Two dimensional semisimple representation spaces of the fundamental groups of algebraic manifolds, Part I

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MPI/90-75

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Abstract. *Let X be a compact algebraic manifold. We construct a natural map from the two dimensional semisimple representation space of the fundamental group of X to the moduli space of line bundles over X. The fibres and the image of this map are discussed.*

Introduction. Let X be a compact algebraic manifold and $x \in X$ a base point. We are interested in the set of the semisimple representations of the fundamental group $\pi_1(X, x)$ modulo conjugation

$$S_B := \mathrm{Hom}(\pi_1(X,x), GL(r,\boldsymbol{C}))^{ss}/GL(r,\boldsymbol{C}).$$

It is well known that $\pi_1(X,x)$ is finitely presented, so there exists naturally a quasi projective variety M_B parametrizing S_B, called the *r-th Betti space* of X and depends only on the topology of X. Since X is an algebraic manifold, one expects that the underlying space of M_B should have additional algebraic structures reflecting the algebraic structure on X. For example, looking at the first Betti space, roughly saying, it is the first cohomology group $H^1(X,\boldsymbol{C})$. On the other hand, one has the Hodge decomposition $H^1(X,C) = H^0(X,\Omega^1_X) \oplus H^1(X,\mathcal{O}_X)$, which depends on the algebraic structure on X and $H^1(X,\mathcal{O}_X)/H^1(X,\boldsymbol{Z})$ parametrizes the flat line bundles over X.
For higher dimensional cases, recently, K. Corlette proved that a representation is semisimple if and only if its associated flat vector bundle V has a *harmonic Hermitian metric* [9], [25]. J. Jost and S-T. Yau [20] gave also some generalizations in this direction. In a canonical way, this metric makes V into a holomorphic vector bundle E and gives E a *Higgs structure* θ, namely, $\theta \in H^0(X, \Omega^1_X\mathrm{End}(E))$ and $\theta \wedge \theta = 0$. The pair (E,θ) is called a *Higgs bundle* [16], [25]. Similar to vector bundles, one defines stable and poly-stable Higgs bundles by looking at θ-invariant subsheaves of E and the set

$$S_{Dol} := \{ \text{ rank-r poly-stable Higgs bundles over } X \}/\text{Isomorphisms}.$$

Most important, there is a one to one correspondence $S_B \simeq S_{Dol}$. This theorem was proved in a succession of generalizations by Corlette, Beilinson-Deligne, Donaldson, Hitchin, Narasimhan-Seshadri, Simpson, Uhlenbeck and Yau cf. [25] for historical comments. Furthermore, Simpson proved that there is a quasi projective variety M_{Dol} parametrizing S_{Dol}, and there exists a locally etale universal family [25], [26]. This moduli space provides an algebraic structure on the underlying space of M_B, called the *r-th Dolbeault space* of X. In general, the Betti space and the Dolbeault space are not isomorphic as algebraic varieties.

The exact sequence of group homomorphisms $\boldsymbol{R}^+ \to GL(r,\boldsymbol{C}) \to SL(r,\boldsymbol{C})$ gives a splitting

$$M_{Dol} = M_{Dol}^{sl} \times H^0(X,\Omega_X^1),$$

the variety M_{Dol}^{sl} parametrizes exactly all poly-stable Higgs bundles (E,θ) with $Tr\,\theta = 0$. For example, if $r = 1$, then $M_{Dol}^{sl} = Pic^\tau(X)$, where $Pic^\tau(X)$ is the moduli space of topological trivial line bundles over X.

In our paper we consider the first non-abelian case, namely , $r = 2$. By the Lefschetz hyperplane section theorem we may assume that X is an algebraic surface with an ample line bundle H. According $\theta = 0$ or $\neq 0$, The moduli space M_{Dol}^{sl} is divided into two subvarieties

$$M_{Dol}^{sl} = M^0 \sqcup M^+.$$

It is easy to see that M^0 parametrizes exactly flat bundles coming from unitary representations.

Theorem A The moduli space M^+ has naturally the following decomposition

$$M^+ = \bigsqcup_i M_i^+ \sqcup \bigsqcup_j M_j^+.$$

Each component M_i^+ corresponds to a fibration $f_i : X \to C_i$ so that for any representation $\rho \in M_i^+$ its restriction to a generic fibre F_i of this fibration

$$\pi_1(F_i,x) \xrightarrow{i_*} \pi_1(X,x) \xrightarrow{\rho} SL(2,\boldsymbol{C}),$$

splits into a direct sum of 1-dimensional unitary representations.
Each component M_j^+ is naturally birational to a product space of a vector space and an abelian variety.

The right way to understand the components in the first part is follows. The unitary splitting of representations $\rho \in M_i$ respect to a generic fibre F_i means that ρ can be separated as a tensor product of a pull back of a 2-dimensional representation of " *the orbiford fundamental group* $\pi^{orb}(C_i,c)$ on the base curve C_i and a direct sum of two 1-dimensional unitary representations of $\pi_1(F_i,x)$. Recently, R. Brussee [7] determined the moduli spaces of semisimple $SL(2,\boldsymbol{C})$ representations of orbiford fundamental groups of curves. Like the original case, the analytic object corresponding the semisimple representations of the orbiford fundamental group is called "*Parapolic poly-stable Higgs bundles*" by N. J. Hitchin [18]. We give the following examples to make this point clear.

1) Suppose $f : X \to P^1$ is a regular elliptic surface with $\chi(\mathcal{O}_X) \geq 1$. S. Bauer [2] proved that any $SU(2)$−representation of $\pi_1(X,x)$ is a pull back of a $SU(2)$−representation of $\pi^{orb}(P^1,p)$. In fact,

one can see, it is still true for any semisimple representation. This means exactly, the second part in the decomposition in theorem A is empty and the first part has only one component.

2) Let X be a hyperelliptic surface of type I. It is constructed as a quotient of product of two elliptic curves $B \times C$ by a Z_2-action, which acts on B as the involusion and on C as the translation. X has two elliptic fibration structures, one is over $B/Z_2 =: P^1$ with four singular fibres of multiplicity 2. We denote $M^+_{Para}(P^1, 4, 2)$ by the moduli of rank-2 poly-stable parapolic Higgs bundles (E, θ) with $\theta \neq 0, \operatorname{Tr}\theta = 0$ over P^1 respcet to four marked points and multiplicity 2. One can show $M^+ \simeq \mathbf{C}^* \times Jac(B) \times Jac(C) \simeq M^+_{Para}(P^1, 4, 2) \times Jac(C)$.

In general, we have the following statement, the proof will be given in part II.
There exists naturally a birational map $M_i^+ \simeq M^+_{Para}(C_i) \times A_i$ up to an etale covering, where $A_i \subset Jac(F_i) \times Jac(F_i)$ is a subabelian variety satisfying some invariant properties under the monodromy action of $\pi_i(C_i, c)$.

For a trivial Higgs bundle $(E, 0) \in M^0$, we may use its deformations to get a nontrivial Higgs bundle $(E, \theta) \in M^+$. More precisely, this is the following construction

Proposition 1 Let $M^0 \to Pic^\tau(X)$ be the map by sending E to $\det E$, suppose $T^*(M^0/Pic^\tau(X))$ is the total space of the relative tangent space. Then there exists naturally an inclusion

$$T^*(M^0/Pic^\tau(X)) \hookrightarrow M_{Dol}$$

by putting E and its relative cotangent vector $\theta \in T^*_{[E]}(M^0/Pic^\tau(X)) \simeq H^0(X, \Omega^1_X \mathrm{End}_0(E))$ togethere.

A consequence from theorem 1 and prop.1 is the following theorem, which descripts the moduli space of 2-dimensional unitary representations

Theorem B The total space of the relative cotangent bundle $T^*(M^0/Pic^\tau(X))$ defined in prop. 1 has a natural decomposition

$$T^*(M^0/Pic^\tau(X)) = \bigsqcup_i T_i \sqcup \bigsqcup_j T_j \sqcup \bigsqcup_k Pic^\tau(X)_k,$$

where the components T_i, T_j have the same properties as for M_i, M_j in theorem A. The last components $\bigsqcup_k Pic^\tau(X)_k$ come form by twisting 1-dimensional unitary representations with k 2-dimensional rigid unitary representations.

The proof of theorem A is divided into two steps. We will also see more geometry meaning of M_{Dol}.

Theorem 1 Suppose $(E,\theta) \in M^+$, then there exists naturally a factor map

$$\begin{tikzcd} E \arrow[r, "\theta"] \arrow[dr, "\theta_L"'] & E \otimes \Omega^1_X \\ & E \otimes L, \arrow[u] \end{tikzcd}$$

where L is a sub invertible sheaf of Ω^1_X, which is numerically semipositive and $L^2 = 0$. The map θ_L does not vanish any where.

The main technique used in the proof is the well known lemma due to F. Bogomolov [23]

Lemma (F. Bogomolov) Suppose X is an algebraic surface and L is a sub invertible sheaf of Ω^1_X, then there is a positive constant a, so that $h^0(X, L^{\otimes n}) \leq an, \quad \forall n \in N$.

Using the above factor map we may naturally define a map

$$M^+ \xrightarrow{g} Pic(X)$$

by sending (E,θ) to the equivalent class $[L] \in Pic(X)$. There is a stratification $M^+ = \bigsqcup_k M^+_k$ so that the restriction $g|_{M^+_k}$ is a morphism. It can be seen by using the locally etale universal family of M^+.

To the fibres of the map g. There are two kind algebraic descriptions to $(E,\theta) \in g^{-1}([L])$ according det $\theta_L = 0$ or $\neq 0$.

1) det $\theta_L = 0$. The $\operatorname{Ker}\theta_L =: M^\vee$ is a line bundle. There is a exact sequence of vector bundles

$$0 \to M^\vee \to E \to M \otimes \det E \to 0,$$

$M^{\otimes 2} \otimes \det E \simeq L$ and $LH > 0$. The map θ_L is given by the composition map

$$E \to M \otimes \det E \simeq M^\vee \otimes L \to E \otimes L.$$

2) $\det\theta_L \neq 0$. On the double covering $\pi : \tilde{X} \to X$ ramified along the zero divisor $(\det\theta_L) \in |L^{\otimes 2}|$ there exists a line bundle $\tilde{M}$, so that $\pi_*\tilde{M} \simeq E$ and θ_L is given by taking direct image

$$\pi_*(\pi^*(\sqrt{\det\theta_L}) : \tilde{M} \to \tilde{M} \otimes \pi^* L).$$

The second description is just a straight generalization of Hitchin's description of generic Higgs bundles over curves [5], [17]. Same as in the curve case, the surface $\tilde{X}$ is called the *spectral surface* of (E,θ).

It is natural to introduce the following abelian varieties; the etale covering $\widetilde{Pic^{\tau}(X)} \to Pic^{\tau}(X)$ by taking 2-torsion points in $Pic^{\tau}(X)$. Let $H^0(X, L^{\otimes 2})^*$ be the nonzero sections space of $H^0(X, L^{\otimes 2})$ and $\tilde{\mathcal{X}} \to X \times H^0(X, L^{\otimes 2})^*$ be the double covering by taking the square root of the universal section in $H^0(X \times H^0(X, L^{\otimes 2})^*, p_1^* L^{\otimes 2})$, we denote its relative Picard group of the projection $\mathcal{X} \to H^0(X, L^{\otimes 2})^*$ by $Pic(\tilde{\mathcal{X}}/H^0(X, L^{\otimes 2})^*)$. Applying the above descriptions we get easily

Corollary 1. There is a natural birational map on the fibre of the map g

$$g^{-1}([L]) \simeq PH^0(X, \Omega^1_X \otimes L^{\vee}) \times (\bigsqcup_i H^1(X, L^{\vee}) \times \widetilde{Pic^{\tau}_i(X)} \sqcup \bigsqcup_j Pic_j(\tilde{\mathcal{X}}/H^0(X, L^{\otimes 2})^*)),$$

where $\widetilde{Pic^{\tau}_i(X)}$ respect $Pic_j(\tilde{\mathcal{X}}/H^0(X, L^{\otimes 2})^*)$ is a connected component of $\widetilde{Pic^{\tau}(X)}$ respect $Pic(\tilde{\mathcal{X}}/H^0(X, L^{\otimes 2})^*)$.

As for the image of the map g we prove

Theorem 2 Let $g(M^+) = \bigcup_i P_i \cup \bigcup_i \{p_i\} \cup \bigcup_j \{p_j\}$ be the irreducible decomposition, where the component P_i has positive dimension and the point p_i has the positive Kodaira-dimension $\kappa(X, L_{p_i})$. Then each component P_i respect each point p_i corresponds a fibration $f_i : X \to C_i$, so that for any representation $\rho \in g^{-1}(P_i)$ respect $\rho \in g^{-1}(p_i)$ its restriction to a generic fibre F_i of f_i splits into a direct sum of 1-dimensional unitary representations of $\pi_1(F_i, x)$.

The main idea of the proof of theorem 2 is based on the following facts. Let us sketch them in the few words. Like classical harmonic form theorem on the trivial vector bundle over a Kähler manifold, the harmonic operator induced by the harmonic Hermitian metric on $\mathrm{End}(E)$ still satisfies *the higher Kähler identity*, but to some special forms. Using the factor map in theorem 1, we get a nontrivial 1-form $\omega_L \in H^0(X, \Omega^1_X \otimes L^{\vee})$. Taking the complex conjucation and using the higher Kähler identity, we obtain a nontrivial class $\bar{\omega}_L \in H^0(X, L)$.
Considering the subvariety of line bundles in $Pic(X)$ with nontrivial $H^1(X, L)$. One has the following lemma, which is a little bit modified version of a theorem due to M. Green and R. Lazarsfeld [14]

Lemma (M. Green and R. Lazarsfeld) Let the subvariety $S := \{ [L] \in Pic(X) | \, H^1(X, L) \neq 0 \}$. Then there exists a stratification $S = \bigsqcup_j S_j$ so that $\forall \, [L] \in S_j$, $\forall \, t \in H^1(X, L)$ and $\forall v \in T_{[L]}(S_j) \subseteq T_{[L]}(Pic(X)) \simeq H^1(X, \mathcal{O}_X)$, it holds $t \cup v = 0$ in $H^2(X, L)$.

Applying this lemma to the line bundles L coming from the factor map in theorem 1 and the classes $\bar{\omega}_L \in H^1(X, L)$, taking the complex conjucation, we get an equality of 1-forms $\omega_L \wedge \bar{v} = 0$, where $v \in H^0(X, \Omega^1_X)$ corresponds a tangent vector $v \in T_{[L]}(P_i) \subseteq H^1(X, \mathcal{O}_X)$. Similar to the Castelnuovo-De Franchis lemma, this equality produces a fibration on X, which we just want.

Acknowlegment I am grateful to R. Brussee and Ch. Okonek for valuable discussions. In fact, lemma 2.1 is due to a discussion with R. Brussee and the proof stated here was suggested by Ch. Okonek. I thank the following people, M. Itoh for his helpful explanation of Corlette and Simpson's works, A. Assadi, S Bauer and J. Jost for very useful conversations with them. Also I thank Professor F. Hirzebruch and the Max-Planck-Institut für Mathematik for the hospitality in the preparation of this work.

1. Harmonic Hermitian metric on flat vector bundles, harmonic forms and a generalizaition of the higher Kähler identity

Suppose (X,ω) is a Kähler manifold. Let V is a rank-r flat vector bundle with the flat connection D. Given a Hermitian metric K on V, there is an unique decomposition $D = D_K + \theta_K + \bar{\theta}_K$, where D_K is a connection on V compatible with K, $\theta_K \in A^{1,0}(\mathrm{End}(E))$ and $\bar{\theta}_K \in A^{0,1}(\mathrm{End}(E))$ is the complex conjucation of θ_K respect to K.

Let $D_K = D'_K + D''_K$ be the (1, 0) and (0, 1) parts decomposition and $\bigwedge$ be the adjoint operator of ω. One has the following

Definition [25] A Hermitian metric K on V is harmonic if $\bigwedge(D''_K + \theta_K)^2 = 0$.

A Hermitian metric on V can be thought of as the multivalued map $\phi_K : X \to GL(r,\boldsymbol{C})/U(r)$. K is harmonic if and only if the map ϕ_K is harmonic.

Theorem (K. Corlette [9]) A flat vector bundle has a harmonic Hermitian metric if and only if it comes from semisimple representaion of $\pi_1(X,x)$. Such a metric is unique.

Using the harmonic metric K, we may make V into a Higgs bundle in a canonical way. First we review the following

Lemma (P. Deligne [25]) K is harmonic if and only if $(D''_K + \theta_K)^2 = 0$.

Given a harmonic hermitia metric K, the above lemma implies $D''^2_K = 0$, $D''_K\theta_K + \theta_K \wedge D''_K = 0$ and $\theta_K \wedge \theta_K = 0$. Hence we get a holomorphic vector bundle $(V, D''_K) =: E$, a section $\theta_K =: \theta \in H^0(X, \Omega^1_X \mathrm{End}(E))$ and $\theta \wedge \theta = 0$, so (E,θ) is a Higgs bundle. We see that the connection D_K is compatible with the metric K and the holomorphic structure on E.

The harmonic Hermitian metric K on E induces canonically a harmonic Hermitian metric $\tilde{K}$ on $\mathrm{End}(E)$. Let $D_{\tilde{K}} = D'_{\tilde{K}} + D''_{\tilde{K}}$ be the connection compatible with $\tilde{K}$ and the holomorphic structure on $\mathrm{End}(E)$. The connection $D_{\tilde{K}}$ and the metric $\tilde{K}$ induce two harmonic operators $\square_{D'_{\tilde{K}}}$ and $\square_{D''_{\tilde{K}}}$. It is clear, the holomorphic form θ is a $\square_{D''_{\tilde{K}}}$-harmonic form. Furthermore, we have

Lemma 1.1 i) θ is a $\square_{D'_{\tilde{K}}}$- harmonic form.

ii) $\forall\,\alpha \in H^0(X,\Omega^1_X)$, $\theta \wedge \alpha$ is a $\square_{D'_{\tilde{K}}}$-harmonic form.

Proof i) Looking at the decomposition $D = (D'_K + \theta) + (D''_K + \bar{\theta})$ of (1, 0) and (0, 1) parts. Since $D^2 = 0$ and $\theta \wedge \theta = 0$, we get $D'^2_K + D'_K\theta + \theta \wedge D'_K = (D'_K + \theta)^2 = 0$. Because D_K is the connection compatible with the metric K and the holomorphic structure on E, we have $D'^2_K = 0$,

hence $D'_K\theta + \theta \wedge D'_K = 0$. This means that θ is a $D'_{\bar{K}}-$closed form.
Because $D'^*_{\bar{K}} = -\sqrt{-1}(\bigwedge D''_{\bar{K}} - D''_{\bar{K}} \bigwedge)$, $D''_{\bar{K}}\theta = 0$ and $\bigwedge \theta = 0$ automatically, we get $D'^*_{\bar{K}}\theta = 0$.
ii) $D'_{\bar{K}}(\theta \wedge \alpha) = D'_{\bar{K}}(\theta) \wedge \alpha - \theta \wedge \partial\alpha = 0$, since $D'_{\bar{K}}(\theta) = 0$ and $\partial\alpha = 0$. Same as i) we show also $D'^*_{\bar{K}}(\theta \wedge \alpha) = -\sqrt{-1}(\bigwedge D''_{\bar{K}}(\theta \wedge \alpha) - D''_{\bar{K}} \bigwedge(\theta \wedge \alpha)) = 0$. ■

In the classical harmonic theorem on Kähler manifolds, it is well known that the harmonic operators $\Box_\partial$ and $\Box_{\bar{\partial}}$ on the trivial vector bundle satisfy the higher Kähler identity, namely $\Box_\partial = \Box_{\bar{\partial}}$. This means that a form is $\Box_\partial-$harmonic if and only if it is $\Box_{\bar{\partial}}-$harmonic. Our lemma says, the harmonical operators induced by the harmonic Hermitian metric still have such property for some special forms.
The next lemma is due to M. Itoh, which will be used in the proof of therem 2.

Lemma 1.2 (M. Itoh) Let E be a holomorphic vector bundle with a Hermitian metric h and $i : F \hookrightarrow E$ be a holomorphic subbundle. The restriction $h|_F$ gives F a Hermitian metric. Let D and D_F denote the connections on E and F compatible with the holomorphic structure and the metric. Suppose $\varphi \in A^{p,q}(F)$. If $i(\varphi) \in A^{p,q}(E)$ is $\Box_{D'}-$harmonic, then φ is $\Box_{D'_F}-$harmonic.

Proof (M. Itoh) Looking at the smooth splitting $E = F \oplus F^\perp$. Let $\sigma \in A^1(X, \mathrm{Hom}(F, F^\perp))$ be the second fundamental form respect to D and D_F. Namely, for each smooth section s of F, $D_v(s) = D_{Fv}(s) + \sigma_v(s)$, v is a tangent vector of X, where $D_{Fv}(s)$ and $\sigma_v(s)$ are the $F-$ and $F^\perp-$components of $D_v(s)$ in E. The equation $\Box_{D'}(i\varphi) = 0$ means exactly $D'(i\varphi) = 0$ and $D'^*(i\varphi) = 0$. Write φ as

$$\varphi = \sum_{i,j} \varphi_{i_1 \ldots i_p \bar{j}_1 \ldots \bar{j}_q} dz^{i_1} \wedge \ldots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \ldots \wedge d\bar{z}^{j_q},$$

with $\varphi_{i_1 \ldots i_p \bar{j}_1 \ldots \bar{j}_q} = \sum_{a=1}^f \varphi^a_{i_1 \ldots i_p \bar{j}_1 \ldots \bar{j}_q} e_a$, where $\{e_1, \ldots, e_f, e_{f+1}, \ldots, e_n\}$ is a unitary frame of E, and $\{e_1, \ldots, e_f\}$ is a unitary frame of F. Then $D'(i\varphi) = \sum_k \sum_{i,j} (D_{\frac{\partial}{\partial z^k}} \varphi_{i_1 \ldots i_p \bar{j}_1 \ldots \bar{j}_q}) dz^k \wedge dz^1 \wedge \ldots d\bar{z}^{j_q}$ is written in terms of D_F and σ as

$$\sum_k \sum_{i,j} (D_{F \frac{\partial}{\partial z^k}} \varphi_{i_1 \ldots i_p \bar{j}_1 \ldots \bar{j}_q} dz^k \wedge dz^1 \ldots \wedge d\bar{z}^{j_q} + \sum_k \sum_{i,j} \sigma_{\frac{\partial}{\partial z^k}} \varphi_{i_1 \ldots i_p \bar{j}_1 \ldots \bar{j}_q} dz^k \wedge dz^1 \ldots d\bar{z}^{j_q}.$$

Hence $D'_F(\varphi) = 0$. On the other hand from the well known formula

$$D'^*(i\varphi) = -\sum_{k,l} \sum_{i,j} (g^{\bar{l}k} D_{\frac{\partial}{\partial \bar{z}^l}} \varphi_{k, i_2 \ldots \bar{j}_q}) dz^{i_2} \wedge \ldots \wedge d\bar{z}^{j_q}$$

(see [21], Th 5.2) we have similarly

$$-\sum_{k,l} \sum_{i,j} (g^{\bar{l}k} D_{F \frac{\partial}{\partial \bar{z}^l}} \varphi_{k, i_2 \ldots \bar{j}_q}) d\bar{z}^{i_2} \wedge \ldots \wedge d\bar{z}^{j_q} = 0,$$

that is $D'^*_F\varphi = 0$. So φ is $\Box_{D'_F}-$harmonic. ■

2. An algebraic geometry description of rank-2 Higgs bundles coming from the semisimple representations of $\pi_1(X,x)$, proofs of theorem 1, corollary 1 and prop.1

Befor the proof of theorem 1 we want to show the following lemma about rank-2 Higgs bundles, which are not necessary coming from representations of $\pi_1(X,x)$.

Lemma 2.1 Let (E,θ) be a rank-2 Higgs bundle and $\theta \neq 0$, $\mathrm{Tr}\,\theta = 0$. Then we have the factor map

$$\begin{tikzcd} E \arrow[r, "\theta"] \arrow[dr, "\theta_L"'] & E\otimes\Omega^1_X \\ & E\otimes L, \arrow[u] \end{tikzcd}$$

where L is a sub invertible sheaf of Ω^1_X. The map θ_L has only isolated zero locus.

Proof Taking local frams of E and $E\otimes\Omega^1_X$, the map θ is written as $M_1dz_1 + M_2dz_2$, here M_i are 2×2 matrix of holomorphic functions. The condition $\bigwedge^2\theta = 0$, and $\mathrm{Tr}\,\theta = 0$ means exactly $M_1M_2 - M_2M_1 = 0$ and $\mathrm{Tr}M_i = 0$. From the fact of linear algebra we see that there are two holomorphic functions λ_1, λ_2 non vanishing at the same time and $\lambda_1M_1 + \lambda_2M_2 = 0$. This follows that the map $\mathrm{End}(E) \to \Omega^1_X$ induced by θ has 1-dimensional image $L\otimes I_z \hookrightarrow \Omega^1_X$, where L is a sub invertible sheaf of Ω^1_X and I_z is the ideal sheaf of a 0-dimensional subscheme z of X. So we complete the proof. ∎

Looking at the maps

$$E \xrightarrow{\theta_L} E\otimes L \xrightarrow{\theta_L\otimes I} E\otimes L^{\otimes 2},$$

these maps is a complex and away from the zero locus of θ_L, is exact as vector bundles map.

Case 1. $\det\theta_L = 0$. Since $\mathrm{Tr}\,\theta_L = 0$. Let $M^\vee := \mathrm{Ker}\,\theta_L$, then we have the exact sequence

$$0 \to M^\vee \to E \to M\otimes\det E\otimes I_z \to 0,$$

where I_z is the ideal sheaf of the zero locus of θ_L and $M^{\otimes 2}\otimes\det E \simeq L$. The map θ_L is exactly the composition map

$$E \to M\otimes\det E\otimes I_z \simeq M^\vee\otimes L\otimes I_z \to E\otimes L.$$

Case 2. $\det\theta_L \neq 0$. The map $\theta_L : E \to E\otimes L$ induces an $\mathrm{Sym}(L^\vee)/\mathcal{I}$-module on E, here $\mathcal{I}$ is the ideal sheaf generated by the map $\det\,\theta_L : L^{\vee\otimes 2} \to \mathcal{O}_X$ and $\mathrm{Spec}(\mathrm{Sym}(L^\vee)/\mathcal{I})) \to X$ is simply the double covering $\pi : \tilde{X} \to X$ by taking the square root of $\det\theta_L$. Using the general correspondence theorem ([12], Ckapter 2, prop. 5.2 and [5] Prop. 3.6.) there is a rank-1 torsion free sheaf N on Y so that $\pi_*N \simeq E$, the singular locus of N is supported at the invers image of the zero locus of θ_L ([12] Lemma (5.21). The map θ_L is just the direct image $\pi_*(\pi^*(\sqrt{\det\theta_L}) : N \to N\otimes\pi^*L)$.

After the above discussion we see, to complete the proof of theorem 1 and corollary 1 we just need to show the follows

Lemma 2.2 Suppose (E,θ) is arising from semisimple representations of $\pi_1(X,x)$, Then the line bundle L is numerically semipositive and the map θ_L does not vanish any where.

Proof **1)** $\det\theta_L=0$. We consider the exact sequence in case 1. Applying the poly-stability of E to the Higgs-subline bundle $M^\vee$ we get $MH\geq 0$ and $MH=0$ iff M is topologically trivial. The calculation of the Chern classes gives $M^2=|z|\geq 0$. If $M^2>0$ and $MH>0$, then by the Riemann-Roch theorem we see $h^0(X,(M^{\otimes 2}\otimes\det E)^{\otimes n})\approx an^2$. On the other hand since $M^{\otimes 2}\otimes\det E\simeq L\hookrightarrow\Omega^1_X$, by the Bogomolov lemma we have $h^0(X,(M^{\otimes 2}\otimes\det E)^{\otimes n})\leq an+b$. So $M^2=|z|=0$.
The poly-stability of (E,θ) implies that M is numerically semipositive. Since $L=M^{\otimes 2}\otimes\det E$, so L is numerically semipositive.
In fact, $MH>0$. Otherwise, we would have $(E,\theta)\simeq(M^\vee,\omega)\oplus(M\otimes\det E,-\omega)$, $\omega_{\neq 0}\in H^0(X,\Omega^1_X)$. This follows $\det\theta_L\neq 0$, a contradiction.

2) $\det\theta_L\neq 0$. We take a blowing up $\sigma:\widehat{X}\to X$ at the singularities of the zero divisorn $(\det\theta_L)_0$ satisfying the condition: Each irreducible component in the pull back $\sigma^*(\det\theta_L)_0$ is a smooth curve and the components with odd multiplisities are disjoint.
The pull back $\sigma^*I_z=I_{z'}\otimes\mathcal{O}_{\hat X}(-\sum_i E_i)$, here z' is a 0-dimensional subscheme of $\widehat{X}$ and $\sum_i E_i$ are some exceptional curves of the blowing up. Let $L':=\sigma^*L\otimes\mathcal{O}_{\hat X}(-\sum_i E_i)$, then we have the factor map

$$\begin{tikzcd}
\mathrm{End}(\sigma^*E) \arrow[r, "\sigma^*\theta"] \arrow[dr, "\theta_{L'}"'] & \Omega^1_{\hat X} \\
& L' \arrow[u]
\end{tikzcd}$$

and $\theta_{L'}$ has only isolated zero locus z'. We look at the double covering $\pi:\tilde{\hat X}\to X$ by taking the square root $\sqrt{\det\theta_{L'}}$ and furthermore, by taking its normalizition we obtain the smooth covering surface $\tilde{X}'$,

$$\begin{tikzcd}
\tilde{X}' \arrow[d, "\nu"'] \arrow[dd, bend left, "\rho'"] \arrow[ddr, "\rho"] & \\
\tilde{\hat X} \arrow[d, "\pi"'] & \\
\widehat{X} \arrow[r, "\sigma"] & X.
\end{tikzcd}$$

Let $\lambda:=\rho'^*\sqrt{\det\theta_{L'}}\in H^0(\tilde{X}',\rho'^*L')$, its zero divisor is exactly supported on $\rho'^{-1}((\det\theta_{L'})_0)$.

Similar to case 1, we have the zero composition map on $\tilde{X}'$

$$\rho^* E \xrightarrow{\rho'^*\theta_{L'}-\lambda} \rho^* E \otimes \rho'^* L' \xrightarrow{(\rho'^*\theta_{L'}+\lambda)\otimes I} \rho^* E \otimes \rho'^* L'^{\otimes 2}.$$

The above two maps have 1-dimensional images and have only 0-dimensional zero locus exactly supported at $\rho'^{-1}(z')$, therefore, away from this 0-dimensional subscheme these two maps are exact as vector bundles maps. Let $\tilde{M}'^{\vee} := \mathrm{Ker}\,(\rho'^*\theta_{L'} - \lambda)$ and i be the involusion on $\tilde{X}'$, then $i^*\tilde{M}'^{\vee} = \mathrm{Ker}(\rho'^*\theta_{L'} + \lambda)$. Similar to case 1, we have an exact sequence on $\tilde{X}'$

$$0 \to \tilde{M}'^{\vee} \to \rho^* E \to \tilde{M}' \otimes \rho^* \det E \otimes I_{z''} \to 0,$$

where $z''_{\mathrm{red}} = (\rho'^* z')_{\mathrm{red}}$ and $\tilde{M}' \otimes i^*\tilde{M}' \otimes \rho^* \det E \simeq \rho'^* L' \hookrightarrow \Omega^1_{\tilde{X}'}$.

Claim $\sum_i E_i = 0$, $L^2 = 0$ and $\tilde{M}'\rho^* L = 0$.

Proof We have $\rho'^* L'^{\otimes 2} \simeq \mathcal{O}_{\tilde{X}'}(\rho'^*(\det \theta_{L'})_0) = \mathcal{O}_{\tilde{X}'}(\sum_j n_j \rho'^*(C_j)_{\mathrm{red}})$, here $\rho'^*(C_j)_{\mathrm{red}}$ are smooth curves on $\tilde{X}'$. Because the harmonic Hermitian metric goes to harmonic Hermitian metric, the pull back $(\rho^* E, \rho^*\theta_{L'})$ is still poly-stable, hence its the restriction to $\rho'^*(C_j)_{\mathrm{red}}$ is again poly-stable. Noting $\tilde{M}'^{\vee}|_{\rho'^*(C_j)_{\mathrm{red}}}$ is a Higgs-subline bundle, we get $\tilde{M}'\rho'^*(C_j)_{\mathrm{red}} \geq 0$. This implies $\tilde{M}'\rho'^* L' \geq 0$ hence $i^*\tilde{M}'\rho'^* L' \geq 0$. Using the above isomorphism we get $\rho'^*(L')^2 \geq 0$.

Because $L'^2 \geq 0$ and $\sum_i E_i$ are exceptional curves of the blowing up σ, noting $L' = \sigma^* L \otimes \mathcal{O}_{\tilde{X}}(-\sum_i E_i)$ we obtain $L^2 = L'^2 - (\sum_i E_i)^2 \geq 0$. On the other hand, because $L \hookrightarrow \Omega^1_X$ and $L^{\otimes 2}$ is effective, the Bogomolov lemma says $L^2 \leq 0$. So we get $\sum_i E_i = 0$, $L^2 = 0$ and $\tilde{M}'\rho^* L = 0$. The claim is proved. ▲

We look at again the above exact sequence, where $\tilde{M}'^2 = |z''| \geq 0$ and $z''_{\mathrm{red}} = (\rho^* z)_{\mathrm{red}}$, since $\sum_i E_i = 0$. We must show $\tilde{M}'^2 = 0$. Suppose $\tilde{M}'^2 > 0$. Since $\tilde{M}'\rho^* L = 0$ and $(\rho^* L)^2 = 0$, the Hodge index theorem follows $\rho^* L$ is topologically trivial line bundle. In particular, $0 = \rho^* H \rho^* L = \rho^* H\tilde{M}' + \rho^* H i^*\tilde{M}' = 2\rho^* H\tilde{M}'$. Using the Hodge index theorem again, we get $\tilde{M}'^2 \leq 0$. This is a contradiction.

To see L is numerically semipositive, it is enough to show $\rho^* L \rho^* C \geq 0, \forall C \subset X$. Looking at again the exact sequence, the poly-stability of $(\rho^* E, \rho^*\theta)$ follows $\tilde{M}'\rho^*(C) \geq 0$, hence $i^*\tilde{M}'\rho^* C \geq 0$. Noting $\rho^* L = \tilde{M}' \otimes i^*\tilde{M}' \otimes \det \rho^* E$, we get $\rho^* L \rho^* C \geq 0$. 2) is also proved. ■

Proof (Prop.1) Let $E \in M^0$, we just need to show $T^*_{[E]}(M^0/Pic^\tau(X)) \subseteq H^0(X, \Omega^1_X \mathrm{End}_0(E))$ and $\forall \theta \in T^*_{[E]}(M^0/Pic^\tau(X))$, it holds $\theta \wedge \theta = 0$. Looking at the Kuranishi obstruction map $\phi : H^1(X, \mathrm{End}_0(E)) \to H^2(X, \mathrm{End}_0(E))$, the relative tangent space $T_{[E]}(M^0/Pic^\tau(X))$ is contained in Kerϕ via the Kodaira-Spence deformation map. Because E is coming from the unitary representation of $\pi_1(X, x)$, $\mathrm{End}_0(E)$ has a flat metric. By taking the Dolbeult-isomorphism and harmonic forms respect to this flat metric we see, the harmonic operator satisfies the higher Kähler-identity, hence under the complex conjucation respect to the flat metric the harmonic form goes to the harmonic form. Therefore we obtain the following anti-isomporphisms

$$\begin{array}{ccc} T_{[E]}(M^0/Pic^\tau(X)) \subseteq H^1(X, \mathrm{End}_0(E)) & \xrightarrow{\phi} & H^2(X, \mathrm{End}_0(E))) \\ -\big\downarrow & & \big\downarrow - \\ T^*_{[E]}(M^0/Pic^\tau(X)) \subseteq H^0(X, \Omega^1_X \mathrm{End}_0(E)) & \xrightarrow{\bar{\phi}} & H^0(X, \Omega^2_X \mathrm{End}_0(E)) \end{array}$$

and $\bar{\phi}$ just sends θ to $\bigwedge^2 \theta$. ■

2. Deformations of sub invertible sheaves in Ω^1_X, the splitting property of two dimensional semisimple representations of $\pi_1(X, x)$ along fibres of fibrations on X and the proof of theorem 2.

The proof of theorem 2 will be divided into two lemmas. Let $g : M^+ \to Pic(X)$ be the map defined in the introduction. Looking at the irreducible decomposition $g(M^+) = \bigcup_i P_i \cup \bigcup_i \{p_i\} \cup \bigcup_j \{p_j\}$, where the component P_i has positive dimension. The point p_i has the positive Kodaira-dimension, i.e. there is an $n \in N$ so that $h^0(X, L_{p_i}^{\otimes n}) \geq 2$.

Lemma 3.1 Let the section $\omega_L \in H^0(X, \Omega^1_X \otimes L^\vee)$ induces by the embedding $L \hookrightarrow \Omega^1_X$ in theorem 1. Then there exists a stratification $P_i = \bigsqcup_j P_{i,j}$ so that $\forall\, L \in f^{-1}(P_{i,j})$ and $\forall\, v \in T_{[L]}(P_{i,j}) \subseteq T_{[L]}(Pic(X)) \simeq H^1(X, \mathcal{O}_X)$ we have $\omega_L \wedge \bar{v} = 0$ in $H^0(X, \Omega^2_X \otimes L^\vee)$.

Proof Looking at the diagram in theorem 1, the dual of θ_L induces a vector bundles embedding $i : L^\vee \hookrightarrow \mathrm{End}(E)$. Let $K_{L^\vee}$ be the Hermitian metric on $L^\vee$ induced by the harmonic Hermitian metric K on $\mathrm{End}(E)$ via the embedding i. Applying lemmas 1.1 and 1.2, we see that ω_L and $\omega_L \wedge \alpha$, $\forall\, \alpha \in H^0(X, \Omega^1_X)$ are $\Box_{D'_{K_{L^\vee}}}$–harmonic forms on $L^\vee$. Hence under the complex conjucation respect to $K_{L^\vee}$, we see that $\bar{\omega}_L$ and $\bar{\omega}_L \wedge \bar{\alpha}$ are $\Box_{D''_{K_L}}$ – harmonic forms on L.
By using the Green-Lazarsfeld lemma to the subvariety $P_i \subseteq S$, we have the stratification $P_i = \bigsqcup_j P_i \cap S_j =: \bigsqcup_{i,j} P_{i,j}$ and $\bar{\omega}_L \cup v = 0$ in $H^2(X, L)$, $\forall\, L \in g^{-1}(P_{i,j})$, $\forall\, v \in T_{[L]}(P_{i,j})$. Taking the complex conjucation, we get $\omega_L \wedge \bar{v} = 0$. ■

Similar to the well known Castelnuovo-De Franchis lemma, through the equality of 1-forms $\omega_L \wedge \bar{v} = 0$ we construct a nontrivial family of curves on X, then show that it induces a fibration on X. (See [4] for the case, L is a topologically trivial line bundle.) Looking at the stratification $g^{-1}(P_{i,j}) =$

$\bigsqcup_k M_{i,j,k}$, where $M_{i,j,k}$ is an irreducible variety and $g : M_{i,j,k} \to Pic(X)$ is a morphism. We take an irreducible curve $B^0 \subset M_{i,j,k}$, whose image $g(B^0) =: D^0$ is a curve. We define two families of curves on X with the parameter space B^0 as follows:

$$\{ F_{\omega_b} := \text{1-dimensional zero locus of } \omega_b : L_b \hookrightarrow \Omega^1_X \,|\, b \in B^0 \},$$

$$\{ F_{\bar{v}_{g(b)}} := \text{1-dimensional zero locus of } \bar{v}_{g(b)} : \mathcal{O}_X \hookrightarrow \Omega^1_X \,|\, v_{g(b)} \in T_{g(b)}(D^0),\, b \in B^0 \}.$$

Claim One of the above families sweeps a nonempty Zariski open set of X.

Proof The full sub invertible sheaf of $\omega_b : L_b \hookrightarrow \Omega^1_X$ respect $\bar{v}_{g(b)} : \mathcal{O}_X \hookrightarrow \Omega^1_X$ is $L \otimes \mathcal{O}_X(F_{\omega_b})$ respect $\mathcal{O}_X(F_{\bar{v}_{g(b)}})$. The equality $\omega_b \wedge \bar{v}_{g(b)} = 0$ in lemma 3.1 follows that their full sub invertible sheaves are linear dependent in Ω^1_X, hence there is an isomorphism $L_b \otimes \mathcal{O}_X(F_{\omega_b}) \simeq \mathcal{O}_X(F_{\bar{v}_{g(b)}})$. Suppose the both families are supported on some curves on X. This would imply that the family of line bundles $\{ L_b \,|\, b \in B^0 \}$ has only finitely many isomorphic classes by using the above isomorphism. On the other hand, we knew already that its isomorphic classes forms the curve D^0. This is a contradiction. ▲

Suppose we get such a family, which sweeps a nonempty Zariski open set of X. Taking the Zariski-cloure of an irreducible component of the graph of this family $Z \subset X \times B$, then $Z \to X$ is a generic finite map. After a suitable extension of function fields $K(Z) \subset K(Z')$ we may assume, $K(X) \subset K(Z')$ is a Galois-extension

$$\begin{tikzcd}
\tilde{Z} \arrow[r, "\tilde{f}"] \arrow[d] \arrow[dd, bend right, "\tau"'] & \tilde{B} \arrow[d] \arrow[dd, dashed, bend left] \\
Z \arrow[r] \arrow[d] & B \\
X \arrow[r, dashed, "f"] & C,
\end{tikzcd}$$

where $\tilde{Z}$ is a smooth resolution of Z and $\tilde{f}$ is the fibration from the Stein-factorlization of the projection $Z \to B$.

Lemma 3.2 i) The fibration $\tilde{f}$ desends to a fibration $f : X \to C$, which satisfies the properties
A) $\forall b \in B^0$, the sub invertible sheaves $\omega_b : L_b \hookrightarrow \Omega^1_X$ and $df : f^*\Omega^1_C \hookrightarrow \Omega^1_X$ are linear dependent.

B) L_b is f-vertical, i.e. $L_b^2 = 0$ and $L_b|_F \simeq \mathcal{O}_F$ for a generic fibre F of f.

ii) The component P_i corresponds a fibration $f_i : X \to C_i$ so that $\forall m \in g^{-1}(P_i)$ the line bundle L_m has the properties A and B respect to f_i.

iii) The point p_i corresponds a fibration $f_i : X \to C_i$ so that $\forall m \in g^{-1}(p_i)$ the line bundle has the properties A and B.

Proof i) Pushing forward the family $\{\tilde{F}_{\tilde{b}} | \tilde{b} \in \tilde{B}\}$ to X, we obtain a family $\{\tau(\tilde{F}_{\tilde{b}}) | \tilde{b} \in \tilde{B}\}$ on X and $(\tau(\tilde{F}_{\tilde{b}}))^2 \geq 0$. We want to show $(\tau(\tilde{F}_{\tilde{b}}))^2 = 0$. It can be seen by the following argument:

1) Suppose $\{F_{\omega_b}\}$ is such a family. Since a generic curve $\tau(\tilde{F}_{\tilde{b}})$ is a component of F_{ω_b}, there is an embedding $L_b \otimes \mathcal{O}_X(\tau(\tilde{F}_{\tilde{b}})) \hookrightarrow L_b \otimes \mathcal{O}_X(F_{\omega_b}) \hookrightarrow \Omega^1_X$. By the Bogomolov lemma we get $h^0(X, (L_b \otimes \mathcal{O}_X(\tau(\tilde{F}_{\tilde{b}}))^{\otimes n}) \leq an + b, \forall n \in N$.
On the other hand, because L_b is numerically semi-positive, $L_b^2 = 0$ and $(\tau(\tilde{F}_{\tilde{b}}))^2 \geq 0$, we have $(L_b + \tau(\tilde{F}_{\tilde{b}}))^2 \geq 0$. The equality holds iff $L_b\tau(\tilde{F}_{\tilde{b}}) = 0$ and $\tau(\tilde{F}_{\tilde{b}})^2 = 0$. Suppose the inequality holds. By the Riemann-Roch theorem we get $h^0(X, (L_b \otimes \mathcal{O}_X(\tau(\tilde{F}_{\tilde{b}})))^{\otimes n}) \approx an^2$. This is a contradiction.

2) Suppose $\{F_{\bar{v}_{g(b)}}\}$ is such a family, then a generic curve $\tau(\tilde{F}_{\tilde{b}})$ is a component in the zero locus of the 1-form $\bar{v}_{g(b)}$. The Bogomolov lemma and Riemann-Roch theorem follows $(\tau(\tilde{F}_{\tilde{b}}))^2 \leq 0$.

The difference $\tau^*\tau(\tilde{F}_{\tilde{b}}) - \sum_{\eta \in Gal(\tilde{Z}/X)} \eta(\tilde{F}_{\tilde{b}}) =: E$ is some exceptional curves contracted to points by τ. Hence $E^2 \leq 0$ and $\tau^*\tau(\tilde{F}_{\tilde{b}})E = 0$. Noting $\eta(\tilde{F}_{\tilde{b}})\eta'(\tilde{F}_{\tilde{b}}) \geq 0, \forall \eta, \eta' \in Gal(\tilde{Z}/X)$ we get

$$0 \geq E^2 = (E - \tau^*\tau(\tilde{F}_{\tilde{b}}))^2 = (\sum_{\eta \in Gal(\tilde{Z}/X)} \eta(\tilde{F}_{\tilde{b}}))^2 \geq 0.$$

Thus $E = 0$ and $\tilde{F}_{\tilde{b}}\eta(\tilde{F}_{\tilde{b}}) = 0$. This implies that the fibration $\tilde{g} : \tilde{Z} \to \tilde{B}$ is $Gal(\tilde{Z}/X)$-invariant and it can be desent to a fibration $f : X \to \tilde{B}/Gal(\tilde{Z}/X) =: C$. Its fibres are just $\tau(\tilde{F}_{\tilde{b}})$.

To prove i) A). Since $\omega_b : L_b \hookrightarrow \Omega^1_X$ and $\bar{v}_{g(b)} : \mathcal{O}_X \hookrightarrow \Omega^1_X$ are linear dependent, we just need to prove the 1-form $\bar{v}_{g(b)}$ is a pull back from C.

1) Suppose the fibration $f : X \to C$ induced by $\{F_{\omega_b}\}$. For a generic fibre F_c of f we have $L_b \otimes \mathcal{O}_X(F_{\omega_b}) = L_b \otimes \mathcal{O}_X(F_c + F'_{\omega_b}) \simeq \mathcal{O}_X(F_{\bar{v}_{g(b)}}) \hookrightarrow \Omega^1_X$, the the Bogomolov lemma follows $h^0(X, (L_b \otimes \mathcal{O}_X(F_c + F'_{g(b)}))^{\otimes n}) \leq an + b$. Looking at $h^0(X, ((L_b \otimes \mathcal{O}_X(F_c))^{\otimes m} \otimes \mathcal{O}_X(F'_{\omega_b}))^{\otimes n}) \leq h^0(X, (L_b \otimes \mathcal{O}_X(F_c + F'_{\omega_b}))^{\otimes mn})$, where m is fixed but sufficiently large. Because L_b and F_c are numerically semi-positive, $(L_b + F_c)^2 = 0$ and $F'_{L_b} \geq 0$, using the Riemann-Roch theorem we see easily $(L_b + F_c)F'_{\omega_b} = 0$, hence $F_cF'_{\omega_b} = 0$ and $F_cF_{\bar{v}_{g(b)}} = 0$. This implies that F'_{ω_b}, $F_{\bar{v}_{g(b)}}$ are contained in fibers of f and $L_b + F_c$ is a f-vertical divisor of positive degree.
Hence $h^0(X, (L_b \otimes \mathcal{O}_X(F_c + F'_{\omega_b}))^{\otimes n}) = h^0(X, F^{\otimes n}_{\bar{v}_{g(b)}}) \approx an$. This follows that $F_{\bar{v}_{g(b)}}$ contains at least one complete fibre of f. Because the 1-form $\bar{v}_{g(b)}$ vanishes along this complete fibre, it is well known that $\bar{v}_{g(b)}$ is pull back of a 1-form on C.

2) Suppose $f : X \to C$ comes from $\{F_{\bar{v}_{g(b)}}\}$. Because $\bar{v}_{g(b)}$ vanishes along a fibre F'_c, Hence $\bar{v}_{g(b)}$ is a pull back from C.

To prove i) B) We saw already that L_b is f-vertical in the above proof.

Proof ii) Looking at the stratifications $P_i = \bigsqcup_j P_{i,j}$ and $g^{-1}(P_{i,j}) = \bigsqcup_k M_{i,j,k}$, suppose $M_{i,0,0}$ is a subvariety so that $g(M_{i,0,0}) \subseteq P_i$ is a Zariski open set. we fix a point $b_0 \in M_{i,0,0}$ and take an irreducible curve $B^0 \subseteq M_{i,0,0}$ passing through b_0. Using i) we get a fibration $f_i : X \to C_i$ so that two sub invertible sheaves $\omega_{b_0} : L_{b_0} \hookrightarrow \Omega^1_X$ and $df_i : f_i^*\Omega^1_{C_i} \hookrightarrow \Omega^1_X$ are linear dependent and L_{b_0} is g_i-vertical. For any $b \in M_{i,0,0}$ we may find an irreducible curve $B^{0\prime}$ connecting b_0 and b. Applying i) again, we get a fibration $f_i' : X \to C_i'$ so that three sub invertible sheaves $\omega_{b_0} : L_{b_0} \hookrightarrow \Omega^1_X$, $\omega_b : L_b \hookrightarrow \Omega^1_X$ and $df_i' : f_i'^*\Omega^1_{C_i'} \hookrightarrow \Omega^1_X$ are linear dependent with each other and L_{b_0}, L_b is f_i'-vertical. Hence the pull back of two cotangent bundles $\Omega^1_{C_i}$ and $\Omega^1_{C_i'}$ are linear dependent in Ω^1_X, this implies $f_i = f_i'$. Therefore, L_b is also f_i-vertical.
For any $[L] \in P_i$ we may find a sequence $\{[L_n]\} \subset M_{i,0,0}$, which convergence to $[L]$. This follows L is also f_i-vertical.
Because $[L] \in P_i$ is f_i-vertical of positive degree, we find a nontrivial section $s \in H^0(X, L^{\otimes n})$ for some $n \in N$. By taking the n-th roots of s we get a covering

$$\begin{array}{ccc} Y & \xrightarrow{\tau} & X \\ {\scriptstyle \tilde{f}_i}\downarrow & & \downarrow{\scriptstyle f_i} \\ \tilde{C}_i & \longrightarrow & C. \end{array}$$

The pull back $\tau^*(\sqrt[n]{s})$ is a section in $H^0(Y, \tau^*L)$ with the zero locus $\sum_i \tilde{F}_i$.
Because $\tau^*\omega_L \in H^0(Y, \Omega^1_Y(-\sum_i \tilde{F}_i))$, so $\tau^*\omega_L$ is a pull back of a 1-form on $\tilde{C}_i$. This means that $\tau^*\omega_L : \tau^*L \hookrightarrow \Omega^1_Y$ and $d\tilde{f}_i : \tilde{f}_i^*\Omega^1_{\tilde{C}_i} \hookrightarrow \Omega^1_Y$ is linear dependent, hence $\omega_L : L \hookrightarrow \Omega^1_X$ and $df_i : f^*\Omega^1_{C_i} \hookrightarrow \Omega^1_X$ are also linear dependent.

Proof iii) Two linear independent sections from $H^0(X, L^{\otimes n})$ give a nontrivial family of curves on X. By using the same arguments in the proof of i), we get also such a fibration. ■

Proof of theorem 2 Suppose $f_i : X \to C_i$ is the fibration in lemma 3.2 ii) or iii). If $\det \theta_L = 0$, then we restrict the exact sequence in corollary 1 to a generic fibre F of f_i. Since L is f_i-vertical, so $\deg(M^\vee|_F) = 0$. Because $(E|_F, i_F^*\theta)$ is poly-stable and $\omega_L : L \hookrightarrow \Omega^1_X$, $df_i : f_i^*\Omega^1_{C_i} \hookrightarrow \Omega^1_X$ are linear dependent, therefore, $(E|_F, i_F^*\theta) = (M^\vee|_F \oplus (M \otimes \det E)|_F, 0)$.
Suppose $\det \theta_L \neq 0$. Looking at the spectral surface $\tilde{X}$ of (E, θ_L) and its resolution of singulatities

$$\begin{array}{ccc} \tilde{X}' & \xrightarrow{\rho} & X \\ {\scriptstyle \tilde{f}_i}\downarrow & & \downarrow{\scriptstyle f_i} \\ \tilde{C}_i & \longrightarrow & C_i \end{array}$$

because L is a f_i-vertical line bundle, there is a splitting $\rho^*F = \tilde{F}_1 + \tilde{F}_2$ for a generic fibre F of

f_i. On $\tilde{X}'$ we have the exact sequence

$$0 \to \tilde{M}'^{\vee} \to \rho^* E \to \tilde{M}' \otimes \rho^* \det E \to 0,$$

where $\tilde{M}'^{\vee} = \mathrm{Ker}(\rho^*(\theta_L) - I \otimes \rho^*(\sqrt{\det \theta_L})$ and $\tilde{M}' \otimes i^* \tilde{M}' \otimes \det E \simeq \rho^* L$. Restricting the exact sequence to $\tilde{F}_1$ we have $\tilde{M}'^{\vee} \tilde{F}_1 = \tilde{M}'^{\vee} \rho^* F/2 = -\rho^* L \rho^* F/4 = 0$. Similar to the first case, we get $(E|_F, i_F^* \theta) \simeq (M'^{\vee}|_{\tilde{F}_1} \oplus (M' \otimes \rho^* \det E)|_{\tilde{F}_1}, 0)$. ■

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