for $c_1(P_A)$. So we have

$$c_1(L) = \sum_{i=1}^{n} x_i \cup l_i,$$

where x_i and l_i are elements of $H^1(T, \mathbb{Z}) \cong \Pi^* = \Gamma_1^* \oplus \Gamma_2^* \oplus \Gamma_1$.

Note that T has projections onto A and B and thus is naturally embedded into the product $j : T \hookrightarrow A \times B$.

10.7.3. Let now τ be a class in $H^n(A \times B, \mathbb{Z})$ such that if we evaluate the formula (20) on the class τ, the resulting transformation sends the monomials of the form $x_R \cup x_n \cup \cdots \cup x_1 \in H^*(A, \mathbb{Z})$ to $x_R \in H^*(B, \mathbb{Z})$, and any other monomials to zero (here $R \subset \{n+1, \ldots, 2n\}$). The class τ corresponds to the homology class $[T]$ in $H_{3n}(A \times B, \mathbb{Z})$ under Poincare isomorphism and suitable choice of an orientation on T.

Any class in $H^*(T, \mathbb{Z})$ is the restriction of a class from $H^*(A \times B, \mathbb{Z})$. In particular, $c_1(L)$ is the restriction of the class $D \in H^2(A \times B, \mathbb{Z})$ given by the following formula:

$$D = \sum_{i=1}^{n} p_1^*(x_i) \cup p_2^*(l_i).$$

Therefore, one has

$$j_*(ch(L)) = j_* j^*(\exp(D)) = \tau \cup \exp(D),$$

where j_* is map from $H^*(T, \mathbb{Z})$ to $H^{*+n}(A \times B, \mathbb{Z})$ that is obtained from evident homology map $H_*(T, \mathbb{Z}) \longrightarrow H_*(A \times B, \mathbb{Z})$ under Poincare isomorphism.

10.8 Proposition. *The map β coincides with the map given by the formula (20) with $\xi = j_*(ch(L^{(-1)^{(n-1)}})$.*

PROOF. Let S and R be index subsets of $\{1, \ldots, n\}$ and $\{n+1, \ldots, 2n\}$, respectively. Take a monomial $x_S \cup x_R$ in the cohomologies of A. We already know the image of this element under β (see (21)). Now we are going to compute its image under the map given by v_ξ.

$$v_\xi(x_S \cup x_R) = p_{2*}(\tau \cup \exp((-1)^{n-1} D) \cup p_2^*(x_S \cup x_R)). \tag{23}$$

Computing the Chern character $\exp((-1)^{n-1} D)$ analogously to Lemma 4.2.3.2, we see that

$$\exp((-1)^{n-1} D) = \sum_{Q \subset \{1, \cdots, n\}} (-1)^{(n-1)|Q|} p_1^*({}_Q x) \cup p_2^*(l_Q)$$

where ${}_Q x$ is a product $x_{i_{|Q|}} \cup \cdots \cup x_{i_1}$ by set $Q = \{i_1, \ldots, i_{|Q|}\}$ in the descending order. Therefore, the entire expression on the right–hand side of (23) rewrites as

$$\begin{aligned} v_\xi(x_S \cup x_R) = (-1)^{(n-1)|\overline{S}|}\, & p_{2*}(\tau \cup p_1^*(x_S) \cup p_1^*(x_R) \cup p_1^*({}_{\overline{S}}x) \cup p_2^*(l_{\overline{S}})) = \\ (-1)^{(n-1)|\overline{S}| + |S||R|}\, & p_{2*}(\tau \cup p_1^*(x_R \cup x_S \cup {}_{\overline{S}}x)) \cup p_2^*(l_{\overline{S}})) = \\ (-1)^{(n-1)|\overline{S}| + |S||R| + \sum_{i \in S}(n-i)}\, & p_{2*}(\tau \cup p_1^*(x_R \cup x_n \cup \cdots \cup x_1) \cup p_2^*(l_{\overline{S}})) = \\ (-1)^{(n-1)(|\overline{S}| + |S|) + |S||R| + \sum_{i \in S}(i-1)}\, & p_{2*}(\tau \cup p_1^*(x_R \cup x_n \cup \cdots \cup x_1)) \cup l_{\overline{S}} = \\ (-1)^{(n-1)n + \epsilon}\, & x_R \cup l_{\overline{S}} = \\ (-1)^{\epsilon}\, & x_R \cup l_{\overline{S}} \end{aligned}$$

in $H^*(B, \mathbb{Z})$ and ϵ was defined by formula (22). Thus, the maps v_ξ and β coincide. $\square$

APPENDIX (the proof of Theorem 8.2).

A.1 Theorem. *Let A be an abelian variety.*

a) The action of $\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$ on C_A is well defined
b) C_A^+ and C_A^- are single orbits under this action.
c) The stabilizer of a point in C_A is a maximal compact subgroup of $\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$

PROOF. Let us prove the assertion about C_A^+. The case of C_A^- is similar. It will be more convenient to work with a different open subset of $NS_A(\mathbb{C})$. Namely, recall that every element $z \in NS_A(\mathbb{R})^0$ defines the corresponding Rosati involution on $\mathrm{End}(A)\otimes\mathbb{R}$:

$$a \mapsto a' := z^{-1}\hat{a}z.$$

Put $NS_A(\mathbb{R})^+ := \{z \in NS_A(\mathbb{R}) \mid \forall a \in \mathrm{End}(A)\otimes\mathbb{R} \quad Tr(aa') > 0\}$. That is $NS_A(\mathbb{R})^+$ is an open subset of $NS_A(\mathbb{R})$ which consists of elements z that define a positive Rosati involution.

A.2 Lemma. *The ample cone C_A^a is a connected component of $NS_A(\mathbb{R})^+$. Hence C_A^+ is a connected component of $NS_A(\mathbb{R}) + iNS_A(\mathbb{R})^+$.*

PROOF. We know [Mum1] that an ample class $z \in NS_A$ belongs to $NS_A(\mathbb{R})^+$. Hence $C_A^a \subset NS_A^+$.

Every $z \in NS_A(\mathbb{R})$ defines a symmetric bilinear form b_z on V_A as follows

$$b_z(x,y) := z(J_Ax)(y).$$

By the Lefschetz theorem ([Mum1]) a class $z \in NS_A$ is ample iff b_z is positive definite. Thus we obtain another characterization of the ample cone

$$C_A^a = \{z \in NS_A(\mathbb{R}) \mid b_z \ is\ positive\ definite\}.$$

It follows that for $z \in \partial C_A^a$ the form b_z is degenerate, i.e. the map z has nontrivial kernel. Thus $\partial C_A^a \subset \partial NS_A(\mathbb{R})^+$. Therefore, C_A^a is a connected component of $NS_A(\mathbb{R})^+$. □

In the next few lemmas we study the $\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$ -action on the set $NS_A(\mathbb{R}) + iNS_A(\mathbb{R})^+$.

A.3 Lemma. *Let $g = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$, $\omega \in NS_A(\mathbb{R}) + iNS_A(\mathbb{R})^+$. Then the endomorphism $(a+b\omega) \in \mathrm{End}(V_A)$ is invertible, i.e. the element $g\omega \in NS_A(\mathbb{C})$ is well defined.*

PROOF. Let $\omega = \eta + iz$, $\eta \in NS_A(\mathbb{R})$, $z \in NS_A(\mathbb{R})^+$. The lemma follows from the following assertion: for every nonzero $x \in \mathrm{End}(V_A \otimes_{\mathbb{R}} \mathbb{C})$

$$Im(Tr(z^{-1}\hat{\bar{x}}(\hat{c} + \hat{\bar{\omega}}\hat{d})(a+b\omega)x)) < 0.$$

The assertion is proved by a straightforward calculation which we omit .

A.4 Remark. Let $\omega = \eta + iz \in NS_A(\mathbb{C})$. Then $\left(\begin{smallmatrix} 1 & 0 \\ -\eta & 1\end{smallmatrix}\right) \in \mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$ and $\left(\begin{smallmatrix} 1 & 0 \\ -\eta & 1\end{smallmatrix}\right)\omega = iz$. Therefore all $\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$ -orbits in $NS_A(\mathbb{C})$ are invariant under translation along $NS_A(\mathbb{R})$.

A.5 Lemma. *Let $\omega \in NS_A(\mathbb{R}) + iNS_A(\mathbb{R})^+$. Let K_ω be its stabilizer in $\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$. Then K_ω is a maximal compact subgroup of $\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$.*

PROOF. By the previous remark we may assume that $\omega = iz$. Denote by $'$ the Rosati involution defined by z. We have

$$K_\omega = \left\{ \begin{pmatrix} a & b \\ -zbz & zaz^{-1}\end{pmatrix} \in \mathrm{U}_{A,\mathbb{Q}}(\mathbb{R}) \middle| \ z^{-1}\hat{a}za + \hat{b}zbz = 1,\ z^{-1}\hat{a}zb = \hat{b}zaz^{-1} \right\}$$

Consider a map $\vartheta : \mathrm{U}_{A,\mathbb{Q}}(\mathbb{R}) \to M_2(\mathrm{End}(A)\otimes\mathbb{R})^*$, given as $\vartheta\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} a & bz \\ z^{-1}c & z^{-1}dz\end{pmatrix}$.

Then

$$Im\vartheta = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta\end{pmatrix} \middle| \begin{pmatrix} \alpha & \beta \\ \gamma & \delta\end{pmatrix}^{-1} = \begin{pmatrix} \delta' & -\beta' \\ -\gamma' & \alpha'\end{pmatrix} \right\} \quad \text{and} \quad \vartheta(K_\omega) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha\end{pmatrix} \middle| \begin{matrix} \alpha'\alpha + \beta'\beta = Id, \\ \alpha'\beta = \beta'\alpha \end{matrix} \right\}$$

Extend the Rosati involution $'$ to $M_2(\mathrm{End}(A))\otimes\mathbb{R}$ by the formula

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta\end{pmatrix}' = \begin{pmatrix} \alpha' & \gamma' \\ \beta' & \delta'\end{pmatrix}$$

and consider the symmetric bilinear form on $M_2(\mathrm{End}(A))\otimes\mathbb{R}$

$$\Psi(X,Y) = Tr(X'Y), \quad \text{i.e.} \quad \Psi(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta\end{pmatrix}, \begin{pmatrix} a & b \\ c & d\end{pmatrix}) = Tr(\begin{pmatrix} \alpha' & \gamma' \\ \beta' & \delta'\end{pmatrix}\begin{pmatrix} a & b \\ c & d\end{pmatrix}) = Tr(\alpha'a + \gamma'c + \beta'b + \delta'd).$$

This form is positive definite.

Consider the action of the group $M_2(\mathrm{End}(A)\otimes\mathbb{R})^*$ on $M_2(\mathrm{End}(A)\otimes\mathbb{R})$ by left multiplication. Let $\mathrm{U}(\Psi) \subset M_2(\mathrm{End}(A)\otimes\mathbb{R})^*$ be the subgroup which preserves the form Ψ. We claim that

$$\vartheta(\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})) \cap \mathrm{U}(\Psi) = \vartheta(K_\omega).$$

Indeed,

$$\mathrm{U}(\Psi) = \{u \in M_2(\mathrm{End}(A) \otimes \mathbb{R})^* \mid u'u = Id\}.$$

Thus $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \vartheta(\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})) \cap \mathrm{U}(\Psi)$ iff $\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) = \begin{pmatrix} d' & -b' \\ -c' & a' \end{pmatrix}$. This means that $a = d$, $b = -c$, $a'a + b'b = 1$, $a'b = b'a$, which in turn means that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \vartheta(K_\omega)$. Thus K_ω is compact.

Notice that the adjoint operator for the left multiplication by $m \in M_2(\mathrm{End}(A) \otimes \mathbb{R})$ with respect to Ψ is the left multiplication by m'. The group $\vartheta(\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R}))$ is self adjoint, i.e. $t \in \vartheta(\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})) \Rightarrow t' \in \vartheta(\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R}))$. Put

$$P_\omega = \{g \in \mathrm{U}_{A,\mathbb{Q}}(\mathbb{R}) \mid \vartheta(g)\ is\ positive\ self\ adjoint\}.$$

The lemma now follows from the following claim.

A.6 Claim.

a) The multiplication map $K_\omega \times P_\omega \to \mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$, such that $(b,p) \mapsto bp$, is a diffeomorphism
b) K_ω is a maximal compact subgroup of $\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$.

Indeed, since $\vartheta(\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R}))$ is self-adjoint, a) follows from theorem 1 in sect.2, ch.5 in [OV]. The assertion b) follows from problem 2 in sect.2, ch.5 in [OV]. □

A.7 Lemma. *The set $NS_A(\mathbb{R}) + iNS_A(\mathbb{R})^+$ is $\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$ -invariant.*

PROOF. Let $\omega \in NS_A(\mathbb{R}) + iNS_A(\mathbb{R})^+, g \in \mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$. We need to show that $g\omega \in NS_A(\mathbb{R}) + iNS_A(\mathbb{R})^+$. By Remark A.4 above we may asssume that $\omega = iz$, $z \in NS_A(\mathbb{R})^+$, and $g\omega = ip$, $p \in NS_A(\mathbb{R})$. Then we need to show that
(i) p is invertible,
(ii) the Rosati involution defined by p is positive definite (i.e. $p \in NS_A(\mathbb{R})^+$.)

Denote by $'$ the Rosati involution defined by z. Put $p = zk$ for $k \in \mathrm{End}(A) \otimes \mathbb{R}$. We have $k' = k$. Let $\lambda \in \mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$ be such that $\lambda(iz) = ip$. Then λ has the form

$$\lambda = \begin{pmatrix} a & b \\ -zkbz & zkaz^{-1} \end{pmatrix}.$$

Under the map $\vartheta : \mathrm{U}_{A,\mathbb{Q}}(\mathbb{R}) \to M_2(\mathrm{End}(A) \otimes \mathbb{R})$, defined in the proof of Lemma A.5, λ goes to

$$\vartheta(\lambda) = \begin{pmatrix} \alpha & \beta \\ -k\beta & k\alpha \end{pmatrix},$$

where

$$1)\ \alpha\alpha'k + \beta\beta'k = Id \quad \text{and} \quad 2)\ -\alpha\beta' + \beta\alpha' = 0$$

1) implies that k is invertible , hence p is such. This proves (i). Let us prove (ii). Note that

$$\begin{pmatrix} k & 0 \\ 0 & \widehat{k}^{-1} \end{pmatrix} \in \mathrm{U}_{A,\mathbb{Q}}(\mathbb{R}) \quad \text{and} \quad \begin{pmatrix} k & 0 \\ 0 & \widehat{k}^{-1} \end{pmatrix} ip = izk^{-1}.$$

Replacing izk by izk^{-1} and applying the previous argument we find $s, t \in \mathrm{End}(A) \otimes \mathbb{R}$ such that $ss'k^{-1} + tt'k^{-1} = 1$, $-st' + ts' = 0$, that is,

$$3)\ k = ss' + tt', \quad \text{and} \quad 4)\ -st' + ts' = 0$$

The Rosati involution defined by p is

$$a \mapsto p^{-1}\widehat{a}p = k^{-1}a'k, a \in \mathrm{End}(A) \otimes \mathbb{R}.$$

We need to show that $Tr(k^{-1}a'ka) > 0$ for all $0 \neq a \in \mathrm{End}(A) \otimes \mathbb{R}$. Or, equivalently, that the following quadratic form on $M_2(\mathrm{End}(A) \otimes \mathbb{R})$ is positive definite

$$Tr(\begin{pmatrix} k^{-1} & 0 \\ 0 & k^{-1} \end{pmatrix} X' \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} X), \quad X \in M_2(\mathrm{End}(A) \otimes \mathbb{R}).$$

Put $Y = \left(\begin{smallmatrix} \alpha & \beta \\ -\beta & \alpha \end{smallmatrix}\right)$ and $Z = \left(\begin{smallmatrix} s & t \\ -t & s \end{smallmatrix}\right)$. Then

$$1), 2) \Leftrightarrow YY' = \begin{pmatrix} k^{-1} & 0 \\ 0 & k^{-1} \end{pmatrix}, \quad \text{and} \quad 3), 4) \Leftrightarrow ZZ' = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}.$$

Hence

$$Tr(\begin{pmatrix} k^{-1} & 0 \\ 0 & k^{-1} \end{pmatrix} X' \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} X) = Tr(YY'X'ZZ'X) = Tr(Y'X'ZZ'XY) = Tr((Z'XY)'(Z'XY)) > 0$$

This proves (ii) and the Lemma.

A.8 Lemma. *The set* $NS_A(\mathbb{R}) + iNS_A(\mathbb{R})^+$ *consists of finitely many* $\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$ *-orbits.*

A.9 Corollary. *The* $\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$ *-orbits in* $NS_A(\mathbb{R}) + iNS_A(\mathbb{R})^+$ *coincide with the connected components of this set.*

Proof of corollary. By the Corollary 5.3.5 the group $\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$ is connected. Hence each $\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$ -orbit is contained in a connected component of $NS_A(\mathbb{R}) + iNS_A(\mathbb{R})^+$. It follows from Lemma A.5 that all $\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$ -orbits in $NS_A(\mathbb{R}) + iNS_A(\mathbb{R})^+$ are isomorphic (all maximal compact subgroups in a reductive Lie group are conjugate). Then by Lemma A.8 each $\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$ -orbit is an open subset in $NS_A(\mathbb{R}) + iNS_A(\mathbb{R})^+$, hence must coincide with a connected component of this set. This proves the corollary.

A.10 Remark. Theorem A.1 now follows from A.2, A.3, A.5 and A.9. So it remains to prove Lemma A.7.

Proof of Lemma A.7. By Remark A.4 above it suffices to show that the set $iNS_A(\mathbb{R})^+$ is contained in a finite number of $\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$ -orbits.

Fix $z \in NS_A(\mathbb{R})^+$ and let $'$ be the corresponding Rosati involution. Let $l \in Aut_A(\mathbb{R}) := (\mathrm{End}(A) \otimes \mathbb{R})^*$. Then iz and $izl'l$ are $\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$ –conjugate. Indeed,

$$\begin{pmatrix} l^{-1} & 0 \\ 0 & \widehat{l} \end{pmatrix} iz = i\widehat{l}zl = izl'l.$$

Consider the space $\mathrm{End}(A) \otimes \mathbb{R}$ with the positive definite symmetric form $\phi(a,b) := Tr(ab')$ and the adjoint action of the group $Aut_A(\mathbb{R})$: $ad_s(a) := sas^{-1}$.

The adjoint with respect to ϕ of the operator ad_s is $ad_{s'}$. Hence the following are equivalent

(i) ad_s is self adjoint;

(ii) $s = s'c$ for some c in the center of $Aut_A(\mathbb{R})$.

Let now $izk \in iNS_A(\mathbb{R})^+$ for some $k \in Aut_A(\mathbb{R})$.

A.11 Claim. *There exist* $l \in Aut_A(\mathbb{R})$ *and* $c \in Z(Aut_A(\mathbb{R}))$ *such that* $c' = c, c^2 = 1$ *and* $k = l'lc$.

Indeed, we have $\widehat{zk} = zk$, i.e. $k' = k$. Thus ad_k is self adjoint with respect to ϕ. The positivity of the Rosati involution defined by zk

$$a \mapsto k^{-1}a'k$$

is equivalent to the positivity of the self adjoint operator ad_k. Hence the operator ad_k has a square root (which is also positive self adjoint). But this square root is also in the image of the homomorphism ad (see ch. 5, sect 2, thm. 1 in [OV]). Thus there exists $l \in Aut_A(\mathbb{R})$ such that $ad_l = ad_{l'}$ and $(ad_l)^2 = ad_k$. We get $l'lc = k$ for some $c \in Z(Aut_A(\mathbb{R}))$. Moreover,

$$l'lc = k = k' = (l'lc)' = c'l'l.$$

Thus $c' = c$.

It follows from the discussion of $\mathrm{End}(A)$ in section 1.8 that the $'$ -invariant part of the center $Z(\mathrm{End}(A) \otimes \mathbb{R})$ is isomorphic to $\mathbb{R} \times \cdots \times \mathbb{R}$. therefore, multiplying c by a positive real number r (and l by $r^{-1/2}$) we may assume that $c^2 = 1$. This proves the claim.

Note that there are finitely many $c \in Z(Aut_A(\mathbb{R}))$ such that $c' = c$ and $c^2 = 1$, say, $c_1, \ldots, c_m$. Then by the argument in the beginning of the proof of the lemma $iNS_A(\mathbb{R})^+$ is contained in the union of $\mathrm{U}_{A,\mathbb{Q}}(\mathbb{R})$ -orbits of $izc_1, izc_2, \ldots, izc_m$. This proves Lemma A.8 and completes the proof of Theorem A.1.

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Institute for Problems of Information Transmission, Russian Academy of Sciences, 19 Bolshoi Karetnyi, Moscow 101447, Russia

E-mail address: golyshev@mccme.ru

Department of Mathematics, Indiana University, Bloomington, IN 47405, USA

E-mail address: vlunts@indiana.edu

Algebra Section, Steklov Mathematical Institute, Russian Academy of Sciences, 8 Gubkin str., GSP-1, Moscow 117966, Russia

E-mail address: orlov@mi.ras.ru