SuperLie and problems (to be) solved with it

Pavel Grozman, Dimitry Leites

Department of Mathematics, University of Stockholm, Roslagsv. 101, Kräftriket hus 6, S-106 91, Stockholm, Sweden; e-mail: mleites@math.su.se; grozman@equa.se or pavel@rixtele.com

Key words and phrases. Lie superalgebras, cohomology, differential equations

PG and DL gratefully acknowledge financial support of the Swedish Institute (1992-94) and NFR (1994-00), and TBSS (2001/02), and MPIM, Bonn (2002).

Abstract. A *Mathematica*-based package **SuperLie** for the study of Lie (super)algebras and their cohomology is offered. Among applications we find:

(1) calculation of the Riemannian tensors needed to write Supergravity Equations for any N-extended Minkowski superspace and to classify possible models for these superspaces;

(2) the possibility to study stability of nonholonomic systems (among which we find: ballbearings, gyroscopes, electro-mechanical devices like rotor collector with a gliding contact; ideal gas and market economy, etc.);

(3) description of the analogue of the curvature tensor for nonlinear nonholonomic constraints and the fields of solids or their surfaces, e.g., cones, as in optimal control;

(4) a new method for the study of integrability of differential equations (numerical methods can provide with individual solutions but are unable, generally, to study qualitative behavior, e.g., stability);

(5) lists of "natural" operators, i.e., the operators between sections of tensor fields (or jets) invariant with respect to the group of diffeomorphism or its subgroup;

(6) list central extensions and deformations.

All the above problems can be expressed in terms of Lie algebra cohomology.

SuperLie makes it possible to

(1) determine Lie algebras as subalgebras, via defining relations, from Cartan matrix, realize via vector fields, as polynomials with Poisson or contact (Legendre) bracket, etc.,

(2) determine various modules over these Lie algebras (tensors; modules with a vacuum vector; submodules, quotients, duals, etc.), and calculate the bracket as well as the action of the Lie algebra on its module,

(3) solve systems of equations with scalar and (linear) vector unknowns; find ker and Im of a linear operator;

(4) calculates exterior differential in various complexes.

A partial list of open problems, currently under study, is offered: the students in search of a topic for a MS or a PhD thesis, as well as other interested, are invited to join forces. Please direct your questions/suggestions to PG if SuperLie/Matehmatica related, to DL if mathematics related.

Preface

This is an expanded version of Leites's Special Invited Lecture delivered at at NATO ASI Special Functions-2000, Tempe, Arizona, June 2000 and a version of the transcript of Leites' lectures at Stockholm University addressed to students from diferent departments who attended an informal "math circle" once a week for about a year. The aim of the circle was to provide with a background necessary to calculate homology and cohomology of Lie algebras and Lie superalgebras needed in various, sometimes unexpected, applications: from supergravity to stability of ballbearings and in market economy, to formal solvability of differential equations.

These calculations can, sometimes, be performed analytically, but most often than not they are too complicated to be performed by bare hands. Besides, there are few general theorems to calculate (co)homology analytically.

SuperLie is a package designated to compute certain Lie algebra cohomology as an initial goal but it soon evolved into a package that makes it possible to

(1) determine Lie algebras and Lie superalgebras via defining relations, or via Cartan matrix, or realize them via vector fields, or as polynomials (or Laurent polynomials) with Poisson or contact bracket, or antibracket, etc.

(2) construct various modules over these Lie algebras and superalgebras, e.g., tensors, with a vacuum vector, etc.,

(3) construct central extensions and deformations.

In the lectures we also tried not just illustrate some calculations with motivations, but explain how to perform such calculations. The task was not easy since students from departments of computer science (not only at Stockholm University; even from the former Soviet Union, where math background was supposed to be solid) vaguely remember what a vector space is and do not know what the Lie algebras, tensor products and quotient spaces are.

So to be accessible to students and engineers, this review was appended with a brief introduction into representation theory of Lie algebras and finite groups [**BKL**].

Rival or not so rival codes. Several packages for computation of Lie algebra and Lie superalgebra homology and cohomology are known. None of them is, so far, sufficiently efficient to overcome D. Kalinin's results [**GKF**] of early 1970's.

After D. Kalinin's untimely death, no trace was left as to how he computed so wonderfully. This is indeed a challenge: compared with modern machines, the computer Kalinin used does not differ much from abacus, still his result was never surpassed!

The known to us other packages that compute Lie algebra (co)homology or, at least, the modules over them, distinguish an irreducible submodule in a module, etc., are as follows.

Their documetation (and some additional manuals) are also available at the homepage of MPIM, Bonn, see

http://www.math.usu.edu/fg_mp/Pages/SymbolicsPage/Symbolics.html

There are at least two codes that complement Grozman's SuperLie. (Ideally, it would be great to join forces, or, at least, make links, and smoothen the seams. But this is a long and expencive job.) These codes are:

1) The package **http://www.mpim-bonn.mpg.de/html/computer/software.html** can calculate cohomology of finite dimensional Lie algebras with trivial coefficients, and relative cohomology (with respect to a subalgebra).

Moreover, this code can compute many things SuperLie can not (the derived series, verification of nilpotency, etc.), and not only for Lie algebras (it can consider tensors, manifolds, differential equations and Spencer cohomology; given time and money, we should have written a comprehensive toc with comments as to pros and cons).

It can not, however, compute anything in *super*setting nor deal directly with infinite dimensional case, whereas SuperLie can.

The package is based on Maple. Documentation is on approximately same level as currently that for SuperLie (in blunt nordic terms: rather pitiful) but in better English.

We did not compare the speeds of performances.

2) **http://www.mpim-bonn.mpg.de/html/computer/software** This is the link to **LiE, A software package for Lie group computations**. The package is designed for work with reductive Lie groups and Lie algebras. **LiE** returns properties of Lie groups, Lie algebras and modules over them in the framework of a reference book, but — unlike the reference book — for any given reductive Lie group or Lie algebra.

LiE can effectively compute, among other things, dimensions of modules, decomposition of the tensor product, symmetric and alternating powers of irreducible representations into irreducible components, systems of weights, characters, orbits, and Dynkin diagrams.

The *elements* of Lie algebras and Lie groups are not available in **LiE**, so there is no (clear) way to compute anything with them. Moreover, **LiE** does not see any difference between Lie groups and Lie algebras, since it only deals with common *properties* of both.

These properties are expressed as integers, vectors, matrices or integer Laurent polynomials. All vectors and matrices are only numerical ones, and **LiE** can perform various calculations with them. Calculations with the same objects but with rational coefficients are not realized.

LiE can only work with finite dimensional modules. (An attempt to compute the dimension of the highest weight $\mathfrak{sl}(2)$-module with highest weight -1 returned 2.)

LiE is an independent package, written in C for Unix. The initial code is available. Moreover, certain computations are available at

http://young.sp2mi.univ-poitiers.fr/˜marc/LiE/form.html.

Summary: **LiE** has few features in common with SuperLie. **LiE** and SuperLie nicely complement each other. It would be useful if somebody would have taught **LiE** deal with superalgebras. In future we should teach SuperLie how to appeal to **LiE** when needed.

Other codes for dealing with certain problems of Lie superalgebras.

1) **http://www-gap.dsc.st-and.ac.uk/ gap** for computations related with Groups, Algorithms and Programming

2) **Macauley2** is also dealing with various aspects of Linear superalgebra. It is a favourite of D. Eisenbud and his colleagues with impressive results. These results and approaches are, so far, transversal to ours, though also deal with superalgebras.

Miscellanea. SuperLie is *Mathematica*-based, hence, rather slow. Moreover, dealing with more voluminous problems we have realized that *Mathematica* has certain inbuilt (but not publicized) restrictions: in can not process data if its volume is greater than 2GB or so. So the motto "if your code is slow, push the button a day earlier" does not help: the machine just aborts. As a result we were unable, regrettably, to use facilities of medics.polytechnique.fr run extremely user-friendly personell directed by Marc Giusti.

So a natural way out is to **rewrite everything in a protho-language**, say, in C.

• V. Kornyak undertook the task but, so far, his code can not produce any result not obtained several years earlier by SuperLie, or easily obtainable by SuperLie.

• Certain aspects of Lie superalgebras are considered (see MR 97j:17006) in

Mikhalev, A A.; Zolotykh, A. A. *Combinatorial aspects of Lie superalgebras.* With 1 IBM-PC floppy disk (3.5 inch; HD). CRC Press, Boca Raton, FL, 1995. viii+260 pp.

• There are several other groups working (pitifully independently in view of scanty funding) on various aspects of representation theory. We would like to mention one: **http://www.mathematik.uni-bielefeld.de/birep**

To be done. We did not (for the lack of resources) try several natural ways to improve the performance of SuperLie:

1) the matrix of the differential for the Lie (super)algebra (co)homology for any algebra and any module is very sparce. We never used this.

2) When the structure constants are integers, one can try to compute cohomology mod p. We did this recently but the going is so slooooooow that clearly this is not THE way.

3) We never tried to parallelize the computations.

PROJECTS: a rough description of selected interesting problems solvable with SuperLie or its future generalization

The following are projects for computer-assisted study of certain tempting mathematical problems. It was not so long ago that some of the physical problems considered in what follows were formulated in purely mathematical terms and, therefore, became solvable, at least theoretically.

Many relations with other branches of mathematics (like calculations of local invariants of nonholonomic manifolds ranging from supergravity to stability of rolling bodies to stability of markets) are far from being clear yet. But even the parts we have understood, lead to a possibility to solve old problems. One "only" has to be able to use computers and interprete the results ALREADY obtainable by means of SuperLie.

The advantage of these problems is that one can start at virtually any level. The only formally needed knowledge is that of how to take partial derivative and definitions of tensor (or symmetric, or exterior) products of linear spaces.

In reality, of course, one has to learn representation theory (the more the better) of Lie algebras (and, to understand it, of finite and compact groups).

Direction 1. Selected applications of Lie superalgebra cohomology.

Background. Invariants of various types help us to describe the physical picture. The knowledge of how to use invariants responsible for stability can be rather vital (Chernobyl or misguided missiles illustrate what we mean). It is known that the sign of the curvature of the space of positions governs stability: if it is positive, as on the sphere, then geodesics (the admissible trajectories of the particle) converge (as meridians do) and if the curvature is negative, then geodesics diverge.

It was only recently understood that the curvature can be interpreted as certain Lie algebra cohomology. Similar cohomology appear as obstructions to formal integrability of differential equations. These obstructions were previously expressed in other terms (Spencer cohomology), difficult to compute, and so far were never used by engineers.

To solve differential equations, people usually use various numerical methods. Here one more, totally new, approach to several types of problems is suggested: in terms of Lie algebra cohomology one can

(1) define criteria of formal integrability of differential equations

(2) calculate the curvature tensor not only for holonomic systems, but — for the first time since Hertz raised the problem — for nonholonomic dynamical systems (which seem to

be omnipresent, unlike the holonomic ones, with which the modern textbooks on analytical mechanics tend to confine themselves). One can even consider *nonlinear* constraints on velocities (example: the car with cruise control switched ON).

This project (to compute (co)homology with a computer) was launched in the talk [**LPo**]. In order to describe the main ingredient of the new approach, recall the main definitions.

1) Having fixed a basis in a space V of dimension n, we identify $\mathrm{End}(V)$, the space of linear operators acting in V, with the space $\mathrm{Mat}(n)$ of $n \times n$ matrices. The subspace $\mathfrak{g}$ of $\mathrm{Mat}(n)$ is called a *Lie algebra* if it is closed with respect to the bracket multiplication $[x, y] = xy - yx$.

2) Suppose that $d : W \longrightarrow W$ is a linear operator whose square vanishes. This implies that $\mathrm{Im}\ d$ is a subspace of $\mathrm{Ker}\ d$. The quotient space $H(d, W) = \mathrm{Ker}\ d/\mathrm{Im}\ d$ is called the *homology of* d.

3) With every pair $(\mathfrak{g}, V)$ one can associate an operator $d : W \longrightarrow W$ for a certain space $W = W(\mathfrak{g}, V)$ canonically recoverable from $(\mathfrak{g}, V)$. The space $H(d, W)$ is called the **co***homology of the Lie algebra* $\mathfrak{g}$ *with coefficients in* V (a slightly different construction yields *homology of the Lie algebra* $\mathfrak{g}$ *with coefficients in* V; furter on we will remind both constructions).

One might think that the problem of calculating $H(d, W)$ is very simple: just find any collection of linearly independent vectors in $\mathrm{Ker}\ d$ complementary to $\mathrm{Im}\ d$. Any student knows how to do this for matrices of small dimension. But our task amounts to solving a HUGE number of systems of linear equations, especially tough to solve if both $\mathrm{Ker}\ d$ and $\mathrm{Im}\ d$ are infinite dimensional even if $H(d, W)$ is, nevertheless, finite dimensional.

Though d corresponds to a rather sparse matrix (which is good for us), d is introduced recursively which requires repeated loops in quantities able to considerably reduce the life expectancy of the researcher.

Selected subprojectss:

1) Purely mathematical goal: calculate $H(d, W)$ for several types of $\mathfrak{g}$'s and V's given. This can be (but seldom is) easy for finite dimensional $\mathfrak{g}$ and V and very tough for infinite dimensional ones. (However, the tougher the problem, the greater recognition :-) .)

E.g., there is a 30 years old problem raised by Gelfand, Kalinin and Fuchs which we belive is ripe to be solved by now. Formally speaking, this problem does not require any preliminary knowledge. In reality, a guided tour along the following textbooks will not hurt:

[**Fu**]. Regrettably it is NOT a text-book for the first reading. [**GM**] most lucidly describes spectral sequences and cohomology.

But even those who do not have these books, can consider a tough old problem, to get the feel of computations of Lie algebra cohomology: compute $H^*(\mathfrak{g})$ for the Lie algebra $\mathfrak{g}$ of Hamiltoniam vector fields with polynomial coefficients on $2n$-diminsional space. Kalinin considered $2n = 2$ but did not get eonugh results to make conjectures. Recently these or similar cohomology appeared in various problems, e.g., in a result by I. Gelfand and M. Kontsevich who considered the Lie algebra of Hamiltoniam vector fields with *Laurent* coefficients; a noncommutative analogs are also of conciderable interest lately.

Though we will deal with Lie algebras, it is adwisable to have some background in groups as well, e.g., from [**NS**].

At the moment the best, perhaps, text book on Lie algebras for beginners is Fulton, William; Harris, Joe *Representation theory. A first course*. Graduate Texts in Mathematics, 129. Readings in Mathematics. Springer-Verlag, New York, 1991. xvi+551 pp.; an old review by Dynkin [**Dy2**] and books by Humphreys and J. F. Adams are worth studying in parallel.

2) Computational problems: test feasibility of computing cohomology as criteria for formal integrability of differential equations. For motivations see [**Br**] (there exists an abridged version of [**Br**] in Russian with useful Appendix by Vershik A. and Gershkovich V. and an abridged version in Chinese).

The paper by Vershik [**V**] is the best (known to us) modern mathematical exposition of nonholonomic problems.

3) Applications to theoretical physics: certain homology $H(d, W)$ for selected pairs $(\mathfrak{g}, V)$ can be interpreted in terms of supergravity, see Grozman P., Leites D. From supergravity to ball bearings in J. Wess and E. Ivanov (eds.). Lecture notes in Physics, 524, 58–67.

Direction 2. Applications of the classical polynomials of several discrete variables. The equation of the form $a(x)y''+b(x)y'+cy=0$, where $a(x)$ is a polynomial of degree ≤ 2, $b(x)$ is a polynomial of degree ≤ 1 and c is a constant, is called a *hypergeometric type equation* and its solutions are called *functions of hypergeometric type*. Examples: Chebychev and Hahn polynomials, in particular, sines and cosines of nx considered as polynomials in x. Polynomial solutions of the hypergeometric equation and of various of its difference approximations are extremely important in various branches of mathematical physics: from thrifty ways to encode signals to various approximations. (Chebyshev discovered his polynomials by studying methods for evaluation and improvement of shell bombarding.)

In tomography, roentgenography (for example) polynomials of SEVERAL variables are desirable. Gauss introduced spherical polynomials to geophysics assuming that Earth can be approximated by a sphere. In the era of satellites this assumption proved to be too rough sometimes; besides, spherical harmonics are out of place in approximation of an arbitrary function without any symmetry.

Modern descriptions of orthogonal polynomials in several variables are usually just sums of products of polynomials in one variable, or a somewhat more sophisticated combinations thereof.

Recently there was found a totally new approach to classical polynomials of (several) discrete variables — solutions of the difference approximation of the above equation and its multidimensional analogs (Leites D., Sergeev A., Orthogonal polynomials of discrete variable and Lie algebras of complex size supermatrices. MPI-1999-36 (http://www.mpim-bonn.mpg.de) 49–70; Theor. and Math. Physics, v. 123, no. 2, 582–609).

First of all, we have to define these polynomials, and study their various properties. In order to do it, we also have to compute certain Lie algebra and Lie superalgebra cohomology that describe relations. We also have to consider realizations of various Lie algebras and superalgebras by means of creation and annihilation operators. This is an "easy" part of this project which, however, seems to be too voluminous to be run on a *Mathematica*-based package.

The "difficult" part is implementation of these polynomials, approximation procedures associated with them, etc.

References: Nikiforov, A. F.; Suslov, S. K.; Uvarov, V. B., *Classical orthogonal polynomials of a discrete variable*. Translated from the Russian. Springer Series in Computational Physics. Springer-Verlag, Berlin, 1991. xvi+374 pp.; Nikiforov A.; Uvarov V., *Special functions of mathematical physics. A unified introduction with applications*. Translated from the Russian and with a preface by Ralph P. Boas. With a foreword by A. A. Samarskii. BirkhSuser Verlag, Basel-Boston, MA, 1988. xviii+427 pp.

Advantages and difficulties of the project. There is a sea of problems leading to calculation of $H(d, W) = \mathrm{Ker}\, d/\mathrm{Im}\, d$ for given $(\mathfrak{g}, V)$ at various levels of difficulty depending on $\mathfrak{g}$ and V. A successful student can continue at Ph.D. level or even go further (apply results to real life problems).

Even if one is interested in actual understanding what one is doing (possible interpretations and applications of the results do require some background in one or several of the following fields: pure mathematics, theoretical physics, analytical mechanics) and to learn a bit will take lots of time, one can start calculating in certain cases immediately. This is, of course, a plus.

However, one can not see connections to real life problems or even compute more efficiently without a course or two in representation theory of Lie algebras. For integrability of differential equations one needs a course in symmetries of differential equations.

A description of the pilot version of SuperLie is given in [**GL5**]. Here we will reproduce the main points of this paper.

1. Cohomology of Lie superalgebras. Their applications

§0. History (a very biased view)

The notion of *Lie algebra cohomology*, though known since late 20's, did not draw much attentions until about 30 years ago. At that time I. M. Gelfand advertised Lie algebra cohomology everywhere, culminating at the Nice Mathematical congress; together with his pupils and colleagues he

(1) gave a purely algebraic proof of Weyl character formula which showed that finite dimensional representations are better understood if considered in the class of certain infinite dimensional modules and interpreted in terms of Lie algebra cohomology,

(2) showed that cohomology of simple infinite dimensional Lie algebras of vector fields are calculable and have application in topology (e.g., characteristic classes of foliations);

(3) realized a deep relation between various cohomology and vacuum weight representations (Borel-Weil-Bott-Kostant-Bernstein-Gelfand-Gelfand's theorem, in what folows breifly BWZ theorem).

Still, Gelfand and Fuchs were, perhaps, the only ones (except their own and their student's Ph.D. students) who devoted a significant part of their time to calculation of Lie algebra cohomology. When Gelfand's interests shifted elsewhere the researchers gradually left the field: almost everything calculable analytically seemed to have been calculated (mainly, by A. Borel, Bott, Kostant and Serre) even before Gelfand and Fuchs entered the field, anyway; Gelfand and Fuchs only considered infinite dimensional Lie algebras (of vector fields) Lepowsky and Wilson (and their followers) considered Kac-Moody algebras. Mainly, only trivial coefficients were discussed in these cases.

Fortunately, there are some calculable problems left for us, as we will see. Besides, having encountered with difficulties calulating cohomology analytically (i.e., with bare hands) we encourage to follow I. M. Gelfand's advice to use computer-aided study for this problem.

So far, the experience with computer-aided calculations of Lie algebra cohomology is not rich: in early 1970's D. Kalinin ([**GKF**]) calcualted some partial results on $H^*(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of Hamiltonian vector fields on the plane; recently Roos and Löfwall had obtained a beautiful interpretation of certain Hochschield cohomology ([**LR**]). Both results are just beginnings of possible fields for study: e.g., even Kalinin's calculations ([**GKF**]) are not surpassed yet, which is a challenge and a shame on modern researchers in the field.

It seems that unexpected negative results [**GKF**] discouraged or even scared researchers; since then nobody did **seriously** applied computers to the problem.

We think it is a shame that researchers had waited for so long: there are several problems which promise to be applicable in industry and economics, to say nothing about applications in pure mathematics and such romantic branches of physics as supergravity [**GL**].

At about the same time as Gelfand-inspired mathematicians studied Lie algebra cohomology (S. Shnider), Goldschmidt gave criteria of formal integrability of differential equations in terms of Spencer cohomology, see [**Br**].

Though such criteria are vital in practice, engineers still use “old-fashioned” numerical methods; nobody uses cohomology. Observe that

(1) Spencer cohomology can be interpreted in terms of Lie algebra ones and the other way round,

(2) with the generalization of the notion of Cartan prolongation, cf. [**Sh5**], one can extend applicability of Goldschmidt's criteria to all differential equations whereas the original criteria embraced only one of the two possible cases.

We prefer to use *Lie algebra* cohomology insted of *Spencer* ones: the answer, as we will see, can be documented in a more succint form; besides, there are (two or even three) theorems that simplify calulations of Lie algebra (co)homology and none for Spencer ones.

One of the reasons the calculation of (co)homology was abandoned and the limelight shifted away from it: the amount of calculations and the necessity to learn new vocabulary and techniques of representation theory did not encourage to indulge into such a project without extra motivations. We hope to change this habit of engineers and other possible customers; for this we need a number of examples computed and a user-friendly package.

During the past decade or two we were constantly aware of ever increasing role of Lie algebra cohomology. In our research we encountered these (co)homologies virtually everywhere: from supergravity to problems of stability of cars and ball bearings or even markets. We are not alone ton appreciate the role of Lie algebra cohomology: recently international conferences entitled "Cohomological Physics" started to mushroom, compare [**HKV**], [**W**] with our manifesto [**LPo**]. Observe that the study of Lie algebra cohomology from "super" point of view clarified various notions, such as semi-infinite cohomology.

These applications, especially, to supergravity, see [**GL**] are sufficient motivations to reconsider the list of cohomology to compute.

More explicitly, the known facts about Lie algebra cohomology and interesting open problems can be reviewed as follows. We follow, mainly, [**FZ**], [**BL**], [**LSc**]. For the proof of the statements see [**FZ**] or more accessible [**OV**], [**K1**].

The main types of Lie algebras $\mathfrak{g}$. Here we only consider the types SuperLie can deal with. Resently we started to teach SuperLie how to deal with characteristic p to get new results in this difficult situation, cf. [**Ja**], see also [**BMR**].

Each type has an important special case: $n = \infty$. (For supespaces, the dimension is represented by a pair of numbers each of which can be equal to ∞.)

1. *Matrix* Lie algebras, i.e., subalgebras of $\mathfrak{gl}(n)$.

2. Lie algebras $\mathfrak{g}(A)$ with Cartan $n \times n$ matrix, defined as $\tilde{\mathfrak{g}}(A)/\mathfrak{m}$ for certain $\tilde{\mathfrak{g}}(A)$ and the maximal ideal $\mathfrak{m}$ whose intersection with the maximal torus is 0.

3. The *vectorial* Lie algebras, i.e., subalgebras of $\mathfrak{vect}(n) = \mathfrak{der}\mathbb{K}[x]$, where $x = (x_1, \dots, x_n)$.

4. Lie algebras with Poisson and contact brackets in the space of "functions" in n indeterminates.

5. Stringy Lie superalgebras on n-extended supercircles.

6. $\mathfrak{diff}(n)_L = \mathbb{K}[x, \partial]$, where $x = (x_1, \dots, x_n)$, with a Lie or a superLie bracket, and its Lie subalgebras.

7. Quotient algebras, in particular, quotients modulo the center.

8.Subalgebras, in particular, nilpotent and solvable subalgebras.

9. Superizations of the above if not already "super".

The main types of $\mathfrak{g}$-modules.

0. Induced and coinduced modules.

1. Modules over matrix Lie algebras considered as subalgebras of $\mathfrak{gl}(V)$:

1a) the standard (identity) module V;

1b) the relatives of V: superspaces V^*, $\Pi(V)$, etc.

1c) the spaces of tensors, e.g., $T^p_q(V) = (\otimes^p(V)) \otimes (\otimes^q(V^*))$ and its subspaces and quotient spaces.
1d) highest weight modules.

2. Modules over the vectorial Lie algebras:

2a) modules of tensor fields and their duals;
2b) more general modules with vacuum vector (of highest or lowest weight).

3. Submodules.
4. Quotient modules.

§1. Background

All vector spaces considered in what follows are defined over a ground field $\mathbb{K}$. Unless otherwise stated $\mathbb{K} = \mathbb{C}$, but we would like to be able to consider sometimes other possibilities: $\mathbb{K} = \mathbb{R}$, $\mathbb{Q}$, $\mathbb{F}_q$, and $\overline{\mathbb{F}_q}$, the algebraic closure of $\mathbb{F}_q$; even the case of a ring instead of the field: $\mathbb{K} = \mathbb{Z}$ will be encountered.

1.1. A *Lie algebra* is a vector space $\mathfrak{g}$ with multiplication $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ called *bracket*. The bracket of two elements is usually denoted by $[x, y]$, not xy, and satisfies the following identitis:

$$[x, x] = 0 \quad \text{for any} \quad x \in \mathfrak{g}; \tag{S-S}$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \text{ for any } x, y, z \in \mathfrak{g} \tag{J.I.}$$

The identity (S-S) signifies anti-symmetry, (J.I.) is the *Jacobi identity*.

By anti-symmetry and bilinearity of $[\cdot, \cdot]$ we have $0 = [x + y, x + y] = [x, y] + [y, x]$, i.e., $[x, y]$ is indeed skew-symmetric, but (S-S) is easier to verify and holds even if characteristics of $\mathbb{K}$ is 2.

1.2. Examples of Lie algebras. Let A be an associative algebra. By means of the subscript L we denote the Lie algebra $\mathfrak{g} = A_L$ whose vector subspace is a copy of A and the bracket is given by the formula $[x, y] = xy - yx$. Clearly, A_L is a Lie algebra: (S-S) and (J.I.) are subject to a direct verification.

Particular cases of the above example are important when A is the algebra of matrices, or differential operators or pseudodifferential symbols.

In particular, let End$V$ be the associative algebra of endomorphisms of a vector space V. The *general linear algebra* $\mathfrak{gl}(V)$ is $(\mathrm{End}V)_L$. We abbreviate $\mathrm{Mat}(n)_L$ to $\mathfrak{gl}(n)$, and, having selected a basis in V, we identify $\mathfrak{gl}(V)$ with $\mathfrak{gl}(n)$.

Let $\mathfrak{g}_+$, $\mathfrak{g}_-$ and $\mathfrak{h}$ be the subsets of $\mathfrak{g} = \mathfrak{gl}(n)$ consisting of all strictly upper triangular, strictly lower triangular or diagonal matrices, respectively. Clearly, $\mathfrak{g}_+$, $\mathfrak{g}_-$ and $\mathfrak{h}$ are Lie algebras.

The classical Lie algebas. The main property of the trace of a matrix is $\mathrm{tr}[x, y] = 0$. Hence, the space of $n \times n$ matrices with trace 0 is a Lie algebra; it is called the *special linear algebra* and denoted by $\mathfrak{sl}(n)$.

Let B be a bilinear form on a vector space V. Clearly, the space $\mathfrak{aut}(B)$ of all operators that preserve B, i.e., $\{x \in \mathfrak{gl}(V) \mid B(xu, v) + B(u, xv) = 0 \text{ for any } u, v \in V\}$ is closed under the bracket and so $\mathfrak{aut}(B)$ is a Lie algebra. If B is non-degenerate we distinguish two important subcases:

B is symmetric, then $\mathfrak{aut}(B)$ is called the *orthogonal Lie algebra* and denoted by $\mathfrak{o}(V, B)$.

B is skew-symmetric, then $\mathfrak{aut}(B)$ is called the *symplectic Lie algebra* and denoted by $\mathfrak{sp}(V, B)$.

It is well-known that over $\mathbb{C}$ a non-degenerate symmetric forms on V are equivalent to each other and the same applies to skew-symmetric forms. So Lie algebras $\mathfrak{o}(V)$ and $\mathfrak{sp}(V)$ actually depend only on dim$V$, and we will sometimes denote them by $\mathfrak{o}(n)$ and $\mathfrak{sp}(2m)$.

The Lie algebras $\mathfrak{gl}(n)$, $\mathfrak{o}(n)$ and $\mathfrak{sp}(2m)$, as well as some (not all!) of their real forms are called *classical*.

Vectorial Lie algebras. Denote by $\mathfrak{vect}(n) = \mathfrak{der}\mathbb{K}[x]$, where $x = (x_1, \dots, x_n)$, the Lie algebra of derivations of $\mathbb{K}[x]$. The elements of $\mathfrak{vect}(n)$ are expressed with the help of partial derivatives as $\sum f_i(x)\partial_i$, where $\partial_i = \frac{\partial}{\partial x_i}$ and $f_i \in \mathbb{K}[x]$ for all i.

In what follows any Lie algebra $\mathfrak{g}$ realized as a subalgebra of $\mathfrak{gl}(n)$ or $\mathfrak{gl}(\infty)$ will be called a *matrix realization* of $\mathfrak{g}$; realized as a subalgebra of $\mathfrak{vect}(n)$ it will be called a *vectorial* Lie algebra. Usually, but not always, the adjective matrix or vectorial is applied to a maximal subalgebra of $\mathfrak{gl}(n)$ (n can be equal to ∞) or $\mathfrak{vect}(n)$, respectively.

For more examples see the list of classical Lie superalgebras [**LSc**] and references therein.

1.3. In view of Example $(\mathrm{End}V)_L$, the bracket in all Lie algebras is often called *commutator*. We say that elements x and y *commute* if $[x, y] = 0$. A Lie algebra $\mathfrak{g}$ is *commutative* (or *abelian*) if any two elements of $\mathfrak{g}$ commute. The *center* of $\mathfrak{g}$ is the set of elements of $\mathfrak{g}$ that commute with all elements of $\mathfrak{g}$.

The *bracket* in the Lie algebra is just another name for the multiplication in it. Therefore, the following concepts for Lie algebras are defined just as for associative algebras: a *subalgebra*; an *ideal* (since the bracket is antisymmetric, any left or right module is two-sided: indeed, the formula $am = -ma$ for a from the algebra and m from the module determines the passage from the left module structure to the right one and vv.); the *quotient algebra* $\mathfrak{g}/\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ modulo the ideal $\mathfrak{h}$; *homomorphisms, monomorphisms* and *epimorphisms* of Lie algebras; the (direct) *sum* of Lie algebras. These notions satisfy all the standard properties.

1.4. Simple and solvable Lie algebras. A nice class of *Lie algebras* with incredibly many applications is that of simple ones and various their subalgebras. Recall that a Lie algebra $\mathfrak{g}$ is *simple*, if $\dim\mathfrak{g} > 1$ and $\mathfrak{g}$ has no nontrivial ideals (distnct from 0 and $\mathfrak{g}$ itself).

Actually, the "simple" algebras are usually replaced with even "better" (i.e., more useful in applications) Lie algebras for which the simple Lie algebras serve as a certain "core": nontrivial central extensions of simple algebras, their algebras of differentiations, etc.

Among simple algebras, the most interesting to study are the ones with sufficiently rich structure for humans to be able to describe their representations, etc. Examples of such subclasses are:

FD) finite dimensional algebras;

PG) $\mathbb{Z}$-graded algebras of polynomial growth;

F) their deformations, such as $\mathfrak{gl}(\lambda)$, see [**LS**], i.e., *filtered* algebras of polynomial growth.

An opposite, in some sence class of Lie algebras is that of solvable ones. Let us describe them. The Lie algebra with the zero bracket is called an *abelian* one. The *center* of $\mathfrak{g}$, denoted by $\mathfrak{z}(\mathfrak{g})$ is the maximal abelian subalgebra. Slightly more complicated than abelian are nilpotent algebras. To describe them, consider series:

$$C^0\mathfrak{g} \subset C^1\mathfrak{g} \subset C^2\mathfrak{g} \dots,$$

where

$$C^0\mathfrak{g} = \{0\}; \quad C^1\mathfrak{g} = \mathfrak{z}(\mathfrak{g}); \quad C^{k+1}\mathfrak{g}/C^k\mathfrak{g} = \mathfrak{z}(\mathfrak{g}/C^k\mathfrak{g}) \text{ for } k \geq 1.$$

The Lie algebra $\mathfrak{g}$ is called *nilpotent* if $C^k\mathfrak{g} = 0$ for some k.

Usually, nilpotent Lie algebra is defined as such that $L^k\mathfrak{g} = 0$ for some k, where $L^k\mathfrak{g} = [L^{k-1}\mathfrak{g}, \mathfrak{g}]$ for $k \geq 1$. This, equivalent, definition is not so transparent as the above one, but resembles the following one: the Lie algebra $\mathfrak{g}$ is *solvable* if $L^{(k)}\mathfrak{g} = 0$ for some k, where $L^{(k)}\mathfrak{g} = [L^{(k-1)}\mathfrak{g}, L^{(k-1)}\mathfrak{g}]$ for $k \geq 1$.

The maximal solvable subalgebra is called the *radical*. The quotient modulo radical is called *semisimple* Lie algebra. The structure of semisimple Lie algebras is especially simple for finite dimensional Llie algebras over any field of characteristic 0: any semisimple Lie algebra is the direct sum of simple ones.

1.5. Modules over Lie algebras. A *representation of a Lie algebra* $\mathfrak{g}$ *in a vector space* V is a Lie algebra homomorphism $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$. The space V or, more exactly, the pair (V, ρ), is called then a $\mathfrak{g}$*-module*. We often write xv instead of $\rho(x) \otimes v$, but dealing with a computer we should be more accurate. Observe that the fact that $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is a Lie algebra homomorphism can be expressed as $\rho([x, y]) = [\rho(x), \rho(y)]$, where $[x, y]$ in the left hand side is the bracket in $\mathfrak{g}$ while $[\rho(x), \rho(y)]$ is the bracket (commutator) in $\mathfrak{gl}(V)$.

Examples of $\mathfrak{g}$-modules. If $\rho(x) = 0$ for any $x \in \mathfrak{g}$, then V is the *trivial* $\mathfrak{g}$-module. Unless otherwise stated, $\mathbb{K}$ is considered as the one-dimensional trivial $\mathfrak{g}$-module, and every trivial module is one-dimensional, unless specified differently.

The *adjoint representation* $\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$. In this case instead of $\rho(x)$ we write $\mathrm{ad}x$; by definition $\mathrm{ad}x(y) = [x, y]$. The Jacobi identity implies that ad is a derivation (satisfies the Leibniz rule with the bracket as the product).

The general linear algebra $\mathfrak{gl}(V)$ and its subalgebras have a natural representation in the space V; this representation is called the *standard* or *identity* representation.

The tensor product $V_1 \otimes V_2$ (over $\mathbb{K}$) of two $\mathfrak{g}$-modules has a natural $\mathfrak{g}$-module structure:

$$x(v_1 \otimes v_2) = xv_1 \otimes v_2 + v_1 \otimes xv_2 \quad \text{for } x \in \mathfrak{g}, v_1 \in V_1, v_2 \in V_2.$$

In this way we define for any $\mathfrak{g}$-module V its tensor, symmetric and exterior powers denoted by $\otimes^n V$, $S^n(V)$ and $\Lambda^n(V)$. (Sometimes we write E^n instead of Λ^n and $V^{\otimes n}$ instead of $\otimes^n V$.)

The formula

$$(xA)(v) = x(Av) - A(xv) \quad \text{for } x \in \mathfrak{g},\ A \in \mathrm{Hom}(V_1, V_2),\ v \in V_1$$

endows $\mathrm{Hom}(V_1, V_2)$ with a $\mathfrak{g}$-module structure. (The origin of this and above formula: consider the action of the Lie group with Lie algebra $\mathfrak{g}$ and differentiate at the unit.)

The *dual module* V' is defined as $V' = \mathrm{Hom}(V, \mathbb{K})$, where $\mathbb{K}$ is the one-dimensional trivial $\mathfrak{g}$-module.

The powers of the volume. The $\mathfrak{gl}(V)$-module $\mathbb{K}_\lambda$ given by $X(k) = \lambda \mathrm{tr}(X) \cdot k$ for any $X \in \mathfrak{gl}(V)$ and $k \in \mathbb{K}_\lambda$ is well-defined for any finite dimensional V. If $\lambda \in \mathbb{Z}$ the formula can be integrated to a representation of the Lie group $GL(V)$ in the λ-th tensor power of the space of volume forms.

The standard concepts of the theory of module over algebras such as *submodules, quotient modules, direct sums of modules, homomorphisms, irreducible modules* and so on, are defined for Lie algebras in just the same way as for associative algebras. Moreover, they can be reduced to the case of associative algebras (see the next subsection). With the help of these notions we construct important induced and coinduced modules over the larger algebra given a module over a subalgebra.

1.6. From Lie algebras to associative ones. Let us construct a functor from the category of Lie algebras to the category of associative algebras with unity. The idea behind this construction is that for the general matrix algebra $\mathfrak{gl}(V)$ the *product* of its elements is defined along with their bracket. For an arbitrary $\mathfrak{g}$, even a matrix one, like $\mathfrak{o}(V)$, the product of it elements does not lie in $\mathfrak{g}$, though for any $\mathfrak{g}$-module V corresponding to a representation ρ the element $\rho(x)\rho(y) \in V$ is well-defined for any $x, y \in \mathfrak{g}$. In order to construct the algebra in which such products lie, we construct an associative algebra $U(\mathfrak{g})$ generated by $\mathfrak{g}$.

Let $\mathfrak{g}$ be a Lie algebra. Let $T(\mathfrak{g}) = \underset{n\geq 0}{\oplus} T^n(\mathfrak{g})$, where $T^0(\mathfrak{g}) = \mathbb{K}$ and $T^n(\mathfrak{g}) = \otimes^n \mathfrak{g}$, be the tensor algebra of $\mathfrak{g}$. Consider also the two-sided ideal $I \subset T(\mathfrak{g})$ generated by the elements $x \otimes y - y \otimes x - [x, y]$ for $x, y \in \mathfrak{g}$. Set $U(\mathfrak{g}) = T(\mathfrak{g})/I$. The correspondence $\mathfrak{g} \mapsto U(\mathfrak{g})$ is the desired functor from the Lie algebras to the associative algebras; $U(\mathfrak{g})$ is called the *universal enveloping algebra* of $\mathfrak{g}$.

We will identify the elements of $\mathfrak{g}$ with their images in $U(\mathfrak{g})$. This allows us to establish a 1-1 correspondence between the $\mathfrak{g}$-modules and $U(\mathfrak{g})$-modules (we consider unitary $U(\mathfrak{g})$-modules, i.e., such that $1 \cdot m = m$ for any $\mathfrak{g}$ module M and $m \in M$).

If $\mathfrak{g}$ is abelian, then $U(\mathfrak{g})$ is by definition the symmetric algebra $S(\mathfrak{g})$. The following theorem shows that, to an extent, this coincidence is true for arbitrary Lie algebras, namely, $U(\mathfrak{g})$ and $S(\mathfrak{g})$ are still isomorphic *as spaces*, though not as algebras.

THEOREM . (Poincaré–Birkhoff–Witt) *Let $\{x_i\}_{i\in I}$ be a basis in $\mathfrak{g}$ indexed by a linearly ordered set I. Then the monomials $x_{i_1}^{n_1} x_{i_2}^{n_2} \dots x_{i_k}^{n_k}$, where $i_1 < i_2 < \dots < i_k$ and $n_1, \dots n_k$ are non-negative integers, form a basis of $U(\mathfrak{g})$.*

COROLLARY . *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be subalgebras of a Lie algebra $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ as a vector space. Then $U(\mathfrak{g}_1)$ and $U(\mathfrak{g}_2)$ are naturally imbedded in $U(\mathfrak{g})$, and the linear mapping $U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_2) \to U(\mathfrak{g})$ sending $x_1 \otimes x_2$ to $x_1 x_2$ is an isomorphism of vector spaces.*

1.7. Induced and coinduced modules. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie algebra and its subalgebra, V a module over $\mathfrak{h}$. From V, we construct the induced and coinduced $\mathfrak{g}$-modules by setting

$$I_{\mathfrak{h}}^{\mathfrak{g}}(V) = U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} V, \qquad C_{\mathfrak{h}}^{\mathfrak{g}}(V) = \mathrm{Hom}_{U(\mathfrak{h})}(U(\mathfrak{g}), V). \tag{1.7.1}$$

Clearly, $U(\mathfrak{g})$ is both left and right $\mathfrak{g}$-module; hence, $I_{\mathfrak{h}}^{\mathfrak{g}}(V)$ and $C_{\mathfrak{h}}^{\mathfrak{g}}(V)$ are $\mathfrak{g}$-modules. Thanks to the PBW theorem, it is not difficult to see that, as spaces,

$$I_{\mathfrak{h}}^{\mathfrak{g}}(V) \simeq \mathbb{K}[\mathfrak{g}/\mathfrak{h}] \otimes V, \qquad C_{\mathfrak{h}}^{\mathfrak{g}}(V) = \mathbb{K}[(\mathfrak{g}/\mathfrak{h})'] \otimes V. \tag{1.7.2}$$

These formulas make lucid either the basis of $I_{\mathfrak{h}}^{\mathfrak{g}}(V)$ and $C_{\mathfrak{h}}^{\mathfrak{g}}(V)$ (from (1.7.2)) or the $\mathfrak{g}$-module structure (from (1.7.1)).

1.8. Presentations: generators and relations. A usual way to study algebraic objects is to present them by generators and defining relations. For $\mathfrak{gl}(n)$ a standard basis consists of matrix units e_{ij}, i.e., matrices with 1 occupying the (i, j)-th slot and 0 other places. It is subject to a direct verification that

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{jk}. \tag{1.8.1}$$

For any simple finite dimensional Lie algebras there is a more convenient shorter presentation via Chevalley generators and Serre relations, see [**K1**] where a wider class of Kac–Moody algebras is considered. The latter can be succinctly encoded in a *Cartan matrix* or, even more graphically, a *Dynkin diagram*. So far, researchers mainly considered Kac–Moody algebras corresponding to extended Dynkin diagrams, until recently Borcherds [**Bo**], and Gritsenko and Nikulin [**GN**] demonstrated importance of a still wider class of algebras of the form $\mathfrak{g}(A)$. The defining relations for Lie algebras and superalgebras with an arbitrary Cartan matrix is still an open problem even for 3×3 matrices, cf. [**GL2**].

Even for simple classical Lie algebras of vector fields a reasonable and explicit presentation was obtained only recently, [**LP3**] (compare with implicit description in [**F1**], [**FF1**], [**FF2**]).

Though superization of this result for the Poisson algebras and description of defining relations for OTHER systems of generators are vital, among other problems, for q-quantization of spinor and oscillator representations, this is still an open problem; so far see [**GL1**],

One can easily obtain a similar explicit description ([**GLP**]) for nilpotent subalgebras of (twisted) loop algebras and superalgebras which recently appeared in a very interesting paper by Löfval and Roos [**LR**].

For simple Lie algebras there exists a more convenient than (1.8.1) shorter presentation. As a first (and typical) exibit, consider $\mathfrak{g} = \mathfrak{sl}(n)$. Then represent $\mathfrak{g}$ as the sum of the Cartan (diagonal) subalgebra $\mathfrak{h}$ and two maximal nilpotent subalgebras (positive and negative): $\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+$. Let us describe the structure of $\mathfrak{g}^+$ (observe that $\mathfrak{g}^- \simeq \mathfrak{g}^+$). For $i = 1, \dots, n-1$ set $X_i^+ = e_{i,i+1}$, $X_i^- = e_{i+1,i}$. Then formula (1.8.1) implies that the elements X_i^+ generate the algebra $\mathfrak{g}^+$ and the X_i^- generate $\mathfrak{g}^-$. It is easy to verify that

$$[X_i^+, X_j^+] = 0 \text{ if } \ i-j \neq \pm 1 \text{ and } [X_j^+, [X_j^+, X_i^+]] = 0 \text{ if } \ i-j = \pm 1; \qquad (SR^\pm) = (1.8.2)$$

(similarly for the X_i^-).

Actually, the Lie algebras $\mathfrak{g}^+$ and $\mathfrak{g}^-$ are *determined* by these relations, i.e., any other relation between the generators is a corollary of (1.8.2) (of course, with the axioms of Lie algebras).

Now consider the interaction of $\mathfrak{g}^+$ with $\mathfrak{g}^-$. Again by (1.8.1), we have $[X_i^+, X_j^-] = \delta_{i,j} H_i$, where $H_i = E_{ii} - E_{i+1,i+1}$. The algebra $\mathfrak{h}$ is abelian, and the elements $H_1, \dots, H_{n-1}$ form its basis. It is easy to verify that

$$[H_i, X_i^+] = 2X_i^+, \ [H_i, X_{i\pm1}^+] = -X_{i\pm1}^+ \text{ and } \ [H_i, X_j^+] = 0 \text{ if } \ |j-i| \geq 2;$$

similarly,

$$[H_i, X_i^-] = -2X_i^-, [H_i, X_{i\pm1}^-] = X_{i\pm1}^- \text{ and } \ [H_i, X_j^-] = 0 \text{ if } \ |j-i| \geq 2.$$

Summing up: we have distinguished in $\mathfrak{g}$ a system of $3(n-1)$ generators $X_1^-, \dots, X_{n-1}^-$, $H_1, \dots, H_{n-1}$ and $X_1^+, \dots, X_{n-1}^+$. All relations given above can be succintly written as follows. Let $A = (a_{ij})$ be the following $(n-1) \times (n-1)$-matrix:

$$A = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & -1 & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix}$$

Then

$$[H_i, H_i] = 0, \quad [X_i^+, X_j^-] = \delta_{ij} H_i, \quad [H_i, X_j^\pm] = \pm a_{ij} X_j^\pm; \qquad (SR^0) = (1.8.3^0)$$

$$(\mathrm{ad} X_i^+)^{1-a_{ij}} X_j^+ = 0 \ \text{ and } (\mathrm{ad} X_i^-)^{1-a_{ij}} X_j^- = 0. \qquad (SR^\pm) = (1.8.3^\pm)$$

1.8.1. It turns out that a good deal of representation theory of classical Lie algebras and their various generalizations can be deduced from the Dynkin diagram and relations (1.8.3), see [**Dy1**], [**Dy2**]. These relations actually form the set of defining relations for $\mathfrak{g}$. Any simple finite dimensional Lie algebra can also be defined in a similar way as $\mathfrak{g}(A)$ for some matrix A. Since it is quite difficult to verify for concrete algebras that given relations form a system of defining relations, it is more convenient to give another definition of the algebras $\mathfrak{g}(A)$.

Let Γ be an abelian group. A Lie algebra $\mathfrak{g}$ is called Γ-*graded* if $\mathfrak{g} = \bigoplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha$, where each $\mathfrak{g}_\alpha$ is a vector subspace of $\mathfrak{g}$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.

Let $A = (a_{ij})$ be an $n \times n$-matrix with entries from $\mathbb{K}$ and Q a free abelian group with a fixed basis $\alpha_1, \dots, \alpha_n$. The Lie algebra $\tilde{\mathfrak{g}}(A)$ of *rank* n corresponding to A is a Q-graded Lie algebra with generators $X_1^-, \dots, X_n^-$, $H_1, \dots, H_n$ and $X_1^+, \dots, X_n^+$, satisfying for any $i, j = 1, \dots, n$ the following relations:

(a) $[H_i, H_j] = 0$, $[H_i, X_j^{\pm}] = \pm a_{ij} X_j^+$, $[X_i^+, X_j^-] = \delta_{ij} H_i$;

(b) $H_1, \dots, H_n$ form a basis of the subspace $\mathfrak{h} = \mathfrak{g}_0$ and $\mathfrak{g}_{\alpha_i} = \mathbb{K} \cdot X_i^+$, $\mathfrak{g}_{-\alpha_i} = \mathbb{K} \cdot X_i^-$;

(c) Every non-zero Q-graded ideal of $\mathfrak{g}$ has a non-zero intersection with $\mathfrak{h}$.

It is easy to verify that $\tilde{\mathfrak{g}}(A)$ exists for any matrix A and is determined by A up to an isomorphism; n is called the rank of $\tilde{\mathfrak{g}}(A)$.

It is known [**K1**] that for certain matrices A there is a maximal Q-graded ideal $\mathfrak{m}$ of $\tilde{\mathfrak{g}}(A)$ whose intersection with $\mathfrak{h}$ is zero and such that $\mathfrak{g}(A) = \tilde{\mathfrak{g}}(A)/\mathfrak{m}$ is simple. It is a very interesting problem to describe the generators of $\mathfrak{m}$ explicitely. If $\mathfrak{g}(A)$ is finite dimensional (or a twisted loop algebra), then these generators are the left hand sides of (1.8.3); for some other cases see [**GL2**].

The definition readily implies that there is a (unique) antiautomorphism[1] $\imath : \mathfrak{g} \to \mathfrak{g}$ defined on the generators by the formula $\imath(X_i^+) = X_i^-$, $\imath(X_i^-) = X_i^+$ and $\imath(H_i) = -H_i$ for $i = 1, \dots, n$.

Roots and weights. Let $\mathfrak{g} = \mathfrak{g}(A)$. Put $R = \{\alpha \in Q \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0\}$; the elements of R are called the *roots* of $\mathfrak{g}$, and the decomposition $\mathfrak{g} = \mathfrak{h} \oplus (\underset{\alpha \in R}{\oplus} \mathfrak{g}_\alpha)$ is called the *root decomposition* of $\mathfrak{g}$. By (b), each α_i is a root for $i = 1, \dots, n$; the elements $\alpha_1, \dots, \alpha_n$ are called *simple roots*.

Let $\mathfrak{g}^+$ be the subalgebra of $\mathfrak{g}$ generated by $X_1^+, \dots, X_n^+$, let $\mathfrak{g}^-$ be generated by $X_1^-, \dots, X_n^-$. It is easy to verify that $\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+$ (in all our examples this will be clear). A root α is called *positive* if $\mathfrak{g}_\alpha \subset \mathfrak{g}^+$ and *negative* if $\mathfrak{g}_\alpha \subset \mathfrak{g}^-$; the set of positive (negative) roots of $\mathfrak{g}$ is denoted by R^+ (resp., R^-).

Put $Q^+ = \{\sum k_i \alpha_i \mid k_i \in \mathbb{Z}^+\}$ and $Q^- = -Q^+$; then $R^+ = R \cap Q^+, R^- = R \cap Q^-$. Though it is clear that $R = R^+ \cup R^-$, it is a wonder that all roots split into positive and negfative ones.

Clearly, if $\alpha = \sum k_i \alpha_i \in R^+$, then the subspace $\mathfrak{g}_\alpha \subset \mathfrak{g}$ is spanned by the elements $[X_{i_1}^+, [X_{i_2}^+, \dots, [X_{i_{r-1}}^+, X_{i_r}^+] \dots]]$, where each X_i^+ occurs k_i times. (A similar statement holds for negative roots.) Therefore, dim $\mathfrak{g}_\alpha < \infty$ for any $\alpha \in R$. Clearly, $\imath(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ for any $\alpha \in Q$. It follows that $-\alpha \in R$ for all $\alpha \in R$ and $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{-\alpha}$; by (b), $\dim \mathfrak{g}_\alpha = 1$ if $\pm\alpha$ is a simple root.

Let $F = Q \otimes_{\mathbb{Z}} \mathbb{C}$ be the vector space with basis $\alpha_1, \dots, \alpha_n$. Define the linear operator $F \to \mathfrak{h}'$ $(\alpha \mapsto \overline{\alpha})$ by the formula $\overline{\alpha}_j(H_i) = a_{ij}$ $(i, j = 1, \dots, n)$. The description of the root subspaces $\mathfrak{g}_\alpha$ given above easily implies that $[h, x] = \overline{\alpha}(h)x$ for any $h \in \mathfrak{h}$, $\alpha \in R$ and $x \in \mathfrak{g}_\alpha$.

Clearly, the map $\alpha \mapsto \overline{\alpha}$ is an isomorphism $F \longrightarrow \mathfrak{h}'$ if and only if the matrix A is invertible.

The assertion that $\mathfrak{sl}(n)$ is a Lie algebra of the form $\mathfrak{g}(A)$ of rank $n-1$ is almost obvious. In this case $R^+ = \{\alpha_i + \alpha_{i+1} + \dots + \alpha_j \mid 1 \leq i \leq j < n\}$; for $\alpha = \alpha_i + \dots + \alpha_j$ we have $\mathfrak{g}_\alpha = \mathbb{C}X_{i,j+1}^+$ and $\mathfrak{g}_{-\alpha} = \mathbb{C}X_{j+1,i}^+$. Properties (a) and (b) are obvious, and (c) follows at once from the equality $[X_{ij}^+, X_{ji}^+] = X_{ii}^+ - X_{jj}^+ \in \mathfrak{h}$.

1.9. Our definition of $\tilde{\mathfrak{g}}(A)$ does not require the relations $(SR^{\pm})$; these relations make sense only if a_{ij} is a nonpositive integer for $i \neq j$. An important class of Lie algebras satisfying these relations is the following one.

[1]Recall that a linear map $a : \mathfrak{g} \to \mathfrak{g}$ is an antiautomorphism if $a([x, y]) = [a(y), a(x)]$.

The matrix $A = (a_{ij})$ is called a *Cartan matrix* if $a_{ii} = 2$ or 0, a_{ij} is a nonpositive integer for $i \neq j$ and $a_{ij} = 0$ implies $a_{ji} = 0$. If A is *symmetrizable*, i.e., if there is an invertible diagonal matrix D such that AD is symmetric, then by (a), D can be chosen so that all its diagonal entries are positive rational numbers.

The Lie algebras $\mathfrak{g}(A)$ or slightly larger algebras, see [**K1**], are called the (generalized) *Kac-Moody algebras*.

The matrix A is called *decomposable* if under some simultaneous permutation of rows and columns it takes the form $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$. Clearly, in this case $\mathfrak{g}(A) = \mathfrak{g}(A_1) \oplus \mathfrak{g}(A_2)$.

PROPOSITION . *Let $\mathfrak{g} = \mathfrak{g}(A)$.*

(a) *The center $\mathfrak{z}$ of $\mathfrak{g}$ consists of the elements of $\mathfrak{h}$ that commute with all X_i^+ and X_i^- $(i = 1, \dots, n)$; the center is 0 if and only if A is invertible.*

(b) *If A is a Cartan matrix then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.*

(c) *If A is an indecomposable Cartan matrix, then any proper graded ideal of $\mathfrak{g}$ is contained in $\mathfrak{z}$. In particular, if A is invertible, then $\mathfrak{g}$ has no proper graded ideals.*

1.10. Let us show that $\mathfrak{o}(n)$ and $\mathfrak{sp}(2n)$ are of the form $\mathfrak{g}(A)$.

For finite dimensional vector spaces V and W we have a natural isomorphism $\mathrm{Hom}(V, W) \simeq V' \otimes W$ (the element $\xi \otimes w \in V' \otimes W$ corresponds to the operator $v \mapsto \xi(v)w$ for any $\xi \in V'$ and $w \in W$). Clearly, if V and W are $\mathfrak{g}$-modules then this isomorphism is compatible with $\mathfrak{g}$-module structures defined in 1.5. In particular, we identify $\otimes^2 V'$ with $\mathrm{Hom}(V, V')$; since $V = (V')'$, we also have $\otimes^2 V = \mathrm{Hom}(V', V)$. Therefore, $\Lambda^2 V$ and $S^2 V$ may be considered not only as quotients of but also as submodules of $\mathrm{Hom}(V', V)$, while $\Lambda^2 V'$ and $S^2 V'$ are submodules of $\mathrm{Hom}(V, V')$.

Let V be an n-dimensional vector space and $\mathfrak{g}$ one of the algebras $\mathfrak{sp}(2n)$, $\mathfrak{o}(2n)$ or $\mathfrak{o}(2n+1)$.

Then $\mathfrak{g}$ admits a $\mathbb{Z}$-grading displayed in the following table:

$\mathfrak{g}$	$k=-2$	$k=-1$	$k=0$	$k=1$	$k=2$
$\mathfrak{sp}(2n)$	–	S^2V'	$\mathfrak{gl}(V)$	S^2V	–
$\mathfrak{o}(2n)$	–	Λ^2V'	$\mathfrak{gl}(V)$	Λ^2V	–
$\mathfrak{o}(2n+1)$	Λ^2V'	V'	$\mathfrak{gl}(V)$	V	Λ^2V

PROPOSITION . *Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra and A the corresponding Cartan matrix (the matrices A are given in* [**OV**]*). Then A is invertible and indecomposable.*

1.11. The *Killing form* B on $\mathfrak{g}$ is defined by

$$B(x, y) = \mathrm{tr}\,(\mathrm{ad}\, x \circ \mathrm{ad}\, y), \quad x, y \in \mathfrak{g}.$$

PROPOSITION . *The form B is symmetric, non-degenerate and invariant (the latter means that $B([x, y], z) = B(x, [y, z])$ for any $x, y, z \in \mathfrak{g}$).*

1.12. The *affine Kac–Moody algebra* $\mathfrak{g}^{(1)}$ (over $\mathbb{C}$) corresponding to $\mathfrak{g}$ is defined as follows. As a vector space $\mathfrak{g}^{(1)}$ is $(\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]) \ni \mathbb{C}c$, where $\mathbb{C}[t, t^{-1}]$ is the ring of Laurent polynomials in t. By definition c lies in the center of $\mathfrak{g}^{(1)}$, and the bracket of two monomials is defined by

$$[x \otimes t^i, y \otimes t^j] = \begin{cases} [x, y] \otimes t^{i+j} & \text{if } \ i + j \neq 0, \\ [x, y] + iB(x, y)c & \text{if } \ i + j = 0 \end{cases} \quad \text{for any } x, y \in \mathfrak{g}.$$

The Lie algebra axioms can be verified directly. We shall write xt^k instead of $x \otimes t^k$ and simply x if $k = 0$.

REMARK . The quotient algebra $\mathfrak{g}^{(1)}/\mathbb{C}c = \mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]$ is called a *loop algebra.* (The meaning of the term stems from the fact that the loop algebra can be naturally realized as the algebra of $\mathfrak{g}$-valued functions on the unit circle with finite Fourier series, and the pointwise bracket.)

1.13. We want to prove that every affine algebra $\mathfrak{g}^{(1)}$ is a Kac–Moody algebra. For this we need some additional information on $\mathfrak{g}$.

PROPOSITION . *Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra and R the set of roots of $\mathfrak{g}$. Then $\mathfrak{g}$ has the maximal root γ, that is, $\gamma-\alpha\in Q^+$ for all $\alpha\in R$.*

In $\mathfrak{h}_\gamma=[\mathfrak{g}_\gamma,\mathfrak{g}_{-\gamma}]$, there is a unique element $h^\vee_\gamma$ such that $\overline{\gamma}(h^\vee_\gamma)=2$. If the elements $X^+_\gamma\in\mathfrak{g}_\gamma$ and $X^-_\gamma\in\mathfrak{g}_{-\gamma}$ are chosen so that $[X^+_\gamma,X^-_\gamma]=h^\vee_\gamma$, then $B(X^+_\gamma,X^-_\gamma)=\frac{1}{2}B(h^\vee_\gamma,h^\vee_\gamma)>0$.

In Table we give the explicit values of the root γ, linear form $\overline{\gamma}$ and vectors $h^\vee_\gamma$, X^+_γ and X^-_γ for all the classical $\mathfrak{g}$ (in the notation introduced above).

$\mathfrak{g}$	γ	$\overline{\gamma}$	$h^\vee X^+_\gamma$	X^+_γ	X^-_γ
$\mathfrak{sl}(n)$	$\alpha_1+\dots+\alpha_{n-1}$	$\xi_1-\xi_n$	$X^+_{11}-X^+_{nn}$	X^+_{1n}	X^+_{n1}
$\mathfrak{sp}(2n)$	$2(\alpha_1+\dots+\alpha_{n-1})+\alpha_n$	$2\xi_1$	X^+_{11}	$v_1\otimes v_1$	$v'_1\otimes v'_1$
$\mathfrak{o}(2n)$	$\alpha_1+2(\alpha_2+\dots+\alpha_{n-2})+\alpha_{n-1}+\alpha_n$	$\xi_1+\xi_2$	$X^+_{11}+X^+_{22}$	$v_1\wedge v_2$	$-v'_1\wedge v'_2$
$\mathfrak{o}(2n+1)$	$\alpha_1+2(\alpha_2+\dots+\alpha_n)$	$\xi_1+\xi_2$	$X^+_{11}+X^+_{22}$	$v_1\wedge v_2$	$-v'_1\wedge v'_2$

The maximal root γ always exists.

1.14. PROPOSITION . *If $\mathfrak{g}$ is a simple finite dimensional Lie algebra of rank n then the affine Lie algebra $\mathfrak{g}^{(1)}$ is a Kac–Moody algebra of rank $n+1$. The corresponding Cartan matrices are given by the extended Dynkin diagrams.*

§2. Homology and cohomology of Lie algebras

We are thankful to A. Dzhumadildaev who drew our attention to the fact that all mathematical textbooks give a "bad" formula for the differential, the one with an extra loop from the computers' point of view.

2.0.1. Cohomology. Let M be a left $\mathfrak{g}$-module. The *space of q-cochains* on $\mathfrak{g}$ with values (coefficients) in M is the space

$$C^q(\mathfrak{g};M)=M\otimes\Lambda^q(\mathfrak{g}')\text{ for }q\geq 0\text{ and }C^q(\mathfrak{g};M)=0\text{ for }q<0$$

of skew-symmetric linear in every variable functions on $\mathfrak{g}$ with values in M. In particular, $C^0(\mathfrak{g};M)=M$. The differential $d_q: C^q(\mathfrak{g};M)\longrightarrow C^{q+1}(\mathfrak{g};M)$ is determined as 0 for $q<0$ and for $q\geq 0$ by the formula (the hatted variable is to be omitted)

$$d_q(f)(g_1,\dots,g_{q+1})=\sum_{1\leq s<t\leq q+1}(-1)^s f(g_1,\dots,g_{s-1},[g_s,g_t],g_{s+1},\dots,\widehat{g_t},\dots,g_{q+1})$$
$$+\sum_{1\leq s\leq q+1}(-1)^{s-1}g_s f(g_1,\dots,\widehat{g_s},\dots,g_{q+1})$$

for any $m\in M$, $g_1,\dots,g_{q+1}\in\mathfrak{g}$.

THEOREM . $d_{q+1}\circ d_q=0$.

(A shorthand of this Theorem: $d^2 = 0$, where $d = \sum d_q$ is considered in $V = \oplus\, C^q$.)

The *space of q-cohomology* of $\mathfrak{g}$ with values (coefficients) in M is

$$H^q(\mathfrak{g}; M) = Ker\ d_{q+1}/\mathrm{Im} d_q.$$

For the trivial module of coefficients we abbreviate $H^q(\mathfrak{g}; \mathbb{K})$ to $H^q(\mathfrak{g})$.

2.0.2. Homology. Let M be a right $\mathfrak{g}$-module. The *space of q-chains* on $\mathfrak{g}$ with values (coefficients) in M is

$$C_q(\mathfrak{g}; M) = M \otimes \Lambda^q(\mathfrak{g}) \text{ for } q \geq 0 \text{ and } C_q(\mathfrak{g}; M) = 0 \text{ for } q < 0.$$

In particular, $C_0(\mathfrak{g}; M) = M$. The differential $d_q : C_q(\mathfrak{g}; M) \longrightarrow C_{q-1}(\mathfrak{g}; M)$ determines as 0 for $q \leq 0$ and for $q > 0$ by the formula

$$d_q(m \otimes (g_1 \wedge \cdots \wedge g_q)) = \sum_{1 \leq s < t \leq q} (-1)^s m \otimes (g_1 \wedge \cdots \wedge g_{s-1} \wedge [g_s, g_t] \wedge g_{s+1} \wedge \cdots \wedge \widehat{g_t} \wedge \cdots \wedge g_q)$$
$$+ \sum_{1 \leq s \leq q} (-1)^{s-1} m g_s \otimes (g_1 \wedge \cdots \wedge \widehat{g_s} \wedge \cdots \wedge g_q)$$

for any $m \in M$, $g_1, \dots, g_q \in \mathfrak{g}$.

THEOREM . $d_q \circ d_{q+1} = 0$.

(A shorthand: $d^2 = 0$, where $d = \sum d_q$ is considered in $\oplus C_q$.)

The *space of q-homology* of $\mathfrak{g}$ with values (coefficients) in M is

$$H_q(\mathfrak{g}; M) = Ker\ d_{q-1}/\mathrm{Im} d_q.$$

For the trivial module of coefficients we abbreviate $H_q(\mathfrak{g}; \mathbb{K})$ to $H_q(\mathfrak{g})$.

If $\mathfrak{g}$ and M are finite dimensional, then

$$H_q(\mathfrak{g}; M) \simeq (H^q(\mathfrak{g}; M'))'.$$

2.0.3. (Co)homology and (co)induced modules. For an $\mathfrak{h}$-module M we have ([**Fu**])

$$H^q(\mathfrak{g}; C^{\mathfrak{g}}_{\mathfrak{h}}(M)) \simeq H^q(\mathfrak{h}; M); \qquad H_q(\mathfrak{g}; I^{\mathfrak{g}}_{\mathfrak{h}}(M)) \simeq H_q(\mathfrak{h}; M).$$

Hereafter $\mathfrak{g}$ is a simple Lie (super)algebra and M an irreducible $\mathfrak{g}$-module unless otherwise stated.

2.1.1. THEOREM . $\dim \mathfrak{g} < \infty$. *Then*

$$H^*(\mathfrak{g}) = \Lambda[\xi_3, \dots, \xi_{2k_n+1}],$$

where the numbers $2k_i$ for $1 \leq i \leq \mathrm{rk}\mathfrak{g}$ are the so-called exponents *listed, e.g., in* [**OV**]*;*

$$H^*(\mathfrak{g}; M) = 0 \quad \textit{if } M = L^\lambda \textit{ is any nontrivial irreducible and } \dim M < \infty$$

This theorem is a classical result, cf. [**FF2**]. For the general modules even irreducible, but not finite dimensional, little is known.

In certain particular cases cohomology with coefficients in irreducible infinite dimensional modules (not only with vacuum vector) are computed.

2.1.2. THEOREM . ([**FF2**]) *Let $\mathfrak{g}$ be vectorial. Then for certain more or less explicitely described ideals I_v and I_k we have*

$$H^*(\mathfrak{vect}(n)) \simeq H^*(\mathfrak{sl}(n+1))/I_v, \quad H^*(\mathfrak{k}(2n+1)) \simeq H^*(\mathfrak{sp}(2n+2))/I_k.$$

These results are rather new. For the remaining two series of simple vectorial Lie superalgebras from Cartan's list we know nothing. In 1971 I. Gelfand, D. Kalinin and D. Fuchs used computer to estimate cohomology of $\mathfrak{h}(2)$, the Lie algebra of Hamiltonian vector fields on the plane, from below. This result is not surpassed even by modern computers, though Fuchs announced a result towards the proof of finite dimension of the cohomology in question. A recent attempt [**KK2**] to compute this cohomology failed: the complexity of calculations grows too steeply and new ideas are a must.

Denote by $\mathfrak{d}(\mathfrak{g})$ the Lie algebra of derivations of $\mathfrak{g}$. The cohomology with trivial coefficients are known for $\mathfrak{d}(\mathfrak{svect}(n))$ [**FF2**]. For $\mathfrak{d}(\mathfrak{h}(2n))$, it is known that they are finite dimensional, but, strangely, no explicit formula is available, though to derive it should not be difficult.

Thanks to the first theorems of cohomology theory, $H^*(\mathfrak{g}; \operatorname{coind}(M)) \simeq H^*(\mathfrak{g}_0; M)$ for any $\mathfrak{g}_0$-module M. Therefore, among nontrivial coefficients in irreducible modules it suffices to consider the exceptional, i.e., not coinduced modules. One of the most interesting nontrivial coefficients for $\mathfrak{svect}(n)$, $\mathfrak{po}(2n)$ and $\mathfrak{h}(2n)$ is the adjoint module and other non-induced modules (of closed differential forms, etc.).

Only in a particular case, $\mathfrak{g} = \mathfrak{vect}(1)$ (with polynomial coefficients), cohomology are computed for various coefficients [**FF2**], e.g.,

$$\dim H^q(\mathfrak{g}; F'_\lambda) = \dim H^q(\mathfrak{g}; F_\lambda) = \begin{cases} 1 & \text{if } \lambda = \frac{1}{2}(3q^2 \pm q) \text{ or } \lambda = \frac{1}{2}(3(q-1)^2 \pm (q-1)) \\ 0 & \text{otherwise.} \end{cases}$$

2.1.3. Cohomology of loop algebras. Feigin's thesis (Selecta Math. Soviet. 7 (1988), no. 1, 49–62) contains the description of these cohomology with trivial coefficients only.

For nontrivial coefficients no systematic treatment is known to us and virtually nothing is known of (co)homology of Kac–Moody algebras or twisted loop algebras, except interesting, though too general (without any down-to-earth applications) results by P. Zusmanovich aborted ca 10 years ago. (We sincerely hope his revitalized interest in the topic will last a while.) Nothing is known even about deformations, i.e., $H^2(\mathfrak{g}; \mathfrak{g})$.)

2.1.4. Cohomology of $\mathfrak{witt}$ and $\mathfrak{vir}$. These were studied by Feigin and Fuchs and Poletaeva, see [**FF2**] and [**P4**]. For cohomology of four series of their superversions ([**GLS1**]) little is known.

2.1.5. Cohomology of $\mathfrak{gl}(\lambda)$. The only result is [**F2**].

2.1.6. Cohomology of "stripy" Lie algebras: $\mathfrak{gl}(\infty)$ and $\mathfrak{o}(2\infty)$, $\mathfrak{o}(2\infty+1)$, $\mathfrak{sp}(2\infty)$. These algebras of infinite matrices with support in a stripe along the main diagonal were investigated by Feigin and Tsygan, see a review in [**FF2**]. Little is known.

2.2. Lie superalgebras $\mathfrak{g}$. Investigated are only trivial coefficients and $H^i(\mathfrak{g}; \mathfrak{g})$ for $I \leq 2$ in certain cases.

For the finite dimensional simple and Lie superalgebras and their "almost simple" relatives the known results with trivial coefficients are announced in [**FL**] (in certain cases proof is given in [**Fu**] and [**Kz**]). In some cases results of [**FL**] should be corrected, cf. [**KK1**], and several series are still to be considered.

The cohomology of exceptional simple Lie superalgebras with trivial coefficients are computed by a wonderful and ingenious method due to Gruson [**Gr1**]–[**Gr3**], a student of M. Duflo. This method has, however, limited applications (at least, so far).

Nothing is known on nontrivial coefficients, except when they constitute a "typical" module and $\mathfrak{g}$ possesses an invariant nondegenerate supersymmetric bilinear form (in which case the arguments from Lie algebra theory are applicable).

Of the simple infinite dimensional vectorial Lie superalgebras results are available only for the general series $\mathfrak{vect}(m|n)$, see [**AF**], for trivial coefficients only and with some restrictions on n and m.

2.2.1. Lie superalgebras. $\dim \mathfrak{g} < \infty$.

$$H^*(\mathfrak{gl}(m|n)) \simeq \begin{cases} H^*(\mathfrak{gl}(m)) & \text{if } m \geq n \\ H^*(\mathfrak{gl}(n)) & \text{otherwise.} \end{cases}$$

$$H^*(\mathfrak{pgl}(n|n)) \simeq H^*(\mathfrak{gl}(n)) \otimes \mathbb{K}[h_2], \quad \text{where } h_2 \text{ represents the extension } \mathfrak{gl}(n|n) \longrightarrow \mathfrak{pgl}(n|n).$$

$$H^*(\mathfrak{sl}(m|n)) \simeq \begin{cases} H^*(\mathfrak{sl}(m)) & \text{if } m \geq n \\ H^*(\mathfrak{sl}(n)) & \text{otherwise.} \end{cases}$$

$$H^*(\mathfrak{psl}(n|n)) \simeq H^*(\mathfrak{sl}(n)) \otimes \mathbb{K}[h_2],$$

where h_2 represents the extension $\mathfrak{sl}(n|n) \longrightarrow \mathfrak{psl}(n|n)$. for $n > 2$

$$H^*(\mathfrak{psl}(2|2)) \simeq \mathbb{C}[x_2, y_2, h_2, \xi_3]/(x_2y_2 - h_2^2).$$

Observe that the cocycle ξ_3 was missed in the original calculations [**FL**] and the answer was rectified at our request by A. Shapovalov with the help of **SuperLie**, see also [**KK1**]. The 2-cocycles represent the following central extensions:

$$\begin{aligned} x_2 : \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} &\mapsto \mathrm{tr} B_1B_2; \\ y_2 : \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} &\mapsto \mathrm{tr} C_1C_2 \\ h_2 : \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} &\mapsto \mathrm{tr}(B_1C_2 + C_1B_2) \end{aligned}$$

Clearly, the *degrees* of the cocycles are:

$$\deg x_2 = 2, \quad \deg y_2 = -2, \quad \deg h_2 = 0$$

and, therefore, resemble $\mathfrak{sl}(2)$ very much. Having defined $\mathfrak{sl}(2)$ structure on the space of these cocycles, we get the superalgebra $\mathfrak{osp}_\alpha(4|2)$ for $\alpha = 0$.

$$H^*(\mathfrak{sl}(1|1)) \simeq \mathbb{C}[\xi_1, \eta_1]/(\xi_1\eta_1).$$

$$H^*(\mathfrak{pe}(n)) \simeq \mathbb{C}[\xi_1, \dots, \xi_{4k+1}, \dots \xi_{2n-1}].$$

$$H^*(\mathfrak{spe}(2n+3)) \simeq H^*(\mathfrak{pe}(2n+3)).$$

$$H^*(\mathfrak{spe}(2n+4)) \simeq H^*(\mathfrak{pe}(2n+4)) \otimes \begin{cases} \mathbb{K}[x_{n+2}] & \text{if } n \equiv 0 \pmod 4 \\ \mathbb{K}[\xi_{n+2}] & \text{if } n \equiv 2 \pmod 4 \end{cases}.$$

$$H^*(\mathfrak{spe}(2)) \simeq \mathbb{K}[x_1, y_2]/(x_1^2 - y_2).$$

To describe the cohomology of $\mathfrak{q}$ series, let $s_1, \dots, s_n$ be the elementary symmetric polynomials in n even indeterminates x, i.e.,

$$\begin{gathered} s_1 = x_1 + \dots + x_n, \\ \dots\dots\dots\dots\dots\dots \\ s_1 = x_1 \dots x_n; \end{gathered}$$

let $S_1, \dots, S_n$ be the elementary symmetric polynomials in the squares of n even indeterminates x, i.e.,

$$\begin{gathered} S_1 = x_1^2 + \dots + x_n^2, \\ \dots\dots\dots\dots\dots\dots \\ S_1 = x_1^2 \dots x_n^2. \end{gathered}$$

Denote by x_i the generators of $S^\bullet(\Pi(\mathfrak{g}_{\bar 1}))^{\mathfrak{g}_{\bar 0}}$. We have then

$$\begin{aligned} &H^*(\mathfrak{q}(n)) \simeq \mathbb{C}[s_1, \dots, s_n]/(S_1, \dots, S_n); \\ &H^*(\mathfrak{sq}(n)) \simeq ??; \\ &H^*(\mathfrak{pq}(n)) \simeq ??; \\ &H^*(\mathfrak{psq}(n)) \simeq ?? \end{aligned}$$

The cohomology of exceptional Lie superalgebras are computed by an ingeneous method due to Caroline Gruson [**Gr1**]-[**Gr3**] and the answer is:

$$\begin{aligned} &H^*(\mathfrak{ag}_2) \simeq H^*(\mathfrak{g}_2); \\ &H^*(\mathfrak{ab}_3) \simeq H^*(\mathfrak{o}(7)); \\ &H^*(\mathfrak{osp}_\lambda(4|2)) \simeq H^*(\mathfrak{o}(4)) \simeq H^*(\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)). \end{aligned}$$

It seems, however, that her method is almost exhausted as far as *simple* algebras are concerned, and, anyway, we have to consider various *nontrivial* coefficients; to do this by hands is hardly possible.

2.2.2. Vectorial Lie superalgebras. Of the vectorial Lie superalgebras the general results are available for the general series only:

$$H^*(\mathfrak{vect}(m|n)) \simeq H^*(\mathfrak{vect}(0|n)) \simeq H^*(S^{2n-1})$$

the first isomorphism being induced by an embedding $\mathfrak{vect}(0|n) \longrightarrow \mathfrak{vect}(m|n)$.

$$H^*(\mathfrak{vect}(n|n)) \simeq H^*(\Sigma^{2n} U(n)),$$

where Σ denotes the suspension, $U(n)$ is the unitary group.

By the same reasons as for Lie algebra case [**Fu**], the cohomology of $\mathfrak{d}(\mathfrak{g})$ for the vectorial Lie superalgebras $\mathfrak{g}$ are finite dimensional; observe that sometimes $\mathfrak{d}(\mathfrak{g}) = \mathfrak{g}$ (e.g., for $\mathfrak{vect}(m|n)$, $\mathfrak{k}(2m+1|n)$, $\mathfrak{m}(n)$). In these cases, especially for the exceptional algebras one can compute the final answer.

2.3. Theorem . Borel-Weil-Bott-Kostant-Bernstein-Gelfand-Gelfand's theorem (BWZ) *Let M be an irreducible finite dimensional $\mathfrak{g}$-module; W the Weyl group of $\mathfrak{g}$, let $l(w)$, the length of w, be the minimal number of generators of W needed to express w. Then*

$$\dim H^i(\mathfrak{n}; M) = \{w \in W(\mathfrak{g}) \mid l(w) = (i)\}.$$

A similar statement holds for smaller nilpotent subalgebras $\mathfrak{n}_{i_1 \dots i_k}$ generated by all the (say positive) Chevalley generators, except $X_{i_1}, \dots, X_{i_k}$. About cohomology of algebras $\mathfrak{n}_{i_1 \dots i_k}$ with coefficients in modules which are not $\mathfrak{g}$-modules, e.g., in the adjoint module of in $\mathfrak{g}/\mathfrak{n}^\pm_{i_1 \dots i_k}$, little is known; mainly these are statements for $i = 0, 1, 2$, cf. Tolpygo's papers [**T**], [**TL**] and Premet's results below for a review, new results and open problems.

2.4. Singular vectors and invariant differential operators. As shown in [**Fu**], description of vacuum (singular) vectors in Verma modules is equivalent to calculation of certain cohomology; in turn these vectors are a first step in the classification of differential operators acting in tensor fields and invariant with respect to various groups of diffeomorphisms ([**BL**]). All physical laws are expressed in terms of invariant differential operators, their classification is important (problem Veblen raised on 1928 Mathematical congress).

2. Analytical results

In this chapter we reproduce some of analytical results (from Premet's letters to Leites in 1989) as a motivation for computer-aded study . Premet's results cover earlier results by Leger and Luks and complement Tolpygo's results [**TL**]. Their complete proof deserves to be published and we hope this note will prompt even further studies.

§1. Background

Hereafter the ground field $\mathbb{K}$ is $\mathbb{C}$, unless otherwise stated. We consider cohomology $H^i(\mathfrak{n}; M)$, where $\mathfrak{n}$ is a nilpotent subalgebra of a simple Lie algebra or superalgebra $\mathfrak{g}$, and M an $\mathfrak{n}$-module which is not necessarily a $\mathfrak{g}$-module.

Our motivations:

— the Riemann tensor can be formulated in terms of $H^2(\mathfrak{n}; \mathfrak{g})$ for a particular pairs $\mathfrak{n}$ and $\mathfrak{g}$, the ones corresponding to the $\mathbb{Z}$-grading of $\mathfrak{g}$ of the form $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ in which case $\mathfrak{n} = \mathfrak{g}_{-1}$. Moreover, the result, i.e., $H^2(\mathfrak{n}; \mathfrak{g})$ is convenient to represent succinctly as a $\mathfrak{g}_0$-module, see [**Go**].

Such a $\mathbb{Z}$-grading ($\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$) corresponds to some minimal nilpotent subalgebras $\mathfrak{n} = \mathfrak{g}_{-1}$ of $\mathfrak{g}$ and pure curiosity requires to consider *other* minimal (and not only minimal) nilpotent subalgebras. This insatible curiosity became grounded on solid applications when the cohomology $H^2(\mathfrak{n}; \mathfrak{g})$ for the remaining minimal and, generally, any other $\mathfrak{n}$ had been inerepreted as analogs of Riemann tensor for non-holonomic structures [**L3**].

When such an interpretation is at hand, one naturally passes to all nilpotent subalgebras $\mathfrak{n}$ of any simple $\mathfrak{g}$; some of them are direct analogs of constructions Manin studied in supersetting [**Ma**].

In a very inspiring paper [**V**] Vershik launched a program: "study non-holonomic structures", in particular, analogs of Riemann tensor for them.

Vershik aborted these studies, presumably because he himself did not believe that the general definition of such a tensor exists (see [**V**]) and nobody followed his research.

M. Zelikin informed us that recently Agrachev also introduced analogs of of Riemann tensor for non-holonomic structures associated with optimal control (so far we did not see anything published). For various non-holonomic structures associated with optimal control see, e.g., [**Ag**].

Apart from appearing in the left hand side of Einstein Equations, the Riemann tensor on a manifold, being a curvature, is responsible for stability of the dynamical system on this manifold ([**Ar**]); the cohomology similar to $H^2(\mathfrak{n}; \mathfrak{g})$ but formualted in equivalent (but less convenient for calcualtions) terms of Spencer cohomology are criteria for formal integrability of differential equations [**Br**].

These areas of possible applications seem to be already sufficient to proceed with calculations for various examples. Still, there are more areas; one of them is economics, see [**SV**].

— desire to represent known results on cohomology of nilpotent subalgebras of simple Lie algebras in more graphic form than classical results. Indeed, H^* possesses several superalgebra structures. What are they? To formulate the problem, we need some preliminaries.

A recent result of Löfvall and Roos [**LR**] is an illustration of what we had in mind as a model of description: in [**LR**] they reduce description of infinite dimensional nilpotent Lie superalgebras $\mathfrak{g}$ to simple finite dimensional ones, $\mathfrak{h}$, having realized that $\mathfrak{g} = \mathfrak{h} \times t\mathbb{C}[t]$. (There answer can be represented still more conveniently if $\mathfrak{g}$ is represented as an $\mathfrak{h}$-module, cf. [**GLP**].)

1.1. Terminological conventions. Let $\mathfrak{g}$ be a simple Lie algebra. The $\mathfrak{g}$-module V with highest (lowest) weight ξ will be denoted by V^ξ (resp. V_ξ) or $R(\xi)$. Let R be the root system of $\mathfrak{g}$, B its base (system of simple roots).

Recall (see 1.8 above) that every $\mathfrak{g}$ can be recovered from an integer $r \times r$ matrix (A_{ij}) (called the *Cartan matrix*) as the quotient of the free Lie algebra generated by $X_1^\pm$, $\dots$, $X_r^\pm$ called *Chevalley generators* of $\mathfrak{g}$ to which one usually adds $H_i = [X_i^+, X_i^-]$ modulo the relations

$$\begin{aligned}
&[X_i^+, X_j^-] = \delta_{ij} H_i; \quad [H_i, H_j] = 0;\\
&[H_i, X_j^\pm] = \pm A_{ij} X_j^\pm;\\
&(\mathrm{ad} X_i^\pm)^{1-A_{ij}}(X_j^\pm) = 0.
\end{aligned}$$

Clearly, by rescaling ($H_i \longrightarrow \lambda H_i$) the Chevalley generators we may assume that $A_{ii} = 2$ or 0 for all the i. For finite dimensional Lie algebras the Cartan matrices are very special, in particular, $A_{ii} = 2$ for all i.

It is easy to see that $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, where $\mathfrak{n}^\pm$ are generated by the $X_i^\pm$ and $\mathfrak{h} = Span(H_1, \dots, H_r)$.

Let $I = \{i_1, \dots, i_k\}$ be a subset of $\mathcal{I} = \{1, \dots r\}$, and $\mathfrak{p}_I^+$ the subalgebra of $\mathfrak{g}$ generated by the $X_i^\pm$ for $i \in I$ and X_i^+ for $i \in \{1, \dots r\} \setminus I$. It is called a (positive) *parabolic* subalgebra.

Let $\mathfrak{n}_I^-$ be the opposite nilpotent algebra, i.e., $\mathfrak{n}_I^-$ is the ideal of $\mathfrak{n}^-$ generated by X_i^- for $i \in \{1, \dots r\} \setminus I$.

In particular, $\mathfrak{n}^\pm = \mathfrak{n}_{\mathcal{I}}^\pm$ are the maximal nilpotent subalgebras. Clearly, $\mathfrak{p}_I^\pm = \mathfrak{l}_I \oplus \mathfrak{n}_I^\pm$, where $\mathfrak{l}_I$ (called the *Levi subalgebra*) is the direct sum of a semisimple Lie algebra and a subalgebra $\mathfrak{h}_{\mathfrak{l}_I}^{comp}$ of $\mathfrak{h}$ complementing the Cartan subalgebra $\mathfrak{h}_{\mathfrak{l}_I}$ of $\mathfrak{l}_I$, so that $\mathfrak{h}_{\mathfrak{l}_I}^{comp} \oplus \mathfrak{h}_{\mathfrak{l}_I} = \mathfrak{h}$.

Having selected the Chevalley generators $X_{i_1}^\pm$, $\dots$, $X_{i_k}^\pm$ corresponding to roots $\alpha_{i_1}^\pm$, $\dots$, $\alpha_{i_k}^\pm$, let $R_I \subset R$ be a subsystem with base B_I corresponding to the remaining simple roots.

Let $W = W(R)$ be the *Weyl group*, i.e., the group generated by the reflections s_i in simple roots. For any I, define the subgroup $W_I \subset W$ corresponding to R_I (with base B_I) as

$$W_I = \{w \in W \mid w^{-1}(B \setminus B_0) > 0\}.$$

Define the *length* $l(w)$ of any element $w \in W$ as the minimal number of the simple reflections s_i required to represent w. Denote the subset of elements of length i in W by $W(B_I)^i$ and its cardinality by $k_I(i)$. In particular, if $B_I = B$, let $k(i)$ be the number of elements from W of length i.

For the definition of the exponents and several other standard notations that might arise see [**OV**] or [**Bo**].

1.2. General statements on cohomology.

1.2.1. Theorem . 1) *Let M be an irreducible finite dimensional $\mathfrak{g}$-module. Then $H^*(\mathfrak{g}; M) = 0$ if $M \neq \mathbb{K}$.*

2) $H^*(\mathfrak{g}) \simeq E[\xi_1, \dots, \xi_r]$, *where* $r = \operatorname{rk}\mathfrak{g}$ *and* $\deg \xi_i = e_i + 1$. (E^i is the operator of the ith exterior power; $E = \sum E^i$.)

1.2.2. Cohomology of nilpotent subalgebras. There are two types of modules: with values in $\mathfrak{g}$-modules and other modules.

Theorem . 1) (Borel-Weil-Bott-Kostant-Bernstein-Gelfand-Gelfand's theorem (BWZ) *Let M be an irreducible finite dimensional $\mathfrak{g}$-module. Then*

$$\dim H^i(\mathfrak{n}; M) = k(i).$$

2) *A similar statement holds for lesser, non-maximal, nilpotent subalgebras $\mathfrak{n}_I^{\pm}$, where $I = \{i_1, \dots, i_k\}$: Let M be an irreducible finite dimensional $\mathfrak{g}$-module. Then*

$$\dim H^j(\mathfrak{n}_I^{\pm}; M) = k_I(j).$$

Observe, that there is a more economic way to describe the spaces $H^j(\mathfrak{n}_I^{\pm}; M)$. Namely, both $\mathfrak{n}_I^{\pm}$ and M are $\mathfrak{l}_I$-modules; hence, so is $H^j(\mathfrak{n}_I^{\pm}; M)$. Since $\mathfrak{l}_I$ is semisimple, every its finite dimensional module is completely reducible and every irreducible component is completely described by its highest (or lowest) weight and the corresponding vector.

For example, let $\mathfrak{n}_I = \mathfrak{n}_i$ be a *minimal* algebra corresponding to $\#(I) = 1$ and $M = \mathfrak{g}$. Suppose that $\mathfrak{n}_i$ is commutative (this is the case when G/P, where G corresponds to $\mathfrak{g}$ and P to $\mathfrak{p}_i$, is a classical domain, e.g., the Grassmannian Gr_k^{n+k}). Then $H^j(\mathfrak{n}_i; M)$ is a $\mathfrak{sl}(k) \oplus \mathfrak{sl}(n+k)$-module. Moreover, since $\dim H^j(\mathfrak{n}_i; M)$ does not depend on M, it suffices to describe it in the simplest case of the trivial module, $M = \mathbb{K}$. In this case the cohomology are easy to compute thanks to commutativity of $\mathfrak{n}_i$:

$$H^j(\mathfrak{n}_i) = E^j(\mathfrak{n}_i).$$

For $j = 1, 2$ these cocomology have interesting applications and they are our prime concern, though the complete information, i.e., $H^* = \oplus H^j$ is also important as a base for computating cohomology of infinite dimensional algebras and for purely mathematical interest.

About cohomology of algebras $\mathfrak{n}^{\pm}_{i_1 \dots i_k}$ with coefficients in modules which are not $\mathfrak{g}$-modules, e.g., in the adjoint module, or in $\mathfrak{g}/\mathfrak{n}^{\pm}_{i_1 \dots i_k}$, little is known, see [**T**] and Premet's theorem below.

Observe that though the BWB theorem is very neat for *generic* irreducible $\mathfrak{g}$-modules M, for $H^*(\mathfrak{n})$ the description is rather unnatural. Indeed, $H^*(\mathfrak{n})$ possesses an algebra structure. Who describes a polynomial or exterior algebra, or their quotients, via dimensions of homogeneous components?!

But here the situation is the same: $H^*(\mathfrak{n})$ and $H^*(\mathfrak{n}; \mathfrak{n})$ possess natural algebraic structures and one of the reasonable descriptions of any algebra is via its generators and defining relations.

The algebraic structures we have in mind are as follows.

If M possesses an associative algebra structure the multiplication in M and the $\mathfrak{g}$-action being related by the Leibniz rule (say, M is the 1-dimensional trivial module), then $H^*(\mathfrak{g} : M)$ possesses an associative superalgebra structure with respect to the product of cochains.

For the adjoint coefficients, for any $a \in C^p(\mathfrak{g}; \mathfrak{g})$, $b \in C^q(\mathfrak{g}; \mathfrak{g})$ define $ab \in C^{p+q-1}(\mathfrak{g}; \mathfrak{g})$ by setting

$$ab(x_1, \dots, x_{p+q-1}) = \sum A(a(b(x_1, \dots, x_q), x_{q+1}, \dots, x_{p+q-1})),$$

where A is the alternation, i.e.,

$$A(f(x_1, \dots, x_q), x_{q+1}, \dots, x_{p+q-1})) =$$
$$\tfrac{1}{(p+q-1)!} \sum_{\sigma \in S_{p+q-1}} \operatorname{sign}(\sigma; x)(f(x_{\sigma(1)}, \dots, x_{\sigma(q)}), x_{\sigma(q+1)}, \dots, x_{\sigma(p+q-1)}),$$

where $\mathrm{sign}(\sigma; x)$ is a cocycle on S_{p+q-1} described in [**L2**], no. 23.

Set

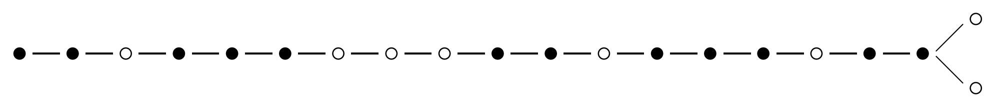
$$[a, b] = ab - (-1)^{(p(a)-1)(p(b)-1)} ba \text{ for any } a, b \in C^*(\mathfrak{g}; \mathfrak{g}). \tag{$*$}$$

THEOREM . $\Pi(H^*(\mathfrak{g}; \mathfrak{g}))$ *with the multiplication* $(*)$ *is a Lie superalgebra.*

Observe that the multiplication is defined on cocycles and the statement is made about their cohomology classes.

§2. Premet's Theorems

2.1. Let Dynkin diagram of B be as follows:



and let B_I consist of black nodes. Let us represent B_I as the union of connected subdiagrams:

$$B_I = B_I^{(1)} \coprod \dots \coprod B_I^{(s)},$$

where s (in our example $s = 5$) is the number of connected components of the Dynkin diagram D_I of B_I and where $B_I^{(i)}$ corresponds to the i-th connected components of D_I (counted from left to right). Set

$$c_0 = \mathrm{card}\ B, \quad c_i = \mathrm{card}\ (\{\alpha \in B \setminus B_I \mid (\alpha, B_I^{(i)}) \neq 0\}) - 1.$$

Clearly, if $B_I \neq B$, then $c_i \in \{0, 1, 2\}$. For example, for the diagram of $\mathfrak{o}(20)$ depicted above we have:

$$c_0 = 20, \quad c_1 = 0, \quad c_2 = 1, \quad c_3 = 1, \quad c_4 = 1, \quad c_5 = 2.$$

The following statement is obvious.

STATEMENT . 1) $c_i = 2$ *if and only if* R *is of type* D_n, E_6, E_7, E_8, *one of the endpoints of* $D_I^{(i)}$ *is a branching point for* D, *and the remaining endpoint of* $D_I^{(i)}$ *is not an endpoint of* D.

2) $c_i = 0$ *if and only if all but one of the end vertices of the diagram of* $B_I^{(i)}$ *are the end vertices for the diagram of* R.

2.2. Let $\mathfrak{g}$ be a simple Lie algebra of rank $r > 1$; let $\mathfrak{p} = \mathfrak{n}_- \oplus \mathfrak{l}$ be its parabolic subalgebra, where $\mathfrak{l}$ is the Levi subalgebra, $\mathfrak{l}^{(1)}$ the derived algebra of the Levi subalgebra, $\mathfrak{z} = \mathfrak{z}(\mathfrak{l})$, the center of $\mathfrak{l}$. (Recall, that $\mathfrak{p}$ is generated by the Chevalley generators $X_i^{\pm}$, $i = 1, \dots, r$, of $\mathfrak{g}$ except several X_i^+; $\mathfrak{l}$ is generated by all the remaining $X_i^{\pm}$; finally, $\mathfrak{l} = \mathfrak{l}^{(1)} \oplus \mathfrak{z}$.)

THEOREM . $H^2(\mathfrak{n}_-; \mathfrak{n}_- \oplus \mathfrak{l}^{(1)}) = H^2(\mathfrak{n}_-; \mathfrak{g}) \oplus H^1(\mathfrak{n}_-; (\mathfrak{n}_- \oplus \mathfrak{z})^*)$.

Let $B_1 = B \setminus B_I$ and $W(B_1)^{(2)} = \{w \in W(B_1) \mid l(w) = 2\}$.

Set $\mathfrak{n}_-^{ab} = (\mathfrak{n}_-/\mathfrak{n}_-^{(1)})^* = H^1(\mathfrak{n}_-)$.

COROLLARY . 1) *The following sequence is exact:*

$$0 \longrightarrow \mathfrak{n}_-^{ab} \longrightarrow \mathfrak{z}^* \otimes \mathfrak{n}_-^{ab} \longrightarrow H^1(\mathfrak{n}_-; \mathfrak{g}/(\mathfrak{n}_- \oplus \mathfrak{l}^{(1)})) \longrightarrow H^1(\mathfrak{n}_-; \mathfrak{n}_-^*) \longrightarrow$$
$$H^2(\mathfrak{n}_-) \oplus \bigoplus_{w \in W(B_1)^{(2)}} R(\rho - w(\rho)) \longrightarrow 0.$$

2) *If* $\dim\ \mathfrak{z} = 1$, *then the sequence*

$$0 \longrightarrow H^1(\mathfrak{n}_-; (\mathfrak{n}_- \oplus \mathfrak{z})^*) \longrightarrow H^1(\mathfrak{n}_-; \mathfrak{n}_-^*) \longrightarrow H^2(\mathfrak{n}_-) \longrightarrow 0$$

is exact. In particular,

a) *if* $\mathfrak{n}_-$ *is abelian, then*

$$H^2(\mathfrak{n}_-;\mathfrak{n}_-\oplus\mathfrak{l}^{(1)})=H^2(\mathfrak{n}_-;\mathfrak{g})\oplus S^2(\mathfrak{n}_-^*);$$

b) *if* $\mathfrak{n}_-$ *is a Heisenberg algebra, then*

$$H^2(\mathfrak{n}_-;\mathfrak{n}_-\oplus\mathfrak{l}^{(1)})=H^2(\mathfrak{n}_-;\mathfrak{g})\oplus S^2(\mathfrak{n}_-/\mathfrak{z}(\mathfrak{n}_-))^*.$$

2.3. Lemma . *Let* $E=L^\lambda$ *be an irreducible (finite dimensional)* $\mathfrak{g}$*-module with highest weight* λ *such that* $E\simeq E^*$. *Let* V *be a* $\mathfrak{p}_-$*-invariant subspace in* E *which contains* $E_-=\bigoplus\limits_{\mu=\sum k_i\alpha_i,\ k_i<0}E_\mu$. *Then for any* $i<\operatorname{rk}\mathfrak{g}$ *we have the* $\mathfrak{l}$*-module isomorphism:*

$$H^i(\mathfrak{n}_-;V)=H^i(\mathfrak{n}_-;E)\oplus H^{i-1}(\mathfrak{n}_-;E/V).$$

Proof of Lemma 2.3. Our main tool is the following version of the BWB theorem.

Theorem . (BWZ). $H^i(\mathfrak{n}_-;E)$ *is the direct sum of* $\mathfrak{l}$*-modules with lowest weghts* $-w_i(\lambda+\rho)+\rho$ *for* $i=1,\ \dots,\ k(i)$*; each such module enters with multiplicity* 1.

As is well known [**FF**], with every short exact sequence of $\mathfrak{n}_-$-modules

$$0\longrightarrow V\longrightarrow E\longrightarrow E/V\longrightarrow 0$$

there is associated the long exact cohomology sequence

$$\begin{gathered}0\longrightarrow H^0(\mathfrak{n}_-;V)\longrightarrow H^0(\mathfrak{n}_-;E)\xrightarrow{\phi_0}H^0(\mathfrak{n}_-;E/V)\longrightarrow\\ H^1(\mathfrak{n}_-;V)\longrightarrow H^1(\mathfrak{n}_-;E)\xrightarrow{\phi_1}H^1(\mathfrak{n}_-;E/V)\longrightarrow\dots\\ \longrightarrow H^i(\mathfrak{n}_-;V)\longrightarrow H^i(\mathfrak{n}_-;E)\xrightarrow{\phi_i}H^i(\mathfrak{n}_-;E/V)\longrightarrow\dots\end{gathered}$$

Let us prove that the weight $-w(\lambda+\rho)+\rho$ can not be a weight of the $\mathfrak{l}$-module $H^i(\mathfrak{n}_-;E/V)$ for $i<\operatorname{rk}\mathfrak{g}$. Indeed, each weight of $H^i(\mathfrak{n}_-;E/V)$, which is a quotient of a submodule of $\wedge^i(\mathfrak{n}_-)^*\otimes E/V$, is of the form

$$\gamma_1+\dots+\gamma_k+\mu,$$

where $\gamma_1,\dots,\gamma_k$ are distinct positive roots which do not belong to the root lattice $Q(B_I)$, and μ is a weight of E of the form $\mu=\sum\limits_{a_i\geq 0}a_i\alpha_i$.

Suppose that

$$-w(\lambda+\rho)+\rho=\gamma_1+\dots+\gamma_k+\mu.$$

Then $-\mu-w(\lambda)-w(\rho)+\rho=\gamma_1+\dots+\gamma_k$. Set

$$R_{W^-}=\{a\in R_+\mid w^{-1}(\alpha)>0\}\text{ and }R_{W^+}=\{a\in R_+\mid w^{-1}(\alpha)>0\}.$$

Suppose that $\gamma_1,\dots,\gamma_s\in R_{W^-},\gamma_{s+1},\dots,\gamma_k\in R_{W^+}$.

Recall that

$$\rho-w(\rho)=\sum_{\beta\in R_{W-}}\beta.$$

Since $E=E^*$, the weight $-\mu$ is a weight of E; but then

$$\begin{aligned}&w^{-1}(\gamma_1+\dots+\gamma_k)=[w^{-1}(\gamma_1)+\dots+w^{-1}(\gamma_s)]+[w^{-1}(\gamma_{s+1})+\dots+w^{-1}(\gamma_k)]=\\&(-\lambda-w^{-1}(\mu))+w^{-1}(\sum_{\beta\in R_{W-}}\beta).\end{aligned}$$

Since $-w^{-1}(\mu)$ is a weight, $\lambda+w^{-1}(\mu)\in Q_+(R)$. Hence,

$$-(\lambda+w^{-1}(\mu))+w^{-1}(\sum_{\beta\in R_{W-}}\beta)\in -Q_+(R).$$

On the other hand,

$$w^{-1}(\gamma_{s+1})+\ldots+w^{-1}(\gamma_k)\in Q_+(R)$$

and all the $w^{-1}(\gamma_i)$ for $i\leq s$ enter, as summands, $\sum\limits_{\beta\in R_{W-}} w^{-1}\beta$ and, therefore, cancell.

We finally get:

$$\begin{aligned}Q_+(R)\ni w^{-1}(\gamma_{s+1})+\ldots+w^{-1}(\gamma_k)=&\\ &-(\lambda+w^{-1}(\mu))+w^{-1}(\sum_{\beta\in R_{W-}\setminus\{\gamma_1,\ldots,\gamma_s\}}\beta)\in -Q_+(R).\end{aligned}$$

Thus, $s=k, \lambda=-w^{-1}(\mu)$. In other words,

$$-\mu=w(\lambda)=-\sum a_i\alpha_i \quad \text{for} \quad a_i\geq 0.$$

Since λ is a dominant weight, then $\lambda=\sum m_i\alpha_i$, where all the $m_i\in\mathbb{Q}$ are positive. This is true for any fundamental weight, as follows from the tables from [**Bo**].

By hypothesis, $l(w)<\mathrm{rk}\ \mathfrak{g}$ and, therefore,

$$w=s_{\alpha_1}\ldots s_{\alpha_r}, \quad \text{where} \quad r<\mathrm{rk}\ \mathfrak{g}, \quad B_W=\{\alpha_{i_1},\ldots,\alpha_{i_r}\}\subset B.$$

For any $x\in Q(B)\otimes_{\mathbb{Z}}\mathbb{R}$ we have $x-w(x)=-\sum\limits_{\alpha\in B_W} n_\alpha\alpha$. But then

$$(x-w(x),\omega_j)=0 \quad \text{for some} \quad j\leq \mathrm{rk}\ \mathfrak{g}.$$

Therefore, $(\lambda,\omega_j)=(w(\lambda),\omega_j)$. Now notice that

$$(\lambda,\omega_j)=m_j(\alpha_j,\omega_j)>0, \quad (w(\lambda),\omega_j)=-a_j(\alpha_j,\omega_j)\leq 0.$$

This contradiction shows that $\phi_i=0$ for $i<\mathrm{rk}\ \mathfrak{g}$. This, in turn, means that there exist short exact sequences of $\mathfrak{l}$-homomorphisms

$$0\longrightarrow H^{i-1}(\mathfrak{n}_-;E/V)\longrightarrow H^i(\mathfrak{n}_-;V)\longrightarrow H^i(\mathfrak{n}_-;E)\longrightarrow 0 \quad \text{for} \quad i=1,\ldots,\mathrm{rk}\mathfrak{g}-1.$$

Lemma 2.3 is proved. □

Proof of Theorem 2.2. Set $E=\mathfrak{g}, V=\mathfrak{n}_-\oplus\mathfrak{l}^{(1)}$. By Lemma 2.3,

$$H^2(\mathfrak{n}_-;\mathfrak{n}_-\oplus\mathfrak{l}^{(1)})=H^2(\mathfrak{n}_-;\mathfrak{g})\oplus H^1(\mathfrak{n}_-;\mathfrak{g}/(\mathfrak{n}_-\oplus\mathfrak{l}^{(1)})).$$

It remains to verify that $\mathfrak{g}/(\mathfrak{n}_-\oplus\mathfrak{l}^{(1)})=(\mathfrak{n}_-\oplus\mathfrak{z})^*$. Indeed, the Killing form B establises an isomorphism $\mathfrak{g}=\mathfrak{g}^*$ and, therefore,

$$(\mathfrak{g}/(\mathfrak{n}_-\oplus\mathfrak{l}^{(1)}))^*=\{x\in\mathfrak{g}\mid B(x,\mathfrak{n}_-\oplus\mathfrak{l}^{(1)})=0\}=\mathfrak{n}_-\oplus\mathfrak{z},$$

where $\mathfrak{z}=\{z\in\mathfrak{l}\mid B(z,h_\alpha)=0 \text{ for any } \alpha\in B_I\}$, as required. □

Proof of Corollary 2.2. From the short exact sequence

$$0\longrightarrow(\mathfrak{n}_-\oplus\mathfrak{l})/(\mathfrak{n}_-\oplus\mathfrak{l}^{(1)})\longrightarrow\mathfrak{g}/(\mathfrak{n}_-\oplus\mathfrak{l}^{(1)}))\longrightarrow\mathfrak{g}/(\mathfrak{n}_-\oplus\mathfrak{l})\longrightarrow 0,$$

where $\mathfrak{g}/(\mathfrak{n}_-\oplus\mathfrak{l})=\mathfrak{n}_-^*$ and $(\mathfrak{n}_-\oplus\mathfrak{l})/(\mathfrak{n}_-\oplus\mathfrak{l}^{(1)})=\mathfrak{z}^*$, we get the long exact sequence

$$\begin{aligned}&0\longrightarrow\mathfrak{n}_-^{ab}\longrightarrow\mathfrak{z}^*\otimes H^1(\mathfrak{n}_-)\longrightarrow H^1(\mathfrak{n}_-;\mathfrak{g}/(\mathfrak{n}_-\oplus\mathfrak{l}^{(1)}))\longrightarrow H^1(\mathfrak{n}_-;\mathfrak{n}_-^*)\xrightarrow{\psi}\\ &\longrightarrow H^2(\mathfrak{n}_-;\mathfrak{z}^*)\ (=H^2(\mathfrak{n}_-)\otimes\mathfrak{z}^*))\longrightarrow\ldots\end{aligned}$$

To compute the image of the map ψ, consider the $\mathfrak{l}$-module

$$M=\{f\in\mathrm{Hom}\ (\mathfrak{n}_-,\mathfrak{g}/(\mathfrak{n}_-\oplus\mathfrak{l}^{(1)}))\mid Xf(Y)-Yf(X)-f([X,Y])\in\mathfrak{z}^* \quad \text{for any} \quad X,Y\in\mathfrak{n}_-\}.$$

From the general properties of the long exact sequences ([**FF**]) we deduce that $\mathrm{Im}\ \psi=M/Z^1(\mathfrak{n}_-;\mathfrak{g}/(\mathfrak{n}_-\oplus\mathfrak{l}^{(1)}))$.

By the BWZ theorem,

$$H^2(\mathfrak{n}_-) = \bigoplus_{l(w)=2,\ w(B_I)>0} E(w(\rho)-\rho).$$

Since M is the direct sum of its weight subspaces relative to $\mathfrak{h}$, let us study the subspaces $M_{\rho-w(\rho)}$. The weight of $f\in M$ is equal to $w(\rho)-\rho$ if and only if it sends $\mathfrak{n}_{-\gamma}$ to $(\mathfrak{g}/(\mathfrak{n}_-\oplus\mathfrak{l}^{(1)}))\rho-w(\rho)-\gamma$. Therefore, either $\rho-w(\rho)-\gamma=\gamma'\in R$ or $\rho-w(\rho)-\gamma=0$.

The second option is ruled out since $\rho-w(\rho)$ is not a root: indeed, $l(w)=2\neq 1$.

In the first case $\rho-w(\rho)=\gamma+\gamma'$, where $\gamma,\gamma'\in R_+$. If $\gamma=\gamma'$, then $\rho-w(\rho)=2\gamma$, implying $w^{-1}(\gamma)<0$. But card $(R_W^-)=2$ and, $\rho-w(\rho)=2\gamma=\gamma_1+\gamma_2$, where $\gamma_1,\gamma_2\in R_W^-$. Since one of the γ_i is equal to γ, so is the other one. This contradicts to the hypothesis: $\gamma_1\neq\gamma_2$.

Thus, $\rho-w(\rho)=\gamma+\gamma'$, where $\gamma\neq\gamma'$. It is not difficult to deduce from this that $w^{-1}(\gamma)<0, w^{-1}(\gamma')<0$. Hence, $f(\mathfrak{n}_{-\delta})\neq 0$ only if $w^{-1}(\delta)<0$ for $f\in M_{\rho-w(\rho)}$. In this case $f(\mathfrak{n}_{-\delta})\in(\mathfrak{g}/(\mathfrak{n}_-\oplus\mathfrak{l}^{(1)}))_{\delta'}$, where $w^{-1}(\delta')<0$. Besides, since $\delta,\delta'\notin R$, we have $[X_{-\delta},X_{-\delta'}]=0$ for the root vectors $X_{-\delta},X_{-\delta'}$. (Indeed, otherwise $\delta,\delta'\in R_W^-$ and $l(w)>2$.)

On the other hand, any map $f_w:\mathfrak{n}_{-\delta}\longrightarrow\mathfrak{g}/(\mathfrak{n}_-\oplus\mathfrak{l}^{(1)})$ which vanishes outside the subspace $\mathfrak{n}_{-\gamma}\oplus\mathfrak{n}_{-\gamma'}$, where $\{\gamma,\gamma'\}=R_W^-$, and such that the weight of $f_w(\mathfrak{n}_{-\gamma})$ is equal to γ' whereas that of $f_w(\mathfrak{n}_{-\gamma'})$ is γ belongs, clearly, to $M_{\rho-w(\rho)}$. This means that dim $M_{\rho-w(\rho)}\leq 2$.

Let $w=s_{\alpha_1}s_{\alpha_2}$, where $\alpha_2\in B_1$ and $(\alpha_1,\alpha_2)\neq 0$. Then dim $M_{\rho-w(\rho)}=1$, as the following calculation shows: let $f(X_{-\alpha_2})=cX'_{\alpha_1+\alpha_2}, f(X_{-(\alpha_1+\alpha_2)})=c'X'_{\alpha_2}$, where the primed vectors belong to the quotient space. Then for appropriate c,c' we have

$$X_{-(\alpha_1+\alpha_2)}f(X_{-\alpha_2})-X_{-\alpha_2}f(X'_{-(\alpha_1+\alpha_2)})= \\ -cH'_{\alpha_1+\alpha_2}+c'H'_{\alpha_2}\in\mathbb{C}H'_{\alpha_1}\subset(\mathfrak{n}_-\oplus\mathfrak{l})/(\mathfrak{n}_-\oplus\mathfrak{l}^{(1)}).$$

If $w=s_{\alpha_1}s_{\alpha_2}$, where $\alpha_1,\alpha_2\in B_I$, then dim $M_{\rho-w(\rho)}=2$. Since for any such w we have $M_{\rho-w(\rho)}\cap Z^2(\mathfrak{n}_-;\mathfrak{g}/(\mathfrak{n}_-\oplus\mathfrak{l}^{(1)}))\neq 0$, we get the desired heading 1) of the corollary.

The general statement of heading 2) is straightforward; its particulart case a) follows from evident remarks: $H^1(\mathfrak{n}_-;\mathfrak{n}_-^*)=\mathfrak{n}'_-\otimes\mathfrak{n}'_-$, $H^2(\mathfrak{n}_-)=\Lambda^2(\mathfrak{n}'_-)$.

Proof of heading 2b). Set $\mathfrak{n}'_-=\mathfrak{n}_-/\mathfrak{z}(\mathfrak{n}_-)$. Clearly, $\mathfrak{n}'_-$ is a trivial $\mathfrak{n}'_-$-module. Therefore, the sequence

$$0\longrightarrow\mathfrak{n}'_-\longrightarrow\mathfrak{n}'_-\longrightarrow\mathbb{C}\longrightarrow 0$$

is exact, hence, so is the sequence

$$0\longrightarrow\mathbb{C}\longrightarrow\mathfrak{n}'_-\otimes H^1(\mathfrak{n}_-)\longrightarrow H^1(\mathfrak{n}_-;\mathfrak{n}'_-)\xrightarrow{f}H^2(\mathfrak{n}_-)\longrightarrow\dots.$$

It is not difficult to notice that the image of any cocycle $\mathfrak{n}_-\longrightarrow\mathfrak{n}'_-$ under f belongs to $\Lambda^2(\mathfrak{n}'_-)$, hence, f is zero. But then $H^1(\mathfrak{n}_-;\mathfrak{n}'_-)$ is isomorphic to the subspace $(\mathfrak{n}'_-\otimes\mathfrak{n}'_-)_0$ of traceless operators in $\mathfrak{n}'_-\otimes\mathfrak{n}'_-$. It remains to notice that $H^2(\mathfrak{n}_-)\oplus\mathbb{C}=\Lambda^2(\mathfrak{n}'_-)$. ☐

2.4. Let IR be the number of irreducible components in the $\mathfrak{g}_0$-module $H^2(\mathfrak{g}_-;\mathfrak{g}_*)$.

THEOREM . $IR=\frac{1}{2}c_0(c_0+1)+\sum\limits_{i\neq 0}c_i$.

Proof. Since $IR=$ card $\{w\in W_I\mid l(w)=2\}$, let us list the length 2 elements of W_I. Let $W(R_I)$ be the Weyl group of the root system R_I. Clearly,

$$W(R_I)_2=\{w\in W(R_I)\mid l(w)=2\}\subset(W_0)_2.$$

If $s_is_j\in(W_0)_2\setminus W(R_I)_2$, then $s_j\in B_I$ and $s_i\notin B_I$. (Indeed, if $s_j\notin B_I$, then $s_is_j(a_j)=-s_i(a_j)<0$ which is false.) Furthermore, $(\alpha_i,\alpha_j)\neq 0$ since otherwise $s_is_j(\alpha_i)=-\alpha_i<0$.

Let rk $R = r$. Let $R = R_1 \coprod \dots \coprod R_s$, be the presentation as the union of connected components. Let $B = \{\alpha_1, \dots, \alpha_r\}$ be a base of R. Then

$$\begin{aligned}&W(R)_2 = \{w \in W(R) \mid l(w) = 2\} = \{s_i s_j \ \text{ for } \ i \neq j\} = \\ &\{s_i s_j \ \text{ for } \ i < j\} \coprod \{s_i s_j \ \text{ for } \ i > j \mid (\alpha_i, \alpha_j) \neq 0\}.\end{aligned}$$

It follows that

$$\begin{aligned}\text{card } W(R)_2 &= \frac{c_0(c_0-1)}{2} + \text{card } \{\text{vertices of the Dynkin diagram of } R\} \\ &= \frac{c_0(c_0-1)}{2} + c_0 - s.\end{aligned}$$

If $\alpha \in R \setminus R_I$ and $(\alpha, R_I^{(i)}) \neq 0$ for some i then there exists a unique $\beta \in R_I^{(i)}$ such that $(\alpha, \beta) \neq 0$. Indeed, if there is two such roots, say, β_1 and β_2, then the Dynkin diagram contains a cycle

$$\alpha - \beta_1 - \gamma_1 - \gamma_2 - \dots - \gamma_s - \beta_2 - \alpha -$$

Indeed: $D_I^{(i)}$ is connected and $\alpha \notin R_I^{(i)}$, hence, $\gamma_1, \dots, \gamma_s \in D_I^{(i)}$.

All this demonstrates that

$$\begin{aligned}\text{card } (W_I)_2 &= \text{card } (W(R_I))_2 + \sum_{i \neq 0} c_i + 1 \\ &= \frac{1}{2} c_0(c_0-1) + c_0 - s + \sum_{i \neq 0} c_i + s = \frac{1}{2} c_0(c_0-1) + \sum_{i \neq 0} c_i. \quad \square\end{aligned}$$

§3. Premet's generalization of a theorem of Leger and Luks

Premet found the following generalization of Lemma 2.3.

Theorem . *Let $E(\lambda) = L_\lambda$ be an irreducible (finite dimensional) $\mathfrak{g}$-module with highest weight λ such that $E = E^*$. Let V be a subspace in E which is $\mathfrak{p}_-$-invariant and contains $E_- = \bigoplus\limits_{\mu = \sum k_i \alpha_i, k_i < 0} E_\mu$.*

Let $P(V)$ be the set of weights of V. Then the following sequence is exact:

$$\begin{aligned}&0 \longrightarrow \bigoplus_{\substack{l(w)=i-1 \\ ww_0(\lambda) \notin P(V)}} E(-w(\lambda+\rho)+\rho) \longrightarrow H^{i-1}(\mathfrak{n}_-; E(\lambda)/V) \\ &\longrightarrow H^i(\mathfrak{n}_-; V) \longrightarrow \bigoplus_{\substack{l(w)=i \\ ww_0(\lambda) \in P(V)}} E(-w(\lambda+\rho)+\rho) \longrightarrow 0 \textit{ for any } i < \text{rk}\mathfrak{g}.\end{aligned}$$

§4. Open problems to start with

4.1. For any simple finite dimensional Lie algebra $\mathfrak{g}$ and its nilponent subalgebra $\mathfrak{N} = \mathfrak{n}_{i_1 \dots i_k}$ for $k = 0, \dots, k = \text{rk}\mathfrak{g}$, describe the associative algebra $H^*(\mathfrak{N})$ and $H^*(\mathfrak{N}; \Lambda^{\boldsymbol{\cdot}}(\mathfrak{N}))$, and the Lie superalgebras $H^*(\mathfrak{N}; \mathfrak{N})$ and $H^*(\mathfrak{N}; \mathfrak{g})$.

Comment. How to describe the algebra (via generators and relations) or as a semidirect product of a radical and a semisimple algebra and how to describe this semisimple Lie superalgebra is not clear at the moment. The result is unpredictable, so one has to test various cases to see what we obtain. So far, particular results are due to Kostant, Leger and Luks, Premet. Tolpygo deduced asymptotic formulas to compute the dimension and generating functions of the above cohomology. Consideration of algebraic structure of these cohomology — a totally new approach — might clarify the picture. Unfortunately, some preliminary results due to I. Shereshevskii demonstrate that the general picture will not be easy to describe even in this approach.

4.2. For every simple finite dimensional Lie superalgebra $\mathfrak{g}$ describe its cohomology with coefficients in simplest nontrivial atypical modules (identity, adjoint, their symmetric powers).

Among most interesting particular cases: $H^*(\mathfrak{g})$ and $H^*(\mathfrak{g};\mathfrak{g})$. Describe associative algebra $H^*(\mathfrak{g})$ and the Lie superalgebras $H^*(\mathfrak{g};\mathfrak{g})$ via generators and relations.

3.3. For every simple $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g} = \underset{i\geq -d}{\oplus}\mathfrak{g}_i$ and its subalgebra $\mathfrak{g}_- = \underset{i<0}{\oplus}\mathfrak{g}_i$ compute $H^i(\mathfrak{g}_-;\mathfrak{g})$ (say, as a $\mathfrak{g}_0$-module) for $i \leq 2$.

Certain interpretations: These cohomology (especially for $i = 2$) can be considered as obstruction to stability of the dynamical system with $\mathfrak{g}$ as its algebra of symmetries; as obstructions of formal integrability of differential equations with $\mathfrak{g}$ as its algebra of symmetries; the $\mathfrak{g}_0$-invariants of $H^2(\mathfrak{g}_-;\mathfrak{g})$ describe filtered deformations of $\mathfrak{g}$.

For vectorial Lie algebras these cohomology are *implicitly* described by Serre, cf. [**St**]. Explicit description is of interest even for Lie algebras, cf. [**LP1**], [**LP2**], [**P1**]. For Lie superalgebras see [**CK**], [**GLS2**].

3. From Supergravity to ball bearings

Disclaimer. The authors are not experts in either analytical mechanics or physics. Therefore our examples are very simple and scanty (Internet lists thousands of entries for nonholonomic and hundreds more on a synonym, sub-riemannian). We were just lucky to observe that various problems listed below (we tried to provide with ample references to start with, we particularly like and suggest [**VG**], [**A2**], [**Br**], [**Gr**], [**H**], [**M**], [**NF**], [**3**], and, especially, a user-friendly paper [**N**]) can be solved (at least theoretically) by means of computing certain Lie algebra cohomology. The Sign Rule inserts superflavour.

The talk will require no preliminary knowledge and will be addressed mainly to students who do not know too much to be prejudiced against unusual methods with only few romantic results to back up our claims of their usefulness.

Here is our story.

§1. Nonholonomic dynamics: History

Supergravity. In 1972 Leites introduced [**L0**] the what is now called *superscheme* or *algebraic supervariety*. (Unfortunately, short communications to Russian Mathematical Surveys were not translated into English at that time, but, even if they were, the text was too short. Even its expansion rewritten from algebraic language to C^∞ one [**BL**] did not get much acclaim.) This (and [**L1**]) was an answer to a question of Berezin together with whom he was slightly ahead of time, see Minlos' recollections in [**D**].

Now supermanifold theory is fully recognized thanks to its part, *supersymmetry*, considered in theoretical physics as the language of the future unified theory of all fundamental forces. Wess and Zumino were the first to realize and explain to others that **we live on a supermanifold** and thus made one of the most fundamental steps in science, not only in physics. Physicists realized that there is not one, but a whole hierarchy of *standard* Minkowski *super*spaces and a number of exsotic models (see [**3**], [**SS**], [**Ma**]), labeled for the standard (for the moment) model by an inner parameter N running (conventionally) 1 to 8. The most interesting was supposed to be the largest model, for $N = 8$.

Ensuing enthusiasm was overwhelming (at least, among physicists). Classification of simple finite dimensional Lie superalgebras performed by Kaplansky, Nahm–Rittenberg–Scheunert, and skillfully rounded up by Kac — strikingly resembling that of simple Lie algebras — convinced many mathematicians that this is “it” in a sense.

Mass media (from Pravda to New York Times, except, perhaps, Ren Min Ribao) heralded that soon Einstein's dream — Grand Unified Theory (now called SUSY GUTs for its supersymmetry) — will come true. But eventually fanfare subsided. The reason: SUGRA equations for $N = 1$ were established during 1974 – 76, $N = 2$ (and to an extent $N = 3$) were supposed to have been written in 1984. After much work in various research centers, it was Ogievetsky's group who realized that to fulfill natural requirements (see §4) one has to *enlarge* the Minkowski space underlying Minkowski superspace by adding the what was called “harmonic” coordinate. How to do it for $N > 3$ was unclear, see [**GI**], [**GIOS**].

To apply for grants subsidizing studies of nonexisting (as far as supersymmetry was concerned) equations became more difficult and most of the researchers left the field. What was the problem? (Not with grants, all is clear here; what is the *mathematics* difficulty?)

One of the best experts, Wess, honestly lectured [**3**]: " *we do not know how to express the Riemann tensor* (the left hand side of the equations to be written) *in the general case*".

In [**Aea**], Witten wrote: "*Direct experimental confirmation of supersymmetry is one of the prime missions of the proposed Superconducting supercollider.*" And then added: "*More fundamentally, I believe that the main obstacle is that the core geometrical ideas — which must underline string theory the way Riemannian geometry underlines general relativity — have not yet been unearthed.*"

We believe that we were lucky to have discovered some of these ideas [**L3**]. They are related to (a) *nonholonomic* mechanics, (b) the presentation of the curvature tensor via cohomology and (c) the classification of simple Lie superalgebras of vector fields — the analog of É. Cartan's classification, [**LS**].

Relation with nonholonomic mechanics is occasioned by the fact that Minkowski *super*spaces (whatever they are, see [**3**], [**SS**], [**Ma**]; in recent dissident studies of M. Vasiliev [**2**] the inner parameter N conventionally running 1 to 8, can tend to ∞; Manin [**Ma**] suggested to consider series of other, nonstandard and and exceptional exotic models) are **nonholonomic**, unlike the Minkowski *space* which is holonomic. (For a holonomic analog of Minkowski superspace see [**GL7**].)

The importance of Lie algebra cohomology for physics in general was in the air lately; conferences entitled "Cohomological Physics" started to mushroom. In particular, in supergravity the importance of cohomology is clearly stated by several authors (e.g., [**CDF**]) but the description of the curvature tensor in nonholonomic case was never given until recently [**LP2**], [**GL3**] and this is the whole point totally missed in attempts to describe Riemannian tensor on *Minkowski* superspaces as if they were holonomic (compare [**LP2**], where these computations are made for a deliberately wrong according to today's views, since it is (holonomic, version of Minkowski superspace, with real SUGRA [**GL3**]) and such attempts stopped at $N = 1$, see [**RS**].

Mechanics. In XIX century Hertz [**H**] in his attempts to get rid of the notion "force" (we recommend to read Feynman's discussions of the problem [**FLS**]) divided the dynamical systems into *holonomic* (no constraints on velocities) and *nonholonomic* (with nonintegrable constraints on velocities). Analytical study of holonomic systems has progressed much further than that of nonholonomic. Moreover, only *linear* constraints are usually considered.

Though nonholonomic systems are important in various branches of applied physics and engineering, they have not been sufficiently studied in mathematics, even the best text books on analytical mechanics just mention several examples and pass to holonomic systems, cf. [**A1**], [**A2**]. One of the reasons: the qualitative study, such as stability problems, require the notion of curvature tensor which is not defined for the nonholonomic systems in the literature, cf. [**St**], the standard textbook for this purpose.

Examples of nonholonomic systems. 1) In **mechanics**. The ball on a rough plane; ballbearings and gyroscopes; any vehicle with wheels or any body rolling without gliding over another body (the point at which the moving body is tangent to the surface of the other body has zero velocity). Any car with cruise control switched ON is an example of nonholonomic system with a *nonlinear* constraint. For further (numerous) examples we refer the reader to Internet, of thousands entries there we would like to point out [**FK**], [**K**], [**E**], [**Gl**] and references therein.

Related problems: general description of the movement and stability of a rapidly rotating hole body with a liquid or gyroscope inside it; orbital stability of a missile with liquid inside it. (The linear approximation does not work here.)

For several interesting practical applications (e.g., how the cat, when dropped, uses nonholonomicity to fall on its paws or how nonholonomicity influences the parking car) see [**M**], [**E**], [**Fe**].

In **electromechanics** (see, e.g., the book by Nejmark and Fufaev [**NF**] or [**BF**]) an early example was provided by Gaponov who, in 1952, found that a conductor connected to a gliding contact (as in a rotor collector) is equivalent to imposing nonholonomic constraints on the distribution of the electric current.

For comprehensive reviews of nonholonomic problems, see the books [**NF**] and [**VG**]. They list numerous problems which go back to Euler, Gauss, Carnot, Hertz, Carathéodory, Appell, Schouten, Synge, and many others.

2) The **magneto-hydrodynamics**: waves in plasma. Study of stability of such waves is extremely important for the development of thermonuclear power plants.

3) Recent studies of Komech, Spohn and Kunze (e.g., [**KS**]) show that electron represents a nonholonomic system with a (Coulomb, Maxwell, etc.) gauge as a nonholonomic constraint.

4) Numerous works on hypoelliptic differential equations (by Hörmander, Melin, Malliavin, Bismut, Bell; for a review see, e.g., [**Ar**]) also lead to nonholonomic distributions.

Statistical thermodynamics and economics. The well-known from school relation between pressure, temperature and volume of the ideal gas is an example of a nonholonomic constraint. This seems to be one of the first non-trivial but easy-to-compute examples (the very first ones are contact and Engel distributions, cf. [**L3**]). Recently V. Sergeev expressed certain market models in thermodynamical terms. Having developped ideas of Hyek, Sergeev suggested mathematical models which gives, among other things, a transparent explanation of such well known phenomena as economic growth in China and Vietnam and economic decline in various post Soviet republics [**SV**]. This is already somthing, but what fascinates us in Sergeev's studies is a clear formulation of economics as a non-holonomic system.

Related problems.

• Vershik was probably the first who noticed similarity of various distributions of solid bodies that appeared in **optimal control** (corresponding to the nonlinear nonholonomic constraints and fields of cones, spheres, etc.) with the "well-known" nonholonomic problems, cf. [**CF**], [**VG**] and Agrachev's plenary talk at the 1994 International Congress of Mathematicians [**Ag**].

• **Supersymmetry** appears even in seemingly non-super questions, like the study of the spectrum of the Schrödinger operator [**L2**] or relation between the Schrödinger operator and the KdV (Korteveg de Vries) operator [**L4**]. It does not always bring in nonholonomicity, but rather often, and usually unexpectedly.

• Recent studies of the Gelfand–Dickey bracket lead us to the discovery of a new class of simple Lie algebras of polynomial growth — generalization of Lie algebras of matrices of "complex size" [**GL**], [**GL2**]. Simultaneously, together with Shchepochkina, we announced two classifications of simple Lie superalgebras: (A) of vector fields [**LS**] and (B) stringy or superconformal superalgebras [**GLS1**].

Solution of these problems [**GLS1**], [**LS**] was considerably speedified by our computer-aided study.

• Related with Lie superalgebras of complex size generalizations of orthogonal polynomials (especially, hopefully, in several variables), see our talk with A. Sergeev at SF-2000 [**LSe**].

Amazingly, all the above topics are interrelated by common mathematics: Lie algebra cohomology and nonholonomic structures. M. Vasiliev augmented our amazement: in one of his papers on super versions of Dunkl operators and Calogero–Sutherland model he (with co-authors) showed [**1**] that generalized Lie *super*algebras of supermatrices of complex supersize play the role of Poincaré algebras in N-extended SUGRAs for $N > 8$.

Accordingly, our research splits into several topics united by the usage of a common *Mathematica*-based package **SuperLie** written by Grozman.

1. Study of nonholonomic mechanics: from supergravity to ballbearings. Attempts to understand applications to stochastic analysis, media with dislocations, and optimal control. Study of stability via cohomology.

2. Study of representation theory; in particular, classification of simple Lie superalgebras of vector fields, stringy or superconformal, and generalizations of Lie algebras of matrices of complex size. Description of defining relations for them via cohomology. Applications to Lie algebras and superalgebras and their representations over fields of prime characteristic and over integers.

3. Application of representation theory to the study of integrability of differential equations (KdV, KP, Schrödinger, Liouville, Toda lattice, etc. and their superizations). Criteria of integrability expressed in terms of Lie algebra cohomology.

4. Classification of invariant differential operators.

In this talk we will briefly report two types of results: on SUGRA and on invariant operators.

§2. Formulation of problems

According to Arnold–Kozlov–Neishtadt [**A2**], the behavior of nonholonomic systems is often “surprising”, and its quantitative study is handicapped by the lack of adequate tools. Let us recall some examples from [**A2**].

Consider a skate on the inclined plane. Where do you think it will move if pushed not directly downwards, but sideways? If we ignore loss of energy due to friction, it will never reach the floor but will oscillate between certain horizontal lines on the plane. Similarly, consider a ball rolling along the wall inside a vertical tube. It seems natural to expect the ball to descend on a spiral trajectory with increasing steepness. In “reality”, however, (if to assume conservation of energy regardless of friction can be considered realistic) the ball will perform oscillations between two fixed vertical planes.

Though individual solutions of nonholonomic systems are usually known, the stability questions are solved *ad hoc*.

Stability problems. The stability of a holonomic system can be studied in terms of the Riemann curvature tensor. The sign of the curvature indicates whether the geodesics converge or diverge: compare meridians on a sphere (positive curvature) with those on a trumpet = Lobachevsky plane (negative curvature).

For nonholonomic manifolds there was no such tool: in the literature the definition of the analogue of the Riemann tensor is only given in a few particular cases of little practical value, cf. [**VG**], [**T**]. For the traditional analytical approach see [**RK**].

Difficulties. In the last decades, dozens of publications on dynamical problems in nonholonomic mechanics appeared each year in technical journals, but very few in Mathematical journals. The main reason: lack of adequate language needed to even formulate the problem of calculating the local curvature tensor (the analogue of the Riemann tensor) in intrinsic terms. The bulky coordinate expressions used by classics in 1920’s, as well as now, hinder further progress. Vershik and Gerschkovich gloomily stated [**VG**] that there is no language to even formulate the integrability problems for nonholonomic systems with linear constraints (to say nothing about nonlinear constraints).

This is one of the reasons why this important field was abandoned by mathematicians, and even the interest and occasional works of such leaders as Faddeev, Griffiths, Chern and Arnold (see [**VG**], [**Br**], [**A2**]) did not attract any followers. Perhaps, only the latest works of Marsden [**Ms**] do not satisfy this pessimistic description.

§3. What is done: a notion, a formula, and a package

We know the importance of the notion of curvature in drawing very exact maps; application to stability were already mentioned. How to compute the curvature? In text books on geometry this is done in terms of Spencer cohomology. Same cohomology are used in Goldschmidt's criteria for (formal) integrability of differential equations.

Observe that these criteria [**Br**] are only given for "a half" of the cases. The point is that by a theorem of Cartan the symmetries of each differential equation are induced by either point transformations or contact transformations [**KLV**]. The latter case belongs to the realm of the simplest nonholonomic manifold — a contact one — and, apart from partial results of [**T**], nothing was known in this case. Our notions allow one to extend Goldschmidt's criteria to all differential equations.

The notion and the formula: Lie Algebra Cohomology. In lectures at a school at ICTP, Trieste, in 1990, Leites reformulated the definition of the Riemann tensor in terms of *Lie algebra cohomology* for any $\mathbb{Z}$-graded Lie algebra of finite depth (see §4). The notion of generalized Cartan prolongation [**LS**] is crutial here. In these terms, the problems discussed above can be posed precisely, and, in principle, solved. In particular, we describe the space of possible values of Riemannian tensor as $H^2(\mathfrak{g}_-; (\mathfrak{g}_-, \mathfrak{g}_0)_*)$; this is "the formula". It even becomes possible to generalize the description of the local curvature tensor for nonholonomic manifolds with any constraints, like the fields of surfaces.

The Package. During 1992–94, Grozman developed a *Mathematica*-based pilot package **SuperLie** for computing Lie algebras and their cohomology. Since then the package was constantly developped and now **SuperLie** makes it possible to determine (1) Lie algebras matrix algebras, algebras of vector fields, algebras defined via generators and relations, via Cartan matrices, via Poisson or contact (Legendre) bracket, etc., as subalgebras of any of the above algebras; (2) various modules over these Lie algebras (tensors, with vacuum vector; submodules, quotients, duals, etc.), and calculate bracket and action on the Lie algebra on its module, (3) solve systems of equations with scalar and (linear) vector unknowns; find ker and Im of a linear operator; (4) calculate exterior differential in various complexes, and their homology and cohomology.

The mentioned above reformulation of Spencer cohomology in terms of Lie algebra cohomology means that much of the time-consuming calculations become considerably simplified and new ways to diminish the amount of calculations appear. However, the computation effort is still overwhelming: to compute the curvature tensor even for the simplest practical examples of, e.g., a ballbearing, or a bike, we need to develop the package further and complete its documentation to make available to engineers. On top of that symbolic computations in *Mathematica* is irritatingly slow (though hardly any of the wide spread computer languages is faster). Still, though it seems that an *ad hoc* written code for each particular task would be faster than our package, none managed so far.

The advantage of **SuperLie** is its universality and versatility. (For example, in [**Gz**], [**LKW**] bilinear differential operators acting in tensor fields invariant with respect to the general and volume preserving diffeomorphism groups are classified; further advancement "by hands" seems hardly possible. Recently with the help of **SuperLie** Grozman obtained new results in hamiltonian and contact cases, cf. §5.)

In spite of some drawbacks of the existing pilot package, we used it to correct several results found analytically by mathematicians (certain cohomology from [**FL**], [**Fu**]) and by physicists (certain Wess–Zumino constraint in supergravity [**3**] turned out to be redundant). The last example is of particular interest: it convinces that computer aded study is a must here since this redundunce was missed by hundreds of physicists working with SUGRA around the globe for two decades.

The results, obtained completely or partly with the help of **SuperLie** are described in [**GL**], [**GL2**], [**GLP**], [**GLS2**]-[**GLS4**] and used also in [**LS**], [**L4**]. Several more are in preparation.

Rival and not so rival teams. An alternative package (not so overwhelming but allegedly much faster) was being developed for several years in JINR, Dubna, by V. Kornyak, our coworker, in C. (So far his package can not do anything **SuperLie** can not.) For results of REDUCE-fans from Twente University see [**LP**] and refs. in [**GLP**].

§4.0. The left hand sides of SUGRAs ([GL3])

Here we briefly explain how to solve the problem in deriving supergravity equations, SUGRA(N) — the analogues of Einstein's equations (EE) on an N-extended Minkowski superspace $\mathcal{M}(N)$ and what are our the criteria for distinguishing suitable Minkowski superspaces among other supermanifolds. Our considerations are in vacuum. As in twistor theory, we consider complex case, the physical reality to be recovered on a suitable real form of $\mathcal{M}(N)$. Our requirements: SUGRA(N) should be

(A) a differential equation of order ≤ 2 on the components;

(B) the component expansion of SUGRA(N) should contain the ordinary Einstein's equations.

For simplicity we assume that the supergroup of motions of $\mathcal{M}(N)$ is $\mathcal{G} = \mathcal{SL}(N|4)$ (though other possibilities can not be eliminated, cf. [**GL2**], [**Ma**]), and we wonder: what is the stationary subgroup $\mathcal{P}$ for which $\mathcal{M}(N) = \mathcal{SL}(N|4)/\mathcal{P}$?

As N grows, it becomes clear (by comparing the spectrum of cohomology for these supergroups of motion with the underlying Riemann tensor, as modules over $O(4)$) that to justify the above requirements (A) and (B) is only possible if we diminish $\mathcal{P}$, as GIKOS did, cf. [**GI**]. Then for $N > 3$ we see that the underlying manifold of $\mathcal{M}(N)$ is the direct product of several copies of the Minkowski space M (times, perhaps, an auxiliary space of a yet unclear merit); so SUGRA splits into the usual Einstein equations on each copy of M glued together by odd superfields. In particular, for $N = 4$ there are two copies of M.

For $N = 8$, in a most symmetric model, there are three copies of M, one of them distinguished (say, "our world"), the other two – perfectly interchangeable (in the model considered here; there are other possibilities) – mirroring, say, "heaven" and "hell"). These extra copies of the Universe (they MUST appear in our approach) embody an idea first, perhaps, voiced by A. Sakharov [**Sa**] on the necessity to consider several copies of our Universe to describe physical reality. Another feature of nonholonomic nature of Minkowski *super*space is the preferred direction of time, an observation we derive by directly looking at the rattleback (Celtic stone); the universality of this observation for nonholonomic systems follows from recent elementary, and therefore wonderful, studies by A. Nordmark [**N**].

We start first with a presentation of Einstein's equations in a form convenient in what follows — as equations on conformally noninvariant components of the Riemann tensor, and represent the Riemann tensor as a section of the bundle whose fiber is certain *Lie algebra cohomology*. This is equivalent to the standard modern treatment of G-structures in differential geometry that uses *Spencer homology* [**St**] but can be generalized to treat nonholonomic structures, such as SUGRA. Then we generalize the notion of the G-structure to embrace *nonholonomic* manifolds, i.e., manifolds with nonintegrable distributions, see [**H**]. The structure of a nonholonomic manifold is encountered quite often. The simplest example is the contact structure on a manifold, cf. [**A1**]. The applications range from the Cat's Problem to electro-mechanical devices [**NF**], [**BF**], [**E**], [**CF**], [**K**], [**KS**], [**M**]. For a moving account of nonholonomic problems and their history see [**VG**].

One can calculate Spencer cohomology for any supermanifold with G-structure (such attempts are numerous in the literature) but the tensors obtained do not match the one physicists consider, cf. [**3**]. for nonholonomic manifolds and supermanifolds we have to consider *generalized* Spencer cohomology which we formulate in terms of *Lie algebra* cohomology.

We also offer a general method to derive constraints — analogues of Wess–Zumino constraints — for any N. Funnily, one of the conventional constraints for $N = 1$ is redundant: its cohomology class is zero. This demonstrates that a computer-aided study [**GL5**] is a must here: even when we know what to do and how, the amount of computations is too vast for a human. Even as skilled one as Poletaeva or physicists [**GI**].

Though the notion of supermanifolds is past its 25-th birthday [**L1**], certain basics are, regrettably, insufficiently well-known yet. So we will recall them.

§4.1. Structure functions for nonholonomic structures

1.1. Nonholonomic (super)manifolds ([VG]). Let M be a manifold with a distribution D. Let

$$D = D_1 \subset D_2 \subset D_3 \subset \ldots \subset D_d \tag{1.0}$$

be the sequence of strict inclusions, where $D_i(x) = D_{i-1}(x) \cup [D_1(x), D_{i-1}(x)]$ for every $x \in M$ and d is the least number for which the sequence (1.0) stabilizes, i.e., such that $D_d(x) \cup [D_1(x), D_d(x)] = D_d(x)$. The number $d = d(M, D)$ is called *the nonholonomicity degree*. In case $D_d = TM$ the manifold M is called *completely nonholonomic*. Each pair: (M, D) with a nonintegrable D will be referred to as a *nonholonomic manifold* if $d \neq 1$. Here we will only consider completely nonholonomic (super)manifolds (the incomplete case corresponds to *nontransitive* generalized Cartan prolongs, cf. below, and the modifications needed to adjust general machinery to it are obvious).

Let $n_i(x) = \dim\ D_i(x)$. The distribution D is called *regular* if all the dimensions n_i are constant functions on M. In what follows we will only consider regular distributions.

With the tangent bundle over a nonholonomic manifold (M, D) we can naturally associate a sheaf of nilpotent Lie algebras as follows. At a point $x \in M$ set

$$\mathfrak{n}(x) = \{X \in \mathfrak{vect}(M) \mid X \text{ preserves the Pfaff system that distinguishes } D\}. \tag{1.1}$$

Clearly, $\mathfrak{n}(x)$ is a well-defined Lie algebra and one can show that, as spaces,

$$\mathfrak{n}(x) = \bigoplus_{-d\leq i\leq -1} \mathfrak{n}_i(x), \quad \text{where } \mathfrak{n}_{-i}(x) \simeq D_i(x)/D_{i-1}(x), \quad D_0 = 0. \tag{1.2}$$

Here d is the nonholonomicity degree of (M, D).

1.1.1. A flat $(G, \mathfrak{n}(x))$-structure. This is a straightforward generalization of the notion of the flat G-structure (the latter implies, in particular, that every $\mathfrak{n}(x)$ is commutative, but not only that: the curvature tensor must also vanish). Namely, let M be just $\mathbb{R}^n$ endowed with a nilpotent $\mathbb{Z}$-graded Lie algebra structure, call it $\mathfrak{n}$. Let G be a subgroup of homogeneous (preserving the grading (1.2)) automorphisms of $\mathfrak{n}$. Let us identify the tangent space at a point $m \in M$ with M by means of a transformation from G. The preimages $\mathfrak{n}_{-1}(m)$ of $\mathfrak{n}_{-1}$ under this identification determine a distribution on M. This distribution together with the G-action on the accompanying flag $\mathfrak{n}(m)$ at each $m \in M$ will be called a *flat* $(G, \mathfrak{n})$-*structure*. Observe that this "nonholonomic flatness" and the obstructions to flatness introduced below differ drastically from their conventional counterparts: any contact manifold is nonholonomically flat (Darboux theorem) whereas it is never flat in terms of G-structures: the torsion is proportional to the symplectic form on the leaves of the distribution.

In order to gauge the measure of this nonholonomic nonflatness, we need an analogue of Spencer cohomological complex, which, in turn, requires the following notion.

1.2. Generalized Cartan's prolongs. Given a $\mathbb{Z}$-graded nilpotent Lie algebra $\mathfrak{g}_- = \mathop{\oplus}\limits_{0>i\geq -d} \mathfrak{g}_i$ and a Lie subalgebra $\mathfrak{g}_0 \subset \mathfrak{der}\ \mathfrak{g}_-$ which preserves the $\mathbb{Z}$-grading of $\mathfrak{g}_-$, define the i-th prolong of $(\mathfrak{g}_-, \mathfrak{g}_0)$ for $i > 0$ to be (here $S^\bullet = \mathop{\oplus}\limits_{k\geq 0} S^k$ and V^* is the dual of V):

$$\mathfrak{g}_i = [(S^\bullet(\mathfrak{g}_-)^* \otimes \mathfrak{g}_0) \cap (S^\bullet(\mathfrak{g}_-)^* \otimes \mathfrak{g}_-)]_i,$$

where the subscript in the right hand side singles out the component of degree i and the intersection is well-defined thanks to the fact that $\mathfrak{g}_0 \subset \mathfrak{der}\ \mathfrak{g}_- \subset \mathfrak{g}_-^* \otimes \mathfrak{g}_-$.

Define the *generalized Cartan's prolong* $(\mathfrak{g}_-, \mathfrak{g}_0)_*$ as $\mathop{\oplus}\limits_{i\geq -d} \mathfrak{g}_i$. By the routine arguments, $(\mathfrak{g}_-, \mathfrak{g}_0)_*$ is a Lie algebra. By the same arguments as for the G-structures, cf. [**St**], [**Gz**], [**T**], the space $H^2(\mathfrak{g}_-; (\mathfrak{g}_-, \mathfrak{g}_0)_*)$ is the space of obstructions to flatness of the nonholonomic supermanifold (M, D) and the elements of $H^2(\mathfrak{g}_-; (\mathfrak{g}_-, \mathfrak{g}_0)_*)$ will be called (in analogy with the case $d = 1$) *structure functions*, though to avoid misunderstanding we will often add adjective *nonholonomic*.

The space of nonholonomic structure functions naturally splits into homogeneous components whose degree is induced by the $\mathbb{Z}$-grading of $(\mathfrak{g}_-, \mathfrak{g}_0)_*$; generally, the minimal order of nonholonomic structure functions is $2 - d$. Let $C^s(\mathfrak{g}_-; (\mathfrak{g}_-, \mathfrak{g}_0)_*) = \mathop{\oplus}\limits_k C^{k,s}(\mathfrak{g}_-; (\mathfrak{g}_-, \mathfrak{g}_0)_*)$ be this splitting on the cochain level; the corresponding cohomology $H^{k,s}(\mathfrak{g}_-; (\mathfrak{g}_-, \mathfrak{g}_0)_*)$ are precisely the analogues of the Spencer cohomology and coincide with them for $\mathfrak{g}_- = \mathfrak{g}_{-1}$.

1.3. Example. Let $\mathfrak{g} = \mathfrak{csp}(2n), \mathfrak{g}_{-1} = R(\pi_1; 1), \mathfrak{g}_{-2} = R(0; 0)$; then $(\mathfrak{g}_-, \mathfrak{g}_0)_* = \mathfrak{k}(2n + 1)$ and (here E is the exterior power operator)

$$C^{k,2}_{(\mathfrak{g}_-, \mathfrak{g}_0)_*} = \mathfrak{g}_{k-2} \otimes E^2(\mathfrak{g}_{-1}^*) \bigoplus \mathfrak{g}_{k-2-1} \otimes E^{2-1}(\mathfrak{g}_{-1}^*) \otimes \mathfrak{g}_{-2}^*.$$

The number k here is the order of the structure functions. Unlike the holonomic case when order k structure functions are obstructions to identification of the k-th infinitesimal neighborhood of the point of M with that on the flat manifold, obstructions to flatness in nonholonomic case are not as lucidly interpreted.

Riemann-like and Weyl-like structures. Let $\hat{\mathfrak{g}}_0$ be the semisimple part of $\mathfrak{g}_0$ and $\hat{\mathfrak{g}}_* = (\mathfrak{g}_-; \hat{\mathfrak{g}}_0)_*$. The elements of $H^2(\mathfrak{g}_-; \hat{\mathfrak{g}}_*)$ are nonholonomic analogues of the *Riemann tensor* whereas the elements of $H^2(\mathfrak{g}_-; (\mathfrak{g}_-, \mathfrak{g}_0)_*)$ are nonholonomic analogues of the *Weyl tensor*, the structure functions of the conformal structure.

Theorem . (Weyl-like structures) *For* $(\mathfrak{g}_-, \mathfrak{g}_0)_* = \mathfrak{k}(2n + 1)$ *all nonholonomic structure functions vanish.*

Remark . This is a conceptual reformulation of Darboux's theorem on the absence of parameters for the contact form.

1.4. Structure functions for projective structures and their analogs. Let $\mathfrak{g} = \mathfrak{gl}(n)$ and $\mathfrak{g}_{-1}$ its standard (identity) representation. Then $(\mathfrak{g}_-, \mathfrak{g}_0)_* = \mathfrak{vect}(n)$ and all structure function vanish ([**LPS**]); but if for coefficients of the above cohomology we take the subalgebra $\mathfrak{h} = \mathfrak{sl}(n+1) \subset \mathfrak{vect}(n)$ instead of $(\mathfrak{g}_-, \mathfrak{g}_0)_*$, then the corresponding structure functions are nonzero and provide us with obstructions to integrability of what is called a *projective connection* ([**Go**]).

More generally, for a $\mathbb{Z}$-graded subalgebra $\mathfrak{h} \subset (\mathfrak{g}_-, \mathfrak{g}_0)_*$, such that $\mathfrak{h}_i = \mathfrak{g}_i$ for $i \leq 0$ we will call $\mathop{\oplus}\limits_k H^{k,2}(\mathfrak{g}_-; \mathfrak{h})$ *nonholonomic structure functions of the generalized projective structure of type* $\mathfrak{h}$. Here are examples:

Theorem . (in notations of [**OV**]; in $R(\varphi; k)$ the value on the center of $\mathfrak{g}_0$ stands after semicolon) 1) *Let* $(\mathfrak{g}_-, \mathfrak{g}_0)_* = \mathfrak{vect}(n)$, $\mathfrak{h} = \mathfrak{sl}(n+1)$. *Then structure functions of order* 1 *and* 2 *vanish, structure functions of order* 3 *constitute the* $\mathfrak{g}_0$*-module* $R(\pi_1 + \pi_2 + \pi_{n-1}; 2)$.

2) *Let* $(\mathfrak{g}_-, \mathfrak{g}_0)_* = \mathfrak{k}(2n+1)$, $\mathfrak{h} = \mathfrak{sp}(2n+2)$. *Then the space of structure functions is* $R(\pi_1 + \pi_2; 3)$ *of order* 3.

1.5. Structure function for contact structures on supermanifolds. Let $\omega_0 = \sum dp_i dq_i + \sum \varepsilon_j (d\xi_j)^2$ and $\omega_1 = \sum d\pi_i dq_i$ be the canonical even and odd, respectively, 2-forms (see [**LPS**]). Denote by $\mathfrak{k}(2n+1|m)$, resp. $\mathfrak{m}(n)$, the Lie superalgebra which preserve the contact structures given by the odd form $dt + \alpha$ and the even $d\tau + \beta$, where $d\alpha = \omega_0$, $d\beta = \omega_1$, respectively.

Theorem . *For* $\mathfrak{k}(2n+1|m)$, $\mathfrak{m}(n)$ *all nonholonomic structure function vanish.*

For the deformation $\mathfrak{sm}_\lambda(n)$ (see [**GLS2**]) of the Lie superalgebra of divergence-free contact fields the analog of the above theorem fails: there are nonzero nonholonomic structure functions.

§4.2. Structure functions of the N-extended Minkowski supermanifold

Recall that the ground field is $\mathbb{C}$, the complex numbers.

The G-structure of the Minkowski space can be viewed as either (pseudo) Riemannian or, equivalently, twistor structure. "Straightforward" superizations of these structures are distinct. They are considered in [**Ma**] (see also refs. therein), or [**LPS**], or [**GL3**]. With some exceptions for $N = 1$, neither of these superizations gives rise to what physicists accept as supergravity. The reason is that a Minkowski superspace is still another superization of the Minkowski space and is naturally endowed with a nonholonomic structure.

2.1. What is an N-extended complexified Minkowski supermanifold. First of all, recall what was the "standard" model of complexified and a compactified Minkowski superspace $\mathcal{M}(N)$ before the GIKOS group (Galperin, Ivanov, Kalitsyn, Ogievetsky, Sokachev) showed that one can not write SUGRA on it. Recall also that the "physical reasons" for the restrictions $N \leq 4$ for the Yang-Mills and $N \leq 8$ for the supergravity theories ([**GI**], [**3**]) were put to doubt in [**2**].

Recapitulations ([**L2**]). A *supermatrix* is a rectangular table with elements from a supercommutative superalgebra C with given sets of parities P_{row} and P_{col} of its rows and columns. The *size* of a matrix is $P_{row} \times P_{col}$. Usually, the parities are chosen so that the even rows and columns come first followed by the odd ones; such matrices are said to be of the *standard format*. For the square matrices we will only consider the cases when

$P_{row} = P_{col}$ and will denote this set of parities by Par. The parity of the matrix unit — the matrix with an element $c \in C$ in the (i, j)-th position and 0's elsewhere — is defined to be $p(c) + P_{row}(i) + P_{col}(j)$. Hereafter $C = \mathbb{C}$.

Let $\mathfrak{gl}\ (Par)$ be the set of square matrices of size $Par \times Par$. It is immediately clear, that for distinct Par's with the same p and q the Lie superalgebras $\mathfrak{gl}(Par)$ have *nonisomorphic* maximal nilpotent (say, upper triangular) subalgebras, though the algebras $\mathfrak{gl}(Par)$ themselves are isomorphic. It often suffices to consider the standard format only; then $\mathfrak{gl}(Par)$ is abbreviated to $\mathfrak{gl}(p|q)$, where p is the number of 0's and q is the number of 1's in Par. In supergravity we MUST consider all formats.

The Lie superalgebra $\mathfrak{gl}(Par)$ is generated by the generators of its maximal nilpotent subalgebras, the upper and the lower triangular ones. Since $P_{row} = P_{col}$, the Lie subsuperalgebra of upper triangular matrices in $\mathfrak{gl}(Par)$ is isomorphic to the Lie subsuperalgebra of lower triangular matrices and we will confine ourselves to one of them, denoted by $\mathfrak{n}$. The generators of $\mathfrak{n}$ are the elements just above (below) the main diagonal; we will denote the even generators of $\mathfrak{n}$ by white nodes and the odd generators by "grey" nodes, the nodes corresponding to commuting generators are disconnected, otherwise they are joined by a segment, cf. [**GL2**].

For instance, for the standard format, i.e., $\mathfrak{gl}(p|q)$, we have:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad \underbrace{0 - \cdots - 0}_{p-1 \text{ nodes}} - \otimes - \underbrace{0 - \cdots - 0}_{q-1 \text{ nodes}} \tag{2.0}$$

Consider the Lie supergroup $\mathcal{SL}(N|4)$ and its parabolic subsupergroup $\mathcal{P}$ corresponding to the two marked odd simple roots in the following system of simple roots (this means that $\mathcal{P}$ is generated by all the simple roots except for the marked negative ones):

$$0 - - - \overset{+}{\otimes} - - - \underbrace{0 - - \cdots - - 0}_{N-1 \text{ nodes}} - - - \overset{+}{\otimes} - - - 0 \tag{2.1}$$

Let $G = SL(N) \times SL_L(2) \times SL_R(2) \times \mathbb{C}^*$ be the Lie group corresponding to the 0-th term of the $\mathbb{Z}$-grading described by diagram (2.1), i.e., the degree of a marked root is equal to 1 the other simple roots being of degree 0; here indices L and R distinguish the "left" copy of $SL(2)$ from its "right" twin. In this case $\mathfrak{g}_0 = \mathfrak{g} = \mathfrak{o}(4) \oplus \mathfrak{gl}(N) = \mathfrak{sl}_L(2) \oplus \mathfrak{sl}_R(2) \oplus \mathfrak{gl}(N)$, $\mathfrak{g}_- = \bigoplus\limits_{-1 \geq i \geq -2} \mathfrak{g}_i$ with $\mathfrak{g}_{-1} = (\mathrm{id}_L \otimes Id) \oplus (\mathrm{id}_R \otimes Id^*)$, $\mathfrak{g}_{-2} = \ id_L \otimes \ id_R^*$, where id_j is the space of the standard (identity) representation of $\mathfrak{sl}_j(2)$; $j = L, R$ and Id is the space of the identity representation of $\mathfrak{sl}(N)$. The corresponding matrix representaion of $\mathfrak{p} = Lie(\mathcal{P})$ is of format $2|N|2$:

$$\begin{pmatrix} A_{11} & 0 & 0 \\ Q_{21} & A_{22} & 0 \\ A_{31} & Q_{32} & A_{33} \end{pmatrix} \ \text{ with } \ A_{11}, A_{33} \in \mathfrak{sl}(2); A_{22} \in \mathfrak{sl}(N) \tag{N_{st}}$$

(the odd elements are denoted by Q); it was first found for a particular real form of $\hat{\mathfrak{g}}_* = \mathfrak{g}_- \oplus \hat{\mathfrak{g}}_0$.

The *N-extended Minkowski superspace* $\mathcal{M}(N)$ is $\mathcal{SL}(N|4)/\mathcal{P}$ endowed with the natural $(G, \mathfrak{g}_-)$-structure. The conventional versions of the Minkowski superspace correspond to a certain real form of the (complex) superspace $\hat{\mathcal{M}}(N)$ with the $(\hat{G}, \mathfrak{g}_-)$-structure, i.e., the reduced $(G, \mathfrak{g}_-)$-structure. Clearly,

$$\hat{\mathcal{M}}(N) = \mathcal{P}/\hat{\mathcal{G}}, \text{ where } \hat{G} = SL(N) \times SL_L(2) \times SL_R(2).$$

GIKOS [**GI**] guessed (and we can prove) that these $\mathcal{M}(N)$ never satisfy our requirements (A) and (B) on SUGRA for $N > 1$. GIKOS considered an enlargement $\hat{\mathcal{R}}(N)$ of $\hat{\mathcal{M}}(N)$ defined as

$$\hat{\mathcal{R}}(N) = \mathcal{P}/\hat{\mathcal{G}}', \text{ where } \hat{G}' = Q \times SL_L(2) \times SL_R(2) \text{ and } Q \text{ is a parabolic subgroup of } SL(N).$$

In other words, from $\mathcal{P}$ we pass to a smaller parabolic subsupergroup, $\mathcal{P}'$, whose diagram has several middle roots marked as well. To satisfy (A) and (B), we have to test various $\mathcal{P}'$s. This is impossible without a computer.

We can prove that the diagram (2.0) regardless of the number of middle roots marked can not satisfy requirements (A) and (B). In particular, it is well known to physicists that even for $N = 1$ the standard model does not satisfy (B). So in order to satisfy requirements (A) and (B) we consider nonstandard formats, some of them will be dubbed as "standard".

For $N = 8$ the following two possibilities seem to be distinguished by their symmetry:

$$0- \overset{+}{\otimes} -0- \overset{+}{0} -0- \overset{+}{0} -0- \overset{+}{0} -0- \overset{+}{\otimes} -0 \quad Par = (001111111100) \qquad (8_{\text{“standard”}})$$

and

$$0- \overset{+}{0} -0- \overset{+}{\otimes} -0- \overset{+}{0} -0- \overset{+}{\otimes} -0- \overset{+}{0} -0 \quad Par = (111100001111) \qquad (8_{\text{triality}})$$

The "standard" model for $N > 0$ was originally of the form (N_{st}).

Even though supergravitationists did use cumbersome notations and worked with bare hands, GIKOS guessed that in (N_{st}) for $N > 1$ there is no chance to get structure functions satisfies (A) and (B) and in order to get EE-like equations, on has to enlarge the Minkowski superspace by taking for $N = 2$ a smaller A_{22}:

$$\begin{pmatrix} A_{11} & 0 & 0 \\ Q_{21} & A_{22} & 0 \\ A_{31} & Q_{32} & A_{33} \end{pmatrix}, \text{ where } A_{22} = \begin{pmatrix} A_{22}^{11} & 0 \\ A_{22}^{21} & A_{22}^{22} \end{pmatrix}.$$

Elsewhere we will discuss the assumptions of Haag-Łopuszanski-Sohnius' theorem which lead to the Poincaré supergroup and its enlargement, $\mathcal{S}L(N|4)$. It could be that by, yet unmentioned, physical reasons some of the quotients of $\mathcal{S}L(N|4)$ have to be ruled out; so far these reasons are not formulated explicitly. At the moment we accept the correctness of the choice of the "twistor supergroup" $\mathcal{S}L(N|4)$ but investigate several possible quotients corresponding to various matrix realizations or, which is the same, to connected components of the flag supermanifolds. The stationary subgroups we will consider in this paper are the simplest ones: the Lie groups, subgroups of $O(4) \times SL(N)$. The stationary Lie *super*groups are to be considered.

Notice that even for the same stationary subgroup $O(4) \times SL(N)$ we can consider several realizations:

$$\begin{pmatrix} A_{11} & 0 & 0 \\ Q_{21} & A_{22} & 0 \\ A_{31} & Q_{32} & A_{33} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ Q_{31} & Q_{32} & A_{33} \end{pmatrix}.$$

The experience with the analogues of Einstein's equations on symmetric spaces [**LPS**] teaches us to consider the models of Minkowski superspace whose stationary subgroup is a product of several copies of $SL(2)$: otherwise the equations will not be of order 2.

With all these conventions we first have to study the following examples of Minkowski superspaces which we will denote more instructively by $\hat{\mathcal{M}}(Par)$ rather than $M(N)$:

For $N=1$:

$$\begin{pmatrix} A_{11} & 0 & 0 \\ Q_{21} & A_{22} & 0 \\ A_{31} & Q_{32} & A_{33} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ Q_{31} & Q_{32} & A_{33} \end{pmatrix}, \quad \text{where } A_{22} \text{ (resp. } A_{33}) = 0$$

and with $Par = (00100)$ and (00001), respectively.

For $N=2$:

$$\begin{pmatrix} A_{11} & 0 & 0 \\ Q_{21} & A_{22} & 0 \\ A_{31} & Q_{32} & A_{33} \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ Q_{31} & Q_{32} & A_{33} \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} A_{11} & 0 & 0 & 0 \\ Q_{21} & A_{22} & 0 & 0 \\ A_{31} & Q_{32} & A_{33} & 0 \\ Q_{41} & A_{42} & Q_{43} & A_{44} \end{pmatrix},$$

where $Par = (001100)$, (000011), and (100001), respectively. In the last case $A_{11} = A_{44} = 0$, otherwise $A_{ii} \in \mathfrak{sl}(2)$.

For $N=4$:

$$\begin{pmatrix} A_{11} & 0 & 0 \\ Q_{21} & A_{22} & 0 \\ A_{31} & Q_{32} & A_{33} \end{pmatrix}, \quad \text{where} \quad A_{22} = \begin{pmatrix} A_{22}^{11} & 0 \\ A_{22}^{21} & A_{22}^{22} \end{pmatrix} \quad \text{and} \quad Par = (00111100), \qquad (N = 4_1)$$

(this corresponds to two copies of the Minkowski space united by a common superstructure) and

$$\begin{pmatrix} A_{11} & 0 & 0 & 0 \\ Q_{21} & A_{22} & 0 & 0 \\ A_{31} & Q_{32} & A_{33} & 0 \\ Q_{41} & A_{42} & Q_{43} & A_{44} \end{pmatrix} \qquad (N = 4_2)$$

There is, naturally, the possibility with $Par = (00001111)$, as always.

For $N=8$ the following cases seem to be the most interesting (symmetric) ones:

$$\begin{pmatrix} A_{11} & 0 & 0 \\ Q_{21} & A_{22} & 0 \\ A_{31} & Q_{32} & A_{33} \end{pmatrix}, \quad \text{where} \quad A_{ii} = \begin{pmatrix} A_{ii}^{11} & 0 \\ A_{ii}^{21} & A_{ii}^{22} \end{pmatrix}, \qquad (N = 8)$$

with $Par = (111100001111)$. This model corresponds to three copies of the Minkowski space appended by a 16-dimensional Gr_4^8, analogue of GIKOS' "harmonic" space and united by a common superstructure. Another possibility is

$$\begin{pmatrix} A_{11} & 0 & 0 \\ Q_{21} & A_{22} & 0 \\ A_{31} & Q_{32} & A_{33} \end{pmatrix}, \quad \text{where} \quad A_{22} = \begin{pmatrix} A_{22}^{11} & 0 & 0 & 0 \\ A_{22}^{21} & A_{22}^{22} & 0 & 0 \\ A_{22}^{31} & A_{22}^{32} & A_{22}^{33} & 0 \\ A_{22}^{41} & A_{22}^{42} & A_{22}^{43} & A_{22}^{44} \end{pmatrix} \qquad (N = 8_1)$$

with $Par = (001111111100)$. Ever less symmetric possibilities are:

$$\begin{pmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ Q_{31} & Q_{32} & A_{33} \end{pmatrix}, \quad \text{where} \quad A_{33} = \begin{pmatrix} A_{33}^{11} & 0 & 0 & 0 \\ A_{33}^{21} & A_{33}^{22} & 0 & 0 \\ A_{33}^{31} & A_{33}^{32} & A_{33}^{33} & 0 \\ A_{33}^{41} & A_{33}^{42} & A_{33}^{43} & A_{33}^{44} \end{pmatrix} \qquad (N = 8_2)$$

with $Par = (000011111111)$ and

$$\begin{pmatrix} A_{11} & 0 & 0 & 0 \\ Q_{21} & A_{22} & 0 & 0 \\ A_{31} & Q_{32} & A_{33} & 0 \\ Q_{41} & A_{42} & Q_{43} & A_{44} \end{pmatrix}, \quad \text{where} \quad A_{44} = \begin{pmatrix} A_{33}^{11} & 0 & 0 \\ A_{33}^{21} & A_{33}^{22} & 0 \\ A_{33}^{31} & A_{33}^{32} & A_{33}^{33} \end{pmatrix} \qquad (N = 8_3)$$

with $Par = (001100111111)$.

2.2. THEOREM . *In Table (below) there are listed all the orders and weights of all the structure functions for the indicated* $\hat{\mathcal{M}}(Par)$.

The corresponding cocycles are listed in a detailed version. We numbered the cocycles so that the i-th cocycle of weight $w = (w_1, \dots, w_k)$ and degree d is denoted by $\mathbf{R}_i^d(w_1, \dots, w_k)$. We drop some of the indices if they are clear from the contents.

Clearly, there are more candidates on the role of $\hat{\mathcal{M}}(Par)$ for the N we have considered and for other N as well; the package allows the reader to perform corresponding calculations.

REMARK . We could have stuck to the yet "standard" model with $\hat{G} = SL_L(2) \times SL(N) \times SL_R(2)$ which preserves $\varepsilon_L \otimes vol \otimes \varepsilon_R$, where ε_i for $i = L, R$ and vol are the volume elements preserved by $SL_i(2)$ and $SL(N)$, respectively. Then we could consider an equation similar to the equations (EE_0) from [**LPS**]; namely, consider a connection compatible with $\varepsilon_L \otimes vol \otimes \varepsilon_R$ and for a component R of weight 0 in the space of nonholonomic structure functions with respect to $\hat{G}$ take

$$\mathbf{R} = \lambda \varepsilon_L \otimes vol \otimes \varepsilon_R. \qquad (EE_0)$$

There might be several obstacles in writing (EE_0): such a tensor R might not exist, or, if it exists, the corresponding equation might be not of order 2, or the valency of R could be not the same as that of the right hand side, the tensor preserved by G. The equations that appear in this way are considered in [**LPS**]; they correspond to higher spins, favorites of M. Vasiliev [**2**].

To interpret the supergravity in the same way as we have treated the Einstein Equations [**LPS**], define the supergravity equations as follows. On $\hat{\mathcal{M}}(N)$, the stationary subgroup (which coincides with $\hat{G}$) of the point preserves $\varepsilon_1 \otimes \varepsilon_2 \otimes \dots \otimes \varepsilon_k$, where ε_i is the volume preserved by $SL_i(2)$, the i-th copy of $SL(2)$, in the 2-dimensional identity representation.

If there are several, say k, tensors of weight 0 we can take for $\mathbf{R}$ their linear combination and the coefficients of this combination determine a parameter which runs over $\mathbb{P}^{k-1}$.

Example: $N = 1$. The tensor $\mathbf{R}$ depends on a parameter, the ratio $a : b$ which runs over the projective line $\mathbb{P}^1$. Physicists call this parameter the *Gates–Sigel* parameter. On $\mathfrak{g}_{-1}$, there is the inner product given by the bracket. Notice that this product is even for $Par = (00100)$ and odd for $Par = (00001)$. The tacit choice was $Par = (00100)$. For $Par = (00100)$ the metric g on the Minkowski space M is the product of spinorial metrics ε_L and ε_R on the the maximal isotropic (with respect to the pairing) subspaces of $\mathfrak{g}_{-1}$, cf. [**Ma**].

The equation (EE_0) takes the form

$$a\mathbf{R}_1(00) + b\mathbf{R}_2(00) = \lambda g. \qquad (EE_0(a : b))$$

From the explicit form of the cocycles it is clear that the component expansion of the above equation does not contain (EE_0) for $Par = (00100)$, and only $\mathbf{R}_1(00)$ for $Par = (00001)$ has the right expansion.

Notice immediately, that for $(EE_0(a : b))$ to be well-defined we must demand that all structure functions of orders > 2 vanish. These conditions are the *Wess-Zumino constraints*. We have fewer of them than, say, in [**3**]: one of the constraints is "harmless", its cohomology class is zero (like torsion of the Levi–Civita connection).

What shall we take for nonholonomic analogs of Ricci flatness? As for $N = 0$, these should be the vanishing conditions on the part of the Riemann tensor which does not belong to the conformal or Weyl tensor. For $N > 0$ there are several such components and we can equate to zero either or all of them. The equations are well-defined as integrability condition provided the constraints vanish.

If all structure functions of order 2 (and of lesser orders) vanish, the higher obstructions are well-defined and we can write an equation on them. For example, for $Par = (00100)$ we can equate to zero one or both of the equations:

$$\mathbf{R}(3,0) = 0, \quad \mathbf{R}(0,3) = 0.$$

Different choices correspond to different supergravities (minimal, flexible, etc.).

On how many parameters do these equation actually depend? The analysis will be also performed in the detailed version, let us only say here that the the above equations should be considered for a *symmetric connection compatible with the structure preserved by the stationary group of point* of $\hat{\mathcal{M}}(N)$. There is only one such connection for all the $\hat{\mathcal{M}}(N)$. For $N = 0$ this suffices to single out the symmetric 10-dimensional part of the vierbein. For $N > 0$ we still have too many components after having singled out the symmetric part of the connection, but the requirement of vanishing of the structure function of orders lesser than that of EE (in our cases such are orders < 2) kills more components.

Table

$\deg(SF)\backslash Par$	(0000)	(00100)	(00001)
$\boxed{0}$	not defined	$(3,1), (1,3)$	—
1	—	$(1,0), (0,1)$	$(1,1)^{\boxed{1}+1}, (0,1)_1, \boxed{(0,3)_1}$
2	$(2,2), (0,0)$ $\boxed{(4,0), (0,4)}$	$(1,1), (0,0)^2$	$(0,0)^2, (2,2),$ $(1,0)_1, \boxed{(3,2)_1}$
3	not defined	$(3,0), (0,3)$	—

$\deg(SF)\backslash Par$	(001100)	(100001)	(000011)
$\boxed{-1}$	not defined	$(0,2), (2,0)$	not defined
$\boxed{0}$	$(222), (123), (321)$	$(0,1)_1, (1,0)_1$	$(222), (123)_1, (301)_1$
1	$(110)_1, (011)_1$	$\boxed{(0,0)}, \boxed{(2,2)^2}$ $(1,1)^2, (1,0)_1^2, (0,1)_1^2$	$\boxed{(110)}, \boxed{(031)_1}$ $(011)_1$
2	$(020)^2, \boxed{\boxed{(200)}}$ $(101), \boxed{\boxed{(002)}}$	$(2,2), (0,0)^3$ $(1,0)_1, (0,1)_1$	$(000), (220), (002)$ $\boxed{(200)}, (020), (101)_1$
≥ 3	—	not defined	—

$\deg(SF)\backslash Par$	(00111100)	(00001111)
$\boxed{-1}$	$(1021), (1023), (1201), (2112),$ $(3201), (1122)_1, (2211)_1$	$(1122), (2211), (1001)_1$ $(1221)_1, (2112)_1, (3001)_1, (1003)_1$
0	—	—
1	$(0110), (0011)_1^{\boxed{2}+1}, (1100)_1^{\boxed{2}+1}$	$(0011)^{\boxed{2}+1}, (1100)^2, (0110)_1^{\boxed{2}+1}$
2	$(0000), (0220), (2000)$ $(1111), (0101)_1, (10010)_1,$ $(0002), (0020), (0200)$	$(0000)^2, (0022), (2200),$ $(0200), (0020)$ $(0101)_1, (1010)_1$
≥ 3	—	—

$\deg(SF)\backslash Par$	(111100001111)
$\boxed{\text{-3}}$	$(100001)^2, (100003), (300001), (100201), (102001)$ $(100221), (122001), (101112), (211101), (210012), (111111)$
$\boxed{\text{-2}}$	$(100122), (221001), (101211), (112101), (111012)$ $(210111), (030001), (100030)$
-1	—
0	—
1	$(000011)^4, (001100)^4, (110000)^4, (000110)^4_1, (011000)^4_1$
2	$(000000)^3, (000020), (000022)$ $(000200), (002200), (002000), (020000)$ $(220000), (000101)_1, (001010)_1, (010100)_1, (101000)_1$
≥ 3	—

Notations. The cocycles which also correspond to the "conformal" case – on shell – are $\boxed{\text{boxed}}$; the cocycles of small orders are all conformally invariant, such $\boxed{\text{orders}}$ are boxed; the cocycles *which only exist in the conformal case* are $\boxed{\boxed{\text{doubleboxed}}}$; the suscript 1 singles out odd cocycles; the exponent denotes the multiplicity of the cocycle; the multiplicity of conformally invariant vectors is boxed.

For structure functions of degree > 2 the entry "not defined" in the Table refers to the "Riemannian" case, i.e., to the $\hat{G}$-structure. It so happened that in these cases there are no conformal cohomologies which facilitates out plotting of the table. The dash — indicates that either there are no structure functions in this order or (if the degree is > 2 and the case is the Riemannian one) they are not defined.

Clearly, the Table is invariant under the change of parities $1 \leftrightarrow 0$ in Par.

4. Invariant differential operators [Gz], [GLS3]

§5. Bilinear differential operators invariant with repect to symplectomorphisms

5.1. Let ρ be a representation of the group $Sp(2m;\mathbb{R})$ in a V_ρ. A *tensor field of type* ρ on a $2m$-dimensional symplectic manifold M is an object t defined in each local coordinate system x, in which the symplectic form is of the canonical form $\omega = \sum\limits_{i\leq m} dx_i \wedge dx_{2m+1-i}$, by the vector $t(x) \in V_\rho$ so that the passage to other coordinates, y (with the same property), is defined by the formula

$$t(y(x)) = \rho\left(\frac{\partial y(x)}{\partial x}\right) t(x).$$

Traditionally (see reviews [**K1**], [**K2**]) the fibers of the tensor bundles were considered finite dimensional, but Leites showed recently [**LKW**] that on supermanifolds it is natural and fruitful to consider infinite dimensional fibers: this leads to semi-infinite cohomology of supermanifolds. Similar problem for symplectic manifolds was not studied yet.

The space of smooth tensor fields of type ρ will be denoted by $T(\rho)$ or by $T(\lambda)$, where $\lambda = (\lambda_1, \dots, \lambda_m)$ is the lowest (or, for finite dimensional representations, highest, for convenience) weight of the irreducible representation ρ.

In what follows the letters ρ, σ, τ will denote irreducible representations of $Sp(2m;\mathbb{R})$ and letters λ, μ, ν their highest weights. (We should have concidered lowest weight only, but in this preliminary report we stick to finite dimensional representations.)

Examples of spaces of tensor fields: a) $T(0) = C^\infty(M)$;

b) $T(1, 0, \dots, 0) \cong \mathfrak{vect} \cong \Omega^1$ is the space of vector fields or (which is the same on any symplectic manifold thanks to the nondegeneracy of ω) the space of 1-forms on M;

c) $\prod^r = T(\underbrace{1, \dots, 1}_{r-\text{many}}, 0, \dots, 0)$ the space of primitive r-forms.

On the space $T_c(\rho)$ of tensor fields of type ρ with compact support, as indicated by the subscript, there is an invariant inner product

$$< \chi, \theta >= \int_M < \chi(x), \theta(x) > \omega_0^m, \tag{IP}$$

where $< \cdot, \cdot >$ in the integrand is the $Sp(2m;\mathbb{R})$-invariant inner product on V_ρ.

A differential operator $B : T(\rho_1) \otimes T(\rho_2) \otimes \dots \otimes T(\rho_n) \longrightarrow T(\tau)$ is called *n-ary* (*un*ary, *bi*nary, etc. for $n = 1, 2$, respectively). Such an operator B is called $Diff(M)$-*invariant* if it is uniquely expressed in all coordinate systems; it is $Diff_\omega(M)$-*invariant* if it is uniquely expressed in all coordinate systems in which the symplectic form is of the standard form.

5.2. The unary $Diff_\omega(M)$-invariant differential operators. All such operators are classifieded by A. N. Rudakov ([**R1**], [**R2**]):

0-th order: the multiplication by a number;

1-st order : the derivations of the primitive forms $d_+ : \prod^r \longrightarrow \prod^{r+1}$ and $d_- : \prod^{r+1} \longrightarrow \prod^r$ $(0 \leq r \leq m-1)$. These operators are compositions of the exterior differential $d : \Omega^p \longrightarrow$

Ω^{p+1} and the projection onto the space of primitive forms; recall that $\Omega^p = \prod^p \oplus \prod^{p-2} \oplus \dots$ for $p \leq m$ and $\Omega^p \cong \Omega^{2m-p}$;

2-nd order: $d_2 = d_+ \circ d_- : \prod^r \longrightarrow \prod^r (1 \leq r \leq m)$.

Remark . Rudakov's theorem implies that other invariant operators that might spring to mind ($d_- \circ \omega d_-$, etc.) are multiples of the described ones.

5.3. The binary $Diff_\omega(M)$-invariant differential operators. Clearly, the operators $B^{*1} : T(\tau) \times T(\rho_2) \longrightarrow T(\rho_1)$ and $B^{*2} : T(\rho_1) \times T(\tau) \longrightarrow T(\rho_2)$ conjugate to (or 1- and 2-duals of) the invariant differential operator $B : T(\rho_1) \otimes T(\rho_2) \longrightarrow T(\tau)$ with respect to the inner product (IP) are also differential and invariant ones.

0-th order operators are obviously those of the form

$$Z(\chi, \theta) = pr(\chi(x) \otimes \theta(x)),$$

where $pr : V_\rho \otimes V_\sigma \longrightarrow V_\tau$ is the projection of the tensor product onto an irreducible component.

1-st order operators:

Theorem . *Any bilinear* 1*-st order (with respect to all arguments)* $Diff_\omega(M)$*-invariant differential operator* $B : T(\lambda) \otimes T(\mu) \longrightarrow T(\nu)$ *is a linear combination of the following cases* P1 – P8 *(some of which host several distinct operators) and the operators obtained from them by* 1*- and* 2*-dualizaton and transposition of the arguments.*

P1) $\lambda = (\underbrace{1, \dots, 1}_{p-\text{many } 1\text{'s}}, 0, \dots, 0)$; weights μ and ν differ by a unit in r places, $r \equiv p+1 \mod 2$. For $r \leq p+1$ there exist operators of the form $Z(d_+\omega, \theta)$ and for $r \leq p-1$ there exist operators of the form $Z(d_-\omega, \theta)$.

P2) The Lie derivative being restricted onto $Sp(2m; \mathbb{R})$-irreducible subspaces splits into several operators of the form $Z(d_\pm\omega, \theta)$ and an operator

$$L : Vect \times T(\rho) \longrightarrow T(\rho)$$

which cannot be reduced to operators of the form P1).

Remark . Observe that if $\xi \in \mathfrak{h}(M) \subset \mathfrak{vect}(M)$ is a Hamiltonian vector field, then, by identifying $\mathfrak{h}$ with $d\Omega^0$, we see that $d_+\xi = d_-\xi = 0$ and in this case L coincides with the Lie derivative. Therefore, L determines a representation of the Lie algebra $\mathfrak{h}(M)$ in the space $T(\rho)$. It is not difficult to show that the invariance of B is equivalent to its $\mathfrak{h}(M)$-invariance:

$$L(\xi, B(\chi, \theta)) = B(L(\xi, \chi), \theta) + B(\chi, L(\xi, \theta))$$

for any $\chi \in T(\rho), \theta \in T(\sigma), \xi \in \mathfrak{h}(M)$.

P3) $S^k\mathfrak{vect} \times S^l\mathfrak{vect} \longrightarrow S^{k+l-1}\mathfrak{vect}$ (clearly $S^k\mathfrak{vect} \cong T(k, 0, \dots, 0)$) is the Poisson bracket (a.k.a. the *symmetric Schouten's concomitant*) on (polynomial in momenta) functions on T^*M.

P4) λ, μ, ν are of the form $(2, 1, \dots, 1, 0, \dots, 0)$ each, with p, q and r non-zero numerical marks, respectively, such that $p+q+r \equiv 0 \mod 2$, $|p-q| \leq r \leq p+q$, and $p+q+r \leq 2m+2$.

If all inequalities are strict, then there exist **four distinct operators** defined on the spaces of such fields, otherwise there exist only **two** distinct operators. For $p+q+r \leq 2m$ two of these four or two operators are obtained as restrictions of the *Nijenhuis bracket*, or its conjugates, onto the subspaces

$$T(2, 1, \dots, 1, 0, \dots, 0) \subset \Omega^p \otimes_{C^\infty(M)} \mathfrak{vect}.$$

Remark . The remaining two operators (i.e., the ones which are not the restrictions of the Nijenhuis bracket) are new. We do not know anything about them except that they exist and the same applies to the following two cases P5) and P6).

P5) λ, μ are of the same form as for P4), $\nu = (3, 1, 1, \dots, 1, 0, \dots, 0)$. There exists one operator for $|p-q|+1 \leq r \leq p+q-1, \quad p+q+r \equiv 1 \mod 2, \quad p+q+r \leq 2m+1$.

P6) λ, μ are the same as in 4), $\nu = (2, 2, 1, \dots, 1, 0, \dots, 0)$ with r non-zero entries. The operator exists under the same conditions on p, q, r as for P5).

P7) $\nu = (1, \dots, 1, 0, \dots, 0)$; whereas λ, μ and conditions on p, q, r are the same as in 5). In this case there exists a unique operator which is not reducible to operators of the form $d_\pm Z$. It is a restriction of the Nijenhuis bracket.

P8) $\lambda = (2, 0, \dots, 0)$; whereas μ and ν differ from each other by a unit at one place. There exists a unique such operator. Further on I'll give arguments which enable one to express it, in principle, explicitly.

5.4. 2nd order operators. Grozman could not *classify* such operators so far. However, he was lucky to find one new invariant operator, denoted in the literature Gz:

$$Gz : T(2, 0, \dots, 0) \times T(2, 0, \dots, 0) \longrightarrow T(2, 0, \dots, 0).$$

For $m = 1$ Grozman got the explicit expression for the operator Gz in 1976. Let me us reproduce it. In coordinates x, y we have $\omega = dx \wedge dy$. Then

$$\begin{aligned}
&Gz : a \cdot dx^2 + 2b \cdot dxdy + c \cdot dy^2, a' \cdot dx^2 + 2b' \cdot dxdy + c' \cdot dy^2 \mapsto \\
&\tfrac{\partial^2 g}{\partial x^2} dx^2 + 2\tfrac{\partial^2 g}{\partial x \partial y} dxdy + \tfrac{\partial^2 g}{\partial y^2} dy^2 + (\{c, a'\} - \{a, c'\}) dxdy + \\
&\left(\tfrac{\partial^2 a}{\partial y^2} - 2\tfrac{\partial^2 b}{\partial x \partial y} + \tfrac{\partial^2 c}{\partial x^2}\right)(a' dx^2 + 2b dxdy + c' dy^2) + (\{a, b'\} - \{b, a'\}) dx^2 + \\
&\left(\tfrac{\partial^2 a}{\partial y^2} - 2\tfrac{\partial^2 b}{\partial x \partial y} + \tfrac{\partial^2 c}{\partial x^2}\right)(a dx^2 + 2b dxdy + c dy^2)(\{b, c'\} - \{c, b'\}) dy^2,
\end{aligned}$$

where $g = ac' - 2bb' + ca'$ and $\{\cdot, \cdot\}$ is the Poisson bracket. An explicit form of Gz for $m > 1$ is to be found.

5.5. Sketch of the proof of Theorem. Set $y_i = x_{2m+1-i} (1 \leq i \leq m), \partial_i = \frac{\partial}{\partial x_i}, \delta_i = \frac{\partial}{\partial y_i}$. Denote the elements of the Lie algebra $\mathfrak{sp}(2m; \mathbb{R}) \subset \mathfrak{h}(M)$ by

$$e^{ii} = y_i \partial_i, \quad e_{ii} = x_i \delta_i, \qquad e^i_j = x_j \partial_i + y_i \delta_j.$$

Then, clearly,

$$e^{ij} = e^{ji} = y_i \partial_j + y_j \partial_i \;\text{ and }\; e_{ij} = e_{ji} = x_i \delta_j + x_j \delta_i \;\text{ for }\; i \neq j.$$

Let $I(\rho)$ be the space of differential operators from $T(\rho)$ into $C^\infty(M)$ with constant coefficients, i.e.,

$$I(\rho) = \{\sum P_i(\partial, \delta) u_i \mid u_i \in V^*_\rho \cong V_\rho\}.$$

The grading in $I(\rho)$ is induced by that in the space of polynomials P_i's, i.e., $I(\rho)_0 \cong V_\rho$. Define the pairing $I(\rho) \times T(\rho) \longrightarrow \mathbb{R}$ by the formula

$$< Pu, x >= P(< u, \chi(x) >)|_{x=0}.$$

On $I(\rho)$, define the $\mathfrak{h}(M)$-action, dual to the action on $T(\rho)$, via L. Now, to describe the invariant operators it suffices to find all the $\mathfrak{h}(M)$-morphisms $I(\tau) \longrightarrow I(\rho) \otimes_{\mathbb{R}} I(\sigma)$. It turns out that such a morphism is completely defined by the image of the highest vector

$v \in V_\tau = I(\tau)_0$. Here we have fixed a Borel subalgebra $\{\sum\limits_{i\leq j} a_{ij}x_j\partial_i\} \bigcap \mathfrak{sp}(2m;\mathbb{R})$ so that $w \in I(\rho) \otimes_{\mathbb{R}} I(\sigma)$ can be the image of a highest weight singular vector if and only if

$$e^i_{i+1}w = 0 \text{ for } 1 \leq i \leq m-1 \text{ and } e^{m,m}w = 0 \qquad \text{(conditions on } w \text{ to be a highest vector)}$$

and

$$(x_1^2\delta_1)w = 0. \qquad \text{(conditions of } \textit{singularity} \text{ of the vector)}$$

The degree of $w \in I(\rho) \otimes_{\mathbb{R}} I(\sigma)$ is equal to the order of the corresponding differential operator. The general form of a vector of degree 1 is

$$w = \sum_{i\leq m} \partial_i^{'} z_i^0 + \delta_i^{'} t_i^0 + \partial_i^{''} z_i^1 + \delta_i^{''} t_i^1,$$

where $z_i^j, t_i^j \in V_\rho \otimes V_\sigma$, $\partial'(u\otimes v) = \partial u \otimes v$, $\partial''(u \otimes v) = u \otimes \partial v$. If w is a highest vector, then all vectors z, t are expressed in terms of z_1^0, z_1^1 which should satisfy

$$e^i_{i+1} z_1^j = 0 \text{ for } 2 \leq i \leq m-1, (e_2^1)^2 z_1^j = 0, e^{m,m} z_1^j = 0.$$

The condition $(x_1^2\delta_1)w = 0$ is equivalent to the equation

$$e^{'}_{1,1} z_1^0 + e^{''}_{1,1} z_1^1 = 0,$$

where (double) prime means that the operator acts only on the first (second) multiple of the tensor product.

Grozman succeeded to define all the cases, where the above system possesses a solution in $V_\rho \otimes V_\sigma$; though in certain cases he was not able to find the solution itself.

Here is an example of a sucessfully solved case (case 8)):

$$\lambda = (2, 0, \dots, 0), \quad \nu = (\mu_1, \dots, \mu_{k-1}, \mu_k + 1, \mu_{k+1}, \dots);$$

the case $\nu_k = \mu_k - 1$ is dual to this one. Let $u_0 \in V_\rho$ be a highest vector, then

$$u_0 \otimes v - \frac{1}{2} \sum_{2\leq i\leq k} e_1^i u_0 \otimes e_i^1 v \in V_\rho \otimes V_\sigma$$

is a highest vector of weight $(\nu_1 + 1, \nu_2, \dots, \nu_m)$. We conclude that

$$z_1^0 = u_0 \oplus e_{11}v - \textstyle\sum_{2\leq i\leq k} e_1^i u_0 \otimes e_{11}e_i^1 v - \frac{1}{2} \quad \sum_{2\leq i<j\leq m} e_1^i e_1^j u_0 \otimes (e_{ij} + e_{1j}e_1^i + e_{1i}e_j^1)v-$$
$$\tfrac{1}{2}\textstyle\sum_{2\leq i\leq k}(e_1^i)^2 u_0 \otimes (e_{ii} + e_{1i}e_i^1)v + \frac{1}{2} \quad \sum_{2\leq i\neq j\leq m} e_1^i e_{1j} u_0 \otimes (e_i^j + e_1^j e_i^1)v + (\nu_i + e_1^i e_i^1)v+$$
$$\tfrac{1}{4}\textstyle\sum_{2\leq i\leq m} e_{ii}(e_1^i)^2 u_0 \otimes + \frac{\nu_1 - 1}{2} \sum_{2\leq i\leq k} e_{11} e_1^i u_0 \otimes e_i^1 v,$$

and

$$z_1^1 = -e_{11}u_0 \otimes v + \sum_{2\leq i\leq k} e_{11}e_1^i u_0 \otimes e_i^1 v.$$

5.6. Conjectures. 1) *The operator Gz is a particular case of a more general operator* :

$$Gz_{r,s} : T(2, \underbrace{1, \dots, 1}_{r-\text{many}}, 0, \dots, 0) \times T(2, \underbrace{1, \dots, 1}_{s-\text{many}}, 0, \dots, 0) \longrightarrow T(2, \underbrace{1, \dots, 1}_{r+s-\text{many}}, 0, \dots, 0).$$

2) *Operators of order > 2 are compositions of operators of orders ≤ 2. There are no operators of order > 5.*

Bibliography

[Ag] Agrachev A. Methods of optimal control in nonholonomic geometry, Proc. Intnatnl. Congress Math., v.2, 1994, 1473–1484; Agrachev A., Vakhrameev S., Gamkrelidze R., Methods of differential geometry and group theory in the optimal control theory. JOSMAR, **28**, 2, 1985, 145–182

[A1] Arnold V., *Mathemetical methods of classical mechanics*, NY ea, Springer, 1978

[A2] Arnold V. (ed.), *Encyclop. of Math. Sci., Dynamical systems*-3 (Arnold V., Kozlov V., Neishtadt A.), Springer, 1988

[AF] Astashkevich, A. B.; Fuchs, D. B. On the cohomology of the Lie superalgebra $W(m|n)$. In: Fuchs, D. B (ed.) *Unconventional Lie algebras*, 1–13, Adv. Soviet Math., 17, Amer. Math. Soc., Providence, RI, 1993

[Aea] Atiyah M., et al. Responses to: A. Jaffe and F. Quinn, "Theoretical mathematics: toward a cultural synthesis of mathematics and theoretical physics" [Bull. Amer. Math. Soc. (N.S.) 29 (1993), no. 1, 1–13; MR 94h:00007]. Bull. Amer. Math. Soc. (N.S.) 30 (1994), no. 2, 178–207

[Ar] Ben Arous G., Développment asymptotique du noyaux de la chaleur hypoelliptique sur la diagonale. Ann. Inst. Fourier, Grenoble, 39 (1988) 1, 73–99; id., Noyau de la chaleur hypoelliptique et géométrie sous-riemannienne. (French) [The hypoelliptic heat kernel and sub-Riemannian geometry] Stochastic analysis (Paris, 1987), 1–16, Lecture Notes in Math., 1322, Springer, Berlin-New York, 1988

[Ba] Baryshnikov Yu. M., The Hamiltonian form of nonholonomic variational problems, Uspehi mat. nauk (=Russian Math. Surveys), **45**, 1, 1989, 167–168

[BL] Bernstein J., Leites D., Invariant differential operators and irreducible representations of Lie superalgebras of vector fields. Sel. Math. Sov., v. 1, N 2, 1981, 143–160

[BMR] Bezrukavnikov R., , Mirković I., Dmitriy Rumynin D., Localization of modules for a semisimple Lie algebra in prime characteristic, math.RT/0205144

[Bo] Borcherds, R. What is Moonshine? *Proceedings of the International Congress of Mathematicians*, Vol. I (Berlin, 1998). Doc. Math. 1998, Extra Vol. I, 607–615

[Br] Bryant R., Chern C., Griffiths Ph., Goldschmidt H., *Exterior differential systems*, Mathematical Sciences Research Institute Publications, 18. Springer-Verlag, New York, 1991. viii+475 pp.;

[BrH] Bryant, R., Griffiths, Ph., Hsu, L., Toward a geometry of differential equations. In: S.-T. Yau (ed.) *Geometry, topology, & physics*, Conf. Proc. in honor of Raoul Bott at Harvard University, Cambridge, Massachusetts, April 1993. Lecture Notes Geom. Topology, VI, Internat. Press, Cambridge, MA, 1995, 1–76

Bryant, R., Hsu, L., Rigidity of integral curves of rank 2 distributions. Invent. Math. 114 (1993), no. 2, 435–461

[BF] Butenin N., Fufaev N., *Introduction to analytical mechanics*, Moscow, Nauka, 1967 (in Russian)

[CF] Cardin F., Favretti M., On nonholonomic and vaconomic dynamics of mechanical systems with non-integrable constraints, J. Geometry and Physics, 18 (1996), 295–325

[CDF] Castellani L., D'Auria R., Fré P., *Supergravity and superstrings*, vv. 1–3, World Scientific, 1991

[D] Dobrushin R., Minlos R., Shubin M. and Vershik A. (eds.) *Contemporary Mathematical Physics* (F. A. Berezin memorial volumes, vv.1, 2), Amer. Math. Soc. Transl. Ser. 2, vol. 174-175, Amer. Math. Soc., Providence, RI (1996) AMS, 1996

[E] Enos M. (ed.) *Dynamics and control of mechanical systems: The falling cat and related problems* (Fields institute communications n.1) AMS, 1993;

[Fe] Fecko M., "Falling cat" connections constructed from the momentum map. Differential geometry and applications (Brno, 1995), 261–265, Masaryk Univ., Brno, 1996; id., "Falling cat" connections and the momentum map. J. Math. Phys. 36 (1995), no. 12, 6709–6719

[CK] Cheng S., Kac V., Generalized Spencer cohomology and filtered deformations of $\mathbb{Z}$-graded Lie superalgebras. Adv. Theor. Math. Phys. 2 (1998), no. 5, 1141–1182

[Di] Dixmier J. *Algèbres envellopentes*, Gautier-Villars, Paris, 1974

[Dy1] Dynkin E.B. Semisimple subalgebras of semisimple Lie algebras. Matem. Sbornik, 1952, v.30 (72), 349–462 (Russian) = English transl. in Moscow Math. Soc. Translations Ser. 2, v.6, 111–244

[Dy2] Dynkin E.B. Maximal subgroups of classical groups. Trudy Moskovskogo Matem. obshchestva, 1952, v.1 (Russian) = English transl. in Moscow Math. Soc. Translations Ser. 2, v.6, 245–378

[F1] Feigin B., Cohomology of the Lie algebras of vector fields on the line and the circle and of the current algebras. In: A. Onishchik (ed.) *Questions of Group Theory and Homological Algebra*, n 2, 1980, 216–221 (Russian)

[F2] Feigin B., Lie algebra $\mathfrak{gl}(\lambda)$ and cohomology of the Lie algebra of differential operators. Russian Math. Surveys. v. 43, n 2, 1988, 157–158

[F3] Feigin B., Cohomology of contact Lie algebras. (Russian) C. R. Acad. Bulgare Sci. 35 (1982), no. 8, 1029–1031

[FF] Feigin B., Fuchs D., Cohomology of Lie groups and Lie algebras. In: Lie groups and Lie algebras – 2. Itogi nauki i theniki. Sovremennye problemy matematiki. Fundamental'nye napravleniya, v. 21, VINITI, 1988, 121–209 (= English translation by Springer in Sov. Math. Encyclop. series)

[FF1] Feigin B., Fuks (Fuchs) D., Stable cohomology of the algebra W_n and relations in the algebra L_1. Funct. Anal. and its Appl. 1984, v.18, n.3, 264–266

[FF2] Feigin B., Fuks (Fuchs) D., Cohomology of Lie groups and Lie algebras. In: *Modern Problems of Mathematics. Fundamental Trends* (Results of sci. and technology), v. 21, VINITI, the USSR Acad. of Sci., Moscow, 1988, 121–209 (Russian; English translation by Springer in Sov. Math. Encycl. series.)

[FZ] Feigin B., Zelevinsky A., Representations of contragredient Lie algebras and the Kac-Macdonald identities. Representations of Lie groups and Lie algebras (Budapest, 1971), 25–77, Akad. Kiadó, Budapest, 1985

[Fu] Fuks (Fuchs) D. B., *Cohomology of Infinite Dimensional Lie Algebras.* Consultants Bureau, NY, 1987

[FL] Fuchs D., Leites D., Cohomology of Lie superalgebras. C. R. Acad. Bulgare Sci. 37 (1984), no. 12, 1595–1596

[FLS] Feynman, R., Leighton, R., Sands, M., *The Feynman lectures on physics. Vol.* 1*: Mainly mechanics, radiation, and heat.* Addison-Wesley Publishing Co., Inc., Reading, Mass.-London, 1963, xii+513 pp.

[FK] Fiziev P., Kleinert H., Europhys. Lett. 35 (1996), 241; id. Anholonomic transformations of mechanical action principle, gr-qc/9605046

[GI] Galperin A.; Ivanov, E.; Ogievetsky, V.; Sokatchev, E. $N = 2$ supergravity in superspace. Proceedings of the Fourth Seminar on Quantum Gravity (Moscow, 1987), 539–547, World Sci. Publishing, Teaneck, NJ, 1988. id., Present status of harmonic superspace. Proceedings of the XXIII International Conference on High Energy Physics, Vol. I, II (Berkeley, CA, 1986), 405–409, World Sci. Publishing, Teaneck, NJ, 1987. id., On a versatile model of supergravity. Group theoretical methods in physics, Vol. 1–3 (Zvenigorod, 1982), 485–491, Harwood Academic Publ., Chur, 1985. id., $N = 2$ supergravity in superspace: different versions and matter couplings. Classical Quantum Gravity **4** (1987), no. 5, 1255–1265.

Galperin, A.; Ivanov, E.; Kalitzin, S.; Ogievetsky, V.; Sokatchev, E. Unconstrained off-shell $N = 3$ supersymmetric Yang–Mills theory. Classical Quantum Gravity **2** (1985), no. 2, 155–166. id., Corrigendum: "Unconstrained $N = 2$ matter, Yang–Mills and supergravity theories in harmonic superspace". Classical Quantum Gravity **2** (1985), no. 1, 127. id., Unconstrained $N = 2$ matter, Yang–Mills and supergravity theories in harmonic superspace. Classical Quantum Gravity **1** (1984), no. 5, 469–498.

[GIOS] Galperin, A. S.; Ivanov, E. A.; Ogievetsky, V. I.; Sokatchev, E. S. *Harmonic superspace.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 2001. xiv+306 pp. [MR: 1 865 518]

[GKF] Gelfand, I. M.; Kalinin, D. I.; Fuks, D. B. The cohomology of the Lie algebra of Hamiltonian formal vector fields. Funkcional. Anal. i Prilozhen. 6 (1972), no. 3, 25–29

[GM] Gelfand, Sergei I.; Manin, Yuri I. *Methods of homological algebra.* Springer-Verlag, Berlin, 1996. xviii+372 pp

[Gl] Gliklikh Yu., *Ordinary and stochastic differential geometry as a tool for Mathematical physics*, Kluwer, 1994; id, *Global analysis in Mathematical methods. Geometric and stochastic methods.* Berlin et al, Springer, 1996

[Go] Goncharov A., Infinitesimal structures related to hermitian symmetric spaces, Funct. Anal. Appl, 15, n3 (1981), 23–24; for a detailed version see: id., Generalized conformal structures on manifolds. In: [L], #11 and Selecta Math. Soviet. 6 (1987), no. 4, 307–340

[GN] Gritsenko, V.; Nikulin, V. Automorphic forms and Lorentzian Kac-Moody algebras. Internat. J. Math. 9 (1998), no. 2, 153–199; id., ibid., Internat. J. Math. 9 (1998), no. 2, 201–275

[Gr] Griffiths P., *Exterior differential systems and the calculus of variations*, Progr. in Math. v.25, Birkhauser, Boston, 1983

[Gz] Grozman P. Classification of bilinear differential operators on tensor fields, Funct. Anal. Appl. 1980, v. 14, n. 2, 58–59; id., Classification of bilinear differential operators on tensor fields invariant with respect to symplectomorphisms, J. Nonlinear Math. Phys., v. 8, 2001, no. 1, 31–37

[GL] Grozman P., Leites D., Defining relations associated with principal $\mathfrak{sl}(2)$-subalgebras. In: R. Dobrushin, R. Minlos, M. Shubin and A. Vershik (eds.) *Contemporary Mathematical Physics* (F.A. Berezin memorial volume), Amer. Math. Soc. Transl. Ser. 2, vol. 175, Amer. Math. Soc., Providence, RI (1996) 57–68;

[GL1] Grozman P., Leites D., Lie superalgebras of supermatrices of complex size. Their generalizations and related integrable systems. In: E. Ramírez de Arellano, M. Shapiro, L. Tovar and N. Vasilevski (eds.) *Proc. Internatnl. Symp. Complex Analysis and related topics*, Mexico, 1996, Birkhauser Verlag, 1999, 73–105 math.RT/0202177 MR 2001a:17008

[GL2] Grozman P., Leites D., Defining relations for classical Lie superalgebras with Cartan matrix, hep-th 9702073; Czech. J. Phys., Vol. 51, no. 1, 1–22 MR 2002e:17008

[GL3] Grozman P., Leites D., From supergravity to ballbearings. In: In: Wess J., Ivanov E. (eds.), *Supersymmetries and quantum symmetries*, Lecture Notes in Phys., 524, 1999, 58–67

[GL4] Grozman P., Leites D., Shapovalov form for the contact Lie superalgebra on $1|6$-dimensional supercircle. Quantum groups and integrable systems, Part I (Prague, 1997). Czech. J. Phys. 47 (1997), 11, 1133–1138; id, The Shapovalov determinant for the Poisson superlagebras, J. Nonlinear Math. Phys., v. 8, 2001, no. 2, 220–228 math.QA/0104287

[GL5] Grozman P., Leites D., *Mathematica*-aided study of Lie algebras and their cohomology. From supergravity to ballbearings and magnetic hydrodynamics In: Keränen V. (ed.) *The second International Mathematica symposium*, Rovaniemi, 1997, 185–192

[GL6] BurdíkČ., Grozman P., Leites D., Sergeev A., Construction of Lie algebras and superalgebras by means of creation and annihilation operators. I. Theor. and Math. Physics, v. 124, no. 2, 227–238 MR 2002b:17021

[GL7] Grozman P., Leites D., An unconventional supergravity, In: S. Duplij and J. Wess (eds.) *Noncommutative structures in mathematics and physics* Proc. NATO Advanced Research Workshop, Kiev, 2000. Kluwer, 41–48 hep-th/0202115

[GLP] Grozman P., Leites D., Poletaeva E., Defining relations for classical Lie superalgebras without Cartan matrix. In: Ivanov E. et. al. (eds.) *Supersymmetries and Quantum Symmetries* (SQS'99, 27–31 July, 1999), Dubna, JINR, 2000, 387–396; math.RT/0202152; Homology, Homotopy and Applications, vol 4 (2), 2002, 259–275

[GLS1] Grozman P., Leites D., Shchepochkina I., Lie superalgebras of string theories., hep-th 9702120; Acta Mathematica Vietnamica, v. 26, 2001, no. 1, 27–63

[GLS2] Grozman P., Leites D., Shchepochkina I., The analogs of the Riemann tensor for exceptional structures on supermanifolds, O. Sheinman et. al. (eds.) Proc. International conference "Fundamental Mathematics Today" (December 26–29, 2001) dedicated to 10th Anniversary of the Independent Univ. of Moscow, to appear; Preprint MPIM-2003

[GLS3] Grozman P., Leites D., Shchepochkina I., Invariant differential operators on supermanifolds and The Standard Model. math.RT/0202193 In: M. Olshanetsky, A. Vainstein (eds.) *Multiple facets of quantization and supersymmetry. Michael Marinov Memorial Volume*, World Sci., 2002.

[GLS4] Grozman P., Leites D., Shchepochkina I., Defining relations for the exceptional Lie superalgebras of vector fields pertaining to The Standard Model, math-ph/0202025 In: C. Duval, L. Guieu and V. Ovsienko (eds.) *The orbit method in geometry and physics* (A. A. Kirillov Festschrift), Progress in Mathematics, Birkhäuser, 2003.

[Gr1] Gruson C., Finitude de l'homologie de certains modules de dimension finie sur une super algbre de Lie. (French) [Finiteness of the homology of certain finite-dimensional modules over a Lie superalgebra] Ann. Inst. Fourier (Grenoble) 47 (1997), no. 2, 531–553

[Gr2] Gruson C., Sur la cohomologie des super algèbres de Lie étranges. (French) [Cohomology of strange Lie superalgebras] Transform. Groups 5 (2000), no. 1, 73–84

[Gr3] Gruson C., Sur l'idéal du cone autocommutant des super algèbres de Lie basiques classiques et étranges. (French) [On the ideal of the self-commuting cone of basic classical and strange Lie superalgebras] Ann. Inst. Fourier (Grenoble) 50 (2000), no. 3, 807–831

[H] Hertz H., *The principles of mechanics in new relation*. NY, Dover, 1956

[Ja] Jantzen J., Restricted Lie algebra cohomology, *Algebraic groups* Utrecht 1986, 91–108, Lecture Notes in Math. 1271, Springer, Berlin, 1987

[Ki1] Kirillov, A. A. Natural differential operations on tensor fields. (Russian) Akad. Nauk SSSR Inst. Prikl. Mat. Preprint No. 56, (1979), 28 pp.

[Ki2] Kirillov, A. A. Invariant operators over geometric quantities. (Russian) Current problems in mathematics, Vol. 16 (Russian), pp. 3–29, 228, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow, 1980.

[HKV] Henneaux M., Krasilshchik J., Vinogradov A. (eds.) *Secondary calculus and cohomological physics*. Proceedings of the conference held at Moscow State University, Moscow, August 24–31, 1997. Contemporary Mathematics, 219. American Mathematical Society, Providence, RI, 1998. xiv+287 pp.

[HP1] van den Hijligenberg N.W., Post G. F., Calculation of Lie algebra cohomology by computer. Memo 833, Faculty of Appl. Math. Univ. of Twente, 1989

[HP2] van den Hijligenberg N.W, Post G.F., Defining relations for Lie algebras of vector fields. Memo 928, Faculty of Appl. Math. Univ. of Twente, 1991; Indag. Mathem. N.S., 2(2), 1991, 207–218

[K1] Kac V.G. *Infinite Dimensional Lie Algebras*. Birkhauser, Boston, 1983; revised 3rd edition: Cambridge Univ. Press, 1991

[K2] Kac V.G. Lie superalgebras. Adv. Math, 1976, 26, 8–96

[Kz] Koszul J-L., Connections and splittings of supermanifolds. Differential Geom. Appl. 4 (1994), no. 2, 151–16; id, Les superalg 'ebres de Lie $\mathbf{W}(n)$ et leurs représentations. *Géométrie différentielle* (Paris, 1986), 161–171, Travaux en Cours, 33, Hermann, Paris, 1988.

[KL] Kotchetkov Yu., Leites D., Simple Lie algebras in characteristic 2 recovered from superalgebras and on the notion of a simple finite group. In: Kegel O. et. al. (eds.) *Proc. Intnat. algebraic conference*, Novosibirsk, August 1989, Contemporary Math. AMS, 1992, (Part 2), v. 131, 59–67

[KK1] Kornyak V., Cohomology of Lie superalgebras of Hamiltonian vector fields: computer analysis. math.SC/9906046 (In: V.G. Ganzha, E.W. Mayr and E.V. Vorozhtsov (eds.),*Computer Algebra in Scientific Computing*, Springer (1999), pp. 241-249); math.SC/0002210

[KK2] Kornyak V., Computation of Cohomology of Lie Superalgebras of Vector Fields math.SC/0002210; Int. J. Mod. Phys. C, Vol.11 No.2 (2000)

[LL] Leger G., Luks E., Cohomology of nilradicals of Borel subalgebras, Trans. AMS, 1974, 195, 305–316

[K] Kleinert H., *Path integrals in quantum mechanics, statistic and polymer physics*, World Scientific, 1995

[KS] Komech A., Spohn H., Soliton-like asymptotics for a classical particle interacting with a scalar wave field. Nonlinear Anal. 33 (1998), no. 1, 13–24; Komech A.; Kunze M.; Spohn H., Effective dynamics for a mechanical particle coupled to a wave field. Comm. Math. Phys. 203 (1999), no. 1, 1–19; Komech A.; Spohn H.; Kunze M., Long-time asymptotics for a classical particle interacting with a scalar wave field. Comm. Partial Differential Equations 22 (1997), no. 1-2, 307–333

[KLV] Krasilshchik J., Lychagin V., Vinogradov A., *Geometry of jet spaces and nonlinear partial differential equations*, Gordon and Breach, NY, 1986

[L0] Leites D. Spectra of graded commutative rings. Russian Math. Surveys, v. 30, N3, 1974, 209–210 (in Russian)

[L1] Leites D., Introduction to Supermanifold theory. Russian Math. Surveys. v. 33, n1,1980, 1–55; an expanded version: *Supermanifold theory*. Karelia Branch of the USSR Acad. of Sci., Petrozavodsk, 1983, 200 pp. (Russian) = in English a still more expanded version is [L2]

[L2] Leites D. (ed.), *Seminar on supermanifolds*, Reports of Dept. of Math. of Stockholm Univ. n1–34,1986–89, 2100 pp. (an edited version is to be preprinted at MPIM-2003 to become accessible)

[L3] Leites D., The Riemann tensor for nonholonomic manifolds. math.RT/0202213 Homology, Homotopy and Applications, vol 4 (2), 2002, 397–407

[L4] Leites D., Supersymmetry of the Sturm–Liouville and Korteveg–de Vries operators. In: S. Albeverio, N. Elander, W. N. Everitt and P. Kurasov (eds.), *Operator Methods in Ordinary and Partial Differential Equations*, (S. Kovalevski Symproium, Univ. of Stockholm, June 2000), Birkhäuser, Basel-Boston-Berlin, 2002 (Operator Methods: Advances and Applications 132) 2002, 267–285 (This is an edited but still unfinished version of a part of hep-th/9710045.)

[LP] Leites D., Poletaeva E., Analogues of the Riemannian structure on supermanifolds. In: Kegel O. et. al. (eds.) Proceedings of the International Conference on Algebra, Part 1 Novosibirsk, August 1989, Contemporary Math. AMS, 1992, 603–612

[LPS] Leites D., Poletaeva E., Serganova V., On Einstein equations on manifolds and supermanifolds, J. Nonlinear Math. Physics, v. 9. no. 4, 1–32

[LPo] Leites, D.; Post, G. Cohomology to compute. In: E. Kaltofen, S. M. Watt (eds.) *Computers and mathematics*. Papers from the International Conference on Computers and Mathematics held at Massachusetts Institute of Technology, Cambridge, Massachusetts, June 13–17, 1989.

[BKL] Leites D., *Chevalley supergroups*, in preparation, a draft is available.

[LKW] Leites D.A., Kotchetkov Yu., Waintrob A., New invariant differential operators on supermanifolds and pseudo-(co)homology. In: Andima S. et. al. (eds.), *General topology and its applications* LN in pure and applied math., v. 134, Marcel Decker, NY, 1991, 217–238

[LP1] Leites, D., Poletaeva, E., Analogues of the Riemannian structure for classical superspaces. Proceedings of the International Conference on Algebra, Part 1 (Novosibirsk, 1989), 603–612, Contemp. Math., 131, Part 1, Amer. Math. Soc., Providence, RI, 1992

[LP2] Leites, D., Poletaeva, E., Supergravities and contact type structures on supermanifolds. Second International Conference on Algebra (Barnaul, 1991), 267–274, Contemp. Math., 184, Amer. Math. Soc., Providence, RI, 1995

[LP3] Leites, D., Poletaeva, E. Defining relations for classical Lie algebras of polynomial vector fields. Math. Scand. 81 (1997), no. 1, 5–19

[LPr] Leites, D., Premet A., The local geometry of flag manifolds, in preparation

[LSS] Leites D., Saveliev M., Serganova V. Embeddings of $\mathfrak{osp}(N|2)$ and integrable nonlinear system. In: Markov., Manko V. (eds) *Group-theoretical methods in phisics. Proc. Internat. Conf, Yurmala,* 1985, Nauka, Moskow 1986, v.1, 377–379; (an enlarged English transl. by VNU Sci Press, Dordrecht, 1986, v.1., 255–297)

[LS] Leites D., Serganova V., Defining relations for classical Lie superalgebras. I. (Lie superalgebras with Cartan matrix). In: J. Mickelsson (ed.) *Proc. Topological and Geometrical Methods in Field Theory*, World Sci., Singapore, 1992, 194–201

[LSV] Leites D., Serganova V., Vinel G., Classical superspaces and related structures. In: Bartocci C. et al. (eds) *Differential Geometric Methods in Theoretical Physics*. Proc. DGM-XIX, 1990, Springer, LN Phys. 375, 1991, 286–297

[LSe] Leites D., Sergeev A., Orthogonal polynomials of discrete variable and Lie algebras of complex size matrices. In: Procedings of M. Saveliev memorial conference, MPI, Bonn, February, 1999, MPI-1999-36, 49–70; ; Theor. and Math. Physics, v. 123, no. 2, 205–236 (Russian), 582–609 MR 2001j:17018

[LSc] Leites D., Shchepochkina I., Classification of the simple Lie superalgebras of vector fields (to appear);

[Le] Lesser M., *The analysis of complex nonlinear mechanical systems. A computer algebra assisted approach*, World Sci., 1995

[LR] Löfwall, C., Roos, J.-E., A nonnilpotent 1-2-presented graded Hopf algebra whose Hilbert series converges in the unit circle. Adv. Math. 130 (1997), no. 2, 161–200

[Ma] Manin Y. I., *Gauge field theory and complex geometry*. Translated from the 1984 Russian original by N. Koblitz and J. R. King. Second edition. With an appendix by Sergei Merkulov. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 289. Springer-Verlag, Berlin, 1997. xii+346 pp

[Ms] Zenkov, D.; Bloch, A.; Marsden, J. The energy-momentum method for the stability of non-holonomic systems. Dynam. Stability Systems 13 (1998), no. 2, 123–165; Koon, Wang Sang; Marsden, J. Poisson reduction for nonholonomic mechanical systems with symmetry. Pacific Institute of Mathematical Sciences Workshop on Nonholonomic Constraints in Dynamics (Calgary, AB, 1997). Rep. Math. Phys. 42 (1998), no. 1-2, 101–134

[M] Montgomery R., Isoholonomic problem and some applications, Comm. Math. Phys. 128:3, 1990, 565–592

[NS] Naimark, M. A., Stern, A. I. *Theory of group representations*. Springer, New York-Berlin, 1982. ix+568 pp. (there is also a French translation from the Russian original).

[NF] Nejmark Yu., Fufaev N., *Dynamics of nonholonomic systems*, Moscow, Nauka, 1967 (in Russian= Transl. Math. Monogr. AMS, 33.IX, 1972)

[N] Nordmark, A.; Essén, H., Systems with a preferred spin direction. R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. 455 (1999), no. 1983, 933–994

[OV] Onishchik A., Vinberg E., *Seminar on algebraic groups and Lie groups*, Springer, 1987

[PS] Penkov I., Serganova V., Generic irreducible representations of finite dimensional Lie superalgebras. Internat. J. Math. 5, 1994, 389–419

[P1] Poletaeva, E. Structure functions on the usual and exotic symplectic and periplectic supermanifolds. Differential geometric methods in theoretical physics (Rapallo, 1990), 390–395, Lecture Notes in Phys., 375, Springer, Berlin, 1991

[P2] Poletaeva, E. Analogues of Riemann tensors for the odd metric on supermanifolds. Acta Appl. Math. 31 (1993), no. 2, 137–169

[P3] Poletaeva, E. Penrose's tensors on super-Grassmannians. Math. Scand. 72 (1993), no. 2, 161–190. Id., Penrose's tensors. II. Math. Scand. 72 (1993), no. 2, 191–211

[P4] Poletaeva, E., Semi-infinite cohomology of the loop algebra of $\mathfrak{sl}(2)$. C. R. Acad. Sci. Paris Sr. I Math. 322 (1996), no. 11, 1019–1023; id., Semi-infinite cohomology and superconformal algebras. C. R. Acad. Sci. Paris Sr. I Math. 326 (1998), no. 5, 533–538

[R1] Rudakov A.N., "Irreducible representations of infinite dimensional Lie algebras of Cartan type," *Math. USSR Izvestiya*, **38**, n. 4, 835-866 (1974).

[R2] Rudakov A.N., "Irreducible representations of infinite dimensional Lie algebras of types S and H," *Math. USSR Izvestiya* **39**, n. 3, 496-511, (1975).

[RK] Rumyantsev V., Karapetyan A., (eds.) *Modern methods of analytical mechanics and their applications.* Lectures from the CISM Advanced School held in Udine, June 16–20, 1997. CISM Courses and Lectures, 387. Springer-Verlag, Vienna, 1998. vii+344 pp.; Rumyantsev, V. V., On stability of steady motions of rigid body with liquid-filled cavity. World Congress of Nonlinear Analysts '92, Vol. I–IV (Tampa, FL, 1992), 1287–1294, de Gruyter, Berlin, 1996; Karapetyan, A. V.,Rumyantsev, V. V. Stability of conservative and dissipative systems. (Russian) General mechanics, Vol. 6, 3–128, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1983.

[RS] Rosly, A. A.; Schwarz, A. S. Geometry of $N = 1$ supergravity. Comm. Math. Phys. 95 (1984), no. 2, 161–184; id., Geometry of $N = 1$ supergravity. II. ibid., 96 (1984), no. 3, 285–309

[Sa] Sakharov A. D., Evaporation of black mini-holes and high energy physics, ZhETPh Lett., **44**, (6), 1986, 295–298 (in Russian)

[SS] Salam A., Sezgin E., *Supergravities in diverse dimensions*, v.v. 1, 2, World Scientific, 1989

[SV] Sergeev V., *The limits of rationality. A thermodynamical approach to the theory of economic equilibrium.* Fazis, Moscow, 1999, v+144 (Russian)

[Sh5] Shchepochkina I., The five exceptional simple Lie superalgebras of vector fields, hep-th 9702121; Five simple exceptional Lie superalgebras of vector fields. (Russian) Funktsional. Anal. i Prilozhen. 33 (1999), no. 3,59–72, 96 translation in Funct. Anal. Appl. 33 (1999), no. 3, 208–219 (a preliminary version of [Sh14])

[Sh14] Shchepochkina I., Five exceptional simple Lie superalgebras of vector fields and their fourteen regradings. Represent. Theory, v. 3, 1999, 3 (1999), 373–415

[Sh-M] Shchepochkina I., Maximal subalgebras of classical Lie superalgebras. In: [L], v. 32, 1992, 1988-15, 1–43; (a shorter but edited version: Maximal subalgebras of matrix Lie superalgebras, hep-th 9702122)

[ShP] Shchepochkina I., Post G., Explicit bracket in an exceptional simple Lie superalgebra. Internat. J. Algebra Comput. 8 (1998), no. 4, 479–495 (physics/9703022)

[St] Sternberg S., *Lectures on differential geometry*, Chelsey, 2nd edition, 1985

[T] Tolpygo, A. K., Deformations of subalgebras in Lie algebras. (Russian) Izv. Akad. Nauk Armjan. SSR Ser. Mat. 3, 1968, no. 6, 470–478; id., The cohomology of parabolic Lie algebras. (Russian) Mat. Zametki 12 (1972), 251–255; id., Cohomology of nilpotent Lie algebras and its generating functions. (Russian) Uspekhi Mat. Nauk 34 (1979), no. 1(205), 245–246; id., Newton polyhedra and the inverse Borel–Weil theorem. (Russian) Funktsional. Anal. i Prilozhen. 17 (1983), no. 3, 55–60; id., Cohomology of infinite-dimensional Borel algebras. (Russian) Ukrain. Mat. Zh. 42 (1990), no. 9,1278–1283; English translation in: Ukrainian Math. J. 42 (1990), no. 9, 1137–1141 (1991); id., Estimation of the cohomology of Verma modules. (Russian) Uspekhi Mat. Nauk 48 (1993), no. 1(289), 185–186

[TL] Tolpygo, A. K., Lie algebra cohomology, its weights and generating functions. In: [L], 29/1988-11, 34p.

[1] V1 Brink, L.; Hansson, T. H.; Konstein, S.; Vasiliev, M. A., The Calogero model—anyonic representation, fermionic extension and supersymmetry. Nuclear Phys. B 401 (1993), no. 3, 591–61;

[2] V2 Vasiliev, M. A., Higher-spin gauge theories in four, three and two dimensions. The Sixth Moscow Quantum Gravity Seminar (1995), Internat. J. Modern Phys. D **5** (1996), no. 6, 763–797; id., Higher-spin-matter gauge interactions in $2 + 1$ dimensions. Theory of elementary particles (Buckow, 1996), Nuclear Phys. B Proc. Suppl. **56B** (1997), 241–252

[V] Vershik, A. M. Classical and nonclassical dynamics with constraints. In: Yu. Borisovich, Yu. Gliklikh (eds.), *Geometry and topology in global nonlinear problems*, Novoe Global. Anal., Voronezh. Gos. Univ., Voronezh, 1984, 23–48 (Russian; a not very understandable English translations In: Yu.

Borisovich, Yu. Gliklikh (eds.), *Global Analysis — Studies and Applications. I.*, LN 1108, 1984, 278–301)

[VG] Vershik A. M., Gershkovich V. Ya., A bundle of nilpotent Lie algebras over a nonholonomic manifold (nilpotentization). (Russian) Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 172 (1989), Differentsialnaya Geom. Gruppy Li i Mekh. Vol. 10,21–40, 169 translation in J. Soviet Math. 59 (1992), no. 5, 1040–1053; id., Nonholonomic manifolds and nilpotent analysis. J. Geom. Phys. 5 (1988), no. 3, 407–452

[3] Wess J and Zumino B, Supergauge Transformations in Four Dimensions, *Nuclear Phys.* **B70** (1974), 39–50;
Wess J, Supersymmetry-Supergravity, in Topics in Quantum Field Theory and Gauge Theories, Editor: de Azcárraga J A, Proc. VIII Internat. GIFT Sem. Theoret. Phys., Salamanca, 1977, Lecture Notes in Phys., Vol. 77, Springer, Berlin – New York, 1978, 81–125;
Wess J and Zumino B, Superspace Formulation of Supergravity, *Phys. Lett.* **B66**, Nr. 4 (1977), 361–364;
Wess J, Supersymmetry/Supergravity, in Concepts and Trends in Particle Physics, Schladming, 1986, Editors: Latal H and Mitter H, Schladming, 1986, Springer, Berlin, 1987, 29–58;
Wess J, Introduction to Supersymmetric Theories, in Frontiers in Particle Physics '83, Dubrovnik, 1983, Editors: Šijački Dj, Bilić N, Dragović B and Popović D, World Sci. Publishing, Singapore, 1984, 104–131;
Wess J and Bagger J, Supersymmetry and Supergravity, Princeton Series in P

[W] Witten, E., Introduction to cohomological field theories. Topological methods in quantum field theory (Trieste, 1990). Internat. J. Modern Phys. A 6 (1991), no. 16, 2775–2792