Construction of universal Thom-Whitney-a stratifications, their functoriality and Sard-type Theorem for singular varieties

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Abstract

Construction. For a dominating polynomial (or analytic) mapping $F: K^n \to K^l$ with an isolated critical value at 0 ($K = \mathbb{R}$ or an algebraically closed field of characteristic zero) we construct a closed *bundle* $G_F \subset T^*K^n$, restrict it over the critical points $\mathrm{Sing}(F)$ of F in $F^{-1}(0)$ and partition $\mathrm{Sing}(F)$ into *'quasistrata'* of points with the fibers of G_F of constant dimension. It turns out that T-W-a (Thom-Whitney-a) stratifications 'near' $F^{-1}(0)$ exist iff the fibers of bundle G_F are orthogonal to the tangent spaces of the quasistrata (e. g. when $l = 1$); and are the orthogonal complements over an irreducible component S of a quasistrata iff S is *universal* for the class of T-W-a stratifications, meaning that for any $\{S'_j\}_j$ in the class there is a component S' of an S'_j with $S \cap S'$ being open and dense in both S and S'. Construction of G_F involves *Glaeser iterations* of replacing the fibers of the successive closures by the respective linear spans and stabilizes after $\rho(F) \leq 2n$ iterations, resulting in $\dim(G_F) = n$ for $K \neq \mathbb{R}$.

Results. We prove that T-W-a stratifications with only universal strata exist iff all fibers of G_F are the orthogonal complements to the respective tangent spaces to the quasistrata, and then the partition of $\mathrm{Sing}(F)$ by the latter yields the coarsest *universal T-W-a stratification.* (We relax condition of smoothness of strata to a continuity of their Gauss maps and show it implies smoothness of their normalizations.) The proof relies on an extension of a singular stratum to a subvariety with a continuous Gauss map and a prescribed tangent bundle over the stratum (assuming a version of Whitney-a condition). The key ingredient is a version of Sard-type Theorem for *singular spaces.* We provide various examples including of $F : K^5 \to K$ that does not admit a universal T-W-a stratification and a family of $F_n\colon K^{4n+1} \to K$ with $\rho(F_n) = n$.

Question. We wonder whether there can ever be an irreducible component of bundle G_F of dimension smaller than n, e. g. for $F\colon \mathbb{C}^n \to \mathbb{C}$?

Introduction

We consider Thom-Whitney-a stratifications of critical points in an isolated critical fiber of a dominating polynomial (or analytic) mapping $F\colon K^n \to K^l$, where $K = \mathbb{R}$ or is an algebraically closed field of characteristic zero. Our main goal is to identify **'universal strata'**, i. e. such that for every stratification of this type their open and dense subsets appear as open dense subsets in appropriate strata of the latter. To that end we consider even a larger class of Thom-Whitney-a stratifications with the condition of smoothness of strata relaxed to a weaker assumption of continuity of their Gauss mappings, which by definition (over the smooth points) send points to the tangent spaces at these points. It turns out that singularities of such strata are mild, in particular their normalizations are smooth. Besides being *Gauss regular* we require strata to be open in their respective closures, pairwise disjoint and, of course, to satisfy classical Thom and Whitney-a conditions (for the definitions of the latter one may consult for instance [7] , [17] , [22]). Construction of the *Glaeser bundle* of the mapping (i. e. of the restriction over the critical points of the subbundle of the cotangent bundle which is minimal by inclusion among closed subbundles containing the differentials of the component functions of our mapping) involves *Glaeser iterations* of replacing fibers of the successive closures by their respective linear spans (see [8]). At the first glance it seemed that the Glaeser bundle of the mapping could serve the purpose of identifying Thom-Whitney-a Gauss regular stratifications with all strata being universal, namely: by means of partitioning of the critical locus by dimension of its fibers. But it does not always work, see example of Subsection 8.3. Nevertheless, the irreducible subsets (we call them *Glaeser components*) over which the fibers of Glaeser bundle are of constant dimension equal their respective codimension are universal even with respect to the class of Thom-Whitney-a Gauss regular stratifications, see Corollary 2.5 . Thom-Whitney-a stratifications 'near' the critical fiber exist iff the fibers of Glaeser bundle are orthogonal to the tangent spaces of the *quasistrata* of points of constant dimension of fibers of Glaeser bundle (e. g. when $l = 1$, see [15]). Our principal result states that Thom-Whitney-a Gauss regular stratifications with all strata being universal essentially coincide with the ones derived from Glaeser bundles by means of the partitioning into the quasistrata described above. The proof relies on our construction of an extension of a smooth stratum of a singular locus of a variety to a Gauss regular subvariety with a prescribed tangent bundle over the stratum under the assumption of Whitney-a condition on the pair. To that end our version of a Sard-type Theorem for *singular varieties* is crucial. We provide various examples of mappings that admit universal Thom-Whitney-a Gauss regular stratifications, but in general the question of recognition of an individual universal stratum we address in a forthcoming manuscript: we will show that the universal strata with respect to Thom-Whitney-a Gauss regular stratifications are precisely the Glaeser components over which Glaeser bundle is of the same dimension as the source of the mapping. The latter Glaeser components we refer to as *Lagrangian components* since off singular locus the restriction of Glaeser bundle over such components is a Lagrangian submanifold of T^*K^n in the natural symplectic structure of the latter.

In abuse of notation we write $\mathrm{Sing}(F)$ for the critical points of F in $F^{-1}(0)$. We say that an algebraic (or analytic respectively) set S *open in its closure* is Gauss regular provided that there is a (unique) continuiation to all of S of the Gauss map from the regular points $\mathrm{Reg}(S)$ of S , i. e. $S \ni x \mapsto T_x(S)$, where $T_x(S)$ denotes the tangent space to S at x . In abuse of notation we will denote (for a Gauss regular S and $a \in \mathrm{Sing}(S) := S \setminus \mathrm{Reg}(S)$) by $T_a(S)$ the unique limiting position at a of the tangent spaces $T_x(S)$ to S at the points $x \in \mathrm{Reg}(S)$. We consider Thom-Whitney-a stratifications $\{S_i\}_i$ of the critical points $\mathrm{Sing}(F) = \cup_i S_i$ with all S_i being Gauss regular (rather than smooth), open in their respective closures and pairwise disjoint, and such that $\{S_i\}_i$ satisfy Thom and Whitney-a conditions. For brevity sake we call them TWG-stratifications and say that $\{S_i\}_i$ is universal if all irreducible components S of S_i are universal, i. e. if for any other TWG-stratification $\{S'_j\}_j$ of $\mathrm{Sing}(F) = \cup_j S'_j$ there exists (a unique) j and an irreducible component S' of S'_j such that $S \cap S'$ is open and dense in both S and S'. Throughout the article by an irreducible component of a constructive set we mean its intersection with an irreducible component of its closure.

Denote by $G_F \subset (\mathrm{Sing}(F)) \times (K^n)^*$ the restriction to $\mathrm{Sing}(F) \subset K^n$ of the minimal by inclusion closed *subbundle* containing bundle $\{(x,\ \mathrm{Span}\{df_j(x)\}_{1\leq j\leq l})\}_{x\in K^n} \subset K^n \times (K^n)^*$ of subspaces of $(K^n)^*$ over K^n , where Span denotes the K-linear hull of a family of vectors in $(T_xK^n)^*$. Let *'quasistrata'* $\mathcal{G}_r \subset K^n$ consist of the points whose fibers of G_F are vector spaces of dimension r . Assuming Thom stratification 'near' $F^{-1}(0)$ exists, cf. [15] (e. g. when $l = 1$) it follows that $r \geq l$ and that the dimensions of quasistrata $\mathcal{G}_r$ are less or equal $n-r$ by virtue of Lemma 2.7 below. Constructed bundle G_F is *functorial* with respect to isomorphisms preserving fibers of F 'near' its critical value 0 (including with respect to C^1 diffeomorphisms when K is $\mathbb{C}$ or $\mathbb{R}$), see Section 2. Construction of Glaeser bundle G_F involves iterations (starting with $\{(x,\ \mathrm{Span}\{df_j(x)\}_{1\leq j\leq l})\}_{x\in K^n}$) of replacing the fibers of the successive closures by their linear spans and stabilizes after $\rho(F) \leq 2n$ iterations (see [4]), resulting in $\dim(G_F) = n$ for $K \neq \mathbb{R}$ (see Claim 2.8 and Remark 2.9).

The principal purpose of the paper is to provide a constructive criterium of the existence of a universal TWG-stratification $\{S_i\}_i$. Our main result states that $\mathrm{Sing}(F)$ admits a universal TWG-stratification if and only if manifolds $\mathrm{Reg}(G_F|_{\mathcal{G}_r})$ are *Lagrangian* in $K^n \times (K^n)^*$ in the natural symplectic structure of the latter. Moreover, for universal TWG-stratifications $\{S_i\}_i$ partitions $\{S_{(m)}\}_m$ of $\mathrm{Sing}(F)$ obtained by replacing all S_i of the same dimension m with their union $S_{(m)}$ results in a universal TWG-stratification and coincides with the functorial partition $\{\mathcal{G}_r\}_{l\leq r\leq n}$ of $\mathrm{Sing}(F)$, which is then the *coarsest* among all universal TWG-stratifications.

A simpler implication that if all $\mathrm{Reg}(G_F|_{\mathcal{G}_r})$ are Lagrangian then $\{\mathcal{G}_k\}_{l\leq k\leq n}$ is a universal TWG-stratification we establish in Section 3. When the latter takes place we would refer to $\{\mathcal{G}_k\}_{l\leq k\leq n}$ as a *functorial TWG-stratification* (with respect to F).

A more difficult converse implication is proved in Sections 4 and 5. It relies on a Proposition 4.10 of interest in its own right. A straightforward generalization of the latter in Theorem 5.1 provides an extension of a (smooth) stratum $\mathcal{G}$ of a singular locus of a variety S (algebraic or analytic, open in its closure and with $\mathcal{G}$ being essentially its boundary) to a Gauss regular subvariety $\mathcal{G}^+$ of $\overline{S}$ with a prescribed tangent bundle $T_{\mathcal{G}}$ over $\mathcal{G}$ (under necessary assumptions of our version of Whitney-a condition for the pair of $T_{\mathcal{G}}$ over $\mathcal{G}$ and S). The key ingredient to both is our version of a Sard-type Theorem 5.3 for singular varieties. Roughly speaking it asserts that for an irreducible Gauss regular algebraic (or analytic) set S its intersection with an appropriate generic hypersurface (of the same class) is Gauss regular and, more importantly, the angles between the tangent spaces to S and to the hypersurface are uniformly separated from 0 on compacts (in a neighborhood of an open dense subset of any irreducible component of $\overline{S} \setminus S$).

In Section 6 we show that the normalization of a Gauss regular variety is smooth.

In Subsection 8.2 we introduce a family of examples of $F_n\colon K^{4n+1} \to K$ and prove that the *index of stabilization* $\rho(F_n)$ of F_n equals n . In Subsection 8.3 we prove that $F := AX^2 + 2B^2XY + CY^2$ does not admit a universal TWG-stratification. Moreover, we show that for an appropriate variation of the former example an arbitrary hypersurface appears as $\mathcal{G}_r$ for some r (see Remark 8.3). We also consider in Subsections 8.1, 8.4 (discriminant-type) examples for which $\{\mathcal{G}_r\}_r$ are functorial TWG-stratifications (and exhibit these stratifications explicitly).

In abuse of notation in the sequel we identify (occasionaly) the dual $(K^n)^*$ with K^n, the cotangent bundle $T^*(K^n)$ with K^{2n} and also denote $dF(x) := \mathrm{Span}\{\{df_i(x)\}_{1\leq i\leq l}\}$. We also denote the variety of zeroes of a polynomial f by $\{f = 0\}$ and for the sake of brevity refer to "Gauss regular" as "G-regular".

1 Canonical Thom-Whitney-a stratifications

We recall that in a *stratification* $\{S_i\}_i$ of the set $\mathrm{Sing}(F) = \cup_i S_i$ of critical points of F in $F^{-1}(0)$ (i. e. the points $x \in F^{-1}(0)$ such that $\dim(dF(x)) < l$) each stratum S_i is assumed to be irreducible (for $K = \mathbb{C}$ and in the analytic case connected in the classical eucleadian topology), open in its closure and assumed to fulfil the frontier condition: for each pair S_i, S_j if $\overline{S}_i \cap S_j \neq \emptyset$ then $S_j \subset \overline{S}_i$, as is e. g. in [7]. Traditionally one assumes each S_i to be smooth.

In the present article for the sake of a concept of universality (and a fortiori functoriality), i. e. of a stronger version of canonicity, we relax condition of smoothness and allow S_i to be G-regular. We consider *Gauss regular stratifications* $Sing(F) = \cup_i S_i$, i. e. all S_i are G-regular, open in their respective closures and pairwise disjoint (but neither necessarily irreducible nor fulfil the frontier condition). The notions of Thom property with respect to a map F and Whitney-a condition on stratifications naturally extend to Gauss regular stratifications.

Lemma 1.1 *i) A Thom stratification exists iff the following condition holds:*

(1) any irreducible constructive set $S \subset \mathrm{Sing}(F)$ *contains an open dense subset* $S^o \subset \mathrm{Reg}(S)$ *such that if a sequence* $\{(x_m\ ,\ dF(x_m)) \subset K^{2n}\}_m$ *has a limit* $\lim_{m\to\infty}(x_m\ ,\ dF(x_m)) = (x_0\ ,\ V)$*, where* $x_0 \in S^o$*,* $x_m \in K^n \setminus \mathrm{Sing}(F)$ *and* V *is an* l*-dimensional linear subspace of* $(K^n)^*$*, then it follows* $V \perp T_{x_0}(S^o)$*;*

ii) A Thom-Whitney-a stratification exists iff (1) and the following condition hold:

(2) for any smooth irreducible constructive set $M \subset Sing(F)$ *and any irreducible constructive set* $S \subset \mathrm{Sing}(F)$ *there is an open dense subset* $S^o \subset \mathrm{Reg}(S)$ *such that if a sequence* $\{(x_m\ ,\ V_m) \subset K^n \times (K^n)^*\}_m$ *has a limit* $\lim_{m\to\infty}(x_m\ ,\ V_m) = (x_0\ ,\ V)$*, where* $x_0 \in S^o$*,* $x_m \in M$ *and subspaces* V_m *in* $(K^n)^*$ *are orthogonal to* $T_{x_m}(M) \subset K^n$*, then it follows that subspace* $V \subset (K^n)^*$ *is orthogonal to* $T_{x_0}(S^o) \subset K^n$*.*

Proof. Since the proofs of i) and ii) are similar, we provide only a proof of ii). First assume that $\{S_i\}_i$ is a Thom-Whitney-a stratification. Once again the proofs of properties (1) and (2) are similar and we provide only a proof of (2). Take a unique S_i (respectively, S_j) such that $M \cap S_i$ (respectively, $S \cap S_j$) is open and dense in M (respectively, in S). If $S \setminus \overline{S_i}$ is open and dense in S then the choice of $S^o := (S_j \cap \mathrm{Reg}(S)) \setminus \overline{S_i}$ is as required in (2). On the other hand the remaining assumptions of (2) can not hold which makes (2) valid, but vacuous. (Property (1) holds due to the Thom property of $\{S_i\}_i$.) Otherwise $S \subset \overline{S_i}$ and the choice of $S^o := S_j \cap \mathrm{Reg}(S)$ is as required in (1) and in (2) due to the Thom and Whitney-a properties of $\{S_i\}_i$ respectively. Indeed, it suffices to replace the sequence of (2) by its subsequence for which exists $\lim_{m\to\infty} T_{x_m}(M) =: W$, and then to choose another sequence $\{x'_m\}_m$ of points in $M \cap S_i$ with the 'distance' between respective $(x_m\ ,\ T_{x_m}(M))$ and $(x'_m\ ,\ T_{x'_m}(M))$ converging to zero. Then $W = \lim_{m\to\infty} T_{x'_m}(M)$ and is orthogonal to V . On the other hand due to the Whitney-a property of the pair S_i , S_j it follows that $W \supset T_{x_0}(S_i) \supset T_{x_0}(S)$ and therefore also $T_{x_0}(S)$ is orthogonal to V , as required.

Now we assume that (1) and (2) are valid. We construct strata $S_1, S_2, \ldots$ by induction on their codimensions, i. e. $\mathrm{codim}(S_1) \leq \mathrm{codim}(S_2) \leq \cdots$. So assume that $S_1, \ldots, S_k$ are already produced with $\mathrm{codim}(S_k) = r$, set $\mathrm{Sing}(F) \setminus (S_1 \cup \cdots \cup S_k) =: Z$ being of $\mathrm{codim}(Z) := r_1 > r$ and that Thom and Whitney-a properties are satisfied for stratification $\{S_i\}_{1\leq i\leq k}$ of $\mathrm{Sing}(F) \setminus Z$. Subsequently for every irreducible component S of Z of $\mathrm{codim}(S) = r_1$ (and by making use of the noetherian property of the Zariski topology of S) we choose a maximal open subset of $\mathrm{Reg}(S)$ which satisfies both property (1) and the property (2) with respect to the choices of sets S_i, for $1 \leq i \leq k$, as the set M of (2). By additionally choosing each subsequent S_j in $\mathrm{Sing}(F) \setminus (S_1 \cup \cdots \cup S_{j-1})$ for $k < j \leq k_1$ we produce strata $S_{k+1}, \ldots, S_{k_1}$ of codimensions r_1 with $\mathrm{codim}((\mathrm{Sing}(F) \setminus (S_1 \cup \cdots \cup S_{k_1})) > r_1$. Such choice ensures Thom and Whitney-a properties of stratification $\{S_i\}_{1 \leq i \leq k_1}$ of set $\cup_{1 \leq i \leq k_1} S_i$, as required in the inductive step, which completes the proof of ii). ▪

Remark 1.2 *It is not true that for* $l > 1$ *and* 0 *being an isolated critical value of a dominating polynomial mapping* $F : K^n \to K^l$ *a stratification that satisfies Thom condition with respect to* F *necessarily exists, e. g consider the 'local' blowing up of the origin:*

$$F : (z_1, ..., z_n) \mapsto (z_1 \ , \ z_1 \cdot z_2 \ , \ \ldots \ , \ z_1 \cdot z_n) \ .$$

The statement (2) holds, see [25], [23], [14], [24]. For $l = 1$ *statement (1) holds, see [15], and for* $l > 1$ *see e. g. [15], [7], [16] for conditions on* F *.*

Remark 1.3 *Fix a class of stratifications. A stratification* $\{S_i\}_i$ *of* $\mathrm{Sing}(F) = \cup_i S_i$ *is called canonical (or minimal), e. g. in [7] and [21], if for any other stratification* $\{S'_i\}_i$ *of* $\mathrm{Sing}(F) = \cup_i S'_i$ *in this class with* $\mathrm{codim}(S_1) \leq \mathrm{codim}(S_2) \leq \cdots$ *and* $\mathrm{codim}(S'_1) \leq \mathrm{codim}(S'_2) \leq \cdots$ *it follows (after possibly renumbering* $\{S'_i\}$*) that* $S'_1 = S_1, \ldots, S'_k = S_k$ *and* $S'_{k+1} \subsetneq S_{k+1}$*. Constructed in the proof of Lemma 1.1 Thom and Thom-Whitney-a stratifications are canonical in the corresponding classes. These respective canonical stratifications are clearly unique. We extend to Gauss regular stratifications the concepts and constructions introduced above for stratifications.*

2 Dual bundles of vector spaces of TWG-stratifications

In the sequel we will repeatedly apply the following construction. Let M , N be constructive sets open in their Zariski closures (by default we consider Zariski topology, sometimes in the case of K being $\mathbb{C}$ or $\mathbb{R}$ we also use euclidean topology). In the analytic case we assume alternatively that M , N are analytic manifolds. Let V , W be vector spaces. For a subset $\mathcal{T} \subset M \times V$ we denote $\mathcal{T}^{(0)} = \mathcal{T}$ and by $\mathcal{T}^{(1)} \subset M \times V$ a bundle of vector spaces whose fiber $\mathcal{T}^{(1)}_x$ at a point $x \in M$ is the linear hull of the fiber $(\overline{\mathcal{T}})_x$ of the closure $\overline{\mathcal{T}} \subset M \times V$ [8]. Defining in a similar way $\mathcal{T}^{(p+1)}$ starting with $\mathcal{T} := \mathcal{T}^{(p)}$, for $p \geq 0$, results in an increasing chain of (not necessary closed) bundles of vector spaces and terminates at $\mathcal{T}^{(\rho)}$ such that $\mathcal{T}^{(\rho)} = \mathcal{T}^{(\rho+1)}$ with $\rho \leq 2\dim(V)$. We denote $Gl(\mathcal{T}) = \mathcal{T}^{(\rho)}$ and refer to the smallest $\rho = \rho(\mathcal{T})$ as the index of stabilization. The so called 'Glaeserization' $Gl(\mathcal{T})$ of $\mathcal{T}$ is the minimal closed bundle of vector spaces which contains $\mathcal{T}$. We apply this construction to $\mathcal{T} = \{(x, dF(x))\}$ where x ranges over all noncritical points of F . The result we denote by $G^{(p)} := G^{(p)}_F := \mathcal{T}^{(p)}|_{\mathrm{Sing}(F)}$, for $p \geq 0$, and $G := G_F := Gl(\mathcal{T})|_{\mathrm{Sing}(F)}$ (and still refer to the smallest $\rho = \rho(F)$ as the index of stabilization). We mention that Thom stratification with respect to F exists iff $\dim(\overline{G^{(0)}}) \leq n$, see [15], [12].

Denote $G_x := \pi^{-1}(x) \cap G$, where $\pi \colon T^*(K^n)|_{\mathrm{Sing}(F)} \to \mathrm{Sing}(F)$ is the natural projection. The proofs of the following Proposition and its corollary are straightforward.

Proposition 2.1 *Let* $\mathcal{T}_M \subset M \times V,\ \mathcal{T}_N \subset N \times W$ *and* $h^{-1} \colon N \to M, H \colon N \times W \to M \times V$ *be homeomorphisms which commute with the natural projections* $N \times W \to N,\ M \times V \to M$ *.*

Assume in addition that H is linear on each fiber of these projections and that $H(\mathcal{T}_N) = \mathcal{T}_M$. Then $H(Gl(\mathcal{T}_N)) = Gl(T_M)$, moreover $H(T_N^{(i)}) = T_M^{(i)}$ for every i.

Corollary 2.2 *Let M , N be nonsingular, $\mathcal{T}_M \subset T^*M$, $\mathcal{T}_N \subset T^*N$. If $h : M \to N$ is an isomorphism such that for the pullback D^*h by h we have $(D^*h)(\mathcal{T}_N) = \mathcal{T}_M$ then $(D^*h)(Gl(\mathcal{T}_N)) = Gl(\mathcal{T}_M)$. Moreover, $(D^*h)(\mathcal{T}_N^{(i)}) = \mathcal{T}_M^{(i)}$ for every i.*

When K is $\mathbb{C}$ or $\mathbb{R}$ it suffices to assume that h is a C^1-diffeomorphism and then constructed bundle G_F and partition $\{\mathcal{G}_r\}_{l\leq r\leq n}$ of $\mathrm{Sing}(F)$ are functorial with respect to C^1 diffeomorphisms preserving fibers of F 'near' its critical value 0 .
(For an arbitrary K replace "C^1 diffeomorphisms" above by "isomorphisms".)

With any Gauss regular stratification $\mathcal{S} = \{S_i\}_i$, where $Sing(F) = \cup_i S_i$, we associate a subbundle $B = B(\mathcal{S})$ of $T^*(K^n)|_{\mathrm{Sing}(F)}$ of vector subspaces of $(K^n)^*$ such that for every i and a smooth point $a \in S_i$ the fiber $B_a := (T_a(S_i))^\perp \subset (K^n)^*$ and for a singular point a of S_i the fiber B_a is defined by continuity, by making use of S_i being G-regular. Note that the dimension of fibers $\dim(B_a) = \mathrm{codim}(S_i)$ for $a \in S_i$.

Remark 2.3 *Note that for any Gauss regular stratification $\mathcal{S} = \{S_i\}_i$ of $Sing(F)$ bundle $B(\mathcal{S}) = \cup_i B(\mathcal{S})|_{S_i}$ and for any irreducible component S of an arbitrary S_i bundle $B(\mathcal{S})|_S$ is an irreducible n-dimensional Gauss regular set open in its closure.*

Proposition 2.4 *A Gauss regular stratification $\mathcal{S}$ satisfies Thom-Whitney-a condition with respect to F iff $G \subset B$ and B is closed.*

Proof. It follows by a straightforward application of definitions that Thom and Whitney-a properties for any Gauss regular stratification $\mathcal{S} = \{S_i\}_i$ of $Sing(F)$ are equivalent to $G^{(1)} \subset B(\mathcal{S})$ and, respectively, that set $B(\mathcal{S})$ is closed. Due to the definition of bundle G proposition follows. ∎

Corollary 2.5 *It follows due to the preceding Remark and Proposition that all n-dimensional irreducible components of G appear as irreducible components of $B(\mathcal{S})$ for any TWG-stratification $\mathcal{S} = \{S_i\}_i$ of $Sing(F)$. Therefore every irreducible component $\mathcal{G}$ of $\mathcal{G}_r$ with $G|_{\mathcal{G}}$ being n-dimensional is a universal stratum.*

Note that $\dim(G) = n$ for $K \neq \mathbb{R}$ (see Claim 2.8 and Remark 2.9).

Remark 2.6 *Let $\{S_i\}_i$ be a TWG-stratification of $\mathrm{Sing}(F)$. Then for every $0 \leq m \leq n$ the union $\bigcup_{\dim(S_i)=m} S_i$ coincides with $(\bigcup_{\dim(S_i)\geq m} S_i) \setminus (\bigcup_{\dim(S_i)>m} S_i)$ and therefore is open in its closure. Also due to Proposition 2.4 it is G-regular. Moreover, if we replace any subfamily of $\{S_i\}_i$ of the same dimension m by its union S, we would again obtain a TWG-stratification if only S is open in its closure.*

Lemma 2.7 *The following three statements are equivalent:*

- *a Thom-Whitney-a stratification exists;*
- *a TWG-stratification exists;*
- *condition (2) of Lemma 1.1 and the following property hold:*

(1') any irreducible constructive set $S \subset \mathrm{Sing}(F)$ contains an open dense subset $S_0 \subset \mathrm{Reg}(S)$ such that for any $x_0 \in S_0$ we have $T_{x_0}(S) \perp G_{x_0}$.

Proof. For the proof of (1') above note that property (1') with G_{x_0} being replaced by $G_{x_0}^{(1)}$ is a straightforward consequence of the Thom property of stratification $\mathcal{S}$ with respect to F and condition (1) of Lemma 1.1 , which Thom property implies. By making use then of condition (2) of Lemma 1.1 consecutively property (1') with G_{x_0} being replaced by $G_{x_0}^{(p)}$, for $p \geq 1$, follows and implies property (1') as stated, since $G = G^{(p)}$ for $p = \rho(F)$. Otherwise the proof is similar to that of Lemma 1.1 with the exception that we replace $\mathrm{Reg}(S)$ with the maximal (by inclusion) open subset U of $\overline{S}$ to which by continuity the Gauss map of S uniquely extends from $\mathrm{Reg}(S)$. ∎

Lemma 1.1 implies (assuming Thom-Whitney-a stratification of $\mathrm{Sing}(F)$ exists) that $r := n - \dim(\mathrm{Sing}(F)) \geq \min_{a \in \mathrm{Sing}(F)}\{\dim(G_a)\} \geq l.$

Claim 2.8 *Assume that Thom stratification of* $\mathrm{Sing}(F)$ *exists (e. g. if* $l = 1$ *, see [15]), and that* $K \neq \mathbb{R}$ *, then* $\mathrm{Sing}(F) = \cup_{j \geq r}\mathcal{G}_j$ *. Also, then quasistrata* $\mathcal{G}_j$ *are open and dense in irreducible components of* $\mathrm{Sing}(F)$ *of dimension* $n-j$ *(if such exist). In particular, appropriate open subsets of the latter are Lagrangian components of the former with their union being dense in* $\mathrm{Sing}(F)$ *, quasistratum* $\mathcal{G}_r \neq \emptyset$ *and* $\dim(G_F) = n$ *.*

Remark 2.9 *In the example of* $F : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $F := x^3 + x \cdot y^4$ *the critical points* $\mathrm{Sing}(F) = \{0\}$*, the fiber at* 0 *of the Glaeser bundle* G_F *is spanned by* dx *, i. e. is* 1*-dimensional, and therefore* $\dim(G_F) = 1 < 2 =: n$ *.*

Proof of Claim. The openness is due to (1') of Lemma 2.7 and the upper semicontinuity of the function $g \colon x \to \dim(G_x)$. Therefore it suffices to verify that a generic point a of an irreducible component Z of $\mathrm{Sing}(F)$ of dimension $n-j$ belongs to $\mathcal{G}_j$.

We first reduce to the case of $l = 1$ by making use of the existence of a TWG-stratification, which is true due to Remark 1.2, Lemma 1.1 and Lemma 2.7. Indeed, let U be an open set such that $U \cap \mathrm{Sing}(F)$ is smooth, irreducible and of dimension $n-j$. We may assume w.l.o.g. that $0 \in U \cap \mathrm{Sing}(F)$ and that for the 1-st component $f := f_1$ of $F : K^n \to K^l$ the differential $df(0) = 0$ (which anyway holds after a linear coordinate change in the target K^l of map F). We may also assume by shrinking U and replacing 0 , if needed, that $0 \in \mathrm{Reg}(\mathrm{Sing}(f))$. Inclusions $\mathrm{Sing}(f) \subset \mathrm{Sing}(F)$ and $(G_f)_a \subset (G_F)_a$, for $a \in \mathrm{Sing}(f)$, are straightforward consequences of the definitions. By making use of (1') of Lemma 2.7 and of the reduction assumption for f (the case of $l = 1$) it follows that $(G_f)_0$ is the orthogonal complement of the tangent space $T_0(\mathrm{Sing}(f)) \subset T_0(\mathrm{Sing}(F))$, while $(G_F)_0$ is orthogonal to $T_0(\mathrm{Sing}(F))$ due to (1') of Lemma 2.7 applied to F . Therefore $(G_F)_0 = (G_f)_0$ and $T_0(\mathrm{Sing}(f)) = T_0(\mathrm{Sing}(F))$, in particular implying that $\dim(U \cap \mathrm{Sing}(f)) = \dim(U \cap \mathrm{Sing}(F))$. Hence also $(U \cap \mathrm{Sing}(f)) = (U \cap \mathrm{Sing}(F))$, which suffices by making use of the established above inclusions.

In the case of $l = 1$ and by once again making use of (1') of Lemma 2.7 it suffices w.l.o.g. to consider the case of the restriction of F to a plane of dimension j intersecting transversally Z at a , thus reducing the proof to the case of $l = 1$ and of a being an isolated critical point. In the latter case it suffices to show that $(G_F)_a = K^n$.

If K is algebraically closed our claim follows since for any $c_2, \dots, c_n \in K$ due to $F_i(a) := \frac{\partial F}{\partial x_i}(a) = 0$, $1 \leq i \leq n$, the germ at a of $\Gamma := \{F_i - c_i \cdot F_1 = 0 , 2 \leq i \leq n\}$ is at least 1-dimensional, thus producing $dx_1 + c_2 \cdot dx_2 + \cdots + c_n \cdot dx_n$ in $(\overline{G_F^{(0)}})_a \subset (G_F)_a$ by means of limits of $dF(a)/||dF(a)||$ along Γ , as required. ∎

3 Universality and Lagrangian bundles

Now we introduce a partial order on the class of TWG-stratifications with respect to F (note that it differs from the order defined in Ch.1 [7]). For any pair $\mathcal{S} = \{S_i\}_i$, $\mathcal{S}' = \{S'_j\}_j$, $Sing(F) = \cup_i S_i = \cup_j S'_j$ of TWG-stratifications of Sing(F) and for every i there exists a unique $j = j(i)$ such that $S_i \cap S'_j$ is open and dense in S_i , reciprocately for every j there exists a unique $i = i(j)$ such that $S_i \cap S'_j$ is open and dense in S'_j . We say that $\mathcal{S}$ is *larger* than $\mathcal{S}'$ (i. e. is 'almost everywhere' finer than $\mathcal{S}$) if for every i equalities $j_0 = j(i)$, $i = i(j_0)$ hold. Thus universal TWG-stratification means the largest one.

Proposition 3.1 *For a pair of TWG-stratifications $\mathcal{S}$ is larger than $\mathcal{S}'$ iff the bundle $B = B(\mathcal{S}) \subset B' = B(\mathcal{S}')$.*

Proof. Let $\mathcal{S}$ be larger than $\mathcal{S}'$. For each i we have that $S_i \cap S'_{j_0}$ (where $j_0 = j(i)$) is open and dense in both S_i, S'_{j_0}, while $\dim(S_i \cap S'_{j_0}) = \dim(S_i) = \dim(S'_{j_0})$. Therefore, for any point $a \in S_i \cap S'_{j_0}$ we have $T_a(S_i) = T_a(S'_{j_0})$, i. e. $B(S_i)_a = B(S'_{j_0})_a$. Hence for any point $b \in S_i$ we obtain $B_b = B(S_i)_b \subset B'_b$ since the Gauss map of $\overline{S_i}$ is continuous on S_i and B' is closed due to Proposition 2.4.

Conversely, let $B \subset B'$. For every S_i take $j_0 = j(i)$, then $S_i \cap S'_{j_0}$ is open and dense in S_i . It follows that for any point $a \in S_i \cap S'_{j_0}$ inclusion $T_a(S_i) \subset T_a(S'_{j_0})$ holds and therefore $B_a \supset B'_a$ implying that $B_a = B'_a$ and $\dim(S_i) = \dim(S'_{j_0})$, hence $S_i \cap S'_{j_0}$ is open and dense in S'_{j_0} , i. e. $i(j_0) = i$. ∎

Proposition 3.1 and Remark 2.6 imply the following corollary.

Corollary 3.2 *i) If for a pair of TWG-stratifications $\mathcal{S} = \{S_i\}_i$ and $\mathcal{S}' = \{S'_j\}_j$ (with respect to F) equality $B(\mathcal{S}) = B(\mathcal{S}')$ holds then the unions $\mathcal{S}_{(m)} := \bigcup_{\dim(S_i)=m} S_i = \bigcup_{\dim(S'_j)=m} S'_j$ coincide and are G-regular;*

ii) If a universal TWG-stratification $\mathcal{S} = \{S_i\}_i$ exists then for every $0 \le m \le n$ the union $\mathcal{S}_{(m)}$ is independent of a choice of a universal TWG-stratification and $\{\mathcal{S}_{(m)}\}_{0 \le m \le n}$ is a universal TWG-stratification and is the coarsest universal in the following sense: for any universal TWG-stratification $\mathcal{S}' = \{S'_j\}_j$ and every $0 \le m \le n$ an equality $\mathcal{S}_{(m)} = \mathcal{S}'_{(m)}$ holds.

For a (constructive) closed subbundle $B \subset T^*(K^n)$ of vector spaces (in the sequel we shortly call them bundles) we consider its *'quasistrata'*, i. e. the constructive sets (open in their respective closures due to the upper-semicontinuity of the function $\dim_K(B_x)$)

$$\mathcal{B}_k := \{x \in K^n : \dim_K(B_x) = k\}, \; 0 \le k \le n.$$

Applying this construction to the bundle G we obtain quasitrata $\mathcal{G}_k$.

Definition 3.3 *We say that irreducible components $\mathcal{B}$ of quasistrata $\mathcal{B}_k$, $0 \le k \le n$, are* **Lagrangian** *if for points $x \in \mathrm{Reg}(\mathcal{B})$ the tangent spaces $T_x(\mathcal{B})$ are the orthogonal complements of B_x . We call bundle B Lagrangian if all irreducible components of $\mathcal{B}_k$, $0 \le k \le n$, are Lagrangian.*

Remark 3.4 *For any bundle B Lagrangian components of its quasistrata $\mathcal{B}_k$ are G-regular (cf. Remark 2.6) and of dimension $n - k$.*

Proposition 3.5 *If bundle B is Lagrangian then there is a bijective correspondence between the irreducible components of its quasistrata $\mathcal{B}_k$, $0 \le k \le n$, and the irreducible components of B. Also, the irreducible components $\tilde{B}$ of B are of dimension n and* $\mathrm{Reg}(\tilde{B})$ *are Lagrangian submanifolds of $T^*(K^n)$ in the natural symplectic structure of the latter.*

Proof. As a straightforward consequence of Definition 3.3 bundle B is a union of n-dimensional (constructive) sets $B|_{\mathcal{B}}$ with $\mathcal{B}$ being the irreducible components of the quasistrata $\mathcal{B}_k$, $0 \le k \le n$, and $\mathrm{Reg}(B|_{\mathcal{B}})$ are Lagrangian submanifolds of $T^*(K^n)$. Therefore the closures of $B|_{\mathcal{B}}$ are the irreducible components $\tilde{B}$ of B implying the remainder of the claims of Proposition 3.5 as well. ∎

Theorem 3.6 *The first two of the following statements are equivalent and imply the third:*

(i) bundle G is Lagrangian;

(ii) Thom stratification of $\mathrm{Sing}(F)$ *exists and each irreducible component of $\mathcal{G}_k$, $r \le k \le n$, is of dimension $n-k$;*

(iii) each irreducible component of G is of dimension n .

Remark 3.7 *In the example of Remark 8.2 there are only* 2 *irreducible components of G and both are of dimension n , but G is not Lagrangian.*

Proof of Theorem 3.6. First (i) implies (ii) since quasistrata $\{\mathcal{G}_k\}_{r\le k\le n}$ form a TWG-stratification due to Proposition 2.4 and Remark 3.4. Now assume (ii). Then (1') of Lemma 2.7 implies that for any irreducible component $\tilde{\mathcal{G}}$ of $\mathcal{G}_k$ there is an open dense subset $\tilde{\mathcal{G}}^{(0)} \subset \tilde{\mathcal{G}}$ such that $T_x(\tilde{\mathcal{G}}) \perp G_x$ holds for any point $x \in \tilde{\mathcal{G}}^{(0)}$. Since $\dim(\tilde{\mathcal{G}}) = n-k$ it follows G_x is the orthogonal complement to $T_x(\tilde{\mathcal{G}})$ for any point $x \in \tilde{\mathcal{G}}^{(0)}$, which implies (i). Finally, (i) implies (iii) is proved in Proposition 3.5. ∎

In the previous section with every TWG-stratification $\mathcal{S}$ (with respect to F) we have associated a bundle $B(\mathcal{S})$ such that $B(\mathcal{S}) \supset G$ (see Proposition 2.4). By construction bundle $B(\mathcal{S})$ is Lagrangian. Conversely, if $B \supset G$ is a Lagrangian bundle then $\mathcal{S}(B) := \{\mathcal{B}_k\}_k$ is a TWG-stratification due to Proposition 2.4 and Remark 3.4. We summarize these observations in the following

Theorem 3.8 *There is a bijective correspondence between TWG-stratifications (with respect to F) and closed Lagrangian subbundles of $T^*(K^n)|_{\mathrm{Sing}(F)}$ (which contain G).*

Moreover Propositions 3.1, 2.4, Theorem 3.6 and Corollary 3.2 imply

Corollary 3.9 *If G is Lagrangian then the corresponding TWG-stratification $\{\mathcal{G}_k\}_{r\le k\le n}$ is functorial and is the coarsest universal.*

In the next section we establish the converse statement.

4 A constructive criterium of universality

Results of this and of the following section essentially depend on the validity of the conclusions of Claim 2.8 (which are, in general, not valid for $K = \mathbb{R}$, cf Remark 2.9). We therefore additionally assume in the case of $K = \mathbb{R}$ for the remainder of this article that bundle $G|_F$ is n-dimensional over open dense subsets of every irreducible component of $\mathrm{Sing}(F)$. The latter assumption replaces references below (for $K \ne \mathbb{R}$) to Claim 2.8.

The following Theorem and its Corollary justify the title of the paper.

Theorem 4.1 *If there exists a universal TWG-stratification of* $\mathrm{Sing}(F)$ *then* G *is Lagrangian.*

Combining with Corollary 3.9 it follows

Corollary 4.2 *If there exists any universal TWG-stratification of* $\mathrm{Sing}(F)$ *then* $\{\mathcal{G}_k\}_{r\leq k\leq n}$ *is the coarsest universal (and is functorial).*

Proof of Theorem 4.1. Assume the contrary and let $\mathcal{G}$ be an irreducible component of some $\mathcal{G}_k$, $r \leq k \leq n$ which is not Lagrangian and with a (lexicographically) maximal possible pair $(n-k\ ,\ m := \dim(\mathcal{G}))$. We recall (see Claim 2.8) that the minimal r for which $\mathcal{G}_r \neq \emptyset$ equals $r = n - \dim(\mathrm{Sing}(F))$. Therefore all irreducible components of $\mathcal{G}_r$ are Lagrangian since $\mathcal{G}_r$ is open in $\mathrm{Sing}(F)$, in particular $k > r$. We have $m = \dim(\mathcal{G}) < n-k$ (see Theorem 3.6) because condition (1') of Lemma 2.7 implies that $\dim(\mathcal{G}_t) \leq n-t$, $r \leq t \leq n$. Denote by $\mathcal{S} = \{S_i\}_i$ a universal TWG-stratification of $\mathrm{Sing}(F) = \cup_i S_i$ whose existence is the assumption of Theorem 4.1 . Below by an irreducible component of $\mathcal{S}$ we mean an irreducible component of an S_i .

Let $R \subset \mathrm{Sing}(F)$. In the sequel we denote by $G^{\perp}|_R \subset T(K^n)|_R$ the bundle of vector spaces whose fibers are the orthogonal complements to the fibers of subbundle $G|_R \subset T^*(K^n)|_R$.

Denote by W the union of all Lagrangian irreducible components of $\{\mathcal{G}_t\}_{r\leq t\leq k}$. Due to the choice of $\mathcal{G}$ we have $\cup_{r\leq t<k}\mathcal{G}_t \subset W$. On the other hand, W is the union of all Lagrangian irreducible components of $\{\mathcal{G}_t\}_{r\leq t\leq n}$ with dimensions greater or equal to $n-k$. Hence $\dim(\mathrm{Sing}(F) \setminus W) < n-k$.

Remark 4.3 *One can produce following the construction in the proof of Lemma 1.1 (cf. Remark 1.3) a TWG-stratification* $\mathcal{S}' = \{S'_j\}_j$ *of* $\mathrm{Sing}(F) = \cup_j S'_j$ *extending the family of all irreducible components contained in* W *. Then* $B(\{S_i\}_i)|_W = G|_W$ *due to Propositions 2.4 and 3.1. Similarly,* $B(\{S_i\}_i)|_L = G|_L$ *for* L *being the union (dense in* $\mathrm{Sing}(F)$*) of all open in* $\mathrm{Sing}(F)$ *Lagrangian components of appropriate quasistrata* $\mathcal{G}_j$ *(cf. Claim 2.8).*

Claim 4.4 *Let* $\mathcal{Q}$ *be an irreducible component of* $\mathcal{S}$ *. Then either* $\mathcal{Q} \cap W = \emptyset$ *or* $\mathcal{Q}$ *is an open and dense subset of a Lagrangian component* $\mathcal{P} \subset W$ *. In particular,* W *coincides with the union of an appropriate subfamily of irreducible components of* $\{S_i\}_i$ *.*

Proof. Indeed, first consider an irreducible component $\mathcal{Q}$ of $\mathcal{S}$ such that $\mathcal{Q} \cap W$ is dense in $\mathcal{Q}$ and denote $t := n - \dim(\mathcal{Q})$. Since $\mathcal{Q}$ is G-regular, $B(\mathcal{S}) \supset G$ and $B(\mathcal{S})|_{\mathcal{Q}\cap W} = G|_{\mathcal{Q}\cap W}$ it follows that $\mathcal{Q} \subset \cup_{q\leq t}\mathcal{G}_q$ and $\mathcal{Q} \cap W \subset \mathcal{G}_t$ (in particular $t \leq k$). On the other hand, set $\mathcal{G}^{(t)} := \cup_{q\geq t}\mathcal{G}_q$ is closed (since function $g\colon x \to \dim(G_x)$ is upper semicontinuous) and therefore $\mathcal{Q} \subset \overline{\mathcal{Q} \cap W} \subset \mathcal{G}^{(t)}$. Hence $\mathcal{Q} \subset \mathcal{G}_t$.

Consider an irreducible component $\mathcal{P}$ of $\mathcal{G}_t$ such that $\mathcal{Q} \cap \mathcal{P}$ is dense in our $\mathcal{Q}$. The latter implies that $\dim(\mathcal{P}) \geq n-t$ and since $\mathcal{P} \subset \mathcal{G}_t$ it follows ($n-t \geq \dim(\mathcal{P})$ and therefore) $\dim(\mathcal{P}) = n-t$. Thus $\mathcal{P}$ is Lagrangian and $\mathcal{P} \subset W$ (since $t \leq k$). We conclude that $\mathcal{Q} \subset (\overline{\mathcal{Q} \cap \mathcal{P}}) \cap \mathcal{G}_t \subset \overline{\mathcal{P}} \cap \mathcal{G}_t = \mathcal{P} \subset W$ and $\dim(\mathcal{Q}) = n-t = \dim(\mathcal{P})$, as required.

Now, assume that an irreducible component $\mathcal{Q}$ of $\mathcal{S}$ has a non-empty intersection with a Lagrangian irreducible component $\mathcal{P} \subset W$ of $\mathcal{G}_t$ (and therefore $\dim(\mathcal{P}) = n-t$ for some $t \leq k$). Then, using $B(\mathcal{S})|_{\mathcal{P}\cap\mathcal{Q}} = G|_{\mathcal{P}\cap\mathcal{Q}}$ and in view of the definition of $B(\mathcal{S})$, it follows that $\dim(\mathcal{Q}) = n-t$. As we have shown above $\dim(\mathrm{Sing}(F) \setminus W) < n-k \leq n-t$. Therefore $\mathcal{Q} \cap W$ is dense in $\mathcal{Q}$. In the latter case we have already proved that $\mathcal{Q} \subset W$, which completes the proof of the claim. ∎

Corollary 4.5 *Let $\mathcal{Q}$ be an irreducible component of $\mathcal{S}$ with $\dim(\mathcal{Q}) > \dim(\mathcal{G})$ and $\overline{\mathcal{Q}} \supset \mathcal{G}$ then $\mathcal{Q} \subset \mathcal{G}_{n-q}$, where $q = \dim(\mathcal{Q}) > n-k > \dim(\mathcal{G})$, and $\mathcal{Q} \subset W$.*

Proof. Due to our assumptions either $\mathcal{G}\cap\mathcal{Q}$ or $\mathcal{G}\cap(\overline{\mathcal{Q}}\setminus\mathcal{Q})$ is dense in $\mathcal{G}$. If $\mathcal{Q}\cap W = \emptyset$ then either $\mathcal{Q} \subset \mathcal{G}^{(k-1)}$ or $\mathcal{Q}\cap(\mathcal{G}_k\setminus W)$ is dense in $\mathcal{Q}$. In the latter case $\dim(\mathcal{Q}) \leq \dim(\mathcal{G}_k\setminus W) = \dim(\mathcal{G})$, which is contrary to the choice of $\mathcal{Q}$. And in the former case $\mathcal{G} \subset \overline{\mathcal{Q}} \subset \mathcal{G}^{(k-1)}$ contrary to $\mathcal{G}$ being an irreducible component of $\mathcal{G}_k$. Hence $\mathcal{Q}\cap W \neq \emptyset$ and due to the claim above $\mathcal{Q} \subset W$. ∎

Consider the union $S^{\cup}$ of all irreducible components $\mathcal{Q}$ of $\mathcal{S}$ of the smallest possible dimension s with $\overline{\mathcal{Q}}\setminus\mathcal{Q}$ containing $\mathcal{G}$.

Remark 4.6 *Due to the upper semi-continuity of function $g : x \to \dim(G_x)$ and Claim 2.8 (or the replacing it assumption when $K = \mathbb{R}$) the following inclusions hold $\mathcal{G} \subset \overline{\cup_{r\leq t<k}\mathcal{G}_t} \subset \overline{W}$. Therefore Claim 4.4, Corollary 4.5 and Remark 2.6 imply repectively that $S^{\cup}$ is not emply, $S^{\cup} \subset (\mathcal{G}_{n-s}\cap W) = \mathcal{G}_{n-s}$ and that $S^{\cup}$ is G-regular.*

Claim 4.7 *Let $\mathcal{W}$ be an irreducible component of $\overline{S^{\cup}}\setminus S^{\cup}$ such that $\mathcal{W}$ contains $\mathcal{G}$. Then $\mathcal{G}$ is dense in $\mathcal{W}$. (Hence such $\mathcal{W}$ is unique). In particular, $\overline{\mathcal{G}}$ is an irreducible component of $\overline{S^{\cup}}\setminus S^{\cup}$ and thus on an appropriate open neighbourhood $\mathcal{G}$ coincides with its own closure and with $\overline{S^{\cup}}\setminus S^{\cup}$.*

Proof. Assume the contrary. Then $\dim(\mathcal{W}) > \dim(\mathcal{G})$. Denote by $t_{\mathcal{W}}$ the minimal value of $g\colon x \to \dim(G_x)$ on $\mathcal{W}$ (attained on an open dense subset of $\mathcal{W}$ in view of the upper semicontinuity of function g). Then $t_{\mathcal{W}} \geq t := n-s = \dim(G_x)$ for $x \in S^{\cup} \subset W$ because $\mathcal{W} \subset (\overline{S^{\cup}}\setminus S^{\cup})$. Pick an irreducible component $\mathcal{Q}$ of $\mathcal{S}$ such that $\mathcal{W}\cap\mathcal{Q}$ is dense in $\mathcal{W}$. Then $\overline{\mathcal{Q}} \supset \mathcal{G}$ and since $\dim(\mathcal{Q}) \geq \dim(\mathcal{W}) > \dim(\mathcal{G})$ inclusion $\mathcal{Q} \subset W$ holds due to Corollary 4.5, implying $(W\cap\mathcal{G}) \supset (\mathcal{Q}\cap\mathcal{G})$. Since $\mathcal{G} \subset (\mathcal{G}_k\setminus W)$ it follows $\mathcal{Q}\cap\mathcal{G}$ is empty, i. e. $\mathcal{G} \subset (\overline{\mathcal{Q}}\setminus\mathcal{Q})$. Since also $\mathcal{Q} \subset W$ and due to the choice of s we conclude that $\dim(\mathcal{Q}) \geq s$. On the other hand $n - \dim(\mathcal{Q}) = \dim(G_x) = t_{\mathcal{W}}$ for $x \in (\mathcal{W}\cap\mathcal{Q})$ by making use of Remark 4.3 and Claim 4.4, which implies $s = n-t \geq n-t_{\mathcal{W}} = \dim(\mathcal{Q})$. Therefore $s = \dim(\mathcal{Q})$ and both $\mathcal{Q} \subset S^{\cup}$ and, due to $\overline{\mathcal{Q}}\cap\overline{\mathcal{W}} \neq \emptyset$, inequality $\mathcal{Q}\cap(\overline{S^{\cup}}\setminus S^{\cup}) \neq \emptyset$ holds, leading to a contradiction. ∎

Corollary 4.8 *Let $\mathcal{Q}$ be an irreducible component of $\mathcal{S}$ of $\dim(\mathcal{Q}) = s$ with $\overline{\mathcal{Q}}\setminus\mathcal{Q} \supset \mathcal{G}$. Let $S_* := \overline{\mathcal{Q}}\cap S^{\cup} \supset \mathcal{Q}$. Then S_* is an irreducible subset of $W\cap\mathcal{G}_{n-s} = \mathcal{G}_{n-s}$ and $\overline{S_*}\setminus S_* = \mathcal{G} = \overline{\mathcal{G}}$ in an open neighbourghood $U_{\mathcal{G}}$.*

Proof. Inclusion $S_* \subset S^{\cup} \subset W\cap\mathcal{G}_{n-s} = \mathcal{G}_{n-s}$ is the main content of Corollary 4.5. Note that S_* is irreducible since $\overline{S_*} = \overline{\mathcal{Q}} \supset \mathcal{G}$ and that sets $\mathcal{G}\cap S^{\cup}$ and $(\overline{S_*}\setminus S_*)\cap S^{\cup}$ are both empty. Therefore $S_*\cap\mathcal{G} = \emptyset$ and $(\overline{S^{\cup}}\setminus S^{\cup}) \supset (\overline{S_*}\setminus S_*) \supset \mathcal{G}$. Hence due to Claim 4.7 also $\overline{S_*}\setminus S_*$ coincides with $\mathcal{G}$ on an open neighbourhood of an open dense subset of $\mathcal{G}$. ∎

Remark 4.9 *We may choose an open neighbourhood $U_{\mathcal{G}}$ of $\mathcal{G}$ so that $\mathcal{G}\cap U_{\mathcal{G}} = \overline{\mathcal{G}}\cap U_{\mathcal{G}}$. Since $\overline{\mathcal{Q}}\cap U_{\mathcal{G}} \supset \mathcal{G}\cap U_{\mathcal{G}} \neq \emptyset$ it follows that $\mathcal{Q}\cap U_{\mathcal{G}} \neq \emptyset$. Consider $S := S_*\cap U_{\mathcal{G}} \supset \mathcal{Q}\cap U_{\mathcal{G}}$ (as in Corollary 4.8). Then $\overline{\mathcal{Q}} \supset \overline{S} \supset \overline{\mathcal{Q}\cap U_{\mathcal{G}}} = \overline{\mathcal{Q}} = \overline{S_*}$ (due to $\mathcal{Q}$ being irreducible) and therefore $\overline{S} = \overline{S_*}$ and S is irreducible. Hence $\mathcal{G}\cap U_{\mathcal{G}} = (\overline{S_*}\setminus S_*)\cap U_{\mathcal{G}} \supset (\overline{S}\setminus S)\cap U_{\mathcal{G}} \supset \mathcal{G}\cap U_{\mathcal{G}}$, which implies*

$$(\overline{S}\setminus S)\cap U_{\mathcal{G}} = \mathcal{G}\cap U_{\mathcal{G}} = \overline{\mathcal{G}}\cap U_{\mathcal{G}} \tag{1}$$

and that S is open in its closure. Finally, S is G-regular (and is a dense subset of a Lagrangian component of $\mathcal{G}_{n-s}$) since $S \subset W \cap \mathcal{G}_{n-s} = \mathcal{G}_{n-s}$.

In the remainder of this and in the following Section we use notation $\mathcal{G}$ for $\mathcal{G} \cap U_{\mathcal{G}}$ and S for $S \cap U_{\mathcal{G}}$ from Remark 4.9.

Proposition 4.10 *There is an irreducible G-regular constructive set $\mathcal{G}^+$ open in its closure such that $\mathcal{G}^+ \subset \overline{S}$, $\dim(\mathcal{G}^+) = n - k$ and $\mathcal{G}^+$ contains an open dense subset of $\mathcal{G}$. Finally*

$$G^{\perp}|_{\mathcal{G}^+ \cap \mathcal{G}} = T(\mathcal{G}^+)|_{\mathcal{G}^+ \cap \mathcal{G}} \ .$$

Deduction of Theorem 4.1 from Proposition 4.10. The bundle of vector spaces associated (as in Section 2) with a family

$$W_1 = \bigcup_{\mathcal{Q} \subset W} (\mathcal{Q} \setminus \overline{\mathcal{G}^+}) \cup \{\mathcal{G}^+\}$$

(where the union ranges, as above, over all irreducible components $\mathcal{Q}$ of $\mathcal{S}$ such that $\mathcal{Q} \subset W$) coincides over $W_1 \setminus \mathcal{G}^+$ with G, is Lagrangian and due to Proposition 4.10 is closed. Since $W \setminus W_1 \subset \overline{\mathcal{G}^+} \setminus \mathcal{G}^+$ and dimensions of $(\overline{\mathcal{G}^+} \setminus \mathcal{G}^+)$ and $(\mathrm{Sing}(F) \setminus W)$ are less than $n - k$ it follows that $\dim(\mathrm{Sing}(F) \setminus W_1) < n - k$. Therefore, as in the Remark 4.3, the latter family extends to a TWG-stratification $\{\tilde{S}_j\}_j$ of $\mathrm{Sing}(F) = \cup_j \tilde{S}_j$.

As we have established above in Claim 4.4 set W and therefore $Sing(F) \setminus W$ are the unions of several irreducible components of $\mathcal{S}$. Hence there exists an irreducible component $\mathcal{P}$ of $\mathcal{S}$ such that $(\mathrm{Sing}(F) \setminus W) \supset \mathcal{P}$ and $\mathcal{G} \cap \mathcal{P}$ is open and dense in $\mathcal{G}$. Since being universal TWG-stratification $\{S_i\}_i$ is larger than $\{\tilde{S}_j\}_j$ it follows by Proposition 3.1 that for any point $x \in \mathcal{G} \cap \mathcal{G}^+ \cap \mathcal{P}$ there is an inclusion $B(\mathcal{P})_x \subset B(\mathcal{G}^+)_x = G_x$ for the fibers of G ; hence $\dim(B(\mathcal{P})_x) \leq \dim(G_x) = k$ and $\dim(\mathcal{P}) \geq n - k$. But on the other hand $\dim(\mathcal{P}) \leq \dim((\mathrm{Sing}(F) \setminus W) < n - k$. Thus the assumption (on the first lines of the proof of Theorem 4.1) of the existence of a non Lagrangian component $\mathcal{G}$ in $\{\mathcal{G}_j\}_j$ leads to a contradiction, i. e. G is Lagrangian. ■

5 Sard-type Theorem for singular varieties

Proof of the more difficult implication of our main result Theorem 4.1 we complete in this section. To that end we prove here Proposition 4.10, which essentially provides an extension of a (smooth) singular locus of an algebraic variety to a smooth subvariety with a prescribed tangent bundle over singularities. The main ingredient is our Sard-type Theorem for singular varieties.

To begin with we introduce a generalization of Whitney-a property for a pair $\mathcal{G}$, S of smooth irreducible algebraic (or analytic respectively) sets closed in a nonsingular ambient variety $U_{\mathcal{G}}$, and in $U_{\mathcal{G}} \setminus \mathcal{G}$ respectively, with $\mathcal{G}$ being the boundary of S in $U_{\mathcal{G}}$. Our generalization requires additional data of a subbundle $T_{\mathcal{G}}$ over $\mathcal{G}$ of the tangent bundle $T(U_{\mathcal{G}})|_{\mathcal{G}}$ of $U_{\mathcal{G}}$ (restricted over $\mathcal{G}$) that contains the tangent bundle of $\mathcal{G}$. (To apply the notion in the setting of Proposition 4.10 we allow S to be Gauss regular.) Then our generalized Whitney-a condition is as follows:

W-a) if a sequence $\{(x_i, T_{x_i}(S)) \subset S \times T(U_{\mathcal{G}})|_S\}_i$ has a limit $\lim_{i \to \infty}(x_i, T_{x_i}(S)) = (x_0, V)$, where $x_0 \in \mathcal{G}$ and subspace $V \subset T_{x_0}(U_{\mathcal{G}})$, then it follows that subspace $V \supset (T_{\mathcal{G}})_{x_0}$.

Theorem 5.1 *Assume $\mathcal{G}$, S , $U_{\mathcal{G}}$ and $T_{\mathcal{G}} \subset T(U_{\mathcal{G}})|_{\mathcal{G}}$ are as in the preceding paragraph and satisfy generalized Whitney-a condition W-a). Then there is an irreducible Gauss regular closed subvariety $\mathcal{G}^+$ of $\overline{S}$ in an open subset $U'_{\mathcal{G}}$ of $U_{\mathcal{G}}$ that contains an open dense subset $\mathcal{G} \cap U'_{\mathcal{G}}$ of $\mathcal{G}$ and such that*

$$T_{\mathcal{G}}|_{\mathcal{G}^+\cap\mathcal{G}} = T(\mathcal{G}^+)|_{\mathcal{G}^+\cap\mathcal{G}} \ .$$

Remark 5.2 *Theorem 5.1 is a straightforward generalization of Proposition 4.10 and a straightforward extension of the proof of the latter below applies to the former.*

Proof of Proposition 4.10. Throughout the proof of the Proposition we assume that the field $K = \mathbb{C}$ or $\mathbb{R}$, and afterwards extend the proposition to an arbitrary algebraically closed field employing the Tarski-Lefschetz principle.

First we construct a $(k+m) \times n$ matrix $M = (M_{j,i})_{1\le j\le k+m, 1\le i\le n}$ with the entries being polynomials over $K = \mathbb{C}$ (or $\mathbb{R}$) in n variables such that for a suitable open subset $U \subset \mathcal{G}$ we have

$$G^{\perp}|_U = T(\mathcal{G})|_U \oplus \mathrm{Ker}(M)|_U \ .$$

In particular, the rank of M equals $k+m$ at all points of U .

Consider a Noether normalisation $\pi : \mathcal{G} \to K^m$ being a restriction of a linear projection $\pi : K^n \to K^m$. Assuming that $K^m \subset K^n$, one can represent $K^n = K^m \oplus K^{n-m}$ with $K^{n-m} = \mathrm{Ker}(\pi)$ and $K^m = \pi(K^n)$. We may assume w.l.o.g. that the first m coordinates are the coordinates of the first summand and the last $n-m$ coordinates are the coordinates of the second summand. We choose in the tangent space to K^n the respective to these X-coordinates a basis of $\frac{\partial}{\partial X_i}$. In abuse of notation we denote $K^{n-m} = T_x(K^{n-m}) \subset T_x(K^n)$ for points $x \in K^{n-m}$.

Take an open subset $\mathcal{U} \subset K^m$ such that for $V := \pi^{-1}(\mathcal{U}) \cap \mathcal{G}$ the dimension of any fiber of the bundle

$$G^{\perp}|_V \cap (V \times K^{n-m})$$

equals $n-k-m$, e. g. any open V over which tangent spaces to $\mathcal{G}$ are mapped onto K^m isomorphically would do. Note that since $\mathcal{G} = \mathcal{G} \cap U_{\mathcal{G}}$ it follows that $\pi(U_{\mathcal{G}} \cap V) = \mathcal{U}$. Then there is a $(k+m) \times n$ matrix M such that

$$\mathrm{Ker}(M)|_V = G^{\perp}|_V \cap (V \times K^{n-m}) \ .$$

Of course we may assume w.l.o.g. that $M_{j,i} = \delta_{j,i}$ for $1 \le j \le m$, $1 \le i \le n$ (where δ denotes the Kronecker's symbol). This provides a required matrix M and a set V .

One can construct (by means of an interpolation in K^{n-m} parametrized by points in $\mathcal{U}'$, see Appendix) rational in the first m (and polynomial in the last $n-m$) coordinates functions $L_j(X)$, $1 \le j \le k$, and an open subset $\mathcal{U}' \subset \mathcal{U}$ such that all L_j , $1 \le j \le k$, vanish on $V' := \pi^{-1}(\mathcal{U}') \cap \mathcal{G}$ (while their denominators do not) and for every point $x \in V'$ equalities

$$\frac{\partial L_j}{\partial X_i}(x) = M_{j+m,i}(x) \ , \ 1 \le j \le k \ , \ m+1 \le i \le n \ ,$$

hold. Multiplying by the common denominator and keeping the same notation for polynomials L_j , $1 \le j \le k$ we conclude that all L_j vanish on $\mathcal{G}$, their differentials $dL_j(x)$, $1 \le j \le k$ are linearly independent for any $x \in V'$ and

$$\bigcap_{1\le j\le k} \mathrm{Ker}(dL_j)|_{V'} \ = \ G^{\perp}|_{V'} \ . \qquad (2)$$

Therefore by shrinking neighbourhood $U_{\mathcal{G}}$ if necessary we may assume w.l.o.g. that $U_{\mathcal{G}} \subset \pi^{-1}(\mathcal{U}')$ and that differentials $dL_1, \dots, dL_k$ are linearly independent at every point in $U_{\mathcal{G}}$.

A collection of varieties forms a *normal crossings* at a point a provided that in appropriate analytic local coordinates centered at this point every variety from this collection and passing through a is a coordinate subspace. Of course this property is open with respect to the choice of points a. Due to our choice above, collection of hypersurfaces $H_j := \{L_j = 0\} \cap U_{\mathcal{G}}$, $1 \leq j \leq k$, forms normal crossings in $U_{\mathcal{G}}$, i. e. at every point of $U_{\mathcal{G}}$. Moreover, since S is irreducible (see Remark 4.9) it follows that the set $\mathrm{Reg}_*(\overline{S})$ of points of $\overline{S} \cap U_{\mathcal{G}}$ at which collection of $\{H_j\}_{1 \leq j \leq k}$ with S forms normal crossings is an open and dense subset of $\mathrm{Reg}(\overline{S} \cap U_{\mathcal{G}})$ (since $\mathrm{Reg}_*(\overline{S}) \supset \mathrm{Reg}(S) \setminus \bigcup_{H_j \not\supset S} H_j \neq \emptyset$). In the sequel we denote $\mathrm{Sing}_*(\overline{S}) := \overline{S} \cap U_{\mathcal{G}} \setminus \mathrm{Reg}_*(\overline{S})$.

To complete the proof of our Proposition we will need a Sard-type Theorem for *singular varieties*. We observe that due to Proposition 2.4 and S being dense in a Lagrangian component of $\mathcal{G}_{n-s}$ (see Remark 4.9) inclusion

$$\overline{T(S)^{\perp}}|_{\mathcal{G}} \subset G|_{\mathcal{G}}$$

holds. In a version of Sard-type Theorem below assuming the latter inclusion and (1) we construct in S a codimension one G-regular subvariety $\hat{S}_{-1} := \hat{S}_{-1}(S) \subset S$ with $(\overline{\hat{S}_{-1}} \cap S) = \hat{S}_{-1}$, such that $\overline{\hat{S}_{-1}} \supset \mathcal{G}$ and inclusion

$$\overline{T(\hat{S}_{-1})^{\perp}}|_{\mathcal{G}} \subset G|_{\mathcal{G}}$$

holds (thus, the pair $\hat{S}_{-1}$, $\mathcal{G}$ behaves similarly to the pair $\hat{S}_{-0} := S$,$\mathcal{G}$, cf. items iii)-vi) below). Our exposition of this Theorem is for the case of $K = \mathbb{C}$ or $\mathbb{R}$ (e. g. items ii) and v)), but there is a straightforward algebraic generalization for an arbitrary K.

In the Sard-type Theorem below $\mathcal{G}$, S , $U_{\mathcal{G}}$ and bundle $T_{\mathcal{G}} := G|_{\mathcal{G}}^{\perp}$ are as constructed above, i. e. satisfy the assumptions of Theorem 5.1. Also functions L_j , $1 \leq j \leq k$, on $U_{\mathcal{G}}$ are as constructed above, i. e. vanish on $\mathcal{G}$ and satisfy (2) with $V' = \mathcal{G}$.

Theorem 5.3 *(***A Sard-type Theorem on singular varieties***)*

For a generic linear combination $L = \sum_{1 \leq j \leq k} c_j L_j$ with coefficients $c = (c_1, \dots, c_k) \in K^k$ the following properties hold:

i) intersection $\{L = 0\} \cap \mathrm{Reg}_*(\overline{S})$ is not empty, dense in $S_{-1} := \{L = 0\} \cap S$ and is smooth of dimension $\dim(S) - 1$ *;*

ii) for any compact (in Euclidean topology on K^n) set $C \subset (\overline{S} \cap U_{\mathcal{G}})$ and all points $a \in \{L = 0\} \cap \mathrm{Reg}_*(\overline{S}) \cap C$ the norms of $d(L|_S)(a) = dL(a)|_{T_a(S)}$ are separated from 0 *by a positive constant (depending on C);*

iii) the boundary $(\overline{S_{-1}} \setminus S_{-1}) \cap U_{\mathcal{G}}$ of set S_{-1} in $U_{\mathcal{G}}$ coincides with $\mathcal{G}$;

iv) $\mathrm{Reg}(S_{-1}) \supset (S_{-1} \cap \mathrm{Reg}(S))$ *and S_{-1} is G-regular in $U_{\mathcal{G}}$;*

v) for every sequence of points in S_{-1} converging to a point $a \in \mathcal{G}$ such that their tangent spaces to S_{-1} converge to a subspace Q in the respective Grassmanian inclusions $T_a(K^n) \supset Q \supset G_a^{\perp}$ are valid and therefore also

$$\overline{T(S_{-1})^{\perp}}|_{\mathcal{G}} \subset G|_{\mathcal{G}} \ ;$$

vi) replacing S_{-1} by an irreducible component $\hat{S}_{-1}$ of S_{-1} whose boundary contains $\mathcal{G}$ the properties iii)-v) remain valid.

Remark 5.4 *For the sake of clarity we include though do not make use of the following:*

• *Of course in ii) of the Lemma above we may equivalently replace "the norms of* $d(L|_S)(a) = dL(a)|_{T_a(S)}$ *are separated from* 0*" by "the angles between gradient* $\operatorname{grad} L(a)$ *of* L *at* a *and tangent spaces* $T_a(S)$ *to* S *at* a *are separated from* $\pi/2$ *".*

• *Due to* S *being irreducible and* $\{L=0\} \cap S \neq S$ *it follows that irreducible components of* S_{-1} *are equidimensional.*

Deduction of Proposition 4.10 from Theorem 5.3. We construct sets $\hat{S}_{-i} := \hat{S}_{-1}(\hat{S}_{-i+1})$, $1 \leq i \leq e := \dim(S) - n + k$, consecutively applying e times Theorem 5.3. Then due to iii) of Theorem 5.3

$$(\overline{\hat{S}_{-e}} \setminus \hat{S}_{-e}) \cap U_{\mathcal{G}} \;=\; \mathcal{G} \tag{3}$$

and, moreover,

$$\overline{T(\hat{S}_{-e})^{\perp}}|_{\mathcal{G}} = G|_{\mathcal{G}} \tag{4}$$

since the Gauss map of $\hat{S}_{-e}$ extends (uniquely) as a continuous map to all of $\mathcal{G}$ (due to v) of Theorem 5.3). Indeed, for every sequence of points from $\hat{S}_{-e}$ converging to a point $a \in \mathcal{G}$ such that their tangent spaces to $\hat{S}_{-e}$ converge to a subspace Q (in the respective Grassmanian), inclusions $T_a(K^n) \supset Q \supset G_a^{\perp}$ hold, but $\dim(Q) = \dim(G_a^{\perp}) = n-k$, and hence $Q = G_a^{\perp}$. Therefore due to (3) $\hat{S}_{-e}$ can be enlarged to an irreducible, G-regular and open in $\overline{\hat{S}_{-e}}$ subset $\mathcal{G}^+ := \hat{S}_{-e} \cup \mathcal{G}$ of dimension $n-k$ satisfying (4), as required in Proposition 4.10. ∎

Proof of Theorem 5.3: **Property vi) follows from iii)-v)** is straightforward using that S_{-1} is open in its closure (see Remark 4.9).

We prove iii) for an arbitrary choice of $c \in K^k$. Inequalities $\dim(\overline{(S_{-1})_a}) \geq \dim(S) - 1 \geq n-k > m = \dim(\mathcal{G})$, where $(S_{-1})_a$ denotes the germ at $a \in \mathcal{G}$ of S_{-1} as an analytic set. Using a similar notation $(\mathcal{G})_a$ for $\mathcal{G}$ it follows that $(\mathcal{G})_a \subset \overline{((\overline{S} \cap \{L=0\}) \setminus \mathcal{G})_a}$. On the other hand, $((\overline{S} \cap \{L=0\}) \setminus \mathcal{G})_a \;=\; ((\overline{S} \setminus \mathcal{G}) \cap \{L=0\})_a \;=\; (S_{-1})_a$, since $(S)_a = (\overline{S} \setminus \mathcal{G})_a$ due to (1). Thus $\mathcal{G} \subset (\overline{S_{-1}} \cap U_{\mathcal{G}})$ and since also $S \cap \mathcal{G} = \emptyset$, it follows that $(\overline{S_{-1}} \setminus S_{-1}) \supset \mathcal{G}$. Using (1) it follows that $\mathcal{G} \;=\; (\overline{S} \setminus S) \cap U_{\mathcal{G}} \;\supset\; (\overline{S_{-1}} \setminus S_{-1}) \cap U_{\mathcal{G}} \;\supset\; \mathcal{G}$, as required in iii).

Properties i) and ii) of Theorem 5.3 imply both iv) and v). Inclusion $\operatorname{Reg}(S_{-1}) \supset \{L=0\} \cap \operatorname{Reg}(S) = (S_{-1} \cap \operatorname{Reg}(S)$ is a straightforward consequence of i) and ii). The remainder is a consequence of the following property: if the limits of two sequences of subspaces of K^n exist, then the limit of the respective intersections of these subspaces also exists and coincides with the intersection of the limits of the sequences, provided that the angles between the respective subspaces in the sequences are separated from 0 by a positive constant.

Thus it remains to prove i) and ii).

Proof of i). We have constructed an open in K^n set $U_{\mathcal{G}}$ and a G-regular irreducible dense subset $S \subset W \cap U_{\mathcal{G}}$ of a Lagrangian component of $\{\mathcal{G}_t\}_{r \leq t \leq k}$ whose boundary $\overline{S} \setminus S = \mathcal{G}$ in $U_{\mathcal{G}}$ (see Remark 4.9). We may assume w.l.o.g. that

$$d(S) := \dim_K(\operatorname{Span}\{L_j|_S\}_{1 \leq j \leq k}) \geq 2 \ ,$$

where Span denotes the K-linear hull of a family of functions. Indeed, since $\dim(S) > n-k$ (Corollary 4.5) it follows that $d(S) > 0$. It remains to exclude the case of $d(S) = 1$. In the latter case we may assume w.l.o.g. that $\dim(\mathrm{Span}\{L_j|_S\}_{2\le j\le k}) \ge 1$ and then change L_1 by adding to it an appropriate generic element of the square of the ideal $I_{\mathcal{G}}$ of all polynomials vanishing on $\mathcal{G}$. This would not change the value of dL_1 at the points of $\mathcal{G}$, but on the other hand $d(S)$ for the new choice of L_1 will increase due to dimension of $I^2_{\mathcal{G}}/I_S$ as a vector space over K being infinite, as required.

We start with an *embedded desingularization* $\sigma : \mathcal{N} \to U_{\mathcal{G}}$ of $\overline{S} \cap U_{\mathcal{G}} \subset U_{\mathcal{G}}$ by means of successive blowings up along smooth admissible centers [13], [1], [3] with 'declared exceptional' hypersurfaces $H_j,\ 1 \le j \le k$, which we may so declare since the latter are smooth and they form normal crossings in $U_{\mathcal{G}}$. In particular, the following properties hold:

0. $\sigma : \mathcal{N} \setminus \sigma^{-1}(\mathrm{Sing}_*(\overline{S})) \to U_{\mathcal{G}} \setminus \mathrm{Sing}_*(\overline{S})$ is an isomorphism;
1. the (so-called) *strict transform* $N := \overline{\sigma^{-1}((\overline{S} \cap U_{\mathcal{G}}) \setminus \sigma(Sing(\sigma)))}$ of $\overline{S} \cap U_{\mathcal{G}}$ is smooth;
2. $\mathrm{Sing}_*(\overline{S}) = \sigma(\mathrm{Sing}(\sigma))$ and $\mathrm{Sing}(\sigma) = \sigma^{-1}(\sigma(Sing(\sigma))) = \cup_{i\ge 1} H_{i+k}$, where each H_{i+k} is a smooth (so-called) *exceptional* hypersurface and in addition each H_{i+k} is the strict transform of the set of the critical points of the successive i-th intermediate blowing up;
3. each $H_i \cap N,\ i \ge 1$, is smooth and $\dim(H_i \cap N) = \dim(N) - 1$ for $i \ge k+1$;
4. the family $\{H_i\}_{i\ge 0}$, where we denote $H_0 := N$, forms a normal crossings in $\mathcal{N}$.

For any hypersurface $\{f = 0\} \subset U_{\mathcal{G}}$ one considers the *strict transform* of $\{f = 0\}$

$$\Lambda_{(f)} = \overline{\sigma^{-1}(\{f = 0\}) \setminus \mathrm{Sing}(\sigma)} \subset \mathcal{N}$$

under map σ.

Remark 5.5 *Due to property 2. above the local equation of $\Lambda_{(f)}$ can be constructed by factoring out from $f \circ \sigma$ the maximal monomial in exceptional hypersurfaces. In particular, assume that f depends on parameter $c \in K^k$ and map $\tilde{\sigma} := \sigma \times id\colon \mathcal{N} \times K^k \to U_{\mathcal{G}} \times K^k$. With $f|_c$ being the evaluation of f at c, hypersurfaces $\Lambda_{(f|_c)} \subset \mathcal{N}$ and $\Lambda_{(f)} \subset \mathcal{N} \times K^k$ being the strict transforms under maps σ and $\tilde{\sigma}$ respectively, it follows that if for a particular value of c hypersurface $\Lambda_{(f)}|_c := \Lambda_{(f)} \cap (\mathcal{N} \times \{c\}) \subset \mathcal{N}$ is smooth then*

$$\Lambda_{(f|_c)} = \Lambda_{(f)}|_c \ , \tag{5}$$

where $\mathcal{N} \times \{c\}$ is identified with $\mathcal{N}$. Of course for a sufficiently generic value of $c \in K^k$ equality (5) holds in any case.

To simplify notation we let $\Lambda_j := \Lambda_{(L_j)}$, $1 \le j \le k$, and $\Lambda := \Lambda_{(L)}$ (all these hypersurfaces being the strict transforms under maps σ and $\tilde{\sigma}$ respectively). Hypersurfaces Λ_j , $1 \le j \le k$, are smooth and together with $\mathrm{Sing}(\sigma)$ form normal crossings in $\mathcal{N}$ due to the choice of admissible centers of blowings up (see e. g. [1] or [3]). In addition, for each j , $1 \le j \le k$, the difference between the divisors of $L_j \circ \sigma$ and Λ_j is the exceptional divisor E_j supported on $Sing(\sigma) = \cup_{i\ge k+1} H_i \subset \mathcal{N}$ (each divisor being of the form $E_j = \sum_i n_{j,i}[H_i]$ and all integers $n_{j,i} \ge 0$).

We now, starting with $\mathcal{N}$, will apply 'combinatorial' blowings up, i. e. with centers of all successive blowings up being the intersections of some of the accumulated exceptional hypersurfaces (possibly including some among Λ_j , $1 \le j \le k$). By means of such blowings up we achieve that the pull back of ideal $\mathcal{I}$ generated by L_j , $1 \le j \le k$, is principal and, moreover, is locally generated at any point a by one of the $L_j \circ \sigma$, $1 \le j \le k$ [1]. (For such $j = j(a)$ it follows that $a \notin \Lambda_j$.) Note that the 'combinatorial part of desingularization' preserves properties 0.-4. (listed above) of embedded desingularization of $\overline{S} \cap U_{\mathcal{G}} \subset U_{\mathcal{G}}$.

It follows that Λ is nonsingular. Indeed, for any point $(x,c) \in \Lambda$ there exists j , $1 \le j \le k$, for which ideal $\mathcal{I} = (L_j \circ \sigma)$ in a neighbourhood of point (x,c) . As a consequence, the partial derivative with respect to c_j of function

$$\lambda := \frac{\sum_{1 \le i \le k} c_i (L_i \circ \sigma)}{L_j \circ \sigma}$$

at (x,c) equals 1 and $\{\lambda = 0\} = \Lambda$.

The standard version of Sard's Theorem implies that for a choice of an appropriate generic $c = (c_1, \ldots, c_k)$ the fiber Λ_c of the restriction to Λ of the natural projection $p \colon \Lambda \to K^k$ is nonsingular in $\sigma^{-1}(U_{\mathcal{G}})$. Note that Sard's Theorem applies because if $x \in \mathcal{N} \setminus \mathrm{Sing}(\sigma)$ and $c \neq 0$ then a straightforward calculation (making use of the linear independence of differentials dL_j , $1 \le j \le k$, in $U_{\mathcal{G}}$) shows the rank of the Jacobian matrix of the projection p at $(x,c) \in \Lambda$ equals k .

To complete the proof of i) we apply Sard's Theorem to the restriction of p to $\Lambda \cap (N \times K^k)$. Note that $\Lambda \cap (N \times K^k) = \{(x,c) \in N \times K^k : \lambda(x,c) = 0\}$ in local coordinates on $N \times K^k$ chosen as above and is nonsingular (since the partial derivative of λ with respect to c_j at (x,c) equals 1). Due to our choice above

$$d(N) := \dim_K(\mathrm{Span}(\{L_j \circ \sigma|_N\}_{1 \le j \le k})) = d(S) \ge 2 \ .$$

Pick $L_{j_1}|_S$, $L_{j_2}|_S$, $1 \le j_1 < j_2 \le k$, being linearly independent over K. It follows that there is a point $x \in N \setminus Sing(\sigma)$ and c_{j_1} , $c_{j_2} \in K$ such that

$$c_{j_1} L_{j_1}(\sigma(x)) + c_{j_2} L_{j_2}(\sigma(x)) = 0 \ , \quad c_{j_1}(dL_{j_1})(\sigma(x)) + c_{j_2}(dL_{j_2})(\sigma(x)) \neq 0$$

holds. Such point $x \in N \setminus Sing(\sigma)$ exists since otherwise it follows that for all $x \in N \setminus Sing(\sigma)$

$$(L_{j_2}(dL_{j_1}) - L_{j_1}(dL_{j_2}))(\sigma(x)) = 0 \ ,$$

which would imply a linear dependence of $L_{j_1}|_S$, $L_{j_2}|_S$ contrary to their choice. Set $c_j = 0$ for all $j \neq j_1$, j_2. Then again by means of a straightforward calculation the rank of the Jacobian at (x,c) of projection $p \colon \Lambda \cap (N \times K^k) \to K^k$ equals k and therefore Sard's Theorem implies that $\Lambda_c \cap N$ is nonsingular for appropriate generic c , where N is identified with $N \times \{c\}$. Since σ is an isomorphism off $\mathrm{Sing}_*(\overline{S})$ (i. e. property 0. of σ) it follows that if $\{L = 0\} \cap \mathrm{Reg}_*(\overline{S}) \neq \emptyset$ then it is a smooth hypersurface of $\mathrm{Reg}_*(\overline{S})$ of dimension $\dim(S) - 1$. To complete the proof of i) it suffices to show that $\Lambda_c \cap N \not\subset \mathrm{Sing}(\sigma) = \cup_{i \ge 1} H_{i+k}$ and that, moreover, $\Lambda_c \cap N \setminus \mathrm{Sing}(\sigma)$ is dense in $\Lambda_c \cap N$.

Both properties follow by specifying an appropriate generic choice of c further, e. g. a choice of c such that Λ_c intersects transversally every $H_J \times \{c\}$ would do, where $H_J = \cap_{j \in J} H_j$ for any acceptable index set $J \subset \{i \ge 0\}$. We achieve the latter by once again applying Sard's Theorem to the restriction of projection p to $\Lambda \cap (H_J \times K^k)$. Of course, for J such that $p(\Lambda \cap (H_J \times K^k))$ is not dense in K^k a generic choice of $c \in K^k$ implies that $\Lambda_c \cap (H_J \times K^k) = \emptyset$, which suffices, and otherwise Sard's Theorem applies and implies for an appropriate generic choice of c the desired transversality, which completes the proof of i).

Proof of ii). We summarize consequences of application of Sard's Theorem in the following

Remark 5.6 *For a choice of an appropriate generic $c \in K^k$ it follows that the family $\{H_i\}_{i\geq 0}$ with Λ_c form a normal crossings in $\mathcal{N} := \mathcal{N} \times \{c\}$.*

For a point $a \in K^n$ denote $\mathcal{L}_a := \mathrm{Span}(\{\mathrm{grad}\, L_j(a)\}_{1\leq j\leq k}) \subset K^n$. Then $\mathcal{L}_a + T_a(S) = K^n$ for all $a \in S$ near any point $b \in \mathcal{G}$. (Indeed, recall that $G_b = \mathcal{L}_b^* := \mathrm{Span}(\{dL_j\}_{1\leq j\leq k})$, due to (2), implying that $k = \dim(\mathcal{L}_b^*) = \dim(\mathcal{L}_a^*)$, and that $T \supset G_b^{\perp}$ if the limit $T = \lim_{a\to b} T_a(S)$ exists, using for the latter inclusion that S is a dense subset of a Lagrangian component of $\{\mathcal{G}_t\}_{r\leq t\leq k}$, see Remark 4.9.) Hence $\dim(\mathcal{L}_a \cap T_a(S)) = k + \dim(S) - n$.

There is a natural isomorphism of

$$\Omega_a := \mathcal{L}_a^*/(\mathcal{L}_a^* \cap T_a(S)^{\perp}) \subset T_a(S)^*$$

with $\mathcal{L}_a \cap T_a(S)$ via realization of the functionals on $T_a(S)$ by means of a scalar product on K^n. In particular, $\dim(\Omega_a) = k + \dim(S) - n$, $\dim(\mathcal{L}_a^* \cap T_a(S)^{\perp}) = n - \dim(S)$ and both dimensions do not depend on a.

We introduce on $\mathcal{L}_a^*$ a metric equivalent to the standard one (over any compact subset of the points $a \in K^n$ with $\dim(\mathcal{L}_a^*) = k$) by declaring $dL_1, \dots, dL_k$ to be an orthonormal basis in $\mathcal{L}_a^*$.

For any point $\tilde{b} \in \Lambda_c \cap \mathrm{Sing}(\sigma) \subset \mathcal{N}$ and points $\tilde{a} \in \mathcal{N} \setminus \mathrm{Sing}(\sigma)$ nearby $\tilde{b}$ we introduce a metric in $T_{\tilde{a}}(\mathcal{N})^*$ as follows. In a neighbourhood of $\tilde{b}$ the smooth variety $\mathcal{N}$ admits a coordinate chart $\mathcal{C}$ with the origin at $\tilde{b}$ and every exceptional hypersurface H intersecting $\mathcal{C}$ by a coordinate hyperplane $\{x_H = 0\}$ of $\mathcal{C}$, unless the intersection is empty (one may use here a traditional complex analytic coordinate chart, or alternatively the notion of an affine 'etale' coordinate chart as in [1], [2]). In a neighbourhood of $\tilde{b}$ the local ideal $\mathcal{I}_{\tilde{b}}$ is generated by a single $L_j \circ \sigma$ for a suitable j (as was achieved by the desingularization above), $1 \leq j \leq k$, and the function $h := \lambda|_c$ has a non-vanishing differential at $\tilde{b}$, since $\Lambda_c \cap N$ is nonsingular due to the choice of c as shown in the proof of i). We shrink the neighbourhood $\mathcal{C}$ so that dh does not vanish at all points of $\mathcal{C}$. In addition, due to Remarks 5.6 and 5.5, we may assume that h is one of the non-exceptional coordinates on $\mathcal{C}$. We define an auxiliary norm on $T_{\tilde{a}}(\mathcal{N})^*$ via imposition of the following:

$$\left\{\frac{dx_H}{x_H}, dx_i\right\}_{H,i} \quad \text{is an orthonormal basis on } T_{\tilde{a}}(\mathcal{N})^* \ , \tag{6}$$

where $\{x_H, x_i\}_{H,i}$ are the coordinates in $\mathcal{C}$ with the former ones corresponding to the exceptional hypersurfaces and the latter $\{x_i\}_i$ being remaining coordinate functions (including function h). A straightforward calculation shows that the Hermitian (Riemannian for $K = \mathbb{R}$) metrics on $\mathcal{C} \setminus \mathrm{Sing}(\sigma)$ that we have introduced by means of (6) do not depend on the coordinate choices preserving exceptional hypersurfaces, i. e. isomorphic for such choices (we do not make use of this fact), for the case of Hermitian metrics cf. [9].

We now will complete the proof of Theorem 5.3 relying on the following lemma, which is stated in the notations of the preceding paragraph.

Lemma 5.7 *The norm of $d(L|_c \circ \sigma)|_{\tilde{a}} \in T_{\tilde{a}}(\mathcal{N})^*$ equals $|L_j \circ \sigma(\tilde{a})|$, which also majorates the norm of the linear map $\sigma_{\tilde{a}}^* \colon \Omega_a \to T_{\tilde{a}}(\mathcal{N})^*$ (up to a constant factor depending only on a choice of $\mathcal{C}$) and where $\tilde{a} \in (\Lambda_c \cap \mathcal{C}) \setminus \mathrm{Sing}(\sigma)$ with $a = \sigma(\tilde{a})$.*

Remark 5.8 *The norm of the map $\sigma_{\tilde{a}}^* \colon \mathcal{L}_a^* \to T_{\tilde{a}}(\mathcal{N})^*$ equals the norm of $\sigma_{\tilde{a}}^* \colon \Omega_a \to T_{\tilde{a}}(\mathcal{N})^*$, because the latter map is the composite of the former one with the natural map $\mathcal{L}_a^* \to \Omega_a$; therefore it suffices to majorate only the norm of the former map by $|L_j(a)|$.*

Lemma 5.7 implies a lower bound depending only on a choice of $\mathcal{C}$ on the norms of $(dL)|_S$ at the points of $\{L = 0\} \cap \mathrm{Reg}_*(\overline{S}) \cap \sigma(\mathcal{C}) = \mathrm{Reg}_*(\overline{S}) \cap \sigma(\Lambda_c \cap \mathcal{C})$. Since σ is a proper map the item ii) of Theorem 5.3 follows. ■

Proof of Lemma 5.7. As mentioned above $L_j \circ \sigma$ coincides (up to an invertible function) with $\prod_{\tilde{b} \in H} x_H^{n_H}$ in $\mathcal{C}$ (w.l.o.g. we may assume that they coincide). Due to Remark 5.5 and using $h(\tilde{a}) = 0$ it follows that

$$d(L|_c \circ \sigma)|_{\tilde{a}} = d((L_j \circ \sigma) \cdot h)|_{\tilde{a}} = L_j(a) \cdot dh|_{\tilde{a}} \ .$$

Due to the choice of the norms on $T_{\tilde{a}}(\mathcal{N})^*$ (see (6)), for $\tilde{a} \in \mathcal{C} \setminus \mathrm{Sing}(\sigma)$, it follows that the norm of $dh|_{\tilde{a}}$ equals 1 . Thus, the norm of $d(L|_c \circ \sigma)|_{\tilde{a}}$ is $|L_j(a)|$, as required.

It remains to bound the norm of $\sigma^*_{\tilde{a}} \colon \mathcal{L}^*_a \to T_{\tilde{a}}(\mathcal{N})^*$ (see Remark 5.8). We observe that the norms of all $d(L_i \circ \sigma)|_{\tilde{a}}$, $1 \le i \le k$, are majorated by $|L_j(a)|$ (up to a constant factor depending only on a choice of $\mathcal{C}$) because $L_j \circ \sigma$ is a common factor of all $L_i \circ \sigma$, $1 \le i \le k$, in $\mathcal{C}$ and the norm of $d(L_j \circ \sigma)|_{\tilde{a}}$ equals $\sqrt{\sum_{\tilde{b} \in H} n_H^2} |L_j(a)|$, see (6). This implies the required upper bound on the norm of $\sigma^*_{\tilde{a}} \colon \mathcal{L}^*_a \to T_{\tilde{a}}(\mathcal{N})^*$, since the latter is bounded by the maximum of the norms of the images of the orthonormal basis $\{dL_i\}_i$ in $\mathcal{L}^*_a$. ■

6 Smoothness of normalisation of a Gauss regular variety

In this section our results are formulated for the case of $K = \mathbb{C}$ or $\mathbb{R}$. One can extend them to an arbitrary algebraically closed field K by means of Tarski-Lefshetz principle.

Theorem 6.1 *Let X be an algebraic (or analytic) variety G-regular at $a \in X$ and let $\phi : \mathcal{X} \to X$ be its normalization. Then $\phi^{-1}(a) \subset \mathrm{Reg}(\mathcal{X})$.*

By definition Nash blowing up of X is the projection $\eta : Y \to X$ of the closure Y of the graph of the Gauss map of X. Variety Y is commonly as well refered to as the Nash blowing up of X.

Lemma 6.2 *If X is G-regular at a then its Nash blowing up $\eta : (Y, \eta^{-1}(a)) \to (X, a)$, where $(Y, \eta^{-1}(a))$ and (X, a) are the germs of Y and of X at $\eta^{-1}(a)$ and a respectively, is dominated by its normalisation, i. e. in germs 'over a' the composite of η with the normalization of Y is the normalisation of X.*

Corollary 6.3 *If X is G-regular at a then its Nash blowing up $\eta : Y \to X$ is finite over $\eta^{-1}(a)$ and the Gauss map of Y is continuous at points of the fiber $\eta^{-1}(a)$.*

Proof of Theorem 6.1. The result of [18] states that η is an isomorphism at a iff X is smooth at a . We use the Noetherian property of the local ring $\mathcal{O}_{X,a}$ of (X, a) and the finiteness of the ring $\mathcal{O}_{\mathcal{X},\phi^{-1}(a)}$ of $(\mathcal{X}, \phi^{-1}(a))$ as a module over $\mathcal{O}_{X,a}$, via ϕ^* . Then it follows from Lemma 6.2 and Corollary 6.3 that for a composite $\psi : Z \to X$ of a finite number of Nash blowings up starting with X the resulting variety Z is smooth at every point $c \in \psi^{-1}(a)$. It also follows that ψ is dominated (over a) by the normalization of X and therefore $\psi : (Z, \psi^{-1}(a)) \to (X, a)$ is finite and, $(Z, \psi^{-1}(a))$ being smooth, is also a normalization of (X, a) , as required. ■

Proof of Lemma 6.2. We may assume w.l.o.g. that $(X,a) \subset (K^n,a)$ and that $\operatorname{codim}(X) =: m$. Then there is a germ of a proper closed subset $(Z,a) \subset (X,a)$ and polynomials (or analytic functions near a respectively) $g_1,\dots,g_m$ such that

$$(X \setminus Z, a) = \{x \in (K^n \setminus Z, a) : g_1(x) = \cdots = g_m(x) = 0\} \neq \emptyset$$

and $\operatorname{rk}\left(\frac{\partial g}{\partial x}\right)(x) = m$ for all $x \in (X \setminus Z) \cap U$, where U is an open in K^n neighborhood of a. In particular, $(X \setminus Z) \cap U \subset \operatorname{Reg}(X)$ and η is an isomorphism over $(X \setminus Z) \cap U$. It suffices to verify that the Gauss map of Y extends to a continuous map on $\eta^{-1}(U)$ for an appropriate U . The Gauss map over $(X \setminus Z) \cap U$ can be defined as sending points x to the tangent spaces $T_x(X)$ of X at x viewed as points of a Grassmanian of all $(n-m)$-dimensional subspaces of K^n , which is a smooth variety. Equivalently (in Plücker coordinates) the Gauss map sends points x to the sequences of all $m \times m$ minors δ_I of matrix $\left(\frac{\partial g}{\partial x}\right)(x)$ as homogeneous coordinates of points in $\mathbb{P}^N$, where $N = \binom{n}{m} - 1$, i. e. $x \to [\cdots : \delta_I : \cdots]$.

The assumption of the continuity at a of the Gauss map of X implies that for one of the minors, say $\delta(x)$, all ratios $|\delta_I(x)/\delta(x)|$ are bounded from above on $(X \setminus Z) \cap U$ (for an appropriate U). It follows that neighborhood $\eta^{-1}(U)$ of Y embeds into respective affine chart $K^n \times K^N := \{\delta \neq 0\}$ of $K^n \times \mathbb{P}^N$ as the closure over $U \subset K^n$ of the graph of the following realization of the Gauss map

$$\Gamma : (X \setminus Z) \cap U \ni x \mapsto (x, \cdots, \delta_I(x)/\delta(x), \cdots) \in K^n \times K^N .$$

The boundedness of ratios $\delta_I(x)/\delta(x)$ implies that each $\delta_I(x)$ is in the integral closure of the ideal generated by $\delta(x)$ in $\mathcal{O}_{(X,a)}$, see a criterium in Appendix to [20], i. e. there are polynomials

$$P_I(\alpha, x) = \alpha^{d_I} + \sum_{1 \le k \le d_I} c_{I,k} \alpha^{d_I - k}$$

such that each coefficient $c_{I,k}$ is in the ideal generated by $(\delta(x))^k$ in the local ring $\mathcal{O}_{(X,a)}$ and $P_I(\delta_I(x), x) = 0$ in $\mathcal{O}_{(X,a)}$. We may assume w.l.o.g. that integers d_I are minimal possible. Then each $\frac{\partial P_I}{\partial z}(\delta_I(x), x) \neq 0$ and therefore each discriminant $\operatorname{disc}_{P_I}(x)$ of $P_I(\alpha, x)$ (with respect to α) does not vanish in $\mathcal{O}_{(X,a)}$. The discriminant of each

$$Q_I(\alpha, x) = \alpha^{d_I} + \sum_{1 \le k \le d_I} \tilde{c}_{I,k}(x) \alpha^{d_I - k} \in \mathcal{O}_{(X,a)}[\alpha] ,$$

where $\tilde{c}_{I,k}(x) := (\delta(x))^{-k} \cdot c_{I,k}$, for $1 \le k \le d_I$, coincides with $(\delta(x))^{-d_I(d_I-1)/2} \cdot \operatorname{disc}_{P_I}(x)$ and therefore does not vanish in $\mathcal{O}_{(X,a)}$. Also $Q_I(\delta_I(x)/\delta(x), x) = 0$ for every I . It follows that for any $b \in \eta^{-1}(a)$ the germ (Y, b) of Y at b is an irreducible component at b of the following equidimensional and reduced space

$$\{(x, w) \in (X, a) \times K^N : Q_I(w_I, x) = 0 \text{ for all } I\} ,$$

and by definition of normalization of (X,a) all ratios $\delta_I(x)/\delta(x) \in \mathcal{O}_{\mathcal{X}, \phi^{-1}(a)}$, i. e. are regular functions on normalization, which implies Lemma 6.2. ∎

Proof of Corollary 6.3. As a straightforward consequence of Lemma 6.2 it follows that map $\eta : \eta^{-1}(X \cap U) \to X \cap U$ dominated over $X \cap U$ by the normalization of X is finite.

Since η is an isomorphism over $(X \setminus Z) \cap U$ and the roots $w_I \in K$ of $Q_I(w_I, x) = 0$ on an open subset $V := \cap_I \{x \in (X \setminus Z) \cap U : \operatorname{disc}_{Q_I}(x) \neq 0\}$ of $X \cap U \subset U \subset K^n$ are (locally) complex analytic functions of x it follows that the equations defining an open in Y set $\eta^{-1}(V)$ in $U \times K^N$ are the equations $g_j = 0$, $1 \le j \le m$, and $Q_I(w_I, x) = 0$, for all I , where w_I are the coordinates of $w \in K^N$. Therefore (in Plücker coordinates) the Gauss map of Y over $\eta^{-1}(V)$ sends points (x, w) from $\eta^{-1}(V)$ to the sequences (as points in $\mathbb{P}^M$, where $M = \binom{n+N}{m+N} - 1$) of all $(m+N) \times (m+N)$ minors $\Delta_J(x, w)$ of matrix

$$\mathcal{M} := \frac{\partial(g, \mathcal{Q})}{\partial(x, w)}(x, w) \ .$$

A straightforward calculation shows that every minor Δ_J which does not include all of the last N columns of the matrix $\mathcal{M}$ vanishes identically and that the remaining $N+1$ minors are proportional to the $N+1$ minors of matrix $\frac{\partial g}{\partial x}(x)$ (the product of all $\frac{\partial Q_I}{\partial w_I}(w_I, x)$ being a common factor) and thus is the value of the Gauss map of X at x in $\mathbb{P}^N$ (in Plücker coordinates). Therefore the continuity of the Gauss map of Y in $\eta^{-1}(X \cap U)$ follows from the continuity of the Gauss map of X in $X \cap U$ provided that the closure of $\eta^{-1}((X \setminus V) \cap U)$ contains $\eta^{-1}(X \cap U)$, which is true due to the finiteness of $\eta : \eta^{-1}(X \cap U) \to X \cap U$ and $X \cap U$ being the closure in U of $(X \setminus V) \cap U$, as required. ∎

7 Complexity of functorial TWG-stratifications

One can construct a chain of bundles of vector spaces $G^{(0)} \subset G^{(1)} \subset \cdots \subset G^{(\rho)} = G$ applying an algorithm for quantifier elimination [10] to proceed from $G^{(p)}$ to $G^{(p+1)}$, $0 \le p < \rho$. This yields an upper bound $R^{(O(1))} d^{n^{O(\rho)}}$ on complexity for construction of G, where $deg(F) < d$ and R majorates the bit-size of the coefficients of components f_i , $1 \le i \le l$, of $F = (f_1, \cdots, f_l)$ assuming that the coefficients are, say, algebraic numbers. Note that $\rho \le 2n$ (see [4]). Then one can construct quasistrata $\mathcal{G}_k$ within the same complexity bound and, if G is Lagrangian, a functorial TWG-stratification as well (see Corollary 3.9). Note that in an example from Subsection 8.2 the index of stabilization ρ grows linearly with n .

We mention that a similar double-exponential complexity bound on stratifications (though without properties of universality nor functoriality) was obtained in [6], [21], [5]. On the other hand, there is an obvious exponential complexity lower bound.

It would be interesting to understand, whether this double-exponential bound is sharp?

8 Examples

8.1 A family of $\mathbf{F} : \mathbf{K^N} \to \mathbf{K}$ which admit functorial TWG-stratifications

First we give an example of a family of polynomials f , i. e. $l = 1$ and $F = (f) : K^N \to K$, that admit functorial TWG-stratifications, which are de facto (in this example) stratifications. (Also, $G^{(1)} = G$, i. e. the index of stabilization $\rho(f) = 1$.)

Let

$$f = f_n = \sum_{1 \le i \le j \le n} A_{i,j} X_i X_j \in K[\{A_{i,j}\}, \{X_i\}].$$

Of course $\mathrm{Sing}(f) = \{X_i = 0\}_{1 \le i \le n}$. For the sake of brevity let B denote the bundle $G^{(1)}$ of the construction in section 2 that corresponds to $F := (f) : K^N \to K$, where $N = n + \binom{n+1}{2}$, and $G := G_F$.

Any nonsingular $n \times n$ matrix C over K induces an isomorphism of $K^N \to K^N$, which for brevity we also denote C , and the latter preserves the rank of quadratic forms. Therefore, for any particular quadratic form $f^{(0)} = \sum_{1\le i\le j\le n} a^{(0)}_{i,j} X_i X_j$ of a rank q the dimension of the fiber $B_{f^{(0)}}$ at a point $a^{(0)} = (\{a^{(0)}_{i,j}\}, \{0\}) \in \mathrm{Sing}(f)$ coincides with the dimension of the fiber $B_{f^{(0)}_q}$ of the quadratic form $f^{(0)}_q = \sum_{1\le i\le q} X_i^2$, e. g. due to Corollary 2.2.

We identify the set of all quadratic forms of rank q with a constructive subset $\mathcal{B}_{k(q)} = (\{a_{i,j}\}, \{0\})$ of $\mathrm{Sing}(f)$. A straightforward calculation shows that $\dim(\mathcal{B}_{k(q)}) = qn - q(q-1)/2$. Once again by means of Corollary 2.2 (and of an appropriate isomorphism $C : K^N \to K^N$) it follows that $\mathcal{B}_{k(q)}$ is smooth and that fibers G_y are of the same dimension $k(q)$ at all the points $y \in \mathcal{B}_{k(q)}$. (Since $l = 1$ Thom stratification of $\mathrm{Sing}(F)$ exists by [15] and therefore due to (1') of Lemma 2.7 inequality $k(q) \le \operatorname{codim} \mathcal{B}_{k(q)}$ holds.) Below we calculate $k(q)$, which would allow us to conclude by making use of Theorem 3.6 that each $\mathcal{B}_{k(q)}$ is Lagrangian and therefore that $B = G$, $\mathcal{B}_{k(q)} = \mathcal{G}_{k(q)}$ and that stratification $\{\mathcal{B}_{k(q)}\}_{k(q)}$, by rank, is a functorial TWG-stratification.

Consider curves $K \ni t \mapsto K^N$ with $f^{(0)}_q$ at $t = 0$ and defined for any $x^{(0)} \in K^n$ as follows:

$$X_i = t^3 x^{(0)}_i \ , \ 1 \le i \le q \ ; \quad X_j = t^2 x^{(0)}_j \ , \ q < j \le n; \quad A_{ii} = 1 \ , \ 1 \le i \le q \ ;$$
$$A_{jj} = t \ , \ q < j \le n \ ; \quad A_{ij} = 0 \ , \ i \ne j$$

A straightforward calculation of the limit along this curve of the normalized differential $df/||df||$ shows that $\sum_{1\le i\le n} x^{(0)}_i dX_i \in B_{f^{(0)}_q}$. Consider similarly limits along curves with the origin at $f^{(0)}_q$ and defined as follows: $A_{ii} = 1$, $1 \le i \le q$, for all the other pairs of i , j with $1 \le i \le j \le n$ we set $A_{ij} = t^2$ and also $X_i = 0$, $1 \le i \le q$ and $X_j = t x^{(0)}_j$, $q < j \le n$. A straightforward calculation implies that the 'coordinate' projection of $B_{f^{(0)}_q}$ to the subspace spanned by $\{dA_{ij}\}_{1\le i\le j\le n}$ contains the image under the degree 2 Veronese map of a point with coordinates $x^{(0)} = (\{0\} , \{x^{(0)}_j\}_{q<j\le n}) \in K^n$. It follows that subspace $B_{f^{(0)}_q}$ of $(K^N)^*$ contains vectors dX_i , $1 \le i \le n$, and $dA_{j,s}$, $q < j \le s \le n$, i. e. $k(q) \ge (n + (n-q)(n-q+1)/2) = \operatorname{codim} \mathcal{B}_{k(q)}$, and therefore $k(q) = \operatorname{codim} \mathcal{B}_{k(q)}$. The latter implying that each (de facto smooth) quasistratum $\mathcal{B}_{k(q)}$ is Lagrangian, $G = B$ and, due to Theorem 3.8 and its Corollary 3.9, partition $\{\mathcal{B}_{k(q)}\}_{k(q)}$, where $0 \le q \le n$, is the functorial Thom-Whitney-a stratification of $\mathrm{Sing}(f)$. We summarize in the following

Proposition 8.1 *For*

$$f = f_n = \sum_{1\le i\le j\le n} A_{i,j} X_i X_j \in K[\{A_{i,j}\}, \{X_i\}]$$

the index of stabilization $\rho(f) = 1$ *and strata* $\mathcal{B}_{k(q)} = \{(\{a_{ij}\}, \{0\}) : rk(f) = q\} \subset \mathrm{Sing}(f)$ *form a functorial Thom-Whitney-a stratification with respect to* f.

8.2 A family of examples of $\mathbf{F_n} : \mathbf{K^{4n+1}} \to \mathbf{K}$ with universal TWG-stratifications and the index of stabilization $\rho(\mathbf{F_n}) = \mathbf{n}$

Let $q(x, y, u, v, w) := u \cdot x^2 + 2w \cdot x \cdot y + v \cdot y^2$ and consider the following polynomials: $q_1 := q(x_1 \ , \ y_1 \ , \ u_1 \ , \ v_1 \ , \ w)$, $q_{k+1} := q(x_{k+1} \ , \ y_{k+1} \ , \ u_{k+1} \ , \ v_{k+1} \ , \ q_k(\cdot))$, $k \ge 1$. Denote

$$f(\vec{x}\ ,\ \vec{y}\ ,\ \vec{u}\ ,\ \vec{v}\ ,\ w) := q_n(\vec{x}\ ,\ \vec{y}\ ,\ \vec{u}\ ,\ \vec{v}\ ,\ w)\ ,$$

where $\vec{x} = (x_1\ ,\ \dots\ ,\ x_n)$ and similarly for $\vec{y}$, $\vec{u}$, $\vec{v}$, i. e. f depends on $N = 4n+1$ independent variables, and let $h_k := u_k \cdot v_k - q_{k-1}^2(\cdot)$, $1 \le k \le n$. Then $f = u_n \cdot x_n^2 + 2q_{n-1} \cdot x_n \cdot y_n + v_n \cdot y_n^2$ and $\mathrm{Sing}(f) = \{x_n = y_n = 0\}$. By making use of Corollary 2.2 and example from Subsection 8.1 it follows that for points $a \in \mathrm{Sing}(f)$ with $dq_{n-1}(a) \neq 0$ the fibers of bundle $G^{(1)}$ are

1. $G_a^{(1)} = \mathrm{Span}\{dx_n\ ;\ dy_n\}$ if $h_n(a) \neq 0$, i. e. $\mathcal{G}_2 = \mathrm{Sing}(f) \setminus \{h_n = 0\}$ off $\{dq_{n-1} = 0\}$;

2. $G_a^{(1)} = \mathrm{Span}\{dx_n\ ;\ dy_n\ ;\ dh_n\}$ if $h_n(a) = 0,\ dh_n(a) \neq 0$, i. e. off $\{dq_{n-1} = 0\}$ quasistratum $\mathcal{G}_3 = \mathrm{Sing}(f) \cap \{h_n = 0\} \setminus \{dh_n \neq 0\}$;

3. $G_a^{(1)} = \mathrm{Span}\{dx_n\ ;\ dy_n\ ;\ du_n\ ;\ dv_n\ ;\ dq_{n-1}\}$, if $h_n(a) = 0$, $dh_n(a) = 0$, i. e. $\mathcal{G}_5 = \mathrm{Sing}(f) \cap \{h_n = 0\ ,\ dh_n = 0\}$ off $\{dq_{n-1} = 0\}$.

4. In the cases 1. and 2. fibers $G_a^{(1)} = (\overline{G^{(0)}})_a$, but in the case 3. fibers $G_a^{(1)} \neq (\overline{G^{(0)}})_a = \big\{\omega = U_n du_n + V_n dv_n + Q_{n-1} dq_{n-1} + X_n dx_n + Y_n dy_n\ :\ U_n \cdot V_n = (Q_{n-1}/2)^2\big\}$, where ω denotes a 1-form at a . Denote $D_1 := \mathrm{Span}\{dx_n\ ;\ dy_n\ ;\ du_n\ ;\ dv_n\}$. Note that

$$df = x_n^2 du_n + y_n^2 dv_n + 2x_n y_n dq_{n-1} + 2(u_n x_n + q_{n-1} y_n) dx_n + 2(q_{n-1} x_n + v_n y_n) dy_n\ .$$

Results above rely on elementary calculations of Subsection 8.1 summarized below: $h_n = \det \begin{pmatrix} u_n & q_{n-1} \\ q_{n-1} & v_n \end{pmatrix}$ and for any sequence of points from K^N converging to a point $a \in \mathrm{Sing}(f)$ the following holds

i) the size of $\{\frac{\partial f}{\partial x_n}\ ;\ \frac{\partial f}{\partial y_n}\}$ dominates $\{x_n^2\ ,\ y_n^2\ ,\ 2x_n \cdot y_n\}$ at a if $h_n \not\to 0$,

ii) the limits of $df/||df||$ are the 1-forms $\omega = U_n du_n + V_n dv_n + Q_{n-1} dq_{n-1} + X_n dx_n + Y_n dy_n$ with $U_n \cdot V_n = Q_{n-1}^2/4$, since the coefficients of df at du_n , dv_n , dq_{n-1} satisfy $x_n^2 \cdot y_n^2 = (2x_n \cdot y_n)^2/4$.

When $h_n(a) = 0$ the latter also follows from the orthogonality of $\omega \in G_a^{(1)}$ to $T_a(\{h_n = 0\})$ (see (1') of Lemma 2.7) and $dh_n = v_n \cdot du_n + u_n \cdot dv_n + 2q_{n-1} \cdot dq_{n-1}$, implying that ω is proportional to dh_n , while $u_n \cdot v_n = q_{n-1}^2$ for points in $\{h_n = 0\}$.

We now turn to a simple, but crucial observation that the coefficients of df at du_n , dv_n , dq_{n-1} satisfy inequality $\sqrt{|x_n|^2 + |y_n|^2} \ge (\sqrt{2})^{-1} \cdot |2x_n \cdot y_n|$ and therefore the limits of $df/||df||$ evaluated at points from K^N that converge to $\mathrm{Sing}(f) \cap \{dq_{n-1} = 0\}$ are the 1-forms with vanishing coefficients at all differentials of the independent variables on which $q_{n-1}(\cdot)$ depends. In particular, combining with the preceding summary of the arguments of Subsection 8.1 properties 1. and 2. follow without making assumption $dq_{n-1}(a) \neq 0$ and also

5. $G_a^{(1)} = D_1$ for $a \in Z_{n-1} := \mathrm{Sing}(f) \cap \{h_n = 0,\ dh_n = dq_{n-1} = 0\} \subset \{q_{n-1} = 0\}$

holds.

Summarizing $\mathcal{G}_2 = \mathrm{Sing}(f) \setminus \{h_n = 0\}$, $\mathcal{G}_3 = \mathrm{Sing}(f) \cap \{h_n = 0\ ,\ dh_n \neq 0\}$ and with $\mathcal{G}_5' := \mathrm{Sing}(f) \cap \{h_n = 0\ ,\ dh_n = 0\ ,\ dq_{n-1}(a) \neq 0\}$ bundle $G^{(1)}|_{\mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_5'} = G|_{\mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_5'}$. Also $\mathcal{G}_5' = \{x_n = y_n = u_n = v_n = q_{n-1} = 0\ ,\ dq_{n-1} \neq 0\}$, and
$Z_{n-1} = \{x_n = y_n = u_n = v_n = x_{n-1} = y_{n-1} = 0\} = \mathrm{Sing}(f) \setminus (\mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_5')$.

Detour. The two Remarks-Examples below are straightforward consequences of the latter observation and the preceding it summary of the arguments of Subsection 8.1.

Remark 8.2 *With notations* $G = G_{\tilde f}$, $G^{(p)} = G^{(p)}_{\tilde f}$ *for a function*

$$\tilde f := u \cdot x^2 + 2w^2 \cdot x \cdot y + v \cdot y^2$$

depending on 5 *variables the following hold:*
inequality $\dim G^{(1)}_a \leq 4$ *for all* $a \in Sing(\tilde f)$; *bundles* G *and* $G^{(1)}$ *coincide; quasistrata* $\mathcal{G}_2 = \{x = y = 0\ ,\ u \cdot v - w^4 \neq 0\}$, $\mathcal{G}_3 = \{x = y = 0\ ,\ u \cdot v - w^4 = 0\ ,\ (u\ ,\ v) \neq 0\}$ *and* $\mathcal{G}_4 = \{0\}$ *are smooth and form Thom-Whitney-a stratification* $\mathcal{S}$ *of* $Sing(\tilde f)$, *but quasistratum* $\mathcal{G}_4$ *is not Lagrangian (*$\dim \mathcal{G}_4 = 0 < 5 - 4$ *!). Also,* $\overline{G|_{\mathcal{G}_2}}$ *and* $\overline{G|_{\mathcal{G}_3}}$ *are* 5*-dimensional irreducible components of* G *and* $G|_{\mathcal{G}_4}$ *is in the closure of* $G|_{\mathcal{G}_3}$.

Remark 8.3 *Let non-zero polynomial* $g \in K[z_1, \dots, z_m]$ *and* $f_g := \tilde f(u, v, x, y, g(z))$, *where* $\tilde f$ *is from the preceding Remark. Denote* $G := G_{f_g}$, $G^{(p)} := G^{(p)}_{f_g}$. *Then for polynomial* f_g *depending on* $m + 4$ *variables the following hold:*
inequality $\dim G^{(1)}_a \leq 4$ *for all* $a \in Sing(f_g)$; *bundles* G *and* $G^{(1)}$ *coincide; the quasistrata are* $\mathcal{G}_2 = \{x = y = 0\ ,\ u \cdot v - g(z)^4 \neq 0\}$, $\mathcal{G}_3 = \{x = y = 0\ ,\ u \cdot v - g(z)^4 = 0\ ,\ (u,\ v) \neq 0\}$ *and* $\mathcal{G}_4 = \{x = y = u = v = g(z) = 0\}$*; only quasistratum* $\mathcal{G}_4$ *is not Lagrangian; the irreducible components* $\overline{G|_{\mathcal{G}_2}}$ *and* $\overline{G|_{\mathcal{G}_3}}$ *of* G *are* $(m+4)$*-dimensional and* $G|_{\mathcal{G}_4}$ *is in the closure of* $G|_{\mathcal{G}_3}$.
Curiously, an arbitrarily chosen hypersurface $\{g = 0\}$ **appears as a quasistratum**.

We now turn to calculation of fibers of $G^{(2)}$ for f . Note that $dq_{n-1} - 2x_{n-1}y_{n-1}dq_{n-2} =$

$$x^2_{n-1}du_{n-1} + y^2_{n-1}dv_{n-1} + 2(u_{n-1}x_{n-1} + q_{n-2}y_{n-1})dx_{n-1} + 2(q_{n-2}x_{n-1} + v_{n-1}y_{n-1})dy_{n-1}$$

and bundles $G = G^{(2)} = G^{(1)}$ off $Z_{n-1} \subset \{x_{n-1} = y_{n-1} = 0\}$. It follows by making use of Corollary 2.2 and of the calculations like in the summary of the arguments of Subsection 8.1 that for points b from $\mathcal{G}'_5$ converging to a point $a \in Z_{n-1} \subset \{q_{n-1} = 0\ ,\ dq_{n-1} = 0\}$ with $dq_{n-2} \neq 0$ the span of the limits of the 1-forms from the fibers G_b of G , which includes the limits of $dq_{n-1}/||dq_{n-1}||$, coincides with the fibers of bundle $G^{(2)}$, namely:

1'. $G^{(2)}_a = \mathrm{Span}\{dx_{n-1}\ ;\ dy_{n-1}\} \oplus D_1$ if $h_{n-1}(a) \neq 0$, i. e. $\mathcal{G}_6 = Z_{n-1} \setminus \{h_{n-1} = 0\}$ off $\{dq_{n-2} = 0\}$;

2'. $G^{(2)}_a = \mathrm{Span}\{dx_{n-1}\ ;\ dy_{n-1}\ ;\ dh_{n-1}\} \oplus D_1$ if $h_{n-1}(a) = 0,\ dh_{n-1}(a) \neq 0$, i. e. off $\{dq_{n-2} = 0\}$ quasistratum $\mathcal{G}_7 = Z_{n-1} \cap \{h_{n-1} = 0\} \setminus \{dh_{n-1} \neq 0\}$;

3'. $G^{(2)}_a = \mathrm{Span}\{dx_{n-1}\ ;\ dy_{n-1}\ ;\ du_{n-1}\ ;\ dv_{n-1}\ ;\ dq_{n-2}\} \oplus D_1$, if $h_{n-1}(a) = 0$, $dh_{n-1}(a) = 0$, i. e. $\mathcal{G}_9 = Z_{n-1} \cap \{h_{n-1} = 0\ ,\ dh_{n-1} = 0\}$ off $\{dq_{n-2} = 0\}$.

4'. In the cases 1'. and 2'. fibers $G^{(2)}_a = (\overline{G^{(1)}})_a$, but in the case 3'. fibers $G^{(2)}_a \not\subset (\overline{G^{(1)}})_a$ and the latter consists of all 1-forms $\omega \in G^{(2)}_a$ with coefficients U_{n-1} , V_{n-1} , Q_{n-2} at du_{n-1} , dv_{n-1} , dq_{n-2} that satisfy equation $U_{n-1} \cdot V_{n-1} = (Q_{n-2}/2)^2$. Denote $D_2 := \mathrm{Span}\{dx_{n-1}\ ;\ dy_{n-1}\ ;\ du_{n-1}\ ;\ dv_{n-1}\} \oplus D_1$.

Once again, due to the observation that the coefficient of dq_{n-1} at dq_{n-2} is dominated by its coefficients at du_{n-1} , dv_{n-1} , it follows that for points $b \in \mathrm{Sing}(f)$ converging to a point $a \in \{dq_{n-2} = 0\}$ the limits of the 1-forms from fibers $G^{(1)}_b$, which include the limits of $dq_{n-1}/||dq_{n-1}||$, consist only of 1-forms with vanishing coefficients at all differentials of the independent variables on which q_{n-2} depends. In particular, properties 1'. and 2'. follow without making assumption $dq_{n-2}(a) \neq 0$ and the fiber of bundle $G^{(2)}$ at a is

5'. $G^{(2)}_a = D_2$ for $a \in Z_{n-2} := Z_{n-1} \cap \{h_{n-1} = 0\ ,\ dh_{n-1} = dq_{n-2} = 0\} \subset \{q_{n-2} = 0\}$.

Summarizing $\mathcal{G}_5 = \mathcal{G}'_5$, $\mathcal{G}_6 = Z_{n-1} \setminus \{h_{n-1} = 0\}$, $\mathcal{G}_7 = Z_{n-1} \cap \{h_{n-1} = 0 \ , \ dh_{n-1} \neq 0\}$ and with $\mathcal{G}'_9 := Z_{n-1} \cap \{h_{n-1} = 0 \ , \ dh_{n-1} = 0 \ , \ dq_{n-2} \neq 0\}$ bundle $G^{(2)}|_{\mathcal{G}_6 \cup \mathcal{G}_7 \cup \mathcal{G}'_9} = G|_{\mathcal{G}_6 \cup \mathcal{G}_7 \cup \mathcal{G}'_9}$. Also $\mathcal{G}'_9 = Z_{n-1} \cap \{u_{n-1} = v_{n-1} = q_{n-2} = 0 \ , \ dq_{n-2} \neq 0\}$, and $Z_{n-2} = Z_{n-1} \cap \{u_{n-1} = v_{n-1} = x_{n-2} = y_{n-2} = 0\} = Z_{n-1} \setminus (\mathcal{G}_6 \cup \mathcal{G}_7 \cup \mathcal{G}'_9)$.

Thus $G^{(1)} \neq G^{(2)}$ and $G = G^{(2)}$ off Z_{n-2} . Calculation of fibers of $G^{(p)}$, $p > 2$ for points from Z_{n-2} is similar (recursively on p), in particular implying that $\mathcal{G}_9 = \mathcal{G}'_9$. Summarizing

Proposition 8.4 *Quasistrata $\{\mathcal{G}_r\}_r$ for polynomial f (in $4n+1$ independent variables) are smooth, Lagrangian, form a Thom-Whitney-a stratification and hence a universal TWG-stratification. The index of stabilization $\rho(f)$ of f equals n .*

8.3 Example of $\mathbf{F} : \mathbf{K}^5 \to \mathbf{K}$ with no universal TWG-stratification

For $\tilde{f}$ from Remark 8.2 we have shown that there is a non Lagrangian quasistratum and therefore $\mathrm{Sing}(\tilde{f})$ by our main Theorem 4.1 does not admit a universal TWG-stratification. For polynomial $\tilde{f}$ we will reprove this claim illustrating the proof of Theorem 4.1. In this example $\mathcal{G} = \mathcal{G}_4$, construction of $\mathcal{G}^+$ is elementary and we provide it explicitly (cf. Section 5). We choose $\mathcal{G}^+$ to be a curve defined parametrically by $\{x = y = 0 \ , \ u = v = t^2 \ , \ w = t\}$. Then partition of $\mathrm{Sing}(\tilde{f})$ by sets $\mathcal{B}_2 := \mathcal{G}_2$, $\mathcal{B}_3 := \mathcal{G}_3 \setminus \mathcal{G}^+$, $\mathcal{B}_4 := \mathcal{G}^+$ forms Thom-Whitney-a stratification $\tilde{\mathcal{S}}$ with the associated bundle $B(\tilde{\mathcal{S}}) \neq B(\mathcal{S})$.

Finally we show that there does not exist a universal TWG-stratification with respect to $\tilde{f}$. Assume the contrary, say $\mathcal{S}^{(0)}$ is a universal TWG-stratification. Denote by $B(\mathcal{S}^{(0)})$ its bundle of vector spaces. Proposition 2.4 and Proposition 3.1 imply that $G \subset B(\mathcal{S}^{(0)}) \subset (B(\mathcal{S}) \cap B(\tilde{\mathcal{S}}))$. It follows that $G_a = B(\mathcal{S}^{(0)})_a = B(\mathcal{S})_a$ for any point $a \in \mathcal{B}_2 \cup \mathcal{B}_3$, while $G_0 = B(\mathcal{S}^{(0)})_0 = B(\tilde{\mathcal{S}})_0$ is 4-dimensional and is orthogonal to vector $\frac{\partial}{\partial w} \in T_0(K^5)$. (On the other hand $B(\mathcal{S})_0 = (K^5)^*$). Therefore, $\mathcal{S}^{(0)}$ being universal should coincide with $\mathcal{S}$, but the origin 0 is not a Lagrangian stratum of $\mathcal{S}^{(0)}$. Thus our assumption leads to a contradiction. Summarizing, we obtain the following proposition.

Proposition 8.5 *There is no universal TWG-stratification with respect to the polynomial $\tilde{f} = u \cdot x^2 + 2w^2 \cdot x \cdot y + v \cdot y^2$.*

8.4 Multiplicities of roots and another functorial TWG-stratification

Let

$$f := f_{q+2} = \sum_{0 \le i \le q} A_i X^i Y^{q-i} \in K[A_0, \ldots, A_q, X, Y] \ ,$$

where $([A_0 : \cdots : A_q], X, Y) \in \mathbb{P}^q(K) \times K^2$. In particular, in this example for every affine chart $\{A_i \neq 0\} \simeq K^q \times K^2$, $0 \le i \le q$ of $\mathbb{P}^q(K) \times K^2$ we consider mapping $F := f : K^n \to K$, where $n := q + 2$. Then $\mathrm{Sing}(F)$ admits Thom stratification and (ii) of Theorem 3.6 applies provided that all irreducible components of $\mathcal{G}_k$, $n - \dim(\mathrm{Sing}(F)) \le k \le n$ are of dimension $n - k$, which we show below.

Similarly to the preceding examples $\mathrm{Sing}(f) = \{X = Y = 0\}$. Here, in the original notations of Section 2, we prove for $G := G_{f_n}$ (and $G^{(p)} := G^{(p)}_{f_n}$) that index of stabilization $\rho(f_n) = 2$, i. e. that $G^{(1)} \neq G^{(2)} = G$, bundle $G = G_{f_n}$ is Lagrangian and that $\{\mathcal{G}_{k+2}\}_{0 \le k \le q/2}$ is a universal (and hence functorial) TWG-stratification with respect to f_n .

Let us fix a point $a^{(0)} = ([a_0^{(0)} : \cdots : a_q^{(0)}], 0, 0) \in \mathrm{Sing}(f)$, for the time being, then polynomial

$$f^{(0)} = \sum_{0 \le i \le q} a_i^{(0)} X^i Y^{q-i} = \prod_j (b_j X - c_j Y)^{m_j}. \tag{7}$$

We first verify that for each factor $b_j X - c_j Y$ with the multiplicity $m_j \geq 2$ the fiber of the closure $(\overline{G^{(0)}})_{a^{(0)}}$ contains

$$v_j := v([c_j : b_j]) = \sum_{0 \le i \le q} c_j^i b_j^{q-i} dA_i.$$

Consider a line defined (parametrically) as follows:

$$A_i(t) = a_i^{(0)} \ , \ 0 \le i \le q \ ; \ X(t) = c_j t \ , \ Y(t) = b_j t \ .$$

Then $\lim_{t \to 0} df/||df||$ along this line equals v_j . Conversely, let $v = \sum_{0 \le i \le q} h_i dA_i + c dX + b dY$ with a non-vanishing $(h_0, \ldots, h_q) \neq 0$ being the $\lim_{t \to 0} df/||df||$ along a curve

$$(\{A_i(t)\}_{0 \le i \le q}, X(t), Y(t)) \subset \mathbb{P}^q(K) \times K^2$$

with the origin at $a^{(0)}$. Making a suitable K-linear homogeneous transformation C of the 2-dimensional plane and applying Corollary 2.2 we may assume w.l.o.g. that $ord_t(X(t)) > ord_t(Y(t))$ and it suffices to show that $X^2 | f^{(0)}$. Assume otherwise, then

$$\mathrm{ord}_t \left\{ \frac{\partial f^{(0)}}{\partial X}, \frac{\partial f^{(0)}}{\partial Y} \right\} = (q-1)\, \mathrm{ord}_t(Y(t)) < \mathrm{ord}_t(X^i Y^{q-i}),\ 0 \le i \le q \ ,$$

which contradicts to $(h_0 \ , \ldots \ , h_q) \neq 0$.

Since vectors $\{v_j\}_j$ form a van-der-Mond matrix and therefore are linearly independent, it follows

Lemma 8.6 *For any point $a^{(0)} \in \mathrm{Sing}(f)$ fiber $(G^{(1)})_{a^{(0)}}$ of bundle $G^{(1)}$ of vector spaces coincides with the linear hull of vectors dX, dY and $\{v_j\}_j$ for all j with the multiplicity of the factor $b_j X - c_j Y$ in $f^{(0)}$ being $m_j \geq 2$ and, moreover, $\dim((G^{(1)})_{a^{(0)}}) - 2$ being the number of such j .*

For every $v = v([c : b])$ let $\mathcal{D}^{(l)}(v)$ denote the linear hull of

$$\left\{ \frac{\partial^l v}{\partial c^i \partial b^{l-i}} \right\}_{0 \le i \le l} .$$

Then $\{v\} = \mathcal{D}^{(0)}(v) \subset \mathcal{D}^{(1)}(v) \subset \cdots$ due to the Euler's formula. W.l.o.g. we may assume that $b = 1$ (if $b = 0$ we exchange the roles of b and c) and then $\mathcal{D}^{(l)}(v)$ is the linear hull of the derivatives $\{\frac{\partial^i v}{\partial c^i}\}_{0 \le i \le l}$, implying $\dim(\mathcal{D}^{(l)}(v)) = l + 1$, $0 \le l \le q$.

Below we calculate the limit $\lim_{t \to 0} (G^{(1)})_{a^{(t)}}$. To that end we consider a curve $\{a^{(t)}\}_t \subset \mathrm{Sing}(f)$ with the origin at $a^{(0)}$, and assume w.l.o.g. that $a_q^{(t)} = 1$ for all t . Due to Lemma 8.6 we may assume (also w.l.o.g.) that for any $t \neq 0$ the multiplicity of every factor of polynomial

$f^{(t)} = \sum_{0\le i\le q} a_i^{(t)} X^i Y^{q-i}$ does not exceed 2 and these multiplicities are independent on $t \neq 0$. We may factorise

$$f^{(t)} = \prod_j \prod_p (X - (c_j + e_{j,p}(t))Y)^{m_{j,p}} \ ,$$

where $1 \le m_{j,p} \le 2$ and $e_{j,p}(t)$ are the appropriate algebraic functions of t with $e_{j,p}(0) = 0$ for all j , p . Then $\sum_p m_{j,p} = m_j$ for each j (see (7)) and we denote $\overline{m_j} = \sum_p [m_{j,p}/2]$, where by $[m_{j,p}/2]$ we mean the integral part of $m_{j,p}/2$. Due to Lemma 8.6 it follows that $\dim((G^{(1)})_{a^{(t)}}) = \sum_j \overline{m_j} + 2$ for any $t \neq 0$ and that collection

$$\{v([c_j + e_{j,p}(t) : 1])\}_{m_{j,p}=2} \cup \{dX,\ dY\} \tag{8}$$

is a basis of the fiber $(G^{(1)})_{a^{(t)}}$.

We claim that

$$\lim_{t\to 0} (G^{(1)})_{a^{(t)}} = \bigoplus_j \mathcal{D}^{(\overline{m_j}-1)}(v([c_j : 1])) \oplus \mathrm{Span}\{dX,\ dY\} \ . \tag{9}$$

To that end we observe that the right-hand side of (9) is indeed the direct sum of the vector spaces due to the Hermite's interpolation (which interpolates uniquely a polynomial in terms of the values of its several consecutive derivatives at the given points, cf. Appendix). Therefore the dimension of the right-hand side equals $\sum_j \overline{m_j} + 2$ and to complete the proof of (9) it suffices to verify that the left-hand side of (9) contains its right-hand side.

To this end fix j , denote $m := \overline{m_j}$ and let

$$E^{(i)} := (\{e^i_{j,p}(t)\}_{1\le p\le m})^T \in K^m \ , \ i \ge 0 \ ,$$

where all p satisfy $m_{j,p} = 2$ (see (8)). Let E be the $m \times m$ van-der-Mond matrix with the columns $E^{(i)}$, $0 \le i \le m-1$. Consider an arbitrary $w = (w_0\ , \ \dots\ , \ w_{m-1}) \in K^m$ and let $u := (\{u_p\}_{1\le p\le m}) := wE^{-1}$. Since $E^{-1}E^{(i)}(0) = 0$ for every $i \ge m$ it follows for $u^{(i)}(t) := u \cdot E^{(i)}(t)$ that $u^{(i)}(0) = 0$. Therefore

$$\sum_{1\le p\le m} u_p v([c_j + e_{j,p}(t) : 1]) = \sum_{0\le s\le m-1} \frac{w_s}{s!} \frac{d^s v([c_j : 1])}{dc^s} + \sum_{m\le i\le q} \frac{u^{(i)}}{i!} \frac{d^i v([c_j : 1])}{dc^i} \ .$$

Claim (9) follows by letting $t = 0$ in the right-hand side of the latter (in view of the choice of w as an 'arbitrary' in K^m).

We now specify the choice of curve $\{a^{(t)}\}_t$ so that for every j equality $\overline{m_j} = [m_j/2]$ holds (see (7)), in other words $m_{j,p} = 2$ for $\overline{m_j}$ number of p's and, moreover, in the case when number m_j is odd that $m_{j,p_0} = 1$ for a single p_0 . Then due to (9) it follows

Proposition 8.7 *For any point $a^{(0)} \in \mathrm{Sing}(f)$ the fiber*

$$(G^{(1)})_{a^{(0)}} = \bigoplus_j \mathcal{D}^{([m_j/2]-1)}(v([c_j : 1])) \oplus \mathrm{Span}\{dX,\ dY\}$$

is a vector space of the dimension $\sum_j [m_j/2] + 2$ (see (7)). In particular, bundle $G := G_f = \overline{G^{(1)}}$.

Finally, we establish that G is Lagrangian. For every k , $0 \le k \le q/2$, let

$$\mathcal{G}^{(0)}_{k+2} := \{a^{(0)} \in \mathrm{Sing}(f) : f^{(0)} = \prod_{1\le j\le k} (X - c_j Y)^2 \cdot \prod_{k<s\le q-k} (X - c_s Y)\} ,$$

i. e. $f^{(0)}$ has k factors of multiplicity 2 and $q-2k$ factors of multiplicity 1 . Proposition 8.7 implies that $\mathcal{G}^{(0)}_{k+2} \subset \mathcal{G}_{k+2}$ (see Definition 3.3) and, moreover, that $\mathcal{G}^{(0)}_{k+2}$ is dense in $\mathcal{G}_{k+2}$. On the other hand, $\mathcal{G}^{(0)}_{k+2}$ is open and is isomorphic to the set of all orbits of the group $\mathrm{Sym}(k) \times \mathrm{Sym}(q-2k)$ acting on a set

$$\mathcal{Z} := K^{q-k} \setminus (\bigcup_{1\le i<j\le q-k} \{Z_i = Z_j\}) ,$$

where $\mathrm{Sym}(k)$ permutes the first k coordinates Z_1 , … , Z_k and $\mathrm{Sym}(q-2k)$ permutes the last $q-2k$ coordinates Z_{k+1} , … , Z_{q-k} . It follows $\dim(\mathcal{G}^{(0)}_{k+2}) = q-k$. Moreover, $\mathcal{G}^{(0)}_{k+2} = H(\mathcal{Z})$, where H maps Z_1 , … , Z_k to double roots of $f^{(0)}$ and Z_{k+1} , … , Z_{q-k} to single roots. It follows that $\mathcal{G}^{(0)}_{k+2}$ is irreducible. Finally, since in this example $\mathrm{Sing}(F)$ admits Thom stratification, quasistrata $\mathcal{G}_{k+2}$ are irreducible and of dimension $n-k-2$ item (ii) of Theorem 3.6 and hence Corollary 3.9 apply and imply the following

Theorem 8.8 *Index of stabilization $\rho(f_{q+2}) = 2$, bundle $G = G_{f_{q+2}}$ is Lagrangian and $\{\mathcal{G}_{k+2}\}_{0\le k\le q/2}$ is a functorial TWG-stratification with respect to f_{q+2} .*

9 Appendix. Complexity of extension to a Gauss regular subvariety with a prescribed tangent bundle over singularities

Here we estimate complexity of an algorithm of extending of a (smooth) singular locus of an algebraic variety to a Gauss regular subvariety with a prescribed tangent bundle over the singularities of the variety (see Section 5). We follow the notations of Sections 4, 5 with an exception that we use K rather than $\mathbb{C}$. The input for this algorithm is a family of polynomials g_p , $M_{j+m,i+m} \in K_0[X_1, \dots, X_n]$ with $p \ge 0$, i , j for a subfield $K_0 \subset K$. For the sake of complexity bounds we assume that elements of K_0 can be represented algorithmically, e. g. one may use here the field of rational or algebraic numbers in place of K_0 , cf. [10]. We assume the following representation for an algebraic variety $S = \{g_0 \cdot g_1 \ne 0 , \ g_p = 0\}_{p\ge 2}$ and its (smooth) singular locus $\mathcal{G} = \{g_0 \ne 0, \ g_p = 0\}_{p\ge 1}$, which also is its boundary in $\{g_0 \ne 0\}$ (see Remark 4.9). The output of the algorithm is a Gauss regular subvariety $\mathcal{G}^+$ of $\overline{S} \cap \{g_0 \ne 0\}$ (see Proposition 4.10).

Basically the algorithm consists of 3 subroutines. The first one is choosing a Noether normalisation π for $\mathcal{G}$. The second one is an *implicit parametric interpolation* of polynomials L_j from Section 5. (We refer to the latter as implicit because the interpolation data are given over the subsets of points from $\mathcal{G}$ and thus the data appear implicitly.) The third subroutine is a construction of $\mathcal{G}^+$ proper. To this end we may exploit a choice of algebraically independent coefficients $c_1, \dots, c_k$ at each consecutive application of Theorem 5.3 and thereafter to construct an irreducible component containing $\mathcal{G}$ of the resulting intersection with $\overline{S} \cap \{g_0 \ne 0\}$ (cf. vi) of Theorem 5.3 and the deduction of Proposition 4.10). Complexity bounds for Noether normalisation and for constructing irreducible components one may find in [19], and in [10] respectively. We observe that the third subroutine depends only on the complexity of finding irreducible components. We therefore focus on an algorithm for a parametric interpolation. In fact, we design an algorithm for interpolation over the parameters varying in K^m , whereas for the purposes of Section 5 it suffices to have the parameters varying in an open subset $\mathcal{U}' \subset K^m$, which would have simplified the algorithm.

To formulate the complexity bounds we assume that $\deg(g_p) < \delta$, $\deg(M_{j+m,i+m}) < \Delta$ for all p , i , j and the total number of bits in representation of the coefficients (in K_0) of polynomials g_p , $M_{j+m,i+m}$ does not exceed R. Our main result here is the following

Proposition 9.1 *One can interpolate polynomials L_j as required in Section 5 and, moreover, under assumptions listed in the preceding paragraph* $\deg(L_j) < \Delta\delta^{O(n)}$ *is a bound on the degrees of the resulting L_j . Complexity bound for this interpolation algorithm is* $(R\Delta^n\delta^{n^2})^{O(1)}$.

Combining with the complexity bounds for the first and the third subroutines it follows

Corollary 9.2 *The complexity of the algorithm constructing $\mathcal{G}^+$ is bounded by*

$$R^{O(1)}(\Delta\delta)^{n^{O(1)}}.$$

Proof of Proposition 9.1. We first consider a *non-parametrical* interpolation.

Lemma 9.3 *Let $v_1, \dots, v_t \in K^{n-m}$ and $w_q^{(i)} \in K$, $1 \le q \le t$, $0 \le i \le n-m$. There exists a polynomial $A \in K[X_{m+1}, \dots, X_n]$ of $\deg(A) < 2t(n-m)$ such that*

$$A(v_q) = w_q^{(0)} \ , \ \frac{\partial A}{\partial X_{i+m}}(v_q) = w_q^{(i)} \ , \ 1 \le q \le t \ , \ 1 \le i \le n-m \ .$$

Proof. By making an appropriate linear change of the coordinates in K^m we may assume w.l.o.g. that $v_{q_1}^{(i)} \ne v_{q_2}^{(i)}$, $1 \le q_1 < q_2 \le t$, $1 \le i \le n-m$, where $v_q = (v_q^{(1)}, \dots, v_q^{(n-m)})$, $1 \le q \le t$. Consider a polynomial

$$A_{q_0} = \prod_{q \ne q_0, 1 \le i \le n-m} (X_{i+m} - v_q^{(i)})^2 \cdot \left(\sum_{1 \le i \le n-m} a_i (X_{i+m} - v_{q_0}^{(i)}) + a_0 \right) \ , \ 1 \le q_0 \le t$$

with indeterminate coefficients a_i , $0 \le i \le n-m$. Then $A_{q_0}(v_q) = \frac{\partial A_{q_0}}{\partial X_{i+m}}(v_q) = 0$, $1 \le i \le n-m$, for every $q \ne q_0$. Equation $A_{q_0}(v_{q_0}) = w_{q_0}^{(0)}$ uniquely determines a_0 . Furthermore equation $\frac{\partial A_{q_0}}{\partial X_{i+m}}(v_{q_0}) = w_{q_0}^{(i)}$ uniquely determines a_i , $1 \le i \le n-m$. Finally we let $A := \sum_{1 \le q \le t} A_q$. ∎

Of course one can in the same vain interpolate the higher derivatives as well.

We now consider a *parametric* interpolation. Due to Bézout inequality $\deg(\overline{\mathcal{G}}) < \delta^n$, we introduce a polynomial

$$\mathcal{A} = \sum_{0 \le e_1 + \cdots + e_{n-m} \le 2(n-m)\delta^n} A_E X_{m+1}^{e_1} \cdots X_n^{e_{n-m}}$$

with indeterminate coefficients $a := \{A_E\}_E$, $E = (e_1, \dots, e_{n-m})$ and a quantifier-free formula $\Phi(u, v, a)$ of the theory of algebraically closed fields which says that

$$\text{if } v \in \mathcal{G} \ , \ \pi(v) = u \in K^m \text{ then } \mathcal{A}(v) = 0 \ , \ \frac{\partial \mathcal{A}}{\partial X_{i+m}}(v) = M_{j+m,i+m}(v) \ , \ 1 \le i \le n-m$$

for some j , $1 \le j \le k$ (we fix j for the time being). Then the formula $\forall u \ \exists a \ \forall v \ \Phi$ is true due to Lemma 9.3.

An algorithm from [11] yields a representation of $\pi^{-1}(u) \cap \mathcal{G}$ commonly refered to as a "shape lemma". Applied to a system $\{g_p = 0, g_0 \neq 0\}_{p>0}$ the output of this algorithm is a partition of $K^m = \cup_\beta U_\beta$ into constructible subsets such that for each β there are a linear combination $\alpha = \sum_{1 \le i \le n-m} \alpha_{i,\beta} v^{(i)}$ of coordinates $v^{(i)}$, $1 \le i \le n-m$, with integer coefficients $\alpha_{i,\beta}$ and rational functions ϕ , $\phi_i \in K_0(X_1, \dots, X_m)[Y]$, $1 \le i \le n-m$, for which the following holds:

- for any $u \in U_\beta$ and any $v = (u, v^{(1)}, \dots, v^{(n-m)}) \in \pi^{-1}(u) \cap \mathcal{G}$ equalities $v^{(i_0)} = \phi_{i_0}(u, \alpha)$, $1 \le i_0 \le n-m$, take place, i. e. α is a primitive element of the field $K_0(u, v^{(1)}, \dots, v^{(n-m)})$ over $K_0(u)$;
- the roots of a univariate polynomial $\phi(u, Y)$ are exactly the values of α while ranging over points $v \in \pi^{-1}(u) \cap \mathcal{G}$.

Furthermore, in formula Φ we replace $v^{(i_0)}$, $1 \le i_0 \le n-m$, by $\phi_{i_0}(u, \alpha)$ and divide the resulting polynomials $\mathcal{A}(\alpha)$ and $\left(\frac{\partial \mathcal{A}}{\partial X_{i+m}}(\alpha) - M_{j+m,i+m}(\alpha)\right)$ by polynomial $\phi(u, \alpha)$ (with the remainders as polynomials in α). Then system Φ_1 obtained by equating to zero all coefficients of the remainders at the powers of α is equivalent to formula $\forall v \ \Phi$, for any $u \in U_\beta$.

One may consider Φ_1 as a linear system with respect to variables a and apply to Φ_1 an algorithm of *parametric Gaussian elimination* (see e. g. [10], [11]). It yields a refinement $K^m = \cup_{\beta'} U'_{\beta'}$ of partition $\cup_\beta U_\beta$ into constructive subsets such that for each β' and for every multiindex E there is rational function $a_E \in K_0(X_1, \dots, X_m)$ such that for any $u \in U'_{\beta'}$ the array of coefficients $a(u) = \{a_E(u)\}_E$ fulfils Φ_1 . For a choice of the unique β' for which $U'_{\beta'}$ is dense in K^m the rational function

$$L_j = \sum_{0 \le e_1 + \cdots + e_{n-m} \le 2(n-m)\delta^n} a_E X_{m+1}^{e_1} \cdots X_n^{e_{n-m}}$$

corresponding to this β' is as required in Section 5.

Finally we address the complexity issue. In the construction of the "shape lemma" above $\deg(\phi)$, $\deg(\phi_i)$ are bounded by $\delta^{O(n)}$ as well as the degrees of the polynomials representing $\{U_\beta\}_\beta$, while the number of $\{U_\beta\}$, the total sum of sizes of the coefficients of these polynomials and the complexity of the algorithm do not exceed $R^{O(1)}\delta^{O(n^2)}$ [11]. Therefore the degrees of the polynomials occuring in Φ_1 are bounded by $\Delta\delta^{O(n)}$, while the number of the polynomials, the total sum of sizes of their coefficients and the complexity of constructing Φ_1 do not exceed $(R\Delta^n\delta^{n^2})^{O(1)}$. At the stage of applying the parametric Gaussian elimination to Φ_1 the bounds are similar. Proposition is proved. ∎

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