CM lifts for Isogeny Classes of Shimura F-crystals over Finite Fields

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ABSTRACT. We extend to large contexts pertaining to Shimura varieties of Hodge type a result of Zink on the existence of CM lifts to characteristic 0 of suitable representatives of certain isogeny classes of abelian varieties endowed with endomorphisms over finite fields. These contexts are general enough in order to apply to the Langlands–Rapoport conjecture for all special fibres of characteristic at least 5 of integral canonical models of Shimura varieties of Hodge type.

Key words: finite fields, p-divisible groups, F-crystals, reductive group schemes, abelian varieties, isogenies, complex multiplication, Newton polygons, Shimura varieties, and integral models.

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Contents

1. Introduction

Let $p \in \mathbb{N}$ be a prime. Let $r \in \mathbb{N}$. Let $q := p^r$. Let $k := \mathbb{F}_q$ be the field with q elements. Let $\bar{k}$ be an algebraic closure of k. Let $W(k)$ be the ring of Witt vectors with coefficients in k. Let $B(k) := W(k)[\frac{1}{p}]$ be the field of fractions of $W(k)$. Let $\sigma := \sigma_k$ be the Frobenius automorphism of k, $W(k)$, and $B(k)$. The Honda–Serre–Tate theory classified the isogeny classes of abelian varieties over k (see [Ta2, Thm. 1]) and in particular showed that each abelian variety over k, up to an extension to a finite field extension of k and up to an isogeny, lifts to an abelian scheme with complex multiplication over a discrete valuation ring of mixed characteristic $(0, p)$ (see [Ta2, Thm. 2]). We recall that an abelian scheme of relative dimension d over an integral scheme is with *complex multiplication* if its ring of endomorphisms has a commutative $\mathbb{Z}$-subalgebra of rank $2d$. Zink generalized [Ta2, Thms. 1 and 2] to contexts which involve suitable abelian varieties endowed with endomorphisms (see [Zi1, Thms. 4.4 and 4.7]). Special cases of loc. cit. were obtained or announced previously (see [Ii1] to [Ii3], [La], and [Mi1]). To detail these contexts and to prepare the background for our paper, we will use the language of reductive group schemes and of crystalline cohomology.

We recall that a group scheme F over an affine scheme $\mathrm{Spec}(R)$ is called *reductive* if it is smooth and affine and its fibres are connected and have trivial unipotent radicals. We denote by F^{der} and F^{ad} the derived group scheme and the adjoint group scheme (respectively) of F. If S is a closed subgroup scheme of F let $\mathrm{Lie}(S)$ be its Lie algebra over R. For a finite, flat monomorphism $R_0 \hookrightarrow R$ let $\mathrm{Res}_{R/R_0} S$ be the group scheme over R_0 obtained from S through the Weil restriction of scalars (see [BT, Subsection 1.5] and [BLR, Ch. 7, Subsection 7.6]). If R is moreover an étale R_0-algebra, then $\mathrm{Res}_{R/R_0} F$ is a reductive group scheme over R_0. The pull back of an object or a morphism $\dagger$ or $\dagger_{R_0}$ (resp. $\dagger_*$ with $*$ an index) of the category of $\mathrm{Spec}(R_0)$-schemes to $\mathrm{Spec}(R)$ is denoted by $\dagger_R$ (resp. $\dagger_{*R}$). If O is a free R-module of finite rank, let $\mathbf{GL}_O$ (resp. $\mathbf{SL}_O$) be the reductive group scheme over R of linear automorphisms (resp. of linear automorphisms of determinant 1) of O. If f_1 and f_2 are two $\mathbb{Z}$-endomorphisms of O let $f_1 f_2 := f_1 \circ f_2$.

1.1. Isogeny classes. Let D be a p-divisible group over k. Let (M, ϕ) be the (contravariant) *Dieudonné module* of D. Thus M is a free $W(k)$-module of finite rank and $\phi : M \to M$ is a σ-linear endomorphism such that we have an inclusion $pM \subseteq \phi(M)$. We denote also by ϕ the σ-linear automorphism of $\mathrm{End}(M[\frac{1}{p}])$ that maps $e \in \mathrm{End}(M[\frac{1}{p}])$ to $\phi(e) := \phi \circ e \circ \phi^{-1} \in \mathrm{End}(M[\frac{1}{p}])$. Let $\mathcal{G}$ be a reductive, closed subgroup scheme of $\mathbf{GL}_M$. We recall from [Va3] and [Va4] that the triple

$$\mathcal{C} := (M, \phi, \mathcal{G})$$

is called a *Shimura F-crystal* over k if there exists a direct sum decomposition $M = F^1 \oplus F^0$ such that the following two axioms hold:

(i) we have identities $\phi(M + \frac{1}{p} F^1) = M$ and $\phi(\mathrm{Lie}(\mathcal{G}_{B(k)})) = \mathrm{Lie}(\mathcal{G}_{B(k)})$, and

(ii) the cocharacter $\mu : \mathbb{G}_m \to \mathbf{GL}_M$ that acts trivially on F^0 and as the inverse of the identical character of $\mathbb{G}_m$ on F^1, factors through $\mathcal{G}$.

Until the end we will assume that $\mathcal{C}$ is a Shimura F-crystal over k and that $M = F^1 \oplus F^0$ is a direct sum decomposition for which the axioms (i) and (ii) hold.

The quadruple $(M, F^1, \phi, \mathcal{G})$ is called a *Shimura filtered F-crystal* over k. Either $(M, F^1, \phi, \mathcal{G})$ or F^1 is called a *lift* of $\mathcal{C}$ (to $W(k)$). By an *endomorphism* of $\mathcal{C}$ (resp. of

$(M, F^1, \phi, \mathcal{G}))$ we mean an element $e \in \mathrm{Lie}(\mathcal{G})$ fixed by ϕ (resp. fixed by ϕ and such that we have an inclusion $e(F^1) \subseteq F^1$). We emphasize that the set of endomorphisms of $\mathcal{C}$ (resp. of $(M, F^1, \phi, \mathcal{G})$) is in general only a Lie algebra over $\mathbb{Z}_p$ (and not a $\mathbb{Z}_p$-algebra).

Let $\mathfrak{P}(\mathcal{C})$ be the set of elements $h \in \mathbf{GL}_M(B(k))$ for which the triple

$$(h(M), \phi, \mathcal{G}(h)) \tag{1}$$

is a Shimura F-crystal over k which can be extended to a Shimura filtered F-crystal $(h(M), \tilde{h}(F^1[\frac{1}{p}]) \cap h(M), \phi, \mathcal{G}(h))$ over k, where $\tilde{h} \in \mathcal{G}(B(k))$ and where $\mathcal{G}(h)$ is the Zariski closure of $\mathcal{G}_{B(k)}$ in $\mathbf{GL}_{h(M)}$. Let $\mathfrak{I}(\mathcal{C}) := \mathfrak{P}(\mathcal{C}) \cap \mathcal{G}(B(k))$. It is easy to see that we have an identity

$$\mathfrak{I}(\mathcal{C}) = \{h \in \mathcal{G}(B(k)) | \exists u \in \mathcal{G}(W(k)) \text{ such that } u^{-1}h^{-1}\phi h u \phi^{-1} \in \mathcal{G}(W(k))\}.$$

The reductive group scheme $\mathcal{G}(h)$ is isomorphic to $\mathcal{G}$ (if $h \notin \mathfrak{I}(\mathcal{C})$, then this follows from [Ti2]). For $i \in \{1, 2\}$ let $h_i \in \mathfrak{I}(\mathcal{C})$ and $g_i \in \mathcal{G}(h_i)(W(k))$. By an *inner isomorphism* between $(h_1(M), g_1\phi, \mathcal{G}(h_1))$ and $(h_2(M), g_2\phi, \mathcal{G}(h_2))$ we mean an element $g \in \mathcal{G}(B(k))$ such that we have $g(h_1(M)) = h_2(M)$ and $gg_1\phi = g_2\phi g$.

By the *isogeny class* of $\mathcal{C}$ we mean the set $\mathcal{I}(\mathcal{C})$ of inner isomorphism classes of Shimura F-crystals over k that are of the form $(h(M), \phi, \mathcal{G}(h))$ with $h \in \mathfrak{I}(\mathcal{C})$. Ideally, one would like to describe the set $\mathcal{I}(\mathcal{C})$ in a way which allows "the reading" of different Lie algebras of endomorphisms of (ramified) lifts of its representatives. Abstract ramified lifts of $\mathcal{C}$ (or of D with respect to $\mathcal{G}$) are formalized in Subsection 3.3. In this introduction we will only mention the abelian varieties counterpart of the ramified lifts.

1.1.1. Two geometric operations. Until Subsubsection 1.4.1 we will assume that D is the p-divisible group of an abelian variety A over k.

By a $\mathbb{Z}[\frac{1}{p}]$-isogeny between two abelian schemes A_1 and A_2 over a given scheme we mean a $\mathbb{Q}$–isomorphism between A_1 and A_2 that induces an isomorphism $A_1[N] \tilde{\to} A_2[N]$ for all natural numbers N that are relatively prime to p. For each $h \in \mathfrak{P}(\mathcal{C})$ there exists a unique abelian variety $A(h)$ over k which is $\mathbb{Z}[\frac{1}{p}]$-isogenous to A and such that under this $\mathbb{Z}[\frac{1}{p}]$-isogeny the Dieudonné module of $A(h)$ is identified with $(h(M), \phi)$. If $h \in \mathfrak{I}(\mathcal{C})$, then we say $A(h)$ is $\mathcal{G}$-*isogenous* to A. In all that follows we study the pair $(A, \mathcal{G})$ only up to the following two operations.

$\mathfrak{O}_1$ *The extension of A to a finite field extension of k.*

$\mathfrak{O}_2$ *The replacement of A by an abelian variety $A(h)$ over k which is $\mathcal{G}$-isogenous to it.*

1.2. Main Problem. *Up to operations $\mathfrak{O}_1$ and $\mathfrak{O}_2$, find conditions which guarantee that there exists a triple $(V, A_V, \mathcal{G}'_V)$, where V is a finite, discrete valuation ring extension of $W(k)$ of residue field k, A_V is an abelian scheme over V that lifts A, and $\mathcal{G}'_V$ is a reductive, closed subgroup scheme of $\boldsymbol{GL}_{H^1_{\mathrm{dR}}(A_V/V)}$, such that the following four conditions hold:*

(a) *the abelian scheme A_V is with complex multiplication;*

(b) *under the canonical identification $M/pM = H^1_{\mathrm{dR}}(A/V)/m_V H^1_{\mathrm{dR}}(A/V)$, the group scheme $\mathcal{G}'_V$ lifts $\mathcal{G}_k$ (here m_V is the maximal ideal of V);*

(c) *under the canonical identification* $H^1_{\mathrm{dR}}(A/V)[\frac{1}{p}] = M\otimes_{W(k)}V[\frac{1}{p}]$ *(see [BO, Thm. 1.3]), the generic fibre of* $\mathcal{G}'_V$ *is the pull back to Spec*$(V[\frac{1}{p}])$ *of* $\mathcal{G}_{B(k)}$*;*

(d) *there exists a cocharacter* $\mathbb{G}_m \to \mathcal{G}'_V$ *that acts on* F^1_V *via the inverse of the identical character of* $\mathbb{G}_m$ *and that fixes* $H^1_{\mathrm{dR}}(A_V/V)/F^1_V$*, where* F^1_V *is the direct summand of* $H^1_{\mathrm{dR}}(A_V/V)$ *which is the Hodge filtration of* A_V*.*

If (c) holds, then the group schemes $\mathcal{G}'_V$ and $\mathcal{G}_V$ are isomorphic (cf. [Ti2]). If only (b) to (d) hold and $V = W(k)$ (resp. and $V \neq W(k)$), then we refer to A_V as a *lift* of A (resp. as a *ramified lift* of A to V) with respect to $\mathcal{G}$.

Let $\mathfrak{e}$ be the $B(k)$-span inside $\mathrm{End}(M[\frac{1}{p}])$ of those endomorphisms of $(M, \phi, \mathcal{G})$ which are crystalline realizations of endomorphisms of A. It is the Lie algebra of a unique connected subgroup $\mathcal{E}$ of $\mathcal{G}_{B(k)}$. The uniqueness of $\mathcal{E}$ follows from [Bo, Ch. II, Subsection 7.1] and the existence of $\mathcal{E}$ is a standard application of the fact that the $\mathbb{Q}$–algebra of $\mathbb{Q}$–endomorphisms of A is semisimple. The triple $(V, A_V, \mathcal{G}'_V)$ does not always exist (simple examples can be constructed with $\mathcal{G}$ a torus). The reason for this is: in general the ranks of $\mathcal{E}$ and $\mathcal{G}_{B(k)}$ are not equal. Thus in order to motivate the Main Problem and to list accurately conditions under which one expects that such a triple exists, next we will recall some terminology pertaining to Hodge cycles and Shimura varieties.

1.2.1. A review. We use the terminology of [De3, Section 2] for Hodge cycles on an abelian scheme B over a reduced $\mathbb{Q}$–scheme Z. Thus we write a *Hodge cycle* v on B as a pair $(v_{\mathrm{dR}}, v_{ét})$, where v_{dR} and $v_{ét}$ are the de Rham component and the étale component (respectively) of v. The étale component $v_{ét}$ as its turn has an l-component $v^l_{ét}$ for each prime $l \in \mathbb{N}$. For instance, if Z is the spectrum of a subfield E of $\bar{\mathbb{Q}} \subseteq \mathbb{C}$, then $v^p_{ét}$ is a suitable $\mathrm{Gal}(E)$-invariant tensor of the tensor algebra of $H^1_{ét}(B_{\bar{\mathbb{Q}}}, \mathbb{Q}_p) \oplus (H^1_{ét}(B_{\bar{\mathbb{Q}}}, \mathbb{Q}_p))^* \oplus \mathbb{Q}_p(1)$, where $(H^1_{ét}(B_{\bar{\mathbb{Q}}}, \mathbb{Q}_p))^*$ is the dual vector space of $H^1_{ét}(B_{\bar{\mathbb{Q}}}, \mathbb{Q}_p)$ (i.e., it is the tensorization with $\mathbb{Q}_p$ of the Tate module of $B_{\bar{\mathbb{Q}}}$) and where $\mathbb{Q}_p(1)$ is the usual Tate twist. If E is a subfield of $\mathbb{C}$, then the Betti realization v_B of v corresponds to v_{dR} (resp. to $v^l_{ét}$) via the standard isomorphism that relates the de Rham (resp. the $\mathbb{Q}_l$ étale) cohomology of $B_{\mathbb{C}}$ with the Betti cohomology of the complex manifold $B(\mathbb{C})$ with $\mathbb{Q}$–coefficients (see [De3, Sections 1 and 2]).

A *Shimura pair* $(G, \mathcal{X})$ consists of a reductive group G over $\mathbb{Q}$ and a $G(\mathbb{R})$-conjugacy class $\mathcal{X}$ of homomorphisms $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_{\mathbb{R}}$ that satisfy Deligne's axioms of [De2, Subsubsection 2.1.1]: the Hodge $\mathbb{Q}$–structure of $\mathrm{Lie}(G)$ defined by any $x \in \mathcal{X}$ is of type $\{(-1,1),(0,0),$
$(1,-1)\}$, $\mathrm{Ad}(x(i))$ defines a Cartan involution of $\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{R}})$, and no simple factor of G^{ad} becomes compact over $\mathbb{R}$. Here $\mathrm{Ad} : G_{\mathbb{R}} \to \mathbf{GL}_{\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{R}})}$ is the adjoint representation. These axioms imply that $\mathcal{X}$ has a natural structure of a hermitian symmetric domain, cf. [De2, Cor. 1.1.17]. The most studied Shimura pairs are constructed as follows. Let W be a vector space over $\mathbb{Q}$ of even dimension $2d$. Let ψ be a non-degenerate alternative form on W. Let $\mathcal{S}$ be the set of all monomorphisms $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \hookrightarrow \mathbf{GSp}(W\otimes_{\mathbb{Q}}\mathbb{R}, \psi)$ that define Hodge $\mathbb{Q}$–structures on W of type $\{(-1,0),(0,-1)\}$ and that have either $2\pi i\psi$ or $-2\pi i\psi$ as polarizations. The pair $(\mathbf{GSp}(W,\psi), \mathcal{S})$ is a Shimura pair that defines a *Siegel modular variety*, cf. [Mi3, p. 161]. See [De1], [De2], [Mi4], and [Va1, Subsection 2.5] for different types of Shimura pairs and for their attached Shimura varieties. We recall that $(G, \mathcal{X})$ is called of

Hodge type, if it can be embedded into a Shimura pair of the form $(\mathbf{GSp}(W,\psi),\mathcal{S})$. We recall that Shimura varieties of Hodge type are moduli spaces of polarized abelian schemes endowed with Hodge cycles, cf. [De1], [De2], [Mi4], and [Va1, Subsection 4.1].

In this paragraph we will assume that the adjoint group G^{ad} is $\mathbb{Q}$–simple. Let θ be the Lie type of any simple factor of $G^{\mathrm{ad}}_{\mathbb{C}}$. If $\theta \in \{A_n, B_n, C_n | n \in \mathbb{N}\}$, then $(G,\mathcal{X})$ is said to be of θ type. If $\theta = D_n$ with $n \geq 4$, then $(G,\mathcal{X})$ is of one of the following three disjoint types: $D_n^{\mathbb{H}}$, $D_n^{\mathbb{R}}$, and D_n^{mixed} (cf. [De2] and [Mi4]). If $(G,\mathcal{X})$ is of $D_n^{\mathbb{R}}$ (resp. of $D_n^{\mathbb{H}}$) type, then all simple, non-compact factors of $G^{\mathrm{ad}}_{\mathbb{R}}$ are isomorphic to $\mathbf{SO}(2,2n-2)^{\mathrm{ad}}_{\mathbb{R}}$ (resp. to $\mathbf{SO}^*(2n)^{\mathrm{ad}}_{\mathbb{R}}$) and the converse of this statement holds for $n \geq 5$ (see [He, p. 445] for the classical groups $\mathbf{SO}(2,2n-2)^{\mathrm{ad}}_{\mathbb{R}}$ and $\mathbf{SO}^*(2n)^{\mathrm{ad}}_{\mathbb{R}}$). If moreover $(G,\mathcal{X})$ is of Hodge type, then $(G,\mathcal{X})$ is of one of the following five possible types: A_n, B_n, C_n, $D_n^{\mathbb{H}}$, and $D_n^{\mathbb{R}}$ (see [Sa1], [Sa2], and [De2, Table 2.3.8]).

1.2.2. Conjecture. *We assume that one of the following two conditions holds:*

(i) *the group $\mathcal{E}$ has the same rank as $\mathcal{G}_{B(k)}$;*

(ii) *there exists an abelian scheme $A_{W(k)}$ over $W(k)$ which lifts A and for which there exists a family $(w_\alpha)_{\alpha\in\mathcal{J}}$ of Hodge cycles on its generic fibre $A_{B(k)}$ such that $\mathcal{G}_{B(k)}$ is the subgroup of $\boldsymbol{GL}_{M[\frac{1}{p}]}$ that fixes the crystalline realization t_α of w_α for all $\alpha \in \mathcal{J}$.*

Then up to the operations $\mathfrak{O}_1$ and $\mathfrak{O}_2$, there exists a triple $(V, A_V, \mathcal{G}'_V)$ such that all conditions 1.2 (a) to (d) hold.

If (i) (resp. (ii)) holds, then we refer to Conjecture 1.2.2 as Conjecture 1.2.2 (i) (resp. Conjecture 1.2.2 (ii)). Conjecture 1.2.2 stems from the Langlands–Rapoport conjecture (see [LR], [Mi2], [Mi3], [Pf], and [Re2]) on $\bar{k}$-valued points of special fibres of (see [Va1] for precise definitions) integral canonical models of Shimura varieties of Hodge type in mixed characteristic $(0,p)$. This motivic conjecture of combinatorial nature is a key ingredient in the understanding of zeta functions of Shimura varieties of Hodge type and of different trace functions that pertain to $\mathbb{Q}_l$-local systems on quotients of finite type of such integral canonical models (for instance, see [LR], [Ko2], and [Mi3]; here l is a prime different from p). Conjecture 1.2.2 (ii) is in fact only a slight refinement of an adequate translation of a part of the Langlands–Rapoport conjecture. The Langlands–Rapoport conjecture is known to be true for Siegel modular varieties (see [Mi2]) and for certain Shimura varieties of A_1 type (see [Ii2], [Ii3], and [Re1]).

We added Conjecture 1.2.2 (i) due to two reasons. First, if one assumes the standard Hodge conjecture for complex abelian varieties (see [Le, Ch. 7]), then $(ii) \Rightarrow (i)$. Second, often due to technical reasons one assumes that $\mathcal{G}^{\mathrm{der}}$ is simply connected and this excludes the cases related to Shimura varieties of $D_n^{\mathbb{H}}$ type (see [De2, Rm. 1.3.10 (ii)]). Thus to handle Conjecture 1.2.2 (ii) in cases related to Shimura varieties of $D_n^{\mathbb{H}}$ type one has to first solve Conjecture 1.2.2 (ii) in cases related to Shimura varieties of C_{2n} type and then to appeal to relative PEL situations as defined in [Va1, Subsubsection 4.3.16] in order to reduce Conjecture 1.2.2 (ii) to Conjecture 1.2.2 (i) for these cases related to the $D_n^{\mathbb{H}}$ type. In what follows we will also refer to the following two Subproblems of the Main Problem.

1.2.3. Subproblem. *Same as Main Problem but with condition 1.2 (a) replaced by the weaker condition that the Frobenius endomorphism of A lifts to an endomorphism of A_V.*

1.2.4. Subproblem. *Same as Main Problem but with condition 1.2 (a) replaced by the weaker condition that the p-divisible group of A_V is with complex multiplication (i.e., the image of the p-adic Galois representation associated to the Tate module $T_p(A_{V[\frac{1}{p}]})$ of $A_{V[\frac{1}{p}]}$, is formed by semisimple elements that commute).*

1.3. The classical PEL context. This is the context in which there exists a principal polarization λ_A of A and there exists a $\mathbb{Z}_{(p)}$-subalgebra $\mathcal{B}$ of $\mathrm{End}(M)$ of crystalline realizations of $\mathbb{Z}_{(p)}$-endomorphisms of A, such that the following two conditions hold:

(i) the $W(k)$-algebra $\mathcal{B}\otimes_{\mathbb{Z}_{(p)}} W(k)$ is semisimple, self dual with respect to the perfect alternating form $\lambda_A : M\otimes_{W(k)} M \to W(k)$ defined by λ_A (and denoted similarly), and is equal to the $W(k)$-algebra $\{e\in \mathrm{End}(M)|e \text{ fixed by } \mathcal{G}\}$;

(ii) the group $\mathcal{G}_{B(k)}$ is the identity component of the subgroup $C_1(\lambda_A)_{B(k)}$ of $\mathbf{GSp}(M[\frac{1}{p}],\lambda_A)$ that fixes each element of $\mathcal{B}[\frac{1}{p}]$.

As A is with complex multiplication and the $\mathbb{Q}$–algebra $\mathrm{End}(A)\otimes_{\mathbb{Z}}\mathbb{Q}$ is semisimple, in this context the group $\mathcal{E}$ is reductive and has the same rank as $\mathcal{G}_{B(k)}$; thus the condition 1.2.2 (i) holds. The existence up to operations $\mathfrak{O}_1$ and $\mathfrak{O}_2$ of a triple $(V, A_V, \mathcal{G}'_V)$ such that all the conditions 1.2 (a) to (d) hold was proved (using a slightly different language) in [Zi1, Thm. 4.4] for the cases when $\mathcal{G}_{B(k)} = C_1(\lambda_A)_{B(k)}$ (strictly speaking, loc. cit. assumes that $\mathcal{B}[\frac{1}{p}]$ is a $\mathbb{Q}$–simple algebra; but the case when $\mathcal{B}[\frac{1}{p}]$ is not $\mathbb{Q}$–simple gets easily reduced to the case when it is so). Loc. cit. also shows that (even if $p = 2$) we can choose the triple $(V, A_V, \mathcal{G}'_V)$ such that $\mathcal{B}$ lifts to a family of $\mathbb{Z}_{(p)}$-endomorphisms of A_V and that λ_A is the crystalline realization of a principal polarization of A_V. Some refinements of loc. cit., which are still weaker than the Langlands–Rapoport conjecture for the corresponding Shimura varieties of PEL type, were obtained in [ReZ] and [Ko2].

1.4. On results and tools. The goal of this paper is to solve Conjecture 1.2.2 (i) and Subproblems 1.2.3 and 1.2.4 in contexts general enough (see Corollary 8.3, Remark 8.4, and Section 9) so that the work in progress of Milne and us can be plugged in to result for $p\geq 5$ in complete proofs of Conjecture 1.2.2 (ii) and of the Langlands–Rapoport conjecture for Shimura varieties of Hodge type. The passage from the mentioned solutions to a solution of Conjecture 1.2.2 (ii) for the case when $p\geq 5$ and $\mathcal{G}^{\mathrm{der}}$ simply connected, is completely controlled by [Va1] to [Va4] and by the following two extra things (see Remarks 9.2.1 (b), 9.4.2, and 9.8 (d) for a brief account):

(i) the weak isogeny property which says that each rational stratification of [Va3, Subsection 5.3] has only one closed stratum;

(ii) announced results of Milne on abelian varieties over finite fields (see [Mi5]).

It is well known that the weak iosgeny property holds for Siegel modular varieties (for instance, see [Oo]): the Newton polygon stratification of the Mumford moduli scheme $\mathcal{A}_{d,1,N}$ over $\bar{k}$ has only one closed stratum (the supersingular one); here $d, N\in\mathbb{N}$, $N\geq 3$, and $g.c.d.(N,p) = 1$. The weak isogeny property requires methods different from the ones of this paper and thus we will prove it (at least for $p\geq 3$) in a future work.

The main tools we use in this paper are the following seven:

T1. The rational classification of Shimura F-crystals over $\bar{k}$ achieved in [Va3].

T2. Approximations of tori of reductive groups over $\mathbb{Q}$ (see [Ha, Lem. 5.5.3]).

T3. A new theory of *admissible cocharacters* of extensions of maximal tori of $\mathcal{G}_{B(k)}$ contained in tori of $\mathbf{GL}_{M[\frac{1}{p}]}$ whose Lie algebras are $B(k)$-generated by crystalline realizations of $\mathbb{Q}_p$–endomorphisms of A. In its abstract form, the theory refines [RaZ, Subsections 1.21 to 1.25] for Shimura F-crystals in two ways. First, it is over k and not only over $\bar{k}$. Second, in many cases it works without assuming that all Newton polygon slopes of $(\mathrm{Lie}(\mathcal{G}_{B(k)}), \phi)$ are 0 and moreover it applies to all such maximal tori of $\mathcal{G}_{B(k)}$. We emphasize that in connection to either this theory or loc. cit., [FR] does not bring anything new.

T4. In some cases we rely on [Zi1, Thm. 4.4] (see Theorem 9.6 and Remark 9.8 (b)).

T5. The classification for $p \geq 3$ of isogeny classes of p-divisible groups over p-adic fields (see [Br, Subsection 5.3]).

T6. The natural $\mathbb{Z}_p$ structure $\mathcal{G}_{\mathbb{Z}_p}$ of $\mathcal{G}$ defined by $\mathcal{C}$ (see Subsection 2.4) and the structure of the pointed set $H^1(\mathbb{Q}_p, \mathcal{G}_{\mathbb{Q}_p})$.

T7. The theory of [Va1, Subsection 4.3] of well positioned families of tensors.

See [Fo] for (weakly admissible or admissible) filtered modules over p-adic fields. Next we exemplify how the tools T1 to T7 work under some conditions. We emphasize that often we do have to perform either the operation $\mathfrak{O}_1$ or the operation $\mathfrak{O}_2$ but this will not be repeated in this paragraph. Based on [Va3, Thm. 3.1.2 (b) and (c)], in connection to Conjecture 1.2.2 (i) and to Subproblems 1.2.3 and 1.2.4 it suffices to refer to the case when all Newton polygon slopes of $(\mathrm{Lie}(\mathcal{G}_{B(k)}), \phi)$ are 0. Assuming that the condition 1.2.2 (i) holds, we show based on [Ha, Lem. 5.5.3] that there exist maximal tori of $\mathcal{G}_{B(k)}$ as mentioned in the tool T3 but with $\mathbb{Q}_p$ replaced by $\mathbb{Q}$. The essence of T3 can be described as follows. Starting from any such maximal torus of $\mathcal{G}_{B(k)}$ we show the existence of suitable cocharacters of its extension to a finite field extension $V[\frac{1}{p}]$ of $B(k)$ such that the resulting filtered modules over $V[\frac{1}{p}]$ are weakly admissible. For this, in some cases related to Shimura varieties of A_n, C_n, and $D_n^{\mathbb{H}}$ types we rely as well on [Zi1, Thm. 4.4] and accordingly some extra assumptions are imposed (roughly speaking we deal with Shimura varieties of Hodge type constructed in [De2, Prop. 2.3.10] but in the integral contexts of [Va1, Sections 5 and 6]). Using the tool T5 we get an isogeny class of p-divisible groups over V (here we require $p \geq 3$). Using the tool T6 we get natural choices of representatives of this isogeny class so that we end up in the étale context with a reductive group scheme over $\mathbb{Z}_p$ that corresponds via Fontaine comparison theory to $\mathcal{G}_{V[\frac{1}{p}]}$ (for this part, we usually assume that the pointed set $H^1(\mathbb{Q}_p, \mathcal{G}_{\mathbb{Q}_p})$ has only one class). Using the tool T7 we "transfer backwards" (as in [Va1, Subsections 5.2 and 5.3]) the mentioned reductive group scheme over $\mathbb{Z}_p$ in order to end up with a reductive group scheme $\mathcal{G}'_V$ in the de Rham context over V (here we require p big enough; for applications to the Langlands–Rapoport conjecture and to Conjecture 1.2.2 (ii) the condition $p \geq 5$ suffices, cf. [Va1]).

1.4.1. On contents. Motivated by general applications, in Sections 2 to 7 we work abstractly. Thus we work with an arbitrary Shimura F-crystal $\mathcal{C}$ over k and, even if by chance (M,ϕ) is the Dieudonné module of the p-divisible group of some abelian variety A over k, most often we do not impose any geometric condition on the group scheme $\mathcal{G}$ over $W(k)$ (of the type of conditions 1.2.2 (i) and (ii) or 1.3 (i) and (ii)). In Section 2 we develop a minute language that pertains to Subsection 1.1 and to the tool T3 which will allow us to solve in many cases stronger versions of Conjecture 1.2.2 (i) and of Subproblems 1.2.3 and 1.2.4. Different abstract CM-isogeny classifications are formalized in Section 3. In particular, Corollary 3.7.3 shows that if $p \geq 3$, then the set of ramified lifts of D with respect to $\mathcal{G}$ (see Definition 3.7.2) are in natural bijection to the ramified lifts of $\mathcal{C}$ (see Definitions 3.3.1). This is a stronger version of the classification of p-divisible groups over V achieved for $p \geq 3$ previously by Faltings, Breuil, and Zink (see [Fa], [Br], and [Zi2]). In Sections 2 and 3 we introduce as well the principally quasi-polarized context (see Subsubsection 3.3.3 for the corresponding variant of the set $\mathfrak{I}(\mathcal{C})$).

In Section 4 we state in the abstract context two basic results that pertain to the tool T3 (see Basic Theorems 4.1 and 4.2) and three Corollaries (see Corollaries 4.3 to 4.5). The basic results implicitly solve Subproblem 1.2.4 under certain conditions. Corollaries 4.3 to 4.5 are the very first situations of general nature where complete ramified CM-classifications as defined in Subsubsection 3.3.3 are accomplished; to "balance" the focus of [Zi1] on Shimura varieties of PEL type (and thus of either A_n or totally non-compact C_n or $D_n^{\mathbb{H}}$ type), they involve cases that pertain to Shimura varieties of B_n and $D_n^{\mathbb{R}}$ types. In Sections 5 to 7 we prove the results 4.1 to 4.5. In Sections 2 to 7 we also refer to the most puzzling aspect (question) of the Subproblems 1.2.3 and 1.2.4: *When we can choose V to be of index of ramification* 1 *(i.e., to be a Witt ring)*? The first applications to abelian varieties are included in Section 8 (see Corollary 8.3 and Remark 8.4 for our partial solutions to Conjecture 1.2.2 (i) and to Subproblems 1.2.3 and 1.2.4).

In Section 9 we introduce the integral context of moduli spaces of polarized abelian varieties endowed with (specializations of) Hodge cycles. See Subsections 9.2 to 9.6 for different properties and how they lead to generalizations of the results of Zink recalled in Subsection 1.3. See Example 9.7 for the very first example of general nature that involve compact Shimura varieties of C_n type which are not of PEL type and for which a stronger isogeny property is implied by [FC, Ch. VII, Prop. 4.3] and therefore to which we can already extend [Zi1, Thm. 4.4] (the extension of [Zi1, Thm. 4.7] is implicitly achieved by [LR] and [Mi3, Subsections 4.1 to 4.6]).

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2. Preliminaries

See Subsection 2.1 for our notations and conventions. Subsection 2.2 recalls some descent properties for connected, affine, algebraic groups in characteristic 0. In Subsections

2.3 and 2.4 we mainly introduce a language. In Subsections 2.5 and 2.6 we recall some definitions and a basic result. In Subsection 2.7 we introduce $W(k)$-algebras that are required for the ramified contexts of Sections 3 to 7.

2.1. Notations and conventions. Let R, F, and O be as before Subsection 1.1. We refer to [Va3, Subsection 2.2] for quasi-cocharacters of F. Let $Z(F)$ be the center of F; we have $F^{\mathrm{ad}} = F/Z(F)$. Let $Z^0(F)$ be the maximal torus of $Z(F)$; the quotient group scheme $Z(F)/Z^0(F)$ is a finite, flat group scheme over R of multiplicative type. Let $F^{\mathrm{ab}} := F/F^{\mathrm{der}}$; it is the maximal abelian quotient of F. Let F^{sc} be the simply connected semisimple group scheme cover of F^{der}. If S is a reductive, closed subgroup scheme of F, let $C_F(S)$ (resp. $N_F(S)$) be the centralizer (resp. the normalizer) of S in F. Thus $C_F(S)$ (resp. $N_F(S)$) is a closed subgroup scheme of F, cf. [DG, Vol. II, Exp. XI, Cor. 6.11]. If R is a finite, discrete valuation ring extension of $W(k)$, then $F(R)$ is called a *hyperspecial* subgroup of $F(R[\frac{1}{p}])$ (see [Ti2]). Let $O^* := \mathrm{Hom}_R(O,R)$. A bilinear form on O is called perfect if it induces naturally an isomorphism $O \xrightarrow{\sim} O^*$. We consider the free O-module

$$\mathcal{T}(O) := \oplus_{s,t\in\mathbb{N}\cup\{0\}} O^{\otimes s} \otimes_R O^{*\otimes t}.$$

We use the same notation for two perfect bilinear forms or tensors of two tensor algebras if they are obtained one from another via either a reduction modulo some ideal or a scalar extension. If $F^1(O)$ is a direct summand of O, then $F^0(O^*) := (O/F^1(O))^*$ is a direct summand of O^*. By the F^0-filtration of $\mathcal{T}(O)$ defined by $F^1(O)$ we mean the direct summand of $\mathcal{T}(O)$ whose elements have filtration degrees at most 0, where $\mathcal{T}(O)$ is equipped with the tensor product filtration defined by the decreasing, exhaustive, and separated filtrations $(F^i(O))_{i\in\{0,1,2\}}$ and $(F^i(O^*))_{i\in\{-1,0,1\}}$ of O and O^* (respectively). Here $F^0(O) := O$, $F^2(O) := 0$, $F^{-1}(O^*) := O^*$, and $F^1(O^*) := 0$. We always identify $\mathrm{End}(O)$ with $O \otimes_R O^*$. Thus $\mathrm{End}(\mathrm{End}(O)) = \mathrm{End}(O \otimes_R O^*) = O \otimes_R O^* \otimes_R O^* \otimes_R O$ is always identified by changing the order with the direct summand $O^{\otimes 2} \otimes_R O^{*\otimes 2}$ of $\mathcal{T}(O)$.

Let $x \in R$ be a non-divisor of 0. A family of tensors of $\mathcal{T}(O[\frac{1}{x}]) = \mathcal{T}(O)[\frac{1}{x}]$ is denoted $(u_\alpha)_{\alpha\in\mathcal{J}}$, with $\mathcal{J}$ as the set of indexes. Let O_1 be another free O-module of finite rank. Let $(u_{1\alpha})_{\alpha\in\mathcal{J}}$ be a family of tensors of $\mathcal{T}(O_1[\frac{1}{x}])$ indexed also by the set $\mathcal{J}$. By an isomorphism $(O,(u_\alpha)_{\alpha\in\mathcal{J}}) \xrightarrow{\sim} (O_1,(u_{1\alpha})_{\alpha\in\mathcal{J}})$ we mean an R-linear isomorphism $O \xrightarrow{\sim} O_1$ that extends naturally to an R-linear isomorphism $\mathcal{T}(O[\frac{1}{x}]) \xrightarrow{\sim} \mathcal{T}(O_1[\frac{1}{x}])$ which takes u_α to $u_{1\alpha}$ for all $\alpha \in \mathcal{J}$.

If K is a field, let $\bar{K}$ be an algebraic closure of K. If K is a p-adic field, see [Fo] for de the Rham ring $B_{\mathrm{dR}}(K)$ and for admissible Galois representations of the Galois group $\mathrm{Gal}(K) := \mathrm{Gal}(\bar{K}/K)$. For the classification of Lie and Dynkin types we refer to [Bou1] and [DG, Vol. III, Exp. XXII and XXIII]. Whenever we use a D_n type, we assume that $n \geq 4$. Let $\mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ at its prime ideal (p).

By a Frobenius lift of a flat $\mathbb{Z}_{(p)}$-algebra R we mean an endomorphism $\Phi_R : R \to R$ which modulo p is the usual Frobenius endomorphism of R/pR. If $\phi_O : O \to O$ is a Φ_R-linear endomorphism such that $O[\frac{1}{p}]$ is $R[\frac{1}{p}]$-generated by $\phi_O(O)$, then we denote also by ϕ_O the Φ_R-linear endomorphism of each R-submodule of $\mathcal{T}(O)[\frac{1}{p}]$ left invariant by ϕ_O. We recall that ϕ_O acts on $O^*[\frac{1}{p}]$ via the rule: if $f \in O^*[\frac{1}{p}]$ and $e \in O[\frac{1}{p}]$, then

$\phi_O(f)(\phi_O(e)) = \Phi_R(f(e)) \in R[\frac{1}{p}]$. If ϕ_O becomes an isomorphism after inverting p and if μ_O is a cocharacter of $\mathbf{GL}_{O[\frac{1}{p}]}$, then let $\phi_O(\mu_R) := \phi_O\mu_O\phi_O^{-1}$.

Always $\mathcal{C} := (M, \phi, \mathcal{G})$ (resp. $(M, F^1, \phi, \mathcal{G})$) is a Shimura (resp. Shimura filtered) F-crystal over $k = \mathbb{F}_q$. We fix a cocharacter $\mu : \mathbb{G}_m \to \mathcal{G}$ of $\mathcal{C}$ as in Subsection 1.1 (thus it normalizes F^1); we call it a *Hodge cocharacter* of $\mathcal{C}$ and we say that it defines F^1. Let $\mathcal{P}$ be the parabolic subgroup scheme of $\mathcal{G}$ which is the normalizer of F^1 in $\mathcal{G}$. Let the sets $\mathfrak{I}(\mathcal{C})$, $\mathfrak{P}(\mathcal{C})$, and $\mathcal{I}(\mathcal{C})$ be as in Subsection 1.1. Let

$$C := C_{\mathbf{GL}_M}(\mathcal{G}).$$

If C is a reductive group scheme over $W(k)$, then let $C_1 := C_{\mathbf{GL}_M}(C)$.

See [Va3, Subsubsections 2.2.1 and 2.2.3] for the Newton quasi-cocharacter of $\mathcal{C}$. Let $P^+_{\mathcal{G}}(\phi)$, $P^-_{\mathcal{G}}(\phi)$, and $L^0_{\mathcal{G}}(\phi)_{B(k)}$ be the non-negative parabolic subgroup scheme, the non-positive parabolic subgroup scheme, and the Levi subgroup (respectively) of $\mathcal{C}$ we defined in [Va3, Lem. 2.3.1 and Def. 2.3.3]. Thus $P^+_{\mathcal{G}}(\phi)$ is the parabolic subgroup scheme of $\mathcal{G}$ which is maximal subject to the property that $\text{Lie}(P^+_{\mathcal{G}}(\phi)_{B(k)})$ is normalized by ϕ and all Newton polygon slopes of $(\text{Lie}(P^+_{\mathcal{G}}(\phi)_{B(k)}), \phi)$ are non-negative, $P^-_{\mathcal{G}}(\phi)$ is defined similarly but by replacing non-negative with non-positive, and $L^0_{\mathcal{G}}(\phi)_{B(k)}$ is the unique Levi subgroup of either $P^+_{\mathcal{G}}(\phi)_{B(k)}$ or $P^-_{\mathcal{G}}(\phi)_{B(k)}$ with the property that $\text{Lie}(L^0_{\mathcal{G}}(\phi)_{B(k)})$ is normalized by ϕ and all Newton polygon slopes of $(\text{Lie}(L^0_{\mathcal{G}}(\phi)_{B(k)}), \phi)$ are 0. We have $P^+_{\mathcal{G}}(\phi)_{B(k)} \cap P^-_{\mathcal{G}}(\phi)_{B(k)} = L^0_{\mathcal{G}}(\phi)_{B(k)}$. Let $U^+_{\mathcal{G}}(\phi)$ be the unipotent radical of $P^+_{\mathcal{G}}(\phi)$. Let $L^0_{\mathcal{G}}(\phi)$ be the Zariski closure of $L^0_{\mathcal{G}}(\phi)_{B(k)}$ in $\mathcal{G}$ (or $P^+_{\mathcal{G}}(\phi)$); we emphasize that it is not always a Levi subgroup scheme of $P^+_{\mathcal{G}}(\phi)$. We say $\mathcal{C}$ is *basic* if all Newton polygon slopes of $(\text{Lie}(\mathcal{G}_{B(k)}), \phi)$ are 0 (i.e., if $P^+_{\mathcal{G}}(\phi) = P^-_{\mathcal{G}}(\phi) = \mathcal{G}$).

Always k_1 is a finite field extension of k. For $l \in \{k_1, \bar{k}\}$, let $W(l)$, $B(l)$, and σ_l be the analogues of $W(k)$, $B(k)$, and σ_k but for l instead of l. Let

$$\mathcal{C} \otimes l := (M \otimes_{W(k)} W(l), \phi \otimes \sigma_l, \mathcal{G}_{W(l)})$$

be the extension of $\mathcal{C}$ to l. We also refer as $\mathfrak{O}_1$ to the operation of replacing k by k_1 and $\mathcal{C}$ by $\mathcal{C} \otimes k_1$, and as $\mathfrak{O}_2$ to the operation of replacing $\mathcal{C}$ by $(h(M), \phi, \mathcal{G}(h))$, where $h \in \mathfrak{I}(\mathcal{C})$ and $\mathcal{G}(h)$ are as in Subsection 1.1. For $g \in \mathcal{G}(W(k))$ let

$$\mathcal{C}_g := (M, g\phi, \mathcal{G}).$$

We have $\mathcal{C} = \mathcal{C}_{1_M}$. Let $\mathcal{F} := \{\mathcal{C}_g | g \in G(W(k))\}$ be the *family* of Shimura F-crystals over k associated naturally to $\mathcal{C}$. Let $\mathcal{Y}(\mathcal{F}) := \cup_{g \in \mathcal{G}(W(k))} \mathcal{I}(\mathcal{C}_g)$. The (inner) isomorphism class of some object $\bigstar$ will be denoted as $[\bigstar]$.

Though in this paper we deal only with Shimura F-crystals, Sections 2 to 4 are organized in such a way that the interested reader can extend their notions to the context of p-divisible objects with a reductive group over k used in [Va3] (often even over an arbitrary perfect field of characteristic p).

2.2. Lemma. *Let $\eta \subseteq \eta_1$ be an extension of fields of characteristic* 0. *Let $\mathfrak{G}$ be a connected, affine, algebraic group over η. Let $\mathcal{L}$ be a Lie subalgebra of Lie*$(\mathfrak{G})$. *We assume* *that there exists a connected (resp. reductive) subgroup $\mathfrak{S}_{\eta_1}$ of $\mathfrak{G}_{\eta_1}$ whose Lie algebra is $\mathfrak{L}\otimes_\eta \eta_1$. We have:*

(a) *there exists a unique connected (resp. reductive) subgroup $\mathfrak{S}$ of $\mathfrak{G}$ whose Lie algebra is $\mathfrak{L}$ (the notations match i.e., its extension to η_1 is $\mathfrak{S}_{\eta_1}$);*

(b) *if $\mathfrak{S}$ is a reductive group and if $\mathfrak{G}$ is the general linear group $\boldsymbol{GL}_W$ of a finite dimensional η-vector space W, then the restriction of the trace form on $End(W)$ to $\mathfrak{L}$ is non-degenerate.*

Proof: We prove (a). The uniqueness part is implied by [Bo, Ch. I, Subsection 7.1]. Loc cit. also implies that if $\mathfrak{S}$ exists, then its extension to η_1 is indeed $\mathfrak{S}_{\eta_1}$. It suffices to prove (a) for the case when $\mathfrak{S}$ is connected. We consider commutative η-algebras κ for which there exists a closed subgroup scheme $\mathfrak{S}_\kappa$ of $\mathfrak{G}_\kappa$ whose Lie algebra is $\mathfrak{L}\otimes_\eta \kappa$. Our hypotheses imply that as κ we can take η_1. Thus as κ we can also take a finitely generated η-subalgebra of η_1. By considering the reduction modulo a maximal ideal of this last η-algebra, we can assume that κ is a finite field extension of η. Even more, (as η has characteristic 0) we can assume that κ is a finite Galois extension of η. By replacing $\mathfrak{S}_\kappa$ with its identity component, we can assume that $\mathfrak{S}_\kappa$ is connected. Due to the mentioned uniqueness part, the Galois group $\mathrm{Gal}(\kappa/\eta)$ acts naturally on the connected subgroup $\mathfrak{S}_\kappa$ of $\mathfrak{G}_\kappa$. As $\mathfrak{S}_\kappa$ is an affine scheme, the resulting Galois descent on $\mathfrak{S}_\kappa$ with respect to $\mathrm{Gal}(\kappa/\eta)$ is effective (cf. [BLR, Ch. 6, 6.1, Thm. 5]). This implies the existence of a subgroup $\mathfrak{S}$ of $\mathfrak{G}$ whose extension to κ is $\mathfrak{S}_\kappa$. As $\mathrm{Lie}(\mathfrak{S})\otimes_\eta \kappa=\mathrm{Lie}(\mathfrak{S}_\kappa)=\mathfrak{L}\otimes_\eta \kappa$, we have $\mathrm{Lie}(\mathfrak{S})=\mathfrak{L}$. The group $\mathfrak{S}$ is connected as $\mathfrak{S}_\kappa$ is so. Thus $\mathfrak{S}$ exists i.e., (a) holds.

To check (b) we can assume that η is algebraically closed. Using isogenies, it suffices to prove (b) in the case when $\mathfrak{S}$ is either $\mathbb{G}_m$ or a semisimple group whose adjoint is simple. If $\mathfrak{S}$ is $\mathbb{G}_m$, then the $\mathfrak{S}$-module W is a direct sum of one dimensional $\mathfrak{S}$-modules. We easily get that there exists an element $x\in\mathfrak{L}\setminus\{0\}$ which is a semisimple element of $\mathrm{End}(W)$ whose eigenvalues are integers. The trace of x^2 is a non-trivial sum of squares of natural numbers and thus it is non-zero. Thus (b) holds if $\mathfrak{S}$ is $\mathbb{G}_m$. If $\mathfrak{S}$ is a semisimple group whose adjoint is simple, then $\mathfrak{L}$ is a simple Lie algebra over η. From Cartan solvability criterion we get that the restriction of the trace form on $\mathrm{End}(W)$ to $\mathfrak{L}$ is non-zero and therefore (as $\mathfrak{L}$ is a simple Lie algebra) it is non-degenerate. Thus (b) holds. $\square$

2.2.1. Example. We take $\eta=\mathbb{Q}_p$ and $\eta_1=B(\star)$, where $\star$ is a perfect field of characteristic p. Let $(\mathcal{W},\varphi)$ be an F-crystal over $\star$. Let $\mathfrak{G}$ be the group over $\mathbb{Q}_p$ which is the group scheme of invertible elements of the $\mathbb{Q}_p$-algebra $\{e\in\mathrm{End}(\mathcal{W})|\varphi(e)=e\}$. Let $\diamond$ be a connected subgroup of $\mathfrak{G}_{\eta_1}$ whose Lie algebra is η_1-generated by elements fixed by φ. From Lemma 2.2 (a) we get that $\diamond$ is the extension to η_1 of the unique connected subgroup $\diamond_{\mathbb{Q}_p}$ of $\mathfrak{G}$ whose Lie algebra is $\{e\in\mathrm{Lie}(\diamond)|\varphi(e)=e\}$. We refer to $\diamond_{\mathbb{Q}_p}$ as the *$\mathbb{Q}_p$-form of $\diamond$ with respect to* $(\mathcal{W},\varphi)$.

2.3. Basic definitions. (a) We say $\mathcal{C}$ has a *lift of quasi CM type* if there exists a maximal torus $\mathcal{T}$ of $\mathcal{G}$ such that we have $\phi(\mathrm{Lie}(\mathcal{T}))=\mathrm{Lie}(\mathcal{T})$.

(b) We say $\mathcal{C}$ is *semisimple* (resp. *unramified semisimple*) if the $B(k)$-linear automorphism ϕ^r of $M[\frac{1}{p}]$ is a semisimple element of $\mathcal{G}(B(k))$ (resp. is a semisimple element of $\mathcal{G}(B(k))$ such that an integral power of it has all its eigenvalues belonging to $B(\bar{k})$).

(c) By a torus of $\mathcal{G}_{B(k)}$ of $\mathbb{Q}_p$-endomorphisms of $\mathcal{C}$ we mean a torus $\mathcal{T}_{1B(k)}$ of $\mathcal{G}_{B(k)}$ whose Lie algebra is $B(k)$-generated by elements fixed by ϕ. Let $\mathcal{T}_{1\mathbb{Q}_p}$ be the $\mathbb{Q}_p$-form of $\mathcal{T}_{1B(k)}$ with respect to $(M[\frac{1}{p}],\phi)$, cf. Example 2.2.1. Let K be the smallest Galois extension of $\mathbb{Q}_p$ over which $\mathcal{T}_{1\mathbb{Q}_p}$ splits. Let K_1 be the smallest unramified extension of K which is unramified over a totally ramified extension K_{1r} of $\mathbb{Q}_p$. Let K_2 be the composite field of K_1 and $B(k)$. Let K_{2u} be the maximal unramified extension of $\mathbb{Q}_p$ included in K_2.

(d) By an *E-pair* of $\mathcal{C}$ we mean a pair $(\mathcal{T}_{1B(k)},\mu_1)$, where $\mathcal{T}_{1B(k)}$ is a maximal torus of $\mathcal{G}_{B(k)}$ of $\mathbb{Q}_p$-endomorphisms of $\mathcal{C}$ and $\mu_1:\mathbb{G}_m\to\mathcal{T}_{1K_1}$ is a cocharacter such that μ_{1K_2}, when viewed as a cocharacter of $\mathcal{G}_{K_2}$, is $\mathcal{G}(K_2)$-conjugate to μ_{K_2}. If μ_1 is definable over an unramified extension of $\mathbb{Q}_p$, then we refer to $(\mathcal{T}_{1B(k)},\mu_1)$ as an *unramified E-pair*. By an *E-triple* of $\mathcal{C}$ we mean a triple $(\mathcal{T}_{1B(k)},\mu_1,\tau)$, where $(\mathcal{T}_{1B(k)},\mu_1)$ is an E-pair and where $\tau=(\tau_1,\ldots,\tau_l)$ is an l-tuple of elements of $\mathrm{Gal}(K_2/\mathbb{Q}_p)$ whose restrictions to K_{2u} are all equal to the Frobenius automorphism $\mathfrak{F}_{2u}$ of K_{2u} whose fixed field is $\mathbb{Q}_p$. Here $l\in\mathbb{N}$ and "E" stands for endomorphisms. For $s\in\mathbb{N}$ and $j\in\{1,\ldots,l\}$ let $\tau_{sl+j}:=\tau_j$.

(e) We say an E-pair $(\mathcal{T}_{1B(k)},\mu_1)$ of $\mathcal{C}$ satisfies the $\mathfrak{C}$ *condition* if there exists an E-triple $(\mathcal{T}_{1B(k)},\mu_1,\tau)$ of $\mathcal{C}$ such that the following condition holds:

(e1) the product of the cocharacters of $\mathcal{T}_{1K_2}$ of the form $\tau_{dl}\tau_{dl-1}\cdots\tau_j(\mu_{1K_2})$ with $j\in\{1,\ldots,dl\}$, factors through $Z^0(\mathcal{G}_{K_2})$; here $d\in\mathbb{N}$ is the smallest number such that μ_{1K_2} is fixed by each element of $\mathrm{Gal}(K_2/\mathbb{Q}_p)$ that can be obtained from the product $\tau_{dl}\tau_{dl-1}\cdots\tau_1$ via a circular rearrangement of it.

If moreover $l=1$ we say $(\mathcal{T}_{1B(k)},\mu_1)$ satisfies the cyclic $\mathfrak{C}$ condition.

(f) We assume that $\mathcal{C}$ is basic. We say $\mathfrak{R}$ (resp. $\mathfrak{U}$) holds for $\mathcal{C}$ if there exists an E-pair (resp. unramified E-pair) of $\mathcal{C}$ that satisfies the $\mathfrak{C}$ condition. We say $T\mathfrak{R}$ (resp. $T\mathfrak{U}$) holds for $\mathcal{C}$ if each maximal torus $\mathcal{T}_{1B(k)}$ of $\mathcal{G}_{B(k)}$ of $\mathbb{Q}_p$-endomorphisms of $\mathcal{C}$ (resp. maximal torus $\mathcal{T}_{1B(k)}$ of $\mathcal{G}_{B(k)}$ of $\mathbb{Q}_p$-endomorphisms of $\mathcal{C}$ which splits over $B(\bar{k})$) is part of an E-pair (resp. of an unramified E-pair) of $\mathcal{C}$ that satisfies the $\mathfrak{C}$ condition. We say $Q\mathfrak{R}$ (resp. $Q\mathfrak{U}$) holds for $\mathcal{C}$ if there exists a k_1 and an E-pair (resp. unramified E-pair) of $\mathcal{C}\otimes k_1$ that satisfies the $\mathfrak{C}$ condition. We say $TT\mathfrak{R}$ (resp. $TT\mathfrak{U}$) holds for $\mathcal{C}$ if for each k_1, $T\mathfrak{R}$ (resp. $T\mathfrak{U}$) holds for $\mathcal{C}\otimes k_1$.

(g) We do not assume that $\mathcal{C}$ is basic. We say $\mathfrak{R}$ (resp. $\mathfrak{U}$, $T\mathfrak{R}$, $T\mathfrak{U}$, $Q\mathfrak{R}$, $Q\mathfrak{U}$, $TT\mathfrak{R}$, $TT\mathfrak{U}$) holds for $\mathcal{C}$ if there exists an element $h\in\mathcal{I}(\mathcal{C})$ such that the triple $(h(M),\phi,L^0_{\mathcal{G}(h)}(\phi))$ is a basic Shimura F-crystal over k and $\mathfrak{R}$ (resp. $\mathfrak{U}$, $T\mathfrak{R}$, $T\mathfrak{U}$, $Q\mathfrak{R}$, $Q\mathfrak{U}$, $TT\mathfrak{R}$, $TT\mathfrak{U}$) holds for it.

(h) We say that an E-pair $(\mathcal{T}_{1B(k)},\mu_1)$ of $\mathcal{C}$ is *admissible* if the filtered module

$$(M[\frac{1}{p}],\phi,F^1_{K_2})$$

over K_2 is admissible. Here $F^1_{K_2}$ is the maximal direct summand of $M\otimes_{W(k)}K_2$ on which μ_{1K_2} acts via the inverse of the identical character of $\mathbb{G}_m$.

(i) We say $\mathfrak{A}$ holds for $\mathcal{C}$ if there exists an E-pair $(\mathcal{T}_{1B(k)},\mu_1)$ of $\mathcal{C}$ which is admissible. We say $T\mathfrak{A}$ holds for $\mathcal{C}$ if each maximal torus $\mathcal{T}_{1B(k)}$ of $\mathcal{G}_{B(k)}$ of $\mathbb{Q}_p$-endomorphisms of $\mathcal{C}$

can be extended to an E-pair $(\mathcal{T}_{1B(k)}, \mu_1)$ of $\mathcal{C}$ which is admissible. As in (f), we speak also about $Q\mathfrak{A}$ or $TT\mathfrak{A}$ holding for $\mathcal{C}$.

(j) We say $\mathcal{C}$ is *U-ordinary* if it has a lift F^1 such that $L^0_{\mathcal{G}}(\phi)_{B(k)}$ normalizes $F^1[\frac{1}{p}]$ (i.e., we have $L^0_{\mathcal{G}}(\phi)_{B(k)} \leqslant \mathcal{P}_{B(k)}$). We say $\mathcal{C}$ is *IU-ordinary* if there exist elements $g \in \mathcal{G}(W(k))$ and $h \in \mathcal{G}(B(k))$ such that $\mathcal{C}_g$ is U-ordinary and we have $h\phi = g\phi h$.

(k) A *principal bilinear quasi-polarization* of $\mathcal{C}$ is a perfect bilinear form $\lambda_M : M \otimes_{W(k)} M \to W(k)$ whose $W(k)$-span is normalized by $\mathcal{G}$ and for which we have $\lambda_M(\phi(x) \otimes \phi(y)) = p\sigma(\lambda_M(x \otimes y))$ for all elements $x, y \in M$.

2.3.1. Example. Let $(\mathcal{T}_{1B(k)}, \mu_1)$ be an E-pair of $\mathcal{C}$ such that the product of the cocharacters of $\mathcal{T}_{1K_1}$ which belong to the $\mathrm{Gal}(K_1/\mathbb{Q}_p)$-orbit of μ_{1K_1}, factors through $Z^0(\mathcal{G}_{K_1})$. We choose an element $\tau_0 \in \mathrm{Gal}(K_2/\mathbb{Q}_p)$ whose restriction to K_{2u} is $\mathfrak{F}_{2u}$ and whose order o is the same as of $\mathfrak{F}_{2u}$. Let $\{e_1, \ldots, e_s\}$ be the elements of $\mathrm{Gal}(K_2/K_{2u})$ listed in such a way that $e_s = 1_K$. Let $e_0 := e_s$. We have $\mathrm{Gal}(K_2/\mathbb{Q}_p) = \{e_a\tau_0^b | 1 \le a \le s,\, 1 \le b \le o\}$. Let $l := os$. We define $\tau = (\tau_1, \ldots, \tau_l)$ as follows. For $i \in \{1, \ldots, l\}$ we define $\tau_i := \tau_0$ if o does not divide i and we define $\tau_i := e_{j-1}^{-1}e_j\tau_0$ if $i = o(s+1-j)$, where $j \in \{1, \ldots, s\}$. As $\tau_l\tau_{l-1}\cdots\tau_1 = 1_K$, let $d := 1$. As we have $\mathrm{Gal}(K_2/\mathbb{Q}_p) = \{\tau_l\tau_{l-1}\cdots\tau_j | 1 \le j \le l\}$, the condition 2.3 (e1) holds (cf. our hypothesis on the E-pair $(\mathcal{T}_{1B(k)}, \mu_1)$ of $\mathcal{C}$). Thus the E-pair $(\mathcal{T}_{1B(k)}, \mu_1)$ of $\mathcal{C}$ satisfies the $\mathfrak{C}$ condition.

2.3.2. Example. Let $m \in \mathbb{N}$. We assume that the rank of M is $2m$, that $\mathcal{G}$ is a product $\mathcal{G}_1 \times \cdots \times \mathcal{G}_m$ of m copies of $\mathbf{GL}_2$, that ϕ permutes transitively the $\mathrm{Lie}(\mathcal{G}_i)[\frac{1}{p}]$'s with $i \in \{1, \ldots, m\}$, that for each $i \in \{1, \ldots, m\}$ the image of μ in $\mathcal{G}_i$ does not factor through $Z(\mathcal{G}_i)$, and that the representation of $\mathcal{G}$ on M is the direct sum of the standard rank 2 representations of the m copies. The rank of F^1 is m and $\mathcal{P} = \mathcal{P}_1 \times \cdots \mathcal{P}_m$ is a Borel subgroup scheme of $\mathcal{G}$. We also assume that there exists a maximal torus $\mathcal{T} = \mathcal{T}_1 \times \cdots \times \mathcal{T}_m$ of $\mathcal{P}$ such that we have $\phi(\mathrm{Lie}(\mathcal{T})) = \mathrm{Lie}(\mathcal{T})$ and $\phi(\mathrm{Lie}(\mathcal{P})) \subseteq \mathrm{Lie}(\mathcal{P})$. This last assumption implies that the Dieudonné module (M, ϕ) is ordinary.

Let $g \in \mathcal{G}(W(k))$ be such that $\mathcal{C}_g$ is not basic. Thus $P^+_{\mathcal{G}}(g\phi) = \prod_{i=1}^m P^+_{\mathcal{G}}(g\phi) \cap \mathcal{G}_i$ is a Borel subgroup scheme of $\mathcal{G}$ and therefore $L^0_{\mathcal{G}}(\phi)_{B(k)}$ is a split maximal torus of $\mathcal{G}_{B(k)}$. We check that $\mathcal{C}_g$ is IU-ordinary. Based on [Va3, Thm. 3.1.2 (b) and (c)], up to a replacement of $g\phi$ by $hg\phi h^{-1}$ with $h \in \mathcal{G}(B(k))$, we can assume that $L^0_{\mathcal{G}}(\phi)$ is a maximal torus of $\mathcal{G}$ through which μ factors. Thus $L^0_{\mathcal{G}}(\phi)$ commutes with μ and therefore it is a maximal torus of $\mathcal{P}$. Thus $\mathcal{C}_g$ is U-ordinary.

We now take $m = 3$. Let $w := (w_1, w_2, 1_M) \in \mathcal{G}_1(W(k)) \times \mathcal{G}_2(W(k)) \times \mathcal{G}_3(W(k))$ be an element that normalizes $\mathcal{T}$ and such that for $i \in \{1, 2\}$ the element w_i takes $\mathcal{P}_i$ to its opposite $\mathcal{P}_i^{\mathrm{opp}}$ with respect to $\mathcal{T}_i$. The Newton polygon slopes of $(M, w\phi)$ are $\frac{1}{3}$ and $\frac{2}{3}$ with multiplicities 3. We have $L^0_{\mathcal{G}}(w\phi) = \mathcal{T} \leqslant \mathcal{P}$. Thus $\mathcal{C}_w$ is U-ordinary. Let $\mathcal{U}_1$ be the unipotent radical of $\mathcal{P}_1^{\mathrm{opp}}$; it is a subgroup scheme of $U^+_{\mathcal{G}}(w\phi)$. Let $g_1 \in \mathcal{U}_1(W(k))$ such that modulo p it is not the identity element. As $g_1 \in U^+_{\mathcal{G}}(w\phi)(W(k))$, we have $P^+_{\mathcal{G}}(g_1w\phi) = P^+_{\mathcal{G}}(w\phi)$. Thus $\mathcal{C}_{g_1w}$ is not basic and therefore (cf. previous paragraph) it is IU-ordinary. We show that the assumption that $\mathcal{C}_{g_1w}$ is U-ordinary leads to a contradiction. It is easy to see that this assumption implies that $L^0_{\mathcal{G}}(g_1w\phi)$ is a maximal torus of $P^+_{\mathcal{G}}(g_1w\phi) = P^+_{\mathcal{G}}(w\phi)$ which normalizes F^1/pF^1 (see proof of Proposition 3.2 below). Let $b \in P^+_{\mathcal{G}}(w\phi)(W(k))$ be an element that normalizes F^1/pF^1 and such that $b(L^0_{\mathcal{G}}(g_1w\phi))b^{-1} = \mathcal{T}$, cf. [Bo, Ch. V, Thm. 19.2] and [DG, Vol. II, Exp. IX, Thm. 3.6 and 7.1]. Thus $bg_1w\phi b^{-1} = g_2w\phi$, where $g_2 \in P_{\mathcal{G}}^+(w\phi)(W(k))$ normalizes $\mathcal{T}$. Therefore $g_2 \in \mathcal{T}(W(k))$. But it is easy to see that the natural images of g_1 and $g_2 = bg_1w\phi(b^{-1})w^{-1}$ in $\mathcal{T}_1(k)\backslash\mathcal{G}_1(k)/\mathcal{T}_1(k)$ are equal. As g_1 modulo p is a non-identity element of $\mathcal{U}_1(k)$), this contradicts the fact that $g_2 \in \mathcal{T}(W(k))$. We conclude that $\mathcal{C}_{g_1w}$ is IU-ordinary without being U-ordinary.

2.4. Some $\mathbb{Z}_p$ structures. Let $\sigma_\phi := \phi \circ \mu(p)$; it is a σ-linear automorphism of M. Let

$$M_{\mathbb{Z}_p} := \{m \in M \otimes_{W(k)} W(\bar{k}) | (\sigma_\phi \otimes \sigma_{\bar{k}})(m) = m\}.$$

We have $M \otimes_{W(k)} W(\bar{k}) = M_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} W(\bar{k})$. Let $\Box$ be a closed subgroup scheme of $\mathbf{GL}_M$ which is an integral scheme. We assume that both μ and ϕ normalize $\mathrm{Lie}(\Box_{B(k)})$. This implies that σ_ϕ normalizes $\mathrm{Lie}(\Box)$. Thus $\Box_{B(\bar{k})}$ is the extension to $B(\bar{k})$ of a connected subgroup of $\mathbf{GL}_{M_{\mathbb{Z}_p}[\frac{1}{p}]}$, cf. Example 2.2.1. If $\Box_{\mathbb{Z}_p}$ is the Zariski closure of $\Box_{\mathbb{Q}_p}$ in $\mathbf{GL}_{M_{\mathbb{Z}_p}}$, then its extension to $W(\bar{k})$ is $\Box_{W(\bar{k})}$. If $\Box$ is a subgroup of $\mathcal{G}$, then $\Box_{\mathbb{Z}_p}$ is a subgroup of $\mathcal{G}_{\mathbb{Z}_p}$.

As μ and ϕ normalize $\mathrm{Lie}(\mathcal{G}_{B(k)})$, σ_ϕ normalizes $\mathrm{Lie}(\mathcal{G})$. Thus from the previous paragraph we get the existence of a unique closed subgroup scheme $\mathcal{G}_{\mathbb{Z}_p}$ of $\mathbf{GL}_{M_{\mathbb{Z}_p}}$ whose extension to $W(\bar{k})$ is $\mathcal{G}_{W(\bar{k})}$; it is a reductive group scheme over $\mathbb{Z}_p$.

2.4.1. Two axioms. We introduce two axioms for $\mathcal{C}$:

(i) there exists a family $(t_\alpha)_{\alpha\in\mathcal{J}}$ of tensors of $\mathcal{T}(M)$ fixed by ϕ and $\mathcal{G}$ and such that $\mathcal{G}$ is the Zariski closure in $\mathbf{GL}_M$ of the subgroup of $\mathbf{GL}_{M[\frac{1}{p}]}$ that fixes t_α for all $\alpha \in \mathcal{J}$;

(ii) there exists a set of cocharacters of $\mathcal{G}_{W(\bar{k})}$ that act on $M \otimes_{W(k)} W(\bar{k})$ via the trivial and the inverse of the identical character of $\mathbb{G}_m$ and whose images in $\mathcal{G}^{\mathrm{ad}}_{W(\bar{k})}$ generate $\mathcal{G}^{\mathrm{ad}}_{W(\bar{k})}$.

Until the end of the paper we will assume that these two axioms hold for $\mathcal{C}$. Axiom (i) implies that $\phi^r \in \mathcal{G}(B(k))$ and that we have $t_\alpha \in \mathcal{T}(M_{\mathbb{Z}_p})$ for all $\alpha \in \mathcal{J}$. Thus the pair $(M_{\mathbb{Z}_p}, (t_\alpha)_{\alpha\in\mathcal{J}})$ is a $\mathbb{Z}_p$ structure of $(M \otimes_{W(k)} W(\bar{k}), (t_\alpha)_{\alpha\in\mathcal{J}})$. The difference between any two such $\mathbb{Z}_p$ structures of $(M \otimes_{W(k)} W(\bar{k}), (t_\alpha)_{\alpha\in\mathcal{J}})$ is measured by a class $\gamma \in H^1(\mathrm{Gal}(\mathbb{Z}_p^{\mathrm{un}}/\mathbb{Z}_p), \mathcal{G}_{\mathbb{Z}_p})$, where $\mathbb{Z}_p^{\mathrm{un}}$ is the maximal unramified, profinite discrete valuation ring extension of $\mathbb{Z}_p$. From Lang theorem (see [Se2, p. 132] and [Bo, Ch. V, Subsections 16.3 to 16.6]) we get that this class is trivial. Thus the iosmorphism class of the triple $(M_{\mathbb{Z}_p}, \mathcal{G}_{\mathbb{Z}_p}, (t_\alpha)_{\alpha\in\mathcal{J}})$ does not depend on the choice of the Hodge cocharacter $\mu : \mathbb{G}_m \to \mathcal{G}$ of $\mathcal{C}$. Also by replacing ϕ with $g\phi$, where $g \in \mathcal{G}(W(k))$, the isomorphism class of $(M_{\mathbb{Z}_p}, \mathcal{G}_{\mathbb{Z}_p}, (t_\alpha)_{\alpha\in\mathcal{J}})$ remains the same. From Lang theorem we also get that each torsor of $\mathcal{G}$ is trivial. This implies that there exists an isomorphism

$$i_M : M_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} W(k) \xrightarrow{\sim} M \tag{2}$$

that takes t_α to t_α for all $\alpha \in \mathcal{J}$. Thus $\mathcal{G} = \mathcal{G}_{W(k)}$ (i.e., our notations match) and we refer to the triple $(M_{\mathbb{Z}_p}, \mathcal{G}_{\mathbb{Z}_p}, (t_\alpha)_{\alpha\in\mathcal{J}})$ as the $\mathbb{Z}_p$ *structure* of $(M, \phi, \mathcal{G}, (t_\alpha)_{\alpha\in\mathcal{J}})$.

Axiom (ii) is inserted for practical reasons i.e., to exclude situations that are not related to Shimura varieties of Hodge type and to get the following properties.

2.4.2. Theorem. *We recall that $C = C_{\mathbf{GL}_M}(\mathcal{G})$.*

(a) *The Lie algebra $Lie(C)$ is $W(k)$-generated by elements fixed by ϕ.*

(b) *The closed subgroup scheme C of $\boldsymbol{GL}_M$ is reductive.*

(c) *We assume that we have a principal bilinear quasi-polarization λ_M of $\mathcal{C}$. Let $C_1(\lambda_M)^0$ be the Zariski closure in $\boldsymbol{GL}_M$ of the identity component $C_{1B(k)}(\lambda_M)^0$ of the maximal subgroup $C_{1B(k)}(\lambda_M)$ of $C_{1B(k)}$ that normalizes the $B(k)$-span of λ_M. Then the Zariski closure $Z^0(C_1(\lambda_M)^0)$ in $\boldsymbol{GL}_M$ of $Z^0(C_{1B(k)}(\lambda_M)^0)$ is a torus over $W(k)$.*

Proof: As $\phi^r \in \mathcal{G}(B(k))$ fixes $\mathrm{Lie}(C)$ and as $\phi(\mathrm{Lie}(C)) = \mathrm{Lie}(C)$, (a) holds. To prove (b) we work only with the $\mathcal{G}$-module M; thus the below reduction steps do not pay any attention to ϕ. To prove (b) we can assume that $\mathcal{G}$ is split and that $Z^0(\mathcal{G}) = Z(\mathbf{GL}_M)$. Let $M[\frac{1}{p}] := \oplus_{i=1}^n M_i[\frac{1}{p}]$ be a direct sum decomposition into irreducible $\mathcal{G}_{B(k)}$-modules, cf. Weyl complete reducibility theorem. Let $M_i := M \cap M_i[\frac{1}{p}]$. Thus $\oplus_{i=1}^n M_i$ is a $\mathcal{G}$-submodule of M. Due to the axiom 2.4.1 (ii), each simple factor of $\mathrm{Lie}(\mathcal{G}^{\mathrm{der}}_{B(k)})$ is of classical Lie type and the representation of $\mathrm{Lie}(\mathcal{G}^{\mathrm{der}}_{B(k)})$ on each $M_i[\frac{1}{p}]$ is a tensor product of irreducible representations which are either trivial or are associated to minuscule weights (see [Se1, Prop. 7 and Cor. 1 of p. 182]). Thus the $\mathcal{G}_k$-module M_i/pM_i is absolutely irreducible and its isomorphism class depends only on the isomorphism class of the $\mathcal{G}_{B(k)}$-module $M_i[\frac{1}{p}]$, cf. the below well known Fact 2.4.3 and [Ja, Part I, 10.9].

By induction on $n \in \mathbb{N}$ we show that we can choose the decomposition $M[\frac{1}{p}] := \oplus_{i=1}^n M_i[\frac{1}{p}]$ such that we have $M = \oplus_{i=1}^n M_i$. The case $n = 1$ is trivial. The passage from n to $n+1$ goes as follows. We have a short exact sequence $0 \to M_1 \to M \to M/M_1 \to 0$ of $\mathcal{G}$-modules. Using induction, it suffices to consider the case $n = 1$; thus the $W(k)$-monomorphism $M_2 \hookrightarrow M/M_1$ becomes an isomorphism after inverting p. If the $\mathcal{G}_{B(k)}$-modules $M_1[\frac{1}{p}]$ and $M_2[\frac{1}{p}]$ are not isomorphic, then the $\mathcal{G}_k$-modules M_1/pM_1 and M_2/pM_2 are not isomorphic and therefore the natural k-linear map $M_1/pM_1 \oplus M_2/pM_2 \to M/pM$ is injective; this implies that we have $M = M_1 \oplus M_2$. We assume now that the $\mathcal{G}_{B(k)}$-modules $M_1[\frac{1}{p}]$ and $M_2[\frac{1}{p}]$ are isomorphic. Thus M_1 and M_2 are isomorphic $\mathcal{G}$-modules. If they are trivial $\mathcal{G}$-modules, then we can replace M_2 by any direct supplement of M_1 in M and thus we have $M = M_1 \oplus M_2$. We now consider the case when M_1 and M_2 are non-trivial $\mathcal{G}$-modules. Let $\tilde{M}$ be a $W(k)$-lattice of $M[\frac{1}{p}]$ which contains M, which is a $\mathcal{G}$-module isomorphic to $M_1 \oplus M_2$, and for which the length of the torsion $W(k)$-module $\tilde{M}/M$ is the smallest possible value $l \in \mathbb{N} \cup \{0\}$. Let $\tilde{M} = \tilde{M}_1 \oplus \tilde{M}_2$ be a direct sum decomposition into irreducible $\mathcal{G}$-modules. We show that the assumption $l \neq 0$ leads to a contradiction. We can assume that the natural $\mathcal{G}_k$-homomorphism $M_1/pM_1 \to \tilde{M}_1/p\tilde{M}_1$ is non-trivial and therefore injective. Thus M_1 is a direct summand of $\tilde{M}$ and therefore we can assume that $M_1 = \tilde{M}_1$. As $M \neq \tilde{M}$, we have $\mathrm{Im}(M_2/pM_2 \to \tilde{M}/p\tilde{M}) \subseteq \mathrm{Im}(M_1/pM_1 \to \tilde{M}/p\tilde{M})$. Thus we can replace $\tilde{M}$ by $\tilde{M}_1 \oplus p\tilde{M}_2$ and this contradicts the minimality of l. Thus $l = 0$. Thus $M = \tilde{M}_1 \oplus \tilde{M}_2$ and therefore as M_i we can take $\tilde{M}_i$. This ends the induction.

Thus to prove (b) we can assume that $M = \oplus_{i=1}^n M_i$. As the isomorphism class of M_i is uniquely determined by the isomorphism class of the $\mathcal{G}_{B(k)}$-module $M_i[\frac{1}{p}]$, we can write $M = \oplus_{j \in J} M_j^{n_j}$, where each M_j is isomorphic to some M_i, where $n_j \in \mathbb{N}$, and where for two distinct elements $j_1, j_2 \in J$ the $\mathcal{G}$-modules M_{j_1} and M_{j_2} are not isomorphic. Also for two distinct elements $j_1, j_2 \in J$, the $\mathcal{G}_k$-modules M_{j_1}/pM_{j_1} and M_{j_2}/pM_{j_2} are not isomorphic. We easily get that the group scheme C is isomorphic to a product $\prod_{j\in J} \mathbf{GL}_{n_j}$ and therefore it is a reductive, closed subgroup scheme of $\mathbf{GL}_M$. Thus (b) holds.

We prove (c). From (b) we get that $C_1 = C_{\mathbf{GL}_M}(C)$ is a reductive, closed subgroup scheme of $\mathbf{GL}_M$. Thus $Z^0(C_1(\lambda_M))$ is the Zariski closure in $Z^0(C_1)$ of a subtorus of $Z^0(C_1)_{B(k)}$ and therefore it is a torus. $\square$

2.4.3. Fact. *Let $\mathcal{H}$ be a split, simply connected group scheme over $\mathbb{Z}$ whose adjoint is absolutely simple and of classical Lie type θ. Let $\mathcal{T}$ be a maximal split torus of $\mathcal{H}$. Let $\rho_\varpi : \mathcal{H} \to \boldsymbol{GL}_{\mathcal{Z}}$ be the representation associated to a minuscule weight ϖ of the root system of the inner conjugation action of $\mathcal{T}$ on Lie($\mathcal{H}$) (thus $\mathcal{Z}$ is a free $\mathbb{Z}$-module of finite rank, cf. [Hu, Subsection 27.1]). Then the special fibres of ρ_ϖ are absolutely irreducible.*

Proof: We use the notations of [Bou1, planches I to IV]. The minuscule weights are: ϖ_i with $i \in \{1, \ldots, n\}$ if $\theta = A_n$, ϖ_n if $\theta = B_n$, ϖ_1 if $\theta = C_n$, ϖ_1, ϖ_{n-1}, and ϖ_n if $\theta = D_n$ (see [Bou2, pp. 127–129] and [Se1, pp. 185–186]). Let $\mathfrak{W}$ be the Weyl group of $\mathcal{H}$ with respect to $\mathcal{T}$. Let $\mathfrak{W}_\varpi$ be the subgroup of $\mathfrak{W}$ that fixes ϖ. We have $\dim_{\mathbb{Z}}(\mathcal{Z}) = [\mathfrak{W} : \mathfrak{W}_\varpi]$, cf. [Bou2, Ch. VIII, §7.3, Prop. 6]. Thus for each prime p, the absolutely irreducible representation of $\mathcal{H}_{\mathbb{F}_p}$ associated to weight ϖ has dimension at least $\dim_{\mathbb{Z}}(\mathcal{Z})$. As it is isomorphic to the representation of $\mathcal{H}_{\mathbb{F}_p}$ on a factor of the composition series of the fibre of ρ_ϖ over $\mathbb{F}_p$, by reasons of dimensions we get that this fibre is absolutely irreducible. $\square$

2.4.4. Extra groups. Let $M_{\mathbb{Q}_p} := M_{\mathbb{Z}_p}[\frac{1}{p}]$. Let $CG_{\mathbb{Q}_p}$ be the identity component of $NG_{\mathbb{Q}_p} := N_{\mathbf{GL}_{M_{\mathbb{Q}_p}}}(\mathcal{G}_{\mathbb{Q}_p})$. The reductive group $CG_{\mathbb{Q}_p}$ is generated by $\mathcal{G}_{\mathbb{Q}_p}$ and by $C_{\mathbb{Q}_p} := C_{\mathbf{GL}_{M_{\mathbb{Q}_p}}}(\mathcal{G}_{\mathbb{Q}_p})$. Let

$$\Pi : \mathrm{End}(M_{\mathbb{Q}_p}) \to \mathrm{End}(M_{\mathbb{Q}_p})$$

be the projector on Lie($\mathcal{G}_{\mathbb{Q}_p}$) along the perpendicular of Lie($\mathcal{G}_{\mathbb{Q}_p}$) with respect to the trace form $\mathfrak{T}$ on End($M_{\mathbb{Q}_p}$), cf. Lemma 2.2 (b). The group $NG_{\mathbb{Q}_p}$ is the subgroup of $\mathbf{GL}_{M_{\mathbb{Q}_p}}$ that fixes Π. Based on (2) the group $CG_{B(k)}$ is naturally a subgroup of $\mathbf{GL}_{M[\frac{1}{p}]}$: it is the identity component of $N_{\mathbf{GL}_{M[\frac{1}{p}]}}(\mathcal{G}_{B(k)})$.

2.4.5. Plus (plus) admissibility. We say that an E-pair $(\mathcal{T}_{1B(k)}, \mu_1)$ of $\mathcal{C}$ is *plus admissible* if it is admissible and if the class

$$\mathfrak{L} \in H^1(\mathbb{Q}_p, \mathcal{G}_{\mathbb{Q}_p}) \tag{3}$$

has a trivial image in $H^1(\mathbb{Q}_p, CG_{\mathbb{Q}_p})$. Here the class $\mathfrak{L}$ is defined as follows. Let ρ : Gal$(K_2) \to \mathbf{GL}_{\mathcal{W}}$ be the admissible Galois representation that corresponds to $(M, \phi, F^1_{K_2})$. Thus $\mathcal{W}$ is a free $\mathbb{Q}_p$-vector space and we have a Gal(K_2)-isomorphism

$$\mathcal{W} \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}(K_2) \xrightarrow{\sim} M \otimes_{W(k)} K_2 \otimes_{K_2} B_{\mathrm{dR}}(K_2) \tag{4}$$

which respects the tensor product filtrations (the filtration of $\mathcal{W}$ is trivial and the filtration of $M \otimes_{W(k)} K_2$ is defined by $F^1_{K_2}$). For $\alpha \in \mathcal{J}$, let $v_\alpha \in \mathcal{T}(\mathcal{W})$ be the tensor that corresponds to t_α via (4) and Fontaine comparison theory. We take $\mathfrak{L}$ to be the class of the right torsor of $\mathcal{G}_{\mathbb{Q}_p}$ that parameterizes isomorphisms between $(M_{\mathbb{Q}_p}, (t_\alpha)_{\alpha\in\mathcal{J}})$ and $(\mathcal{W}, (v_\alpha)_{\alpha\in\mathcal{J}})$ (such a torsor exists, cf. (2) and (4)). If the class $\mathfrak{L}$ is trivial, then we say that the E-pair $(\mathcal{T}_{1B(k)}, \mu_1)$ of $\mathcal{C}$ is *plus plus admissible*.

We say $+\mathfrak{A}$ (resp. $++\mathfrak{A}$) holds for $\mathcal{C}$ if there exists an E-pair of $\mathcal{C}$ which is plus (resp. plus plus) admissible. As in Definition 2.3 (i) we speak about $Q+\mathfrak{A}$, $Q++\mathfrak{A}$, $T+\mathfrak{A}$, $T++\mathfrak{A}$, $TT+\mathfrak{A}$, or $TT++\mathfrak{A}$ holding for $\mathcal{C}$.

2.4.6. Lemma. *Let $ZG_{\mathbb{Q}_p}$ be the subgroup of $CG_{\mathbb{Q}_p}$ generated by $\mathcal{G}_{\mathbb{Q}_p}$ and the torus $Z(CG_{\mathbb{Q}_p})$. If $\mathcal{G}^{\mathrm{der}}$ is simply connected and if the torus $Z(CG_{\mathbb{Q}_p})/Z(\mathcal{G}^{\mathrm{der}}_{\mathbb{Q}_p}) = ZG_{\mathbb{Q}_p}/\mathcal{G}^{\mathrm{der}}_{\mathbb{Q}_p}$ is isomorphic to $Z(CG_{\mathbb{Q}_p})$, then the pointed set $H^1(\mathbb{Q}_p, ZG_{\mathbb{Q}_p})$ has only one class.*

Proof: The group $Z(CG_{\mathbb{Q}_p})$ is the group scheme of invertible elements of an étale $\mathbb{Q}_p$-subalgebra of $\mathrm{End}(M_{\mathbb{Q}_p})$. Thus it is a torus over $\mathbb{Q}_p$ and moreover the (abstract) group $H^1(\mathbb{Q}_p, Z(CG_{\mathbb{Q}_p}))$ is trivial. Therefore the group $H^1(\mathbb{Q}_p, ZG_{\mathbb{Q}_p}/\mathcal{G}^{\mathrm{der}}_{\mathbb{Q}_p})$ is also trivial. The pointed set $H^1(\mathbb{Q}_p, \mathcal{G}^{\mathrm{der}}_{\mathbb{Q}_p})$ has also only one class, cf. [Kn, Thm. 1]. As we have an exact complex $H^1(\mathbb{Q}_p, \mathcal{G}^{\mathrm{der}}_{\mathbb{Q}_p}) \to H^1(\mathbb{Q}_p, ZG_{\mathbb{Q}_p}) \to H^1(\mathbb{Q}_p, ZG_{\mathbb{Q}_p}/\mathcal{G}^{\mathrm{der}}_{\mathbb{Q}_p})$ of pointed sets, the Lemma follows. □

2.4.7. The polarized context. We assume that there exists a principal bilinear quasi-polarization $\lambda_M : M \otimes_{W(k)} M \to W(k)$ of $\mathcal{C}$; it give birth naturally to a symmetric bilinear form λ_M on $M_{\mathbb{Z}_p}$ and therefore we can speak about the $\mathbb{Q}_p$-span of λ_M (inside $(M_{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} M_{\mathbb{Q}_p})^*$). Let $\mathcal{G}^0_{\mathbb{Q}_p}$ be the identity component of the subgroup of $\mathcal{G}_{\mathbb{Q}_p}$ that fixes λ_M. Let $DG_{\mathbb{Q}_p}$ (resp. $DG^0_{\mathbb{Q}_p}$) be the identity component of the subgroup of $CG_{\mathbb{Q}_p}$ that normalizes the $\mathbb{Q}_p$ of λ (resp. that fixes λ_M). We have $\mathcal{G}_{\mathbb{Q}_p} \leqslant DG_{\mathbb{Q}_p}$ (resp. $\mathcal{G}^0_{\mathbb{Q}_p} \leqslant DG^0_{\mathbb{Q}_p}$). Either $\mathcal{G}^0_{\mathbb{Q}_p} = \mathcal{G}_{\mathbb{Q}_p}$ or we have a short exact sequence $0 \to \mathcal{G}^0_{\mathbb{Q}_p} \to \mathcal{G}_{\mathbb{Q}_p} \to \mathbb{G}_m \to 0$. Thus the class $\mathfrak{L}$ is the image of a class $\mathfrak{L}^0 \in H^1(\mathbb{Q}_p, \mathcal{G}^0_{\mathbb{Q}_p})$. We say that an E-pair $(\mathcal{T}_{1B(k)}, \mu_1)$ of $\mathcal{C}$ is plus (resp. plus plus) admissible with respect to λ_M if and only if it is admissible and moreover the image of $\mathfrak{L}^0$ in $H^1(\mathbb{Q}_p, DG^0_{\mathbb{Q}_p})$ (resp. and moreover $\mathfrak{L}^0$) is the trivial class. As in Definition 2.3 (i) and Subsubsection 2.4.5 we speak about $Q+\mathfrak{A}$, $Q++\mathfrak{A}$, $T+\mathfrak{A}$, $T++\mathfrak{A}$, $TT+\mathfrak{A}$, or $TT++\mathfrak{A}$ holding for $(\mathcal{C}, \lambda_M)$. If $Q+\mathfrak{A}$ (or $Q++\mathfrak{A}$, etc.) holds for $(\mathcal{C}, \lambda_M)$, then it also holds for $\mathcal{C}$.

2.4.8. A reduction. We assume that there exists a non-trivial product decomposition $\mathcal{G}^{\mathrm{ad}}_{\mathbb{Z}_p} = \mathcal{V}_1 \times_{\mathbb{Z}_p} \mathcal{V}_2$. Let $\phi_0 = i_M^{-1}\phi i_M : M_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} W(k) \to M_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} W(k)$ and $\mu_0 = i_M^{-1}\mu i_M : \mathbb{G}_m \to \mathcal{G}_{W(k)} \leqslant \mathbf{GL}_{M_{\mathbb{Z}_p}\otimes_{\mathbb{Z}_p} W(k)}$. We have $\phi_0 = g(1_{M_{\mathbb{Z}_p}} \otimes \sigma)\mu_0(\frac{1}{p})$, where $g \in \mathcal{G}_{\mathbb{Z}_p}(W(k))$. Let $\mathcal{G}^1_{\mathbb{Z}_p}$ be a reductive, closed subgroup scheme of $\mathcal{G}_{\mathbb{Z}_p}$ of the same rank and the same $\mathbb{Z}_p$-rank as $\mathcal{G}_{\mathbb{Z}_p}$ and whose adjoint group scheme is $\mathcal{V}_1$. By replacing i_M with its composite with an automorphism of $M_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} W(k)$ defined by an element of $\mathcal{G}_{\mathbb{Z}_p}(W(k))$, we can assume that the cocharacter μ_0 factors through $\mathcal{G}^1_{W(k)}$, cf. [Bo, Ch. V, Thm. 19.2] and [DG, Vol. II, Exp. IX, Thm. 3.6 and 7.1]. Let $g_0 \in \mathcal{G}^{\mathrm{der}}_{\mathbb{Z}_p}(W(k))$ be an element whose image in $\mathcal{V}_1(W(k))$ is trivial and such that we have $g_0 g \in \mathcal{G}^1_{\mathbb{Z}_p}(W(k))$. From the last two sentences we get that the triple $(M_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} W(k), g_0\phi_0, \mathcal{G}^1_{W(k)})$ is a Shimura F-crystal over k. Both axioms 2.4.1 (i) and (ii) hold for $(M_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} W(k), g_0\phi_0, \mathcal{G}^1_{W(k)})$. Argument: axiom

2.4.1 (i) holds as $\mathcal{G}^1_{\mathbb{Z}_p}$ is the closed subgroup scheme of $\mathcal{G}_{\mathbb{Z}_p}$ that fixes $\mathrm{Lie}(Z^0(\mathcal{G}^1_{\mathbb{Z}_p}))$ and axiom 2.4.1 (ii) holds as a maximal torus of $\mathcal{G}^1_{W(\bar{k})}$ is a maximal torus of $\mathcal{G}_{W(\bar{k})}$. Thus from many "adjoint" points of view, one can assume that $\mathcal{G}^{\mathrm{ad}}_{\mathbb{Z}_p}$ is simple. We will use this fact in Section 6.

2.5. Definitions. Let $\mathcal{J}_0$ be a subset of $\mathcal{J}$.

(a) Let $a \in \mathbb{N}$. If we have $t_\alpha \in \oplus_{s,t\in\{0,\dots,a\}} M^{\otimes s} \otimes_R M^{*\otimes t}$ for all $\alpha \in \mathcal{J}_0$, then we say that the family $(t_\alpha)_{\alpha\in\mathcal{J}_0}$ of tensors of $\mathcal{T}(M)$ (or $\mathcal{T}(M_{\mathbb{Z}_p})$) is of *partial degrees at most* a.

(b) The family $(t_\alpha)_{\alpha\in\mathcal{J}_0}$ of tensors of $\mathcal{T}(M_{\mathbb{Z}_p})$ is called $\mathbb{Z}_p$-*very well positioned* for $\mathcal{G}_{\mathbb{Z}_p}$ if the following condition holds:

(*) *For each faithfully flat, integral $\mathbb{Z}_p$-algebra R and for every free R-module O that satisfies $O[\frac{1}{p}] = M_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} R[\frac{1}{p}]$ and such that we have $t_\alpha \in \mathcal{T}(O)$ for all $\alpha \in \mathcal{J}_0$, the Zariski closure $\tilde{\mathcal{G}}_R$ of $\mathcal{G}_{R[\frac{1}{p}]}$ in $\boldsymbol{GL}_O$ is a reductive, closed subgroup scheme of $\boldsymbol{GL}_O$.*

Definition (b) is only a particular case of [Va1, Def. 4.3.4 and Rm. 4.3.7 1)].

2.6. Reduction to the basic context. Let $g \in \mathcal{G}(W(k))$ and $h \in \mathcal{G}(B(k))$ be such that $L^0_{\mathcal{G}}(g\phi)$ is a reductive, closed subgroup scheme of $\mathcal{G}$ through which $\mu : \mathbb{G}_m \to \mathcal{G}$ factors and we have an equality $hg\phi = \phi h$, cf. [Va3, Subsubsection 3.1.1 and Thm. 3.1.2]. Thus by performing the operation $\mathfrak{D}_2$ (i.e., by replacing $\mathcal{C}$ with $(h(M), \phi, \mathcal{G}(h))$), in this Subsubsection we will also assume that $L^0_{\mathcal{G}}(\phi)$ is a Levi subgroup scheme of $P^+_{\mathcal{G}}(\phi)$ and that μ factors through it. Therefore $(M, F^1, \phi, L^0_{\mathcal{G}}(\phi))$ is a Shimura filtered F-crystal over k and $(M, \phi, L^0_{\mathcal{G}}(\phi))$ is basic. Thus in connection to Conjecture 1.2.2 and to Subproblems 1.2.3 and 1.2.4, we can always replace $\mathcal{G}$ by a Levi subgroup scheme of $P^+_{\mathcal{G}}(\phi)$. However, often we will not perform this replacement in Sections 5 to 7, as by keeping track of $\mathcal{G}$ we get extra information on $L^0_{\mathcal{G}}(\phi)$ as follows. Let $L^0_{\mathcal{G}}(\phi)_{\mathbb{Z}_p}$ be the reductive, closed subgroup scheme of $\mathcal{G}_{\mathbb{Z}_p}$ which is the $\mathbb{Z}_p$ structure of $L^0_{\mathcal{G}}(\phi)$ obtained as in Subsection 2.4 (for $\square = Ł^0_{\mathcal{G}}(\phi)$).

2.6.1. Fact. *We have $L^0_{\mathcal{G}}(\phi) = C_{\mathcal{G}}(Z^0(L^0_{\mathcal{G}}(\phi)))$. Moreover, $L^0_{\mathcal{G}}(\phi)_{\mathbb{Z}_p}$ is the centralizer of a $\mathbb{G}_m$ subgroup scheme of $\mathcal{G}_{\mathbb{Z}_p}$ in $\mathcal{G}_{\mathbb{Z}_p}$ and (M, ϕ) is a direct sum of F-crystals over k which have only one Newton polygon slope.*

Proof: Both $L^0_{\mathcal{G}}(\phi)$ and $C_{\mathcal{G}}(Z^0(L^0_{\mathcal{G}}(\phi)))$ are reductive, closed subgroup schemes of $\mathcal{G}$ (cf. [DG, Vol. III, Exp. XIX, Subsection 2.8] for $C_{\mathcal{G}}(Z^0(L^0_{\mathcal{G}}(\phi))))$. Thus they coincide if and only if their generic fibres coincide. But this follows from the fact that $L^0_{\mathcal{G}}(\phi)_{B(k)}$ is the centralizer of the cocharacter $\nu_{B(k)}$ of $\mathcal{G}_{B(k)}$ which factors through $Z^0(L^0_{\mathcal{G}}(\phi)_{B(k)})$ and which is the Newton cocharacter of $\mathcal{C}$ (see [Va3, Subsection 2.3]). The cocharacter $\nu_{B(k)}$ is fixed by ϕ and μ and thus it is the extension to $B(k)$ of a cocharacter ν of $Z^0(L^0_{\mathcal{G}}(\phi)_{\mathbb{Q}_p})$. As $Z^0(L^0_{\mathcal{G}}(\phi))$ is a torus, ν extends to a cocharacter of $Z^0(L^0_{\mathcal{G}}(\phi)_{\mathbb{Z}_p})$. Its centralizer in $\mathcal{G}_{\mathbb{Z}_p}$ is a reductive, closed subgroup scheme of $\mathcal{G}_{\mathbb{Z}_p}$ (cf. [DG, Vol. III, Exp. XIX, Subsection 2.8]) whose generic fibre is $L^0_{\mathcal{G}}(\phi)_{\mathbb{Q}_p}$ and therefore it is $L^0_{\mathcal{G}}(\phi)_{\mathbb{Z}_p}$ itself. As ν extends to a cocharacter of $Z^0(L^0_{\mathcal{G}}(\phi)_{\mathbb{Z}_p})$, $\nu_{B(k)}$ extends also to a cocharacter of $Z^0(L^0_{\mathcal{G}}(\phi))$. This implies that$(M, \phi)$ is a direct sum of F-crystals over k which have only one Newton polygon slope $\square$

2.6.2. Corollary. *We assume that* $(\mathcal{T}_{1B(k)}, \mu_1)$ *is an* E*-pair of* $\mathcal{C}$ *which is plus plus admissible. Then* $(\mathcal{T}_{1B(k)}, \mu_1)$ *is also an* E*-pair of* $(M, \phi, L^0_{\mathcal{G}}(\phi))$ *which is plus plus admissible.*

Proof: Let $u \in \mathrm{Lie}(Z^0(L^0_{\mathcal{G}}(\phi)_{\mathbb{Z}_p}))$ be the image via $d\nu$ of the standard generator of $\mathrm{Lie}(\mathbb{G}_m)$. Thus $L^0_{\mathcal{G}}(\phi)_{\mathbb{Q}_p}$ is the subgroup of $\mathcal{G}_{\mathbb{Q}_p}$ that fixes u and $(t_\alpha)_{\alpha\in\mathcal{J}}$. We use the notations of Subsubsection 2.4.5. Let $u_{\acute{e}t} \in \mathrm{End}(\mathcal{W})$ correspond to u via (4). We consider an isomorphism $i : M_{\mathbb{Q}_p} \xrightarrow{\sim} \mathcal{W}$ that takes t_α to v_α for all $\alpha \in \mathcal{J}$. Two cocharacters of $\mathcal{G}_{\mathbb{Q}_p}$ which over $\overline{\mathbb{Q}_p}$ are $\mathcal{G}_{\mathbb{Q}_p}(\overline{\mathbb{Q}_p})$-conjugate, are $\mathcal{G}_{\mathbb{Q}_p}(\mathbb{Q}_p)$-conjugate. Thus we can choose i such that $i(u) = u_{\acute{e}t}$. Thus the E-pair $(\mathcal{T}_{1B(k)}, \mu_1)$ of $(M, \phi, L^0_{\mathcal{G}}(\phi))$ is plus plus admissible. $\square$

Let now Z be the center of $C_{\mathbf{GL}_M}(Z^0(L^0_{\mathcal{G}}(\phi)))$. Let $Z_{\mathbb{Z}_p}$ be the $\mathbb{Z}_p$ structure of Z obtained as in Subsection 2.4; it is a reductive, closed subgroup scheme of $\mathbf{GL}_{M_{\mathbb{Z}_p}}$. As $\phi^r \in \mathcal{G}(B(k))$ (cf. Subsubsection 2.4.1) normalizes $P^+_{\mathcal{G}}(\phi)_{B(k)}$ and $P^-_{\mathcal{G}}(\phi)_{B(k)}$, we have $\phi^r \in L^0_{\mathcal{G}}(\phi)(B(k))$. Thus $\mathrm{Lie}(Z)$ is normalized by ϕ and fixed by ϕ^r. Therefore $\mathrm{Lie}(Z)$ is $W(k)$-generated by its elements fixed by ϕ and thus we can identify naturally $\mathrm{Lie}(Z) = \mathrm{Lie}(Z_{\mathbb{Z}_p}) \otimes_{\mathbb{Z}_p} W(k)$. Let $(t_\alpha)_{\alpha\in\mathcal{J}(0)}$ be the family of all tensors which are elements of $\mathrm{Lie}(Z_{\mathbb{Z}_p})$. The group scheme $C_{\mathcal{G}}(Z^0(L^0_{\mathcal{G}}(\phi))) = L^0_{\mathcal{G}}(\phi)$ is the Zariski closure in $\mathbf{GL}_M$ of the subgroup of $\mathbf{GL}_{M[\frac{1}{p}]}$ that fixes t_α for all $\alpha \in \mathcal{J} \cup \mathcal{J}(0)$.

2.6.3. Fact. *If the family* $(t_\alpha)_{\alpha\in\mathcal{J}_0}$ *of tensors of* $\mathcal{T}(M_{\mathbb{Z}_p})$ *is* $\mathbb{Z}_p$*-very well positioned for* $\mathcal{G}_{\mathbb{Z}_p}$*, then the family* $(t_\alpha)_{\alpha\in\mathcal{J}_0\cup\mathcal{J}(0)}$ *of tensors of* $\mathcal{T}(M_{\mathbb{Z}_p})$ *is* $\mathbb{Z}_p$*-very well positioned for* $L^0_{\mathcal{G}}(\phi)_{\mathbb{Z}_p}$*.*

Proof: Let R, O, and $\tilde{\mathcal{G}}_R$ be as in Definition 2.5 (b). We assume that we have $t_\alpha \in \mathcal{T}(O)$ for all $\alpha \in \mathcal{J}_0 \cup \mathcal{J}(0)$. We know that $\tilde{\mathcal{G}}_R$ is a reductive, closed subgroup scheme of $\mathbf{GL}_O$. From [Va1, Subsubsection 4.3.13] applied in the context of $Z^0(L^0_{\mathcal{G}}(\phi))_{\mathbb{Z}_p}$ and $Z_{\mathbb{Z}_p}$, we get that the Zariski closure of $Z^0(L^0_{\mathcal{G}}(\phi))_{R[\frac{1}{p}]}$ in $\mathbf{GL}_O$ is a torus. Its centralizer in $\tilde{\mathcal{G}}_R$ is a reductive, closed subgroup scheme of $\tilde{\mathcal{G}}_R$ (cf. [DG, Vol. III, Exp. XIX, Subsubsection 2.8]) and thus it is the Zariski closure of $L^0_{\mathcal{G}}(\phi)_{R[\frac{1}{p}]}$ in $\mathbf{GL}_O$. Thus the Fact holds, cf. Definition 2.5 (b). $\square$

2.7. Some $W(k)$-algebras. Let $e \in \mathbb{N}$. Let X be an independent variable. Let $R := W(k)[[X]]$. Let $\widetilde{Re}$ (resp. Re) be the $W(k)$-subalgebra of $B(k)[[X]]$ formed by formal power series $\sum_{n=0}^{\infty} a_n X^n$ for which we have $a_n\left[\frac{n}{e}\right]! \in W(k)$ for all n (resp. for which the sequence $b_n := a_n\left[\frac{n}{e}\right]!$ is formed by elements of $W(k)$ and converges to 0). Thus Re is a $W(k)$-subalgebra of $\widetilde{Re}$. Let Φ_R, Φ_{Re}, and $\Phi_{\widetilde{Re}}$ be the Frobenius lifts of R, Re, and $\widetilde{Re}$ (respectively) that are compatible with σ and that take X to X^p. For $m \in \mathbb{N}$ let $I(m)$ be the ideal of $\widetilde{Re}$ formed by formal power series with $a_0 = a_1 = \cdots = a_{m-1} = 0$. The proof of the following elementary Fact is left as an exercise.

2.7.1. Fact. *We assume that* $p \geq 3$ *(resp.* $p = 2$*). Let* V *be a finite, totally ramified discrete valuation ring extension of* $W(k)$ *of degree at most* e*. Let* π_V *be a uniformizer of* V*. Then there exist* $W(k)$*-epimorphisms* $R \twoheadrightarrow V$*,* $Re \twoheadrightarrow V$*, and* $\widetilde{Re} \twoheadrightarrow V$ *(resp.* $R \twoheadrightarrow V$ *and* $Re \twoheadrightarrow V$*) that map* X *to* π_V*. Also, by mapping* X *to* 0 *we get* $W(k)$*-epimorphisms* $R \twoheadrightarrow W(k)$*,* $Re \twoheadrightarrow W(k)$*, and* $\widetilde{Re} \twoheadrightarrow W(k)$ *that respect the Frobenius lifts.*

3. Unramified and ramified CM-isogeny classifications

Let $\mathcal{F} = \{\mathcal{C}_g | g \in \mathcal{G}(W(k))\}$ be as in Subsection 2.1. By the *strong CM-isogeny* (resp. by the *CM-isogeny*) *classification* of $\mathcal{F}$ we mean the description of the subset $SZ(\mathcal{Y}(\mathcal{F}))$ (resp. $Z(\mathcal{Y}(\mathcal{F}))$) of $\mathcal{Y}(\mathcal{F})$ formed by inner isomorphism classes of those $\mathcal{C}_g$ with $g \in \mathcal{G}(W(k))$ which, up to the operation $\mathfrak{O}_2$ (resp. up to the operations $\mathfrak{O}_1$ and $\mathfrak{O}_2$), have a lift of quasi CM type.

Fact 3.1 and Proposition 3.2 present some necessary and sufficient conditions that pertain to the statement "$\mathcal{C} \in Z(\mathcal{Y}(\mathcal{F}))$"; their main goal is to motivate why such CM-isogenies classifications are too restrictive and often too difficult to be accomplished and therefore why in Subsection 3.3 we also introduce ramified lifts of $\mathcal{C}$ (or of D with respect to $\mathcal{G}$) and the (strong) ramified CM-isogeny classification of $\mathcal{F}$. In Subsections 3.4 to 3.8 we include different properties required in Sections 5 to 9 and some remarks. In particular, Corollary 3.7.3 checks that for $p \geq 3$ (resp. for $p = 2$) the ramified lifts of D with respect to $\mathcal{G}$ (see Definition 3.7.2) are in natural bijection to (resp. define naturally) abstract ramified lifts of $\mathcal{C}$. In this Section we will use the notations of Subsections 2.1 and 2.4. We recall that the axioms 2.4.1 (i) and (ii) hold.

3.1. Fact. (a) *If $\mathcal{C}$ has a lift of quasi CM type, then it is unramified semisimple.*

(b) *If there is a maximal torus of $\mathcal{G}_{B(k)}$ of $\mathbb{Q}_p$-endomorphisms of $\mathcal{C}$, then $\mathcal{C}$ is semisimple.*

Proof: We prove only (a) as the proof of (b) is very much the same. Let $\mathcal{T}$ be a maximal torus of $\mathcal{G}$ such that we have $\phi(\mathrm{Lie}(\mathcal{T})) = \mathrm{Lie}(\mathcal{T})$. Thus $\phi^r \in \mathcal{G}(B(k))$ (see Subsubsection 2.4.1) normalizes $\mathcal{T}$. Therefore we have $\phi^r \in N_{\mathcal{G}}(\mathcal{T})(B(k))$. Let $m \in \mathbb{N}$ be such that $\phi^{rm} \in \mathcal{T}(B(k))$. As the torus $\mathcal{T}_{W(\bar{k})}$ is split, part (a) follows. $\square$

3.2. Proposition. *Let $(M_{\mathbb{Z}_p}, \mathcal{G}_{\mathbb{Z}_p}, (t_\alpha)_{\alpha \in \mathcal{J}})$ be as in Subsubsection 2.4.1. We assume that $\mathcal{C}$ is unramified semisimple and quasi IU-ordinary. If all simple factors of $\mathcal{G}^{\mathrm{ad}}_{\mathbb{Z}_p}$ are Weil restrictions of* ***PGL*** *group schemes and if $\mathcal{G}_{\mathbb{Z}_p}(\mathbb{Q}_p)$ surjects onto $\mathcal{G}^{\mathrm{ad}}_{\mathbb{Z}_p}(\mathbb{Q}_p)$, then $[\mathcal{C}] \in Z(\mathcal{Y}(\mathcal{F}))$.*

Proof: It suffices to prove the Proposition under the extra assumptions that $\mathcal{C}$ is U-ordinary, that $L^0_{\mathcal{G}}(\phi)_{B(k)}$ is a subgroup of $\mathcal{P}_{B(k)}$, and that μ is the inverse of the canonical split cocharacter of (M, F^1, ϕ) defined in [Wi, p. 512]. The Lie algebra $\mathrm{Lie}(L^0_{\mathcal{G}}(\phi)_{B(\bar{k})})$ is $B(\bar{k})$-generated by elements which are fixed by $\phi \otimes \sigma_{\bar{k}}$ and which leave invariant $F^1[\frac{1}{p}]$. Due to the functorial aspect of [Wi, p. 513] these elements as well as the t_α's are fixed by $\mu_{B(\bar{k})}$. Thus $\mu_{B(k)}$ factors through $Z^0(L^0_{\mathcal{G}}(\phi)_{B(k)})$. We check that $L^0_{\mathcal{G}}(\phi)$ is a reductive, closed subgroup scheme of $\mathcal{G}$. Let T^0 be the image of μ; it is a torus of the center of $L^0_{\mathcal{G}}(\phi)$. By induction on $i \in \mathbb{N}$ we get the existence of a unique torus T^i of the center of $L^0_{\mathcal{G}}(\phi)$ such that we have $\mathrm{Lie}(T^i) = \phi^i(\mathrm{Lie}(T^0))$. Let T_0 be the torus of $\mathcal{G}$ generated by T^i's. We claim that $L^0_{\mathcal{G}}(\phi)$ is $C_0 := C_{\mathcal{G}}(T_0)$. Obviously $L^0_{\mathcal{G}}(\phi)$ is a closed subgroup scheme of C_0. As μ factors through $Z^0(C_0)$ we have $\phi(\mathrm{Lie}(C_0)) = \mathrm{Lie}(C_0)$. Thus $\mathrm{Lie}(C_{0B(k)}) \subseteq \mathrm{Lie}(L^0_{\mathcal{G}}(\phi)_{B(k)})$ i.e., $C_{0B(k)}$ is a subgroup of $L^0_{\mathcal{G}}(\phi)_{B(k)}$ (cf. [Bo, Ch. II, Subsection 7.1]). Therefore $L^0_{\mathcal{G}}(\phi) = C_0$ is a reductive, closed subgroup scheme of $\mathcal{G}$.

We have $\phi^r \in C_0(B(k))$, cf. paragraph before Fact 2.6.3. By performing the operation $\mathfrak{O}_1$ we can assume that C_0 is split and that all eigenvalues of ϕ^r as an automorphism of either $M[\frac{1}{p}]$ or $\mathrm{End}(M[\frac{1}{p}])$ belong to $B(\bar{k})$ and are not roots of unity different from 1. Let I_0 be the image of ϕ^r in $C_0^{\mathrm{ad}}(B(k))$. Let C_2 be the centralizer of I_0 in $C^{\mathrm{ad}}_{0B(k)}$; it is a split reductive group over $B(k)$. As $\mathrm{Lie}(C_2^{\mathrm{der}})$ is fixed by ϕ^r and normalized by ϕ, the Lie algebra $\mathrm{Lie}(C_2^{\mathrm{der}})$ is $B(k)$-generated by elements fixed by ϕ. Let $C^{\mathrm{ad}}_{2\mathbb{Q}_p}$ be the adjoint group over $\mathbb{Q}_p$ whose Lie algebra is formed by such elements; its extension to $B(k)$ is C_2^{ad}.

Let $T_{2\mathbb{Q}_p}$ be a maximal torus of $C^{\mathrm{ad}}_{2\mathbb{Q}_p}$ which splits over $B(k)$, cf. [Ti2, Subsection 1.10]. Let $\mathcal{T}_{1B(k)}$ be the maximal torus of $\mathcal{G}_{B(k)}$ which contains $Z^0(C_0)_{B(k)}$ and whose image in $C^{\mathrm{ad}}_{0B(k)}$ is generated by $Z^0(C_2)$ and by the maximal torus of $C^{\mathrm{der}}_{2B(k)}$ that is naturally isogenous to $T_{2B(k)}$. The torus $\mathcal{T}_{1B(k)}$ is split and its Lie algebra is normalized by ϕ. As ϕ^r acts trivially on $\mathrm{Lie}(\mathcal{T}_{1B(k)})$, $\mathrm{Lie}(\mathcal{T}_{1b(k)})$ is $B(k)$-generated by elements fixed by ϕ. Let $\mathcal{T}_{1\mathbb{Q}_p}$ be the $\mathbb{Q}_p$-form of $\mathcal{T}_1$ with respect to $(M[\frac{1}{p}],\phi)$; it splits over $B(k)$. Thus let $\mathcal{T}_{1\mathbb{Z}_p}$ be the torus over $\mathbb{Z}_p$ whose generic fibre is $\mathcal{T}_{1\mathbb{Q}_p}$, cf. [Ti2]. We can identify naturally $\mathcal{T}_{1\mathbb{Q}_p}$ with a maximal torus of the subgroup $\mathcal{G}_{\mathbb{Q}_p}$ of $\mathbf{GL}_{M_{\mathbb{Q}_p}}$. Let $\tilde{\mathcal{T}}_{1\mathbb{Q}_p}$ be the inverse image of $\mathcal{T}_{1\mathbb{Q}_p}$ in $Z^0(\mathcal{G}_{\mathbb{Q}_p})\times\mathcal{G}^{\mathrm{sc}}_{\mathbb{Q}_p}$. As above, let $\tilde{\mathcal{T}}_{1\mathbb{Z}_p}$ be the torus over $\mathbb{Z}_p$ whose generic fibre is $\tilde{\mathcal{T}}_{1\mathbb{Q}_p}$. We check that there exists a reductive group scheme $\tilde{\mathcal{G}}_{\mathbb{Z}_p}$ over $\mathbb{Z}_p$ whose generic fibre is $Z^0(\mathcal{G}_{\mathbb{Q}_p})\times\mathcal{G}^{\mathrm{sc}}_{\mathbb{Q}_p}$ and which has $\tilde{\mathcal{T}}_{1\mathbb{Z}_p}$ as a maximal torus. As $\mathcal{G}^{\mathrm{sc}}_{\mathbb{Q}_p}$ is a product of Weil restrictions of $\mathbf{SL}$ groups, it suffices to check that if k_3 is a finite field and if T_3 is a torus over $W(k_3)$ such that $T_{3B(k_3)}$ is a maximal torus of $\mathbf{SL}_{B(k_3)^n}$, then there exists a $W(k_3)$-lattice M_3 of $B(k_3)^n$ such that T_3 is a maximal torus of $\mathbf{SL}_{M_3}$. We take M_3 such that it is normalized by T_3, cf. [Ja, Part I, 10.4]. It is easy to see that T_3 is a maximal torus of $\mathbf{SL}_{M_3}$ (for instance, cf. [Va2, Thm. 1.1 (d)]).

As $\mathcal{G}_{\mathbb{Z}_p}(\mathbb{Q}_p)$ surjects onto $\mathcal{G}^{\mathrm{ad}}_{\mathbb{Z}_p}(\mathbb{Q}_p)$, there exists an element $h\in\mathcal{G}_{\mathbb{Q}_p}(\mathbb{Q}_p)$ such that we have $h\tilde{\mathcal{G}}^{\mathrm{sc}}_{\mathbb{Z}_p}(\mathbb{Z}_p)h^{-1}=\mathcal{G}^{\mathrm{sc}}_{\mathbb{Z}_p}(\mathbb{Z}_p)$ (cf. [Ti2, Subsection 1.10]). Thus by performing the operation $\mathfrak{O}_2$ (i.e., by replacing $M_{\mathbb{Z}_p}$ with $h^{-1}(M_{\mathbb{Z}_p})$) we can assume that the Zariski closure of $\mathcal{T}_{1\mathbb{Q}_p}$ in $\mathbf{GL}_{M_{\mathbb{Z}_p}}$ is the torus $\mathcal{T}_{1\mathbb{Z}_p}$. Obviously $\mathcal{T}_{1W(\bar{k})}$ is the extension to $W(\bar{k})$ of a maximal torus of $\mathcal{G}$ whose Lie algebra is normalized by ϕ. Thus from the very definitions we get that $[\mathcal{C}]\in Z(\mathcal{Y}(\mathcal{F}))$. $\square$

3.2.1. Remark. If $\mathcal{G}^{\mathrm{ad}}_{\mathbb{Z}_p}$ has simple factors of C_n Dynkin type ($n\geq 2$), then in general we can not assume that up to the operation $\mathfrak{O}_2$ the Zariski closure of $\mathcal{T}_{1\mathbb{Q}_p}$ in $\mathcal{G}_{\mathbb{Z}_p}$ is a torus (cf. [Va1, Rm. 3.1.2.2 1)]). This can be adapted to the B_n and D_n Dynkin types.

3.3. The ramified context. Let V be a finite, totally ramified discrete valuation ring extension of $W(k)$. Let $K:=V[\frac{1}{p}]$ and let π_V be a uniformizer of V. We assume that $e:=[V:W(k)]\geq 2$. Let the $W(k)$-algebras R, Re, and $\tilde{R}e$ be as in Subsection 2.7. For $m\in\mathbb{N}$ let Φ_{R_m} be the Frobenius lift of $R_m:=W(k)[[X_1,\ldots,X_m]]$ which is compatible with σ and which takes X_i to X_i^p for all $i\in\{1,\ldots,m\}$. If $m=1$ we drop it as an index; thus $R_1=R$. The p-adic completion $\Omega^{\wedge}_{R_m}$ of Ω_{R_m} is a free R_m-module that has $\{dX_1,\ldots,dX_m\}$ as an R_m-basis. Let $m_e:R\twoheadrightarrow V$ be the $W(k)$-epimorphism which takes X to π_V. If $p\geq 3$ (resp. $p\geq 2$), we denote also by m_e the $W(k)$-epimorphism $Re\twoheadrightarrow V$ (resp. $\tilde{R}e\twoheadrightarrow V$) defined by m_e (cf. Fact 2.7.1).

3.3.1. Definitions. (a) By a lift of $\mathcal{C}$ to R_m we mean a quadruple

$$(M_{R_m}, F^1_{R_m}, \phi_{M_{R_m}}, \mathcal{G}_{R_m}), \tag{5}$$

where M_{R_m} is a free R_m-module of the same rank as M, $F^1_{R_m}$ is a direct summand of M_{R_m}, $\mathcal{G}_{R_m}$ is a reductive, closed subgroup scheme of $\mathbf{GL}_{M_{R_m}}$, and $\phi_{M_{R_m}} : M_{R_m} \to M_{R_m}$ is a Φ_{R_m}-linear endomorphism, such that the following three axioms hold:

(i) the R_m-module M_{R_m} is generated by $\phi_{M_{R_m}}(M_{R_m} + p^{-1}F^1_{R_m})$;

(ii) there exists a family of tensors $(t_\alpha^{R_m})_{\alpha\in\mathcal{J}}$ of the F^0-filtration of $\mathcal{T}(M_{R_m})$ defined by $F^1_{R_m}$ such that we have $\phi_{M_{R_m}}(t_\alpha^{R_m}) = t_\alpha^{R_m}$ for all $\alpha \in \mathcal{J}$ and $\mathcal{G}_{R_m}$ is the Zariski closure in $\mathbf{GL}_{M_{R_m}}$ of the closed subscheme of $\mathbf{GL}_{M_{R_m}[\frac{1}{p}]}$ that fixes $t_\alpha^{R_m}$ for all $\alpha \in \mathcal{J}$;

(iii) the extension of $(M_{R_m}, \phi_{M_{R_m}}, \mathcal{G}_{R_m})$ via the $W(k)$-epimorphism $m_0 : R_m \twoheadrightarrow W(k)$ that maps each X_i to 0, is $\mathcal{C}$.

(b) Let $m_{m;e} : R_m \twoheadrightarrow V$ be a $W(k)$-epimorphism; if $m = 1$ we take $m_{1;e} := m_e$. Let $F^1_V := F^1_{R_m} \otimes_{R_m\, m_{m;e}} V$. We refer to $(M_{R_m}, F^1_V, \phi_{M_{R_m}}, \mathcal{G}_{R_m})$ as a lift of $\mathcal{C}$ to R_m with respect to V.

(c) We say $(M_{R_m}, F^1_V, \phi_{M_{R_m}}, \mathcal{G}_{R_m})$ is a lift of $\mathcal{C}$ to R_m of quasi CM (resp. of CM) type with respect to V, if there exists a maximal torus $\mathcal{T}_{R_m[\frac{1}{p}]}$ of $\mathcal{G}_{R_m[\frac{1}{p}]}$ such that the following two axioms hold:

(i) $\mathrm{Lie}(\mathcal{T}_{R_m[\frac{1}{p}]})$ is normalized (resp. is $R_m[\frac{1}{p}]$-generated by elements fixed) by $\phi_{M_{R_m}}$;

(ii) $F^1_V[\frac{1}{p}]$ is a $\mathrm{Lie}(\mathcal{T}_{R_m[\frac{1}{p}]})/\mathrm{Ker}(m_{m;e})\mathrm{Lie}(\mathcal{T}_{R_m[\frac{1}{p}]})$-module.

(d) We have variants of (a) to (c), where we replace R_m by Re or $\widetilde{Re}$ (the $W(k)$-epimorphisms from either Re or $\widetilde{Re}$ onto V being m_e). A lift of $\mathcal{C}$ to $\widetilde{Re}$ (which is of quasi CM or of CM type) with respect to V is also called a ramified lift of $\mathcal{C}$ to V (of quasi CM or of CM type).

(e) If we have a principal bilinear quasi-polarization $\lambda_M : M \otimes_{W(k)} M \to W(k)$ of $\mathcal{C}$, then by a lift of $(\mathcal{C}, \lambda_M)$ to R_m we mean a quintuple

$$(M_{R_m}, F^1_{R_m}, \phi_{M_{R_m}}, \mathcal{G}_{R_m}, \lambda_{M_{R_m}}),$$

where $(M_{R_m}, F^1_{R_m}, \phi_{M_{R_m}}, \mathcal{G}_{R_m})$ is as in (a) and $\lambda_{M_{R_m}}$ is a perfect bilinear form on M_{R_m} which lifts λ_M, whose R_m-span is normalized by $\mathcal{G}_{R_m}$, and for which we have an identity $\lambda_M(\phi_{M_{R_m}}(x) \otimes \phi_{M_{R_m}}(y)) = p\Phi_{R_m}(\lambda_{M_{R_m}}(x \otimes y))$ for all elements $x, y \in M_{R_m}$. Similarly, definitions (b) to (d) extend to the principal bilinear quasi-polarized context.

3.3.2. Remarks. (a) Let $(M_{R_m}, F^1_V, \phi_{M_{R_m}}, \mathcal{G}_{R_m})$ be a lift of $\mathcal{C}$ to R_m of CM type with respect to V. Let $\mathcal{T}_{R_m[\frac{1}{p}]}$ be a maximal torus of $\mathcal{G}_{R_m[\frac{1}{p}]}$ such that the two axioms of Definition 3.3.1 (c) hold for it. Let $\mathcal{T}_{1B(k)}$ be the pull back of $\mathcal{T}_{R_m[\frac{1}{p}]}$ via the $B(k)$-epimorphism $R_m[\frac{1}{p}] \twoheadrightarrow B(k)$ that takes each X_i to 0. As $\mathrm{Lie}(\mathcal{T}_{R_m[\frac{1}{p}]})$ is $R_m[\frac{1}{p}]$-generated by elements fixed by $\phi_{M_{R_m}}$, $\mathcal{T}_{1B(k)}$ is a maximal torus of $\mathcal{G}_{B(k)}$ of $\mathbb{Q}_p$-endomorphisms of $\mathcal{C}$.

Similarly, if $\mathcal{C}$ has a ramified lift to V of CM type, then there exist maximal tori of $\mathcal{G}_{B(k)}$ of $\mathbb{Q}_p$-endomorphisms of $\mathcal{C}$.

(b) We refer to Definition 3.3.1 (a). The reductive group scheme $\mathcal{G}_{R_m}$ over R_m lifts $\mathcal{G}$ (cf. axiom (iii) of Definition 3.3.1 (a)) and thus (as R_m is complete in the $(X_1, \ldots, X_m)$ topology) it is isomorphic to $\mathcal{G} \times_{\mathrm{Spec}(W(k))} \mathrm{Spec}(R_m)$ (i.e., our notations match).

3.3.3. Ramified CM-isogeny classifications. By the *strong ramified* (resp. by the *ramified*) *CM-isogeny classification* of $\mathcal{F}$ we mean the description of the subset

$$SZ^{\mathrm{ram}}(\mathcal{Y}(\mathcal{F})) \text{ (resp. } Z^{\mathrm{ram}}(\mathcal{Y}(\mathcal{F})))$$

of $\mathcal{Y}(\mathcal{F})$ formed by inner isomorphism classes of those $\mathcal{C}_g$ with $g \in \mathcal{G}(W(k))$ for which, up to the operation $\mathfrak{O}_2$ (resp. up to operations $\mathfrak{O}_1$ and $\mathfrak{O}_2$), there exists a discrete valuation ring V as in Subsection 3.3 and a ramified lift of $\mathcal{C}_g$ to V of quasi CM type. We have $Z(\mathcal{Y}(\mathcal{F})) \subseteq Z^{\mathrm{ram}}(\mathcal{Y}(\mathcal{F}))$ and $SZ(\mathcal{Y}(\mathcal{F})) \subseteq SZ^{\mathrm{ram}}(\mathcal{Y}(\mathcal{F}))$.

Let $PSZ^{\mathrm{ram}}(\mathcal{Y}(\mathcal{F}))$ (resp. $PZ^{\mathrm{ram}}(\mathcal{Y}(\mathcal{F}))$) be the subset of $\mathcal{Y}(\mathcal{F})$ formed by inner isomorphism classes of those $\mathcal{C}_g$ with $g \in \mathcal{G}(W(k))$ for which (resp. for which, up to the operation $\mathfrak{O}_1$,) there exists a discrete valuation ring V as in Subsection 3.3 and an element $h \in \mathfrak{P}(\mathcal{C}_g)$ such that the Shimura F-crystal $(h(M), \phi, \mathcal{G}(h))$ over k has a ramified lift to V of quasi CM type.

We assume now that we have a principal bilinear quasi-polarization $\lambda_M : M \otimes_{W(k)} M \to W(k)$ of $\mathcal{C}$. Let

$$\mathfrak{I}(\mathcal{C}, \lambda_M) := \mathfrak{I}(\mathcal{C}) \cap \mathbf{Aut}(M, \lambda_M)(B(k)).$$

Let $\mathcal{I}(\mathcal{C}, \lambda_M)$ be the set of inner isomorphism classes of quadruples of the form $(h(M), \phi, \mathcal{G}(h), \lambda_M)$ with $h \in \mathfrak{I}(\mathcal{C}, \lambda_M)$ (i.e., the set of such quadruples up to isomorphisms defined by elements of $(\mathcal{G} \cap \mathbf{Aut}(M, \lambda_M))(W(k))$). Let $\mathcal{Y}(\mathcal{F}, \lambda_M) := \cup_{g \in (\mathcal{G} \cap \mathbf{Aut}(M, \lambda_M))(W(k))} \mathcal{I}(\mathcal{C}_g, \lambda_M)$. As above, let $SZ^{\mathrm{ram}}(\mathcal{Y}(\mathcal{F}, \lambda_M))$ (resp. $Z^{\mathrm{ram}}(\mathcal{Y}(\mathcal{F}, \lambda_M))$) be the subset of $\mathcal{Y}(\mathcal{F}, \lambda_M)$ formed by inner isomorphism classes of those $(\mathcal{C}_g, \lambda_M)$ with $g \in (\mathcal{G} \cap \mathbf{Aut}(M, \lambda_M))(W(k))$ for which, up to the operation $\mathfrak{O}_1$ (resp. up to operations $\mathfrak{O}_1$ and $\mathfrak{O}_2$), there exists a discrete valuation ring V as in Subsection 3.3 and a ramified lift of $(\mathcal{C}_g, \lambda_M)$ to V of quasi CM type.

3.4. Lemma. *Let $s \in \mathbb{N}$. Let $g \in \boldsymbol{GL}_{M_{R_m}}(R_m)$ be congruent to $1_{M_{R_m}}$ modulo $p^s R_m$. From [Fa, Thm. 10] we deduce the existence of a unique connection $\nabla_0 : M_{R_m} \to M_{R_m} \otimes_{R_m} \Omega^{\wedge}_{R_m}$ (resp. $\nabla_1 : M_{R_m} \to M_{R_m} \otimes_{R_m} \Omega^{\wedge}_{R_m}$) on M_{R_m} such that $\phi_{M_{R_m}}$ (resp. $g\phi_{M_{R_m}}$) is horizontal with respect to it; it is integrable and nilpotent modulo p. Let D_0 (resp. D_1) be the unique (up to a unique isomorphism) p-divisible group over R_m/pR_m whose F-crystal is $(M_{R_m}, \phi_{M_{R_m}}, \nabla_0)$ (resp. $(M_{R_m}, g\phi_{M_{R_m}}, \nabla_1)$ (the uniqueness part is implied by [BM, Thm. 4.1.1] while the existence part is implied by [Fa, Thm. 10]). Then we have $D_0[p^s] = D_1[p^s]$.*

Proof: Let $d\Phi_{R_m*/p}$ be the differential map of Φ_{R_m} divided by p. Let $F^0_{R_m}$ be a direct supplement of $F^1_{R_m}$ in M_{R_m}. We have $\nabla_0(\phi_{M_{R_m}}(x)) = p(\phi_{M_{R_m}} \otimes d\Phi_{R_m*/p}) \circ \nabla_0(x)$ if $x \in F^0_{R_m}$ and $\nabla_0(\phi_{M_{R_m}}(x/p)) = (\phi_{M_{R_m}} \otimes d\Phi_{R_m*/p}) \circ \nabla_0(x)$ if $x \in F^1_{R_m}$. Similar equations are satisfied by ∇_1. Let $\nabla_{01}[p^s]$ be $\nabla_0 - \nabla_1$ modulo p^s; it is an $R_m/p^s R_m$-linear map $M_{R_m}/p^sM_{R_m} \to M_{R_m}/p^sM_{R_m} \otimes_{R_m/p^sR_m} \Omega_{R_m/p^sR_m}$. As M_{R_m} is R_m-generated by $\phi_{M_{R_m}}(F^0_{R_m} \oplus \frac{1}{p}F^1_{R_m})$ and as we have $\Phi_{R_m}(X_i) = X_i^p$, by induction on $l \in \mathbb{N}$ we get that $\nabla_{01}[p^s]$ is zero modulo the ideal $(X_1, \ldots, X_m)^l$ of R_m/p^sR_m. Thus the connections on $M_{R_m}/p^sM_{R_m}$ defined by ∇_0 and ∇_1 coincide. Therefore we have $D_0[p^s] = D_1[p^s]$, cf. [BM, Prop. 1.3.3 and Thm. 4.1.1]. $\square$

3.5. Theorem. *We assume that $p \geq 3$ and $\mathcal{G} = \boldsymbol{GL}_M$. Then the ramified lifts of $\mathcal{C}$ to V are in natural bijection to lifts of D to p-divisible groups over V.*

Proof: To a p-divisible group D_V over V that lifts D one associates uniquely a ramified lift of $\mathcal{C}$ to V as follows. Let

$$(M_{\tilde{R}e}, \phi_{M_{\tilde{R}e}}, \nabla)$$

be the extension via the $W(k)$-monomorphism $Re \hookrightarrow \tilde{R}e$ of (the projective limit indexed by $n \in \mathbb{N}$ of the evaluation at the thickening naturally attached to the closed embedding $\mathrm{Spec}(V/pV) \hookrightarrow \mathrm{Spec}(Re/p^nRe)$ of) the Dieudonné F-crystal over V/pV of $D_V \times_{\mathrm{Spec}(V)} \mathrm{Spec}(V/pV)$ (see [Me], [BBM], [BM], and [dJ, Subsection 2.3]). Thus $M_{\tilde{R}e}$ is a free $\tilde{R}e$-module of the same rank as D, $\nabla : M_{\tilde{R}e} \to M_{\tilde{R}e} \otimes_{\tilde{R}e} \tilde{R}edX$ is an integrable and nilpotent modulo p connection on $M_{\tilde{R}e}$, and $\phi_{M_{\tilde{R}e}}$ is a $\Phi_{\tilde{R}e}$-linear endomorphism of $M_{\tilde{R}e}$ which is horizontal with respect to ∇. If $F^1(M_{\tilde{R}e})$ is the inverse image in $M_{\tilde{R}e}$ of the Hodge filtration F^1_V of $M_{\tilde{R}e}/\mathrm{Ker}(m_e)M_{\tilde{R}e} = H^1_{\mathrm{dR}}(D_V/V)$ defined by D_V, then the restriction of $\phi_{M_{\tilde{R}e}}$ to $F^1(M_{\tilde{R}e})$ is divisible by p and $M_{\tilde{R}e}$ is $\tilde{R}e$-generated by $\phi_{M_{\tilde{R}e}}(M_{\tilde{R}e} + \frac{1}{p}F^1(M_{\tilde{R}e}))$. Thus the quasruple $(M_{\tilde{R}e}, F^1_V, \phi_{M_{\tilde{R}e}}, \mathbf{GL}_{M_{\tilde{R}e}})$ is the ramified lift of $\mathcal{C}$ to V associated to D_V. Due to this and to the fully faithfulness part of [Fa, Thm. 5], to prove the Theorem it suffices to show that every ramified lift of $\mathcal{C}$ to V is associated to a p-divisible group over V which lifts D. As $p \geq 3$, each lift of $(M, \phi, \mathbf{GL}_M)$ is associated to (i.e., it is the filtered F-crystal of) a unique p-divisible group over $W(k)$ that lifts D (cf. Grothendieck–Messing deformation theory of [Me, Chs. IV and V]). Thus each lift of $\mathcal{C}$ to R_m is associated to a unique p-divisible group over R_m that lifts D, cf. [Fa, Thm. 10]. Thus the Theorem follows from the following general result (applied with $p \geq 3$, $\mathcal{G} = \mathbf{GL}_M$, and $\mathcal{J} = \emptyset$). $\square$

3.6. Theorem. *We assume that $p \geq 2$ but do not assume that $\mathcal{G}$ is $\boldsymbol{GL}_M$. We take m to be* $\dim(\mathcal{G}^{\mathrm{der}}_{B(k)})$. *Then each lift $(M_{\tilde{R}e}, F^1_{\tilde{R}e}, \phi_{M_{\tilde{R}e}}, \mathcal{G}_{\tilde{R}e})$ of $\mathcal{C}$ to $\tilde{R}e$ is the extension via a $W(k)$-homomorphism $R_m \to \widetilde{Re}$ of a lift of $\mathcal{C}$ to R_m (this makes sense, cf. the uniqueness of connections in Lemma 3.4).*

Proof: Let $(t^{\tilde{R}e}_\alpha)_{\alpha \in \mathcal{J}}$ be a family of tensors of the F^0-filtration of $\mathcal{T}(M_{\tilde{R}e})$ defined by $F^1_{\tilde{R}e}$ which has the analogue meaning of the family of tensors $(t^{R_m}_\alpha)_{\alpha \in \mathcal{J}}$ of Definition 3.3.1 (a). Not to introduce extra notations, we can assume that the extension of $\mathcal{C}^{\tilde{R}e} := (M_{\tilde{R}e}, F^1_{\tilde{R}e}, \phi_{M_{\tilde{R}e}}, (t^{\tilde{R}e}_\alpha)_{\alpha \in \mathcal{J}})$ via the $W(k)$-epimorphism $m_0 : R_m \twoheadrightarrow W(k)$ of Definition 3.3.1 (a) is of the form $\mathcal{C}^{W(k)} = (M, F^1_0, \phi, (t_\alpha)_{\alpha \in \mathcal{J}})$. Let

$$\mathcal{C}^{R_m} := (M \otimes_{W(k)} R_m, F^1_0 \otimes_{W(k)} R_m, g^{\mathrm{der}}_{\mathrm{univ}}(\phi \otimes \Phi_{R_m}), (t_\alpha)_{\alpha \in \mathcal{J}}),$$

where $g_{\mathrm{univ}}^{\mathrm{der}} : \mathrm{Spec}(R_m) \to \mathcal{G}^{\mathrm{der}}$ is a universal morphism which identifies $\mathrm{Spf}(R_m)$ with the formal completion of $\mathcal{G}^{\mathrm{der}}$ along the identity section. Let ∇_{univ} be the unique connection on $M \otimes_{W(k)} R_m$ such that $g_{\mathrm{univ}}^{\mathrm{der}}(\phi \otimes \Phi_{R_m})$ is horizontal with respect to it. Let δ_0 be the flat connection on $M \otimes_{W(k)} R_m$ that annihilates $M \otimes 1$.

We have $\gamma_{\mathrm{univ}} := \nabla_{\mathrm{univ}} - \delta_0 \in \mathrm{Lie}(\mathcal{G}) \otimes_{W(k)} \Omega^{\wedge}_{R_m}$, cf. [Fa2, §7, Rm. ii)]. For the sake of completeness, we include a proof of the last result. We view $\mathcal{T}(M)$ as a module over the Lie algebra (associated to) $\mathrm{End}_{W(k)}(M)$ and we denote also by ∇_{univ} the connection on $\mathcal{T}(M \otimes_{W(k)} R_m[\frac{1}{p}])$ which extends naturally the connection ∇_{univ} on $M \otimes_{W(k)} R_m$. Each tensor $t_\alpha \in \mathcal{T}(M \otimes_{W(k)} R_m[\frac{1}{p}])$ is fixed under the natural action of $g_{\mathrm{univ}}^{\mathrm{der}}(\phi \otimes \Phi_{R_m})$ on $\mathcal{T}(M \otimes_{W(k)} R_m[\frac{1}{p}])$. Thus we have $\nabla_{\mathrm{univ}}(t_\alpha) = (g_{\mathrm{univ}}^{\mathrm{der}}(\phi \otimes \Phi_{R_m}) \otimes d\Phi_{R_m})(\nabla_{\mathrm{univ}}(t_\alpha))$. As we have $d\Phi_{R_m}(X_i) = pX_i^{p-1}dX_i$ for each $i \in \{1, \ldots, m\}$, by induction on $s \in \mathbb{N}$ we get that $\nabla_{\mathrm{univ}}(t_\alpha) = \gamma_{\mathrm{univ}}(t_\alpha) \in \mathcal{T}(M) \otimes_{W(k)} (X_1, \ldots, X_m)^s \Omega^{\wedge}_{R_m}[\frac{1}{p}]$. As R_m is complete with respect to the $(X_1, \ldots, X_m)$-topology, we get that $\nabla_{\mathrm{univ}}(t_\alpha) = \gamma_{\mathrm{univ}}(t_\alpha) = 0$. But $\mathrm{Lie}(\mathcal{G}_{B(k)}) \cap \mathrm{End}_{W(k)}(M)$ is the Lie subalgebra of $\mathrm{End}_{W(k)}(M)$ that centralizes t_α for all $\alpha \in \mathcal{J}$. From the last two sentences we get that $\gamma_{\mathrm{univ}} \in \mathrm{Lie}(\mathcal{G}) \otimes_{W(k)} \Omega^{\wedge}_{R_m}$.

Next we list three basic properties of the $W(k)$-algebra $\tilde{R}e$:

(i) we have $\tilde{R}e = \mathrm{proj.lim.}_{m \in \mathbb{N}} \tilde{R}e/I(m)$, the transition $W(k)$-epimorphisms being the logical ones (see Subsection 2.7 for $I(m)$'s);

(ii) the $W(k)$-module $I(m)/I(m+1)$ is free of rank 1 for all $m \in \mathbb{N}$;

(iii) we have an inclusion $I(m)^2 + \Phi_{\tilde{R}e}(I(m)) \subseteq I(m+1)$ for all $m \in \mathbb{N}$.

Thus the arguments of [Fa, Thm. 10 and Rm. (iii) of p. 136] apply entirely to give us that $\mathcal{C}^{\tilde{R}e}$ is the extension of $\mathcal{C}^{R_m}$ through a $W(k)$-homomorphism $R_m \to \tilde{R}e$ that maps the ideal $(X_1, \ldots, X_m)$ to $I(1)$ (this extension is well defined as the connection ∇_{univ} exists and is unique). Strictly speaking, loc. cit. is stated in terms of a universal element of $\mathcal{G}$ and not of $\mathcal{G}^{\mathrm{der}}$. But the image of the Kodaira–Spencer map of ∇_{univ} is the same regardless if we work with $\mathcal{G}^{\mathrm{der}}$ or $\mathcal{G}$ (this follows easily from the relation $\gamma_{\mathrm{univ}} \in \mathrm{Lie}(\mathcal{G}) \otimes_{W(k)} \Omega^{\wedge}_{R_m}$) and therefore loc. cit. applies in our present context of $\mathcal{G}^{\mathrm{der}}$ as well. $\square$

3.6.1. Corollary. *Let $(M_{\tilde{R}e}, F^1_{\tilde{R}e}, \phi_{M_{\tilde{R}e}}, \mathcal{G}_{\tilde{R}e})$ be a lift of $\mathcal{C}$ to $\tilde{R}e$. If $p \geq 3$, let D_V be the p-divisible group over V that lifts D and that corresponds to $(M_{\tilde{R}e}, F^1_{\tilde{R}e} \otimes_{\tilde{R}e\, m_e} V, \phi_{M_{\tilde{R}e}})$ via the natural bijection of Theorem 3.5. If $p = 2$, we assume that there exists a p-divisible group D_V over V which lifts D and such that the triple $(M_{\tilde{R}e}, F^1_{\tilde{R}e} \otimes_{\tilde{R}e\, m_e} V, \phi_{M_{\tilde{R}e}})$ is associated to it as in the proof of Theorem 3.5. Let $\mathcal{T}_{\tilde{R}e[\frac{1}{p}]}$ be a maximal torus of $\mathcal{G}_{\tilde{R}e[\frac{1}{p}]}$ such that $\phi_{M_{\tilde{R}e}}$ leaves invariant Lie$(\mathcal{T}_{\tilde{R}e[\frac{1}{p}]})$ and $F^1_K := F^1_{\tilde{R}e} \otimes_{\tilde{R}e[\frac{1}{p}]} K$ is a Lie$(\mathcal{T}_{\tilde{R}e[\frac{1}{p}]}) \otimes_{\tilde{R}e[\frac{1}{p}]} K$-module. Then by performing the operation $\mathfrak{O}_1$ we can assume that Lie$(\mathcal{T}_{\tilde{R}e[\frac{1}{p}]})$ is $\tilde{R}e[\frac{1}{p}]$-generated by elements fixed by $\phi_{M_{\tilde{R}e}}$ and thus that $\mathcal{C}$ has ramified lifts to V of CM type. Thus, up to the operation $\mathfrak{O}_1$, the p-divisible group D_V is with complex multiplication.*

Proof: There exists a canonical and functorial (in D_V) identification

$$(6) \qquad (M \otimes_{W(k)} \tilde{R}e[\frac{1}{p}], \phi \otimes \Phi_{\tilde{R}e}) = (M_{\tilde{R}e}[\frac{1}{p}], \phi_{M_{\tilde{R}e}})$$

under which the pull back of the natural $B(k)$-epimorphism $\tilde{R}e[\frac{1}{p}] \twoheadrightarrow B(k)$ that takes X to 0 is the identity automorphism of $(M[\frac{1}{p}], \phi)$ (see [Fa, Section 6] for the existence part; the uniqueness part follows from the fact that no element of $\mathrm{End}(M[\frac{1}{p}] \otimes_{B(k)} I(1)$ is fixed by $\phi \otimes \Phi_{\tilde{R}e}$). Therefore via (6), we can identify F^1_K with a direct summand of $M \otimes_{W(k)} K = H^1_{\mathrm{dR}}(D_V/V)[\frac{1}{p}] = [M_{\tilde{R}e}/\mathrm{Ker}(m_e)M_{\tilde{R}e}][\frac{1}{p}]$.

Let $\mathcal{T}_{1B(k)}$ be the pull back of $\mathcal{T}_{\tilde{R}e[\frac{1}{p}]}$ to a maximal torus of $\mathcal{G}_{B(k)}$. Under the identification (6), $\mathrm{Lie}(\mathcal{T}_{1B(k)}) \otimes_{B(k)} \tilde{R}e[\frac{1}{p}]$ gets identified with $\mathrm{Lie}(\mathcal{T}_{\tilde{R}e[\frac{1}{p}]})$. Thus F^1_K is a $\mathrm{Lie}(\mathcal{T}_{1K})$-module. The triple $(M[\frac{1}{p}], \phi, F^1_K)$ is the filtered Dieudonné module of D_V and thus it is an admissible filtered module over K. The triple $(\mathrm{Lie}(\mathcal{T}_{1B(k)}), \phi, 0)$ is an admissible filtered submodule over K of $\mathrm{End}(M[\frac{1}{p}], \phi, F^1_K)$. Therefore all Newton polygon slopes of $(\mathrm{Lie}(\mathcal{T}_{1B(k)}), \phi)$ are 0. As ϕ normalizes $\mathrm{Lie}(\mathcal{T}_{1B(k)})$, it is easy to see that by performing the operation $\mathfrak{O}_1$ we can assume that $\mathrm{Lie}(\mathcal{T}_{1B(k)})$ is $B(k)$-generated by elements fixed by ϕ.

Let $\tilde{t} \in \mathrm{Lie}(\mathcal{T}_{\tilde{R}e[\frac{1}{p}]})$ be an element which lifts an element $t \in \mathrm{Lie}(\mathcal{T}_{1B(k)})$ fixed by ϕ and for which we have $\tilde{t}(M_{\tilde{R}e}) \subseteq M_{\tilde{R}e}$. As $\Phi_{\tilde{R}e}(X) = X^p$, the sequence $(\phi^s_{M_{\tilde{R}e}}(t))_{s \in \mathbb{N}}$ converges in the topology of the $\tilde{R}e[\frac{1}{p}]$-module $\mathrm{End}(M_{\tilde{R}e})[\frac{1}{p}]$ defined by the sequence $(I(m)\mathrm{End}(M_{\tilde{R}e})[\frac{1}{p}])_{m \in \mathbb{N}}$ of $\tilde{R}e[\frac{1}{p}]$-submodules to an element $t_0 \in \mathrm{Lie}(\mathcal{T}_{\tilde{R}e[\frac{1}{p}]})$ which is fixed by $\phi_{M_{\tilde{R}e}}$ and which lifts t. This implies that $\mathrm{Lie}(\mathcal{T}_{\tilde{R}e[\frac{1}{p}]})$ is $\tilde{R}e[\frac{1}{p}]$-generated by elements fixed by $\phi_{M_{\tilde{R}e}}$. This proves the first part. As t_0 leaves invariant F^1_K, an integral p-power of it corresponds naturally to an endomorphism of D_V (even if $p = 2$). Thus the second part follows from this and the first part. $\square$

3.7. Connection to the Main Problem. Let D_V be a p-divisible group over k that lifts D. Let $(M_{\tilde{R}e}, F^1_V, \phi_{M_{\tilde{R}e}})$ be associated to D_V as in the proof of Theorem 3.5 (even if $p = 2$). Under the identification (6), we can naturally view $\mathcal{G}_{\tilde{R}e[\frac{1}{p}]}$ as a subgroup scheme of $\mathbf{GL}_{M_{\tilde{R}e}[\frac{1}{p}]}$. Let $\mathcal{G}'_{\tilde{R}e}$ be the Zariski closure in $\mathbf{GL}_{M_{\tilde{R}e}}$ of $\mathcal{G}_{\tilde{R}e[\frac{1}{p}]}$ (in general it is not a closed subgroup scheme of $\mathbf{GL}_{M_{\tilde{R}e}}$). We have the following Corollary of Theorem 3.6.

3.7.1. Corollary. *We assume that the Zariski closure $\tilde{\mathcal{G}}'_V$ of $m_e^*(\mathcal{G}'_{\tilde{R}e})_K$ in $\boldsymbol{GL}_{M_{\tilde{R}e} \otimes_{\tilde{R}e} m_e V}$ is a reductive group scheme over V whose special fibre, under the canonical identification $M_{\tilde{R}e} \otimes_{\tilde{R}e} k = M/pM$, is $\mathcal{G}_k$. We also assume that there exists a cocharacter $\tilde{\mu}_V : \mathbb{G}_m \to \tilde{\mathcal{G}}'_V$ that acts on F^1_V via the inverse of the identical character of $\mathbb{G}_m$ and that fixes $[M_{\tilde{R}e}/Ker(m_e)M_{\tilde{R}e}]/F^1_V = H^1_{\mathrm{dR}}(D_V/V)/F^1_V$. Then $\mathcal{G}'_{\tilde{R}e}$ is a reductive, closed subgroup scheme of $\boldsymbol{GL}_{M_{\tilde{R}e}}$ isomorphic to $\mathcal{G}_{\tilde{R}e}$ and $(M_{\tilde{R}e}, F^1_V, \phi_{M_{\tilde{R}e}}, \mathcal{G}'_{\tilde{R}e})$ is a ramified lift of $\mathcal{C}$ to V.*

Proof: For $s \in \{1, \ldots, e\}$ we have $R_{1,s} := \tilde{R}e/I(s) = W(k)[[X]]/(X^s)$. By induction on $s \in \{1, \ldots, e\}$ we show that the Zariski closure $\tilde{\mathcal{G}}'_{R_{1,s}}$ of $\mathcal{G}_{B(k)[[X]]/(X^s)}$ in $\mathbf{GL}_{M_{\tilde{R}e}/I(s)M_{\tilde{R}e}}$ is a reductive, closed subgroup scheme of $\mathbf{GL}_{M_{\tilde{R}e}/I(s)M_{\tilde{R}e}}$. The case $s \leq p-1$ is obvious as the ideal $(X)/(X^s)$ of $R_{1,s}$ has a nilpotent divided power structure. More precisely, the reduction modulo $I(s)[\frac{1}{p}]$ of the identification (6) gives birth to a canonical identification

$(M\otimes_{W(k)} R_{1,s},\phi\otimes\Phi_{R_{1,s}})=(M_{\tilde Re},\phi_{M_{\tilde Re}})\otimes_{\tilde Re}\tilde Re/I(s)$, where $\Phi_{1,s}$ is the Frobenius lift of $R_1/(X^s)=R_{1,s}$ which is compatible with σ and which annihilates X modulo (X^s). If $p-1\le s\le e-1$, then the passage from s to $s+1$ goes as follows.

Let m_V be the maximal ideal of V. Under the canonical identification $M_{\tilde Re}\otimes_{\tilde Re} k=M/pM$, we can also identify $F^1_V/m_VF^1_V=F^1/pF^1$ and $\tilde{\mathcal{G}}'_V=\mathcal{G}_k$; therefore we can assume view both $\tilde\mu_V$ modulo m_V and μ_k as cocharacters of $\mathcal{P}_k$. By replacing μ with a $\mathcal{P}(W(k))$-conjugate of it, we can assume that $\tilde\mu_V$ modulo m_V commutes with μ_k. As $\tilde\mu_V$ modulo m_V and μ_k are two commuting cocharacters of $\mathcal{P}_k$ that act in the same way on F^1/pF^1 and $(M/pM)/(F^1/pF^1)$, they coincide. Let $\tilde\mu_{R_{1,s}}:\mathbb{G}_m\to\tilde{\mathcal{G}}'_{R_{1,s}}$ be a cocharacter that lifts both $\tilde\mu_V$ modulo m_V and μ, cf. [DG, Vol. II, Exp. IX, Thms. 3.6 and 7.1]. Let $F^1_{R_{1,s}}$ be the direct summand of $M_{\tilde Re}/I(s)M_{\tilde Re}$ which lifts $F^1_V/m_VF^1_V=F^1/pF^1$ and which is normalized by $\tilde\mu_{R_{1,s}}$.

Let $m:=\dim(\mathcal{G}^{\mathrm{der}}_{B(k)})$. From [Fa, proof of Thm. 10 and Rm. (iii) of p. 136] we get that the quadruple $(M_{\tilde Re}/I(s)M_{\tilde Re},F^1_{R_{1,s}},\phi_{M_{\tilde Re}},(t_\alpha)_{\alpha\in\mathcal{J}})$ is induced from $\mathcal{C}^{R_m}$ via a $W(k)$-homomorphism $j_s:R_m\to R_{1,s}$ that maps the ideal $(X_1,\ldots,X_m)$ to the ideal $(X)/(X^s)$. Here we denote also by $\phi_{M_{\tilde Re}}$ its reduction modulo $I(s)$. As the ideal $(X^s)/(X^{s+1})$ of $R_{1,s+1}$ has naturally a trivial divided power structure and as j_s lifts to a $W(k)$-homomorphism $j_{s+1}:R_m\to R_{1,s+1}$ that maps the ideal $(X_1,\ldots,X_m)$ to the ideal $(X^s)/(X^{s+1})$, the triple $(M_{\tilde Re}/I(s+1)M_{\tilde Re},\phi_{M_{\tilde Re}},(t_\alpha)_{\alpha\in\mathcal{J}})$ is the extension of $(M_{R_m},\phi_{M_{R_m}},(t^{R_m}_\alpha)_{\alpha\in\mathcal{J}})$ via such a homomorphism j_{s+1}. Thus $\tilde{\mathcal{G}}'_{R_{1,s+1}}$ is the pull back of $\mathcal{G}_{R_m}$ via the morphism $j_{s+1}:\mathrm{Spec}(R_{1,s+1})\to\mathrm{Spec}(R_m)$ defined by j_{s+1} (and denoted in the same way) and it is therefore a reductive, closed subgroup scheme group scheme of $\mathbf{GL}_{M_{\tilde Re}/I(s+1)M_{\tilde Re}}$. This ends the induction.

Let $\tilde\mu_{R_{1,e}}:\mathbb{G}_m\to\tilde{\mathcal{G}}'_{R_{1,e}}$ be a cocharacter that lifts both μ and the reduction modulo p of μ_V, cf. [DG, Vol. II, Exp. IX, Thms. 3.6 and 7.1]. Let $F^1_{R_{1,e}}$ be the direct summand of $M_{\tilde Re}/I(e)M_{\tilde Re}$ which lifts $F^1_V/m_VF^1_V=F^1/pF^1$ and which is normalized by $\tilde\mu_{R_{1,e}}$. From [Fa, proof of Thm. 10 and Rm. (iii) of p. 136] we get that $(M_{\tilde Re}/I(e)M_{\tilde Re},F^1_{R_{1,e}},\phi_{M_{\tilde Re}},(t_\alpha)_{\alpha\in\mathcal{J}})$ is induced from $\mathcal{C}^{R_m}$ via a $W(k)$-homomorphism $j_e:R_m\to R_{1,e}$ that maps the ideal $(X_1,\ldots,X_m)$ to the ideal $(X)/(X^e)$. Here we denote also by $\phi_{M_{\tilde Re}}$ its reduction modulo $I(e)$.

Let D_{univ} be the p-divisible group over R_m/pR_m whose F-crystal is $(M\otimes_{W(k)} R_m,g^{\mathrm{der}}_{\mathrm{univ}}(\phi\otimes\Phi_{R_m}),\nabla_{\mathrm{univ}})$ (see proof of Theorem 3.6). A second induction on $s\in\{1,\ldots,e\}$ shows (based on Grothendieck–Messing deformation theory) that there is a $W(k)$-homomorphism $j_s:R_m\to R_{1,s}$ that maps the ideal $(X_1,\ldots,X_m)$ to the ideal $(X)/(X^s)$ and such that the pull back of D_{univ} via the morphism $\mathrm{Spec}(k[[X]]/(X^s))\to\mathrm{Spec}(R_m)$ defined by j_s is $D_V\times_{\mathrm{Spec}(V)}\mathrm{Spec}(k[[X]]/(X^e))$. Taking $s=e$ we get that we can assume that $D_V\times_{\mathrm{Spec}(V)}\mathrm{Spec}(V/pV)$ is the extension via j_e modulo p of D_{univ}. This implies that the triple $(M_{\tilde Re},\phi_{M_{\tilde Re}},(t_\alpha)_{\alpha\in\mathcal{J}})$ is the extension of $(M\otimes_{W(k)} R_m,g^{\mathrm{der}}_{\mathrm{univ}}(\phi\otimes\Phi_{R_m}),(t_\alpha)_{\alpha\in\mathcal{J}})$ via a (any) $W(k)$-homomomorphism $R_m\to\tilde Re$ which lifts j_e modulo p (the fact that under such an extension and the identification (6), each t_α is maped into t_α follows from the fact that no element of $\mathcal{T}(M[\frac{1}{p}])\otimes_{B(k)} I(e)[\frac{1}{p}]$ is fixed by $\phi\otimes\Phi_{\tilde Re}$). Thus the closed embedding $\mathcal{G}'_{\tilde Re}\hookrightarrow\mathbf{GL}_{M_{\tilde Re}}$ is the pull back of the closed embedding $\mathcal{G}_{R_m}\hookrightarrow\mathbf{GL}_{M\otimes_{W(k)}R_m}$

via a morphism $\mathrm{Spec}(\tilde{R}e) \to \mathrm{Spec}(R_m)$ which lifts j_e modulo p. As $\mathcal{G}_{R_m}$ is a reductive, closed subgroup scheme of $\mathbf{GL}_{M\otimes_{W(k)}R-m}$, we conclude that $\mathcal{G}'_{\tilde{R}e}$ is a reductive, closed subgroup scheme of $\mathbf{GL}_{M_{\tilde{R}e}}$.

As $\mathcal{G}'_{\tilde{R}e}$ is smooth over $\tilde{R}e$ and due to the property (i) of the proof of Theorem 3.6, there exists a cocharacter $\mu_{\tilde{R}e}$ of $\mathcal{G}'_{\tilde{R}e}$ that lifts both μ and $\tilde{\mu}_V$ (to be compared with [Va1, Lem. 5.3.2]). Let $F^1_{\tilde{R}e}$ be the direct summand of $M_{\tilde{R}e}$ that lifts F^1_V and that is normalized by $\mu_{\tilde{R}e}$. The quadruple $(M_{\tilde{R}e}, F^1_{\tilde{R}e}, \phi_{M_{\tilde{R}e}}, \tilde{\mathcal{G}}_{\tilde{R}e})$ is a lift of $\mathcal{C}$ to $\tilde{R}e$ (the analogue for $\tilde{R}e$ of the axiom (iii) of Definition 3.3.1 holds for this quadruple due to the very definition of $\tilde{\mathcal{G}}_{\tilde{R}e}$). Thus $(M_{\tilde{R}e}, F^1_V, \phi_{M_{\tilde{R}e}}, \tilde{\mathcal{G}}_{\tilde{R}e})$ is a ramified lift of $\mathcal{C}$ to V. $\square$

3.7.2. Definition. Let D_V be a p-divisible group over V that lifts D. We say that D_V is a ramified lift of D to V with respect to $\mathcal{G}$ if (to be compared with the Manin Problem 1.2) the following three axioms hold:

(a) under the canonical identification $H^1_{\mathrm{dR}}(D_V/V)[\frac{1}{p}] = M \otimes_{W(k)} V[\frac{1}{p}]$ (see proof of Theorem 3.6), the Zariski closure $\mathcal{G}'_V$ of $\mathcal{G}_K$ in $\mathbf{GL}_{H^1_{\mathrm{dR}}(D_V/V)}$ is a reductive, closed subgroup scheme of $\mathbf{GL}_{H^1_{\mathrm{dR}}(D_V/V)}$;

(b) under the canonical identification $M/pM = H^1_{\mathrm{dR}}(D_V/V)/m_V H^1_{\mathrm{dR}}(A/V)$, the group scheme $\mathcal{G}'_V$ lifts $\mathcal{G}_k$ (here m_V is the maximal ideal of V);

(c) there exists a cocharacter $\mathbb{G}_m \to \mathcal{G}'_V$ that acts on F^1_V via the inverse of the identical character of $\mathbb{G}_m$ and that fixes $H^1_{\mathrm{dR}}(D_V/V)/F^1_V$, where F^1_V is the direct summand of $H^1_{\mathrm{dR}}(D_V/V)$ which is the Hodge filtration of D_V.

3.7.3. Corollary. *We assume that $p \geq 3$. Then the ramified lifts of $\mathcal{C}$ to V are in natural bijection to the ramified lifts of D to V with respect to $\mathcal{G}$.*

Proof: Let $(M_{\tilde{R}e}, F^1_{\tilde{R}e}, \phi_{M_{\tilde{R}e}}, \mathcal{G}_{\tilde{R}e})$ be a ramified lift of $\mathcal{C}$ to V. Let $F^1_V := F^1_{\tilde{R}e} \otimes_{\tilde{R}e\ m_e} V$. Let D_V be the p-divisible group over V that corresponds to $(M_{\tilde{R}e}, F^1_V, \phi_{M_{\tilde{R}e}})$ via Theorem 3.5. From Definitions 3.3.1 (a) and (d) we get that D_V is a ramified lift of D to V with respect to $\mathcal{G}$. Thus the Corollary follows from Theorem 3.5 and Corollary 3.7.1. $\square$

3.8. Remarks. (a) Sections 2.7, 3.5, and 3.7 hold with k replaced by an arbitrary perfect field of characteristic p. Theorem 3.5 was first obtained in [Br] and [Zi2] (strictly speaking these references worked with Re instead of $\widetilde{R}e$ but as $\Phi_{\tilde{R}e}(\widetilde{R}e) \subseteq Re$ it is easy to see that for $p \geq 2$ there exists a natural bijection between lifts of $\mathcal{C}$ to Re and lifts of $\mathcal{C}$ to $\widetilde{R}e$). The results 3.6, 3.7.1, and 3.7.3 are not in the reach of either [Br] or [Zi2].

(b) Often in this paper the principally quasi-polarized contexts are treated as variants of non–polarized contexts. This is so due to the following two reasons. First, often the principally quasi-polarized context is handled by making small (if any at all) modifications to the contexts that involve only $\mathcal{C}$. There exists no element of $M \otimes_{W(k)} I(1)$ fixed by $\phi \otimes \Phi_{\tilde{R}e}$. Thus if we have a principal quasi-polarization $\lambda_M : M \otimes_{W(k)} M \to W(k)$ of $\mathcal{C}$ and if in Theorem 3.6 we have a lift $(M_{\tilde{R}e}, F^1_{\tilde{R}e}, \phi_{M_{\tilde{R}e}}, \mathcal{G}_{\tilde{R}e}, \lambda_{M_{\tilde{R}e}})$ of $(\mathcal{C}, \lambda_M)$ to $\tilde{R}e$, then the $W(k)$-homomorphism $R_m \to \tilde{R}e$ of Theorem 3.6 takes automatically λ_M to $\lambda_{M_{\tilde{R}e}}$.

To explain the second reason we assume that $\mathcal{G}$ is generated by $Z(\mathbf{GL}_M)$ and by a reductive, closed subgroup scheme $\mathcal{G}^0$ of $\mathbf{SL}_M$ and that the intersection $Z(\mathbf{GL}_M)\cap\mathcal{G}^0$ is either $\boldsymbol{\mu}_2$ or $\mathrm{Spec}(W(k))$. Then for most applications we can replace $\mathcal{C}$ by the direct sum $\mathcal{C}\oplus\mathcal{C}^*(1):=(M\oplus M^*(1),\phi\oplus p1_{M^*}\circ\phi,\mathcal{G})$, where $M^*(1):=M^*$ and where $\mathcal{G}$ is a reductive, closed subgroup scheme of $\mathbf{GL}_{M\oplus M^*(1)}$ such that $\mathcal{G}^0$ acts on $M^*(1)$ via its action on M^* and $Z(\mathbf{GL}_M)$ is naturally identified with $Z(\mathbf{GL}_{M\oplus M^*(1)})$. Let $\lambda_{M\oplus M^*(1)}$ be the natural principal alternating quasi-polarization of $\mathcal{C}\oplus\mathcal{C}^*(1)$. Then each ramified lift of $\mathcal{C}$ to V gives birth naturally to a unique ramified lift of $(\mathcal{C}\oplus\mathcal{C}^*(1),\lambda_{M\oplus M^*(1)})$ to V.

4. The basic results

In this Section we state our basic results pertaining to the (ramified) CM-classifications of Section 3 (see Basic Theorems 4.1 and 4.2). Corollaries 4.3 to 4.5 are practical applications of Basic Theorems 4.1 and 4.2 for contexts related to Shimura varieties of either B_n or $D_n^{\mathbb{R}}$ type. Let $(M_{\mathbb{Z}_p},\mathcal{G}_{\mathbb{Z}_p},(t_\alpha)_{\alpha\in\mathcal{J}})$ be as in Subsubsection 2.4.1.

4.1. Basic Theorem. *We assume that $\mathcal{C}$ is semisimple and basic. We have:*

(a) *If $[\mathcal{C}]\in Z(\mathcal{Y}(\mathcal{F}))$, then $Q\mathfrak{U}$ holds for $\mathcal{C}$.*

(b) *$\mathfrak{R}$ (resp. $Q\mathfrak{R}$, $T\mathfrak{R}$, or $TT\mathfrak{R}$) holds for $\mathcal{C}$ if and only if $\mathfrak{A}$ (resp. $Q\mathfrak{A}$, $T\mathfrak{A}$, or $TT\mathfrak{A}$) holds for $\mathcal{C}$.*[1]

(c) *We assume that $p\geq 3$, that $Q+\mathfrak{A}$ holds for $\mathcal{C}$, and that there exists a subset $\mathcal{J}_0$ of $\mathcal{J}$ such that the family $(t_\alpha)_{\alpha\in\mathcal{J}_0}$ of tensors of $\mathcal{T}(M_{\mathbb{Z}_p})$ is of partial degrees at most $p-2$ and is $\mathbb{Z}_p$-very well position for $\mathcal{G}$ (see Definitions 2.5 (a) and (b)). Then $[\mathcal{C}]\in PZ^{\mathrm{ram}}(\mathcal{Y}(\mathcal{F}))$. Moreover, if either $Q++\mathfrak{A}$ holds for $\mathcal{C}$ and $Z^0(\mathcal{G})=Z^0(C_1)$ or $\mathcal{G}=N_{C_1}(\mathcal{G})$, then in fact we have $[\mathcal{C}]\in Z^{\mathrm{ram}}(\mathcal{Y}(\mathcal{F}))$.*

4.2. Basic Theorem. (a) *We assume that $\mathcal{C}$ is basic and semisimple. We also assume that each simple factor of $\mathcal{G}^{\mathrm{ad}}_{W(\bar{k})}$ is of B_n, C_n, or D_n Lie type. Then $TT\mathfrak{R}$ holds for $\mathcal{C}$.*

(b) *We assume that $\mathcal{C}$ is semisimple and that each simple factor of $\mathcal{G}^{\mathrm{ad}}_{W(\bar{k})}$ is of B_n or D_n Lie type. If $\mathcal{G}^{\mathrm{ad}}_{W(\bar{k})}$ has a simple factor $\mathcal{V}$ of D_n Lie type, then we also assume that the centralizer in $\mathcal{V}$ of the image of $\mu_{W(\bar{k})}$ in it is either $\mathcal{V}$ itself or it is of D_{n-1} Lie type and, in the case $n=4$, that $\mathcal{V}^{\mathrm{sc}}$ is naturally a normal, closed subgroup scheme of $\mathcal{G}^{\mathrm{der}}_{W(\bar{k})}$. Then $TT\mathfrak{R}$ holds for $\mathcal{C}$.*

In Sections 5 and 6 we prove Basic Theorems 4.1 and 4.2 (respectively). The following three Corollaries are abstract extensions of [Zi1, Thm. 4.4] for $p>3$ and for contexts related to Shimura varieties of either B_n or $D_n^{\mathbb{R}}$ type. They are also the very first situations where complete ramified CM-classifications are accomplished. Their proofs are presented in Section 7. Let $\mathfrak{T}$ be the restriction to $\mathrm{Lie}(\mathcal{G}^{\mathrm{der}})$ of the trace form on $\mathrm{End}(M)$. Let $\mathfrak{K}$ be the Killing form on $\mathrm{Lie}(\mathcal{G}^{\mathrm{der}})$.

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1 *We expect that the (b) part is well known.*

4.3. Corollary. *We assume that all simply factors of $\mathcal{G}^{\mathrm{ad}}_{W(\bar{k})}$ are of B_n or D_{2n+1} Lie type, that $\mathcal{G}^{\mathrm{der}}$ is simply connected, that $Z^0(\mathcal{G}) = Z^0(C_1)$, that the natural isogeny $Z^0(\mathcal{G}) \to \mathcal{G}^{\mathrm{ab}}$ can be identified with the square isogeny $2 : Z^0(\mathcal{G}) \to Z^0(\mathcal{G})$, that $p > 3$ and that the symmetric forms $\mathfrak{T}$ and $\mathfrak{K}$ on Lie($\mathcal{G}^{\mathrm{der}}$) are perfect. Then for an element $g \in \mathcal{G}(W(k))$ we have $[\mathcal{C}_g] \in Z^{\mathrm{ram}}(\mathcal{Y}(\mathcal{F}))$ if and only if $\mathcal{C}_g$ is semisimple.*

4.3.1. Example. Let $n \in \mathbb{N}$. We assume that the representation $\mathcal{G} \to \mathbf{GL}_M$ is a product of spin representations of $\mathbf{GSpin}_{2n+1}$ group schemes. Thus all simply factors of $\mathcal{G}^{\mathrm{ad}}_{W(\bar{k})}$ are of B_n Lie type, $\mathcal{G}^{\mathrm{der}}$ is simply connected, and $Z^0(\mathcal{G}) = Z^0(C_1)$. We also assume that $p > 3$ does not divide $n - 1$. Then the symmetric forms $\mathfrak{T}$ and $\mathfrak{K}$ on Lie($\mathcal{G}^{\mathrm{der}}$) are perfect, cf. [Va1, Lem. 5.7.2.1]. Thus for an element $g \in \mathcal{G}(W(k))$ we have $[\mathcal{C}_g] \in Z^{\mathrm{ram}}(\mathcal{Y}(\mathcal{F}))$ if and only if $\mathcal{C}_g$ is semisimple, cf. Corollary 4.3.

4.4. Corollary. *We assume that $p > 3$, that all simply factors of $\mathcal{G}^{\mathrm{ad}}_{W(\bar{k})}$ are of D_n Lie type, that $\mathcal{G}^{\mathrm{der}}$ is simply connected, that the symmetric forms $\mathfrak{K}$ and $\mathfrak{T}$ on Lie($\mathcal{G}^{\mathrm{der}}$) are perfect, and that $Z^0(\mathcal{G}) = Z^0(C_1)$ is a torus of rank 2 times the number of simply factors of $\mathcal{G}^{\mathrm{ad}}_{W(\bar{k})}$. If n is odd we also assume that $(M, \phi, \tilde{\mathcal{G}})$ is a Shimura F-crystal over k, where $\tilde{\mathcal{G}}$ is the closed subgroup scheme of $\mathcal{G}$ generated by $\mathcal{G}^{\mathrm{der}}$ and by the maximal subtorus $Z^{00}(\mathcal{G})$ of $Z^0(\mathcal{G})$ with the property that the representation on $M \otimes_{W(k)} W(\bar{k})$ of each normal, semisimple, closed subgroup scheme of $\mathcal{G}_{W(\bar{k})}$ whose adjoint is simple, is a direct sum of trivial and of spin representations on which $Z^{00}(\mathcal{G})_{W(\bar{k})}$ acts via scalar multiplications. Then for an element $g \in \mathcal{G}(W(k))$ we have $[\mathcal{C}_g] \in Z^{\mathrm{ram}}(\mathcal{Y}(\mathcal{F}))$ if and only if $\mathcal{C}_g$ is semisimple.*

4.5. Corollary. *We assume that $p > 3$, that $\mathcal{G}^{\mathrm{der}}$ is simply connected, that each simple factor of $\mathcal{G}^{\mathrm{ad}}_{W(\bar{k})}$ is of B_n or D_n Lie type, that the symmetric forms $\mathfrak{K}$ and $\mathfrak{T}$ on Lie($\mathcal{G}^{\mathrm{der}}$) are perfect, and that we have a principal bilinear quasi-polarization λ_M of $\mathcal{C}$. Let $C_1(\lambda_M)^0$ be as in Theorem 2.4.2 (c). Let $\mathcal{G}^0_{\mathbb{Z}_p}$ be the maximal reductive, closed subgroup scheme of $\mathcal{G}_{\mathbb{Z}_p}$ that fixes λ_M. We also assume that $Z^0(\mathcal{G}) = Z^0(C_1(\lambda_M)^0)$ and that the group $H^1(\mathbb{Q}_p, \mathcal{G}^{0\mathrm{ab}}_{\mathbb{Q}_p})$ is trivial. Then for an element $g \in (\mathcal{G} \cap \boldsymbol{Sp}(M, \lambda_M))(W(k))$ we have $[(\mathcal{C}_g, \lambda_M)] \in Z^{\mathrm{ram}}(\mathcal{Y}(\mathcal{F}, \lambda_M))$ if and only if $\mathcal{C}_g$ is semisimple.*

5. Proof of Basic Theorem 4.1

In this Section we assume that $\mathcal{C}$ is basic and semisimple. In Subsections 5.1, 5.2, and 5.3 we prove Theorems 4.1 (a), 4.1 (b), and 4.1 (c) (respectively).

5.1. Proof of 4.1 (a). To prove Theorem 4.1 (a) we can assume that there exists a maximal split torus $\mathcal{T}$ of $\mathcal{G}$ such that we have $\phi(\mathrm{Lie}(\mathcal{T})) = \mathrm{Lie}(\mathcal{T})$. We can also assume that Lie($\mathcal{T}$) is generated by elements fixed by ϕ (see proof of Proposition 3.2). Thus $C_{\mathcal{T}} := C_{\mathbf{GL}_M}(\mathcal{T})$ is a reductive, closed subgroup scheme of $\mathbf{GL}_M$ such that we have $\phi(\mathrm{Lie}(C_{\mathcal{T}})) = \mathrm{Lie}(C_{\mathcal{T}})$. This implies that F^1/pF^1 is a Lie($C_{\mathcal{T}k}$)-module. As $C_{\mathcal{T}}(W(\bar{k}))$ is naturally a subset of Lie($C_{\mathcal{T}W(\bar{k})}$), the group $C_{\mathcal{T}k}$ normalizes F^1/pF^1. Thus $\mathcal{T}_k$ is a maximal torus of $\mathcal{P}_k$. Let $\mathcal{T}_0$ be a maximal torus of $\mathcal{P}$ through which the cocharacter $\mu : \mathbb{G}_m \to \mathcal{G}$ factors.

From [DG, Vol. II, Exp. IX, Thms. 3.6 and 7.1] and [Bo, Ch. V, Thm. 15.14] we deduce the existence of an element $g \in \mathcal{G}(W(k))$ which modulo p belongs to $\mathcal{P}(k)$ and for which we have an identity $g(\mathcal{T}_0)g^{-1} = \mathcal{T}$. By replacing μ with its inner conjugate through g we can assume that μ factors through $\mathcal{T}$. Thus $\mathrm{Lie}(\mathcal{T})$ is $W(k)$-generated by elements fixed by $\sigma_\phi := \phi\mu(p)$. Let $\mathcal{T}_{\mathbb{Z}_p}$ be the torus of $\mathcal{G}_{\mathbb{Z}_p}$ whose Lie algebra is formed by elements of $\mathrm{Lie}(\mathcal{T})$ fixed by σ_ϕ (cf. beginning of Subsection 2.4 applied with $\square = \mathcal{T}$); its extension to $W(\bar{k})$ is $\mathcal{T}_{W(\bar{k})}$ and $\mathcal{T}_{\mathbb{Q}_p}$ is the $\mathbb{Q}_p$-form of $\mathcal{T}_{B(k)}$ with respect to $(M[\frac{1}{p}, \phi)$. As $\mathcal{C}$ is basic, the product of cocharacters of $\mathcal{T}$ of the orbit of μ under integral powers of σ_ϕ (equivalently of ϕ) factors through $Z^0(\mathcal{G})$. Thus $(\mathcal{T}_{B(k)}, \mu_{B(k)})$ is an unramified E-pair of $\mathcal{C}$ that satisfies the cyclic $\mathfrak{C}$ condition. This proves Theorem 4.1 (a). $\square$

5.2. Proof of 4.1 (b). To prove Theorem 4.1 (b) we can assume that $\mathcal{C}$ is basic. It is enough to show that an E-pair $(\mathcal{T}_{1B(k)}, \mu_1)$ of $\mathcal{C}$ is admissible if and only if $\mathfrak{U}$ holds for it. If an E-pair $(\mathcal{T}_{1B(k)}, \mu_1)$ is admissible, then (as $\mathcal{C}$ is basic) from [Ko1, Subsections 2.4 and 2.5] and [RaZ, Prop. 1.21] (see also [RR, Thm. 1.15]) we get that the product of the cocharacters of $\mathcal{T}_{1K_1}$ which belong to the $\mathrm{Gal}(K_1/\mathbb{Q}_p)$-orbit of μ_1 factors through $Z^0(\mathcal{G}_{K_1})$. Thus the E-pair $(\mathcal{T}_{1B(k)}, \mu_1)$ of $\mathcal{C}$ satisfies the $\mathfrak{C}$ condition, cf. Example 2.3.1. Thus $\mathfrak{R}$ holds for $\mathcal{C}$.

We now show the the converse holds i.e., we prove that if

$$(\mathcal{T}_{1B(k)}, \mu_1, \tau = (\tau_1, \tau_2, \ldots, \tau_l))$$

is an E-triple of $\mathcal{C}$ such that the condition 2.3 (e1) holds and if K_2 and $F^1_{K_2}$ are as in Definitions 2.3 (d) and (h), then the filtered module $(M[\frac{1}{p}], \phi, F^1_{K_2})$ over K_2 is admissible. It is enough to show that the filtered module $(M[\frac{1}{p}], \phi, F^1_{K_2})$ over K_2 is weakly admissible, cf. [CF, Thm. A]. Let $\bar{K}_2$ be the subfield of $\overline{B(\bar{k})}$ generated by $B(\bar{k})$ and K_2. For $i \in \{1, \ldots, l\}$ let $M_i := M$. Let

$$O := (\oplus_{i=1}^l M_i) \otimes_{W(k)} K_2.$$

Let $\bar{O} := O \otimes_{K_2} \bar{K}_2$. Let $K_{00} := \{x \in K_2 | \tau_i(x) = x \; \forall i \in \{1, \ldots, l\}\}$; it is a totally ramified finite field extension of $\mathbb{Q}_p$. Let K_0 be the smallest subfield of K_2 which contains K_{00} and such that the cocharacter $\mu_{1K_2} : \mathbb{G}_m \to \mathcal{T}_{1K_2}$ is fixed by all elements of $\mathrm{Gal}(K_2/K_0)$. Let $d \in \mathbb{N}$ be as in the condition 2.3 (e1).

We denote also by τ the σ-linear automorphism

$$\tau : O \xrightarrow{\sim} O$$

which takes $m_i \otimes v_2 \in M_i \otimes_{W(k)} K_2$ to $\phi(m_i) \otimes \tau_i(v_2) \in M_{i+1} \otimes_{W(k)} K_2$, where $M_{l+1} := M_1$. We view naturally $\mathcal{T}^l_{1K_2} := \prod_{i=1}^l \mathcal{T}_{1K_2}$ (resp. $\mathcal{G}^l_{K_2} := \prod_{i=1}^l \mathcal{G}_{K_2}$) as a subtorus (resp. as a reductive, closed subgroup scheme) of $\mathbf{GL}_O$. We embed $\mathcal{T}_{1K_2}$ (resp. $\mathcal{G}_{K_2}$) diagonally into $\mathcal{T}^l_{1K_2}$ (resp. $\mathcal{G}^l_{K_2}$). Let μ_2 be the cocharacter of $\mathcal{T}^l_{1K_2}$ which normalizes each $M_i \otimes_{W(k)} K_2$ and which acts on $M_i \otimes_{W(k)} K_2$ identified with $M \otimes_{W(k)} K_2$ as μ_{1K_2} does. We consider the σ-linear automorphism

$$\sigma_2 := \tau\mu_2(p) : O \xrightarrow{\sim} O.$$

We denote also by τ and σ_2 their $\sigma_{\bar{k}}$-linear extensions to $\bar{O}$. The actions of τ and σ_2 on cocharacters of $\mathcal{T}^l_{1K_2}$ are the same. As $\mathcal{C}$ is basic, the Newton quasi-cocharacter ν of $\mathcal{C}$ factors through $Z^0(\mathcal{G}_{B(k)})$ (see [Va3, Cor. 2.3.2]). We consider the quasi-cocharacter ν_2 of $\mathcal{T}^l_{1K_2}$ which is the mean average of the orbit of μ_2 under integral powers of τ. It factors through $Z^0(\mathcal{G}^l_{1K_2})$, cf. property 2.3 (e1). Strictly speaking we get directly this only for the first factor $\mathcal{G}_{1K_2}$ of $\mathcal{G}^l_{1K_2}$. However, due to the circular aspect of τ and of the condition 2.3 (e1), this extends automatically to all the other $l-1$ factors $\mathcal{G}_{1K_2}$ of $\mathcal{G}^l_{1K_2}$. As each τ_i extends the Frobenius automorphism $\mathfrak{F}_{2u}$ of K_{2u}, the image of ν_2 in $(\mathcal{G}^l_{K_2})^{\mathrm{ab}} = \prod_{i=1}^l \mathcal{G}^{\mathrm{ab}}_{K_2}$ is such that its natural projections on $\mathcal{G}^{\mathrm{ab}}_{K_2}$ are all the same and equal to the composite of ν with the natural epimorphism $\mathcal{G}_{K_2} \twoheadrightarrow \mathcal{G}^{\mathrm{ab}}_{K_2}$. As ν_2 factors through $Z^0(\mathcal{G}^l_{1K_2})$, from the last sentence we get that in fact ν_2 factors through $\mathcal{T}_{1K_2}$ and this factorization coincides with the factorization of ν through $\mathcal{T}_{1K_2}$. As for each $u \in \mathbb{N}$ we have $\sigma_2^u = [(\prod_{m=1}^u \tau^m(\mu_2))(p)]\tau^u$, we easily get that all Newton polygon slopes of $(O[\frac{1}{p}], \sigma_2)$ are 0. Thus $\bar{O}$ is $\bar{K}_2$-generated by elements fixed by σ_2. Let O_{00} be the K_{00}-vector subspace of $\bar{O}$ formed by such elements. Its dimension t equals to $\dim_{K_2}(O) = l\mathrm{rk}_{W(k)}(M)$, cf. the definitions of K_{00} and τ. Let $O_0 := O_{00} \otimes_{K_{00}} K_0$. The torus $\mathcal{T}_{1K_{00}}$ is naturally a subtorus of $\mathbf{GL}_{O_{00}}$ and therefore we have $\tau(O_0) \subseteq O_0$. We denote also by τ its restriction to O_0.

5.2.1. Proposition. *There exists a K_0-basis $\mathcal{B} = \{e_1, \ldots, e_t\}$ of O_0 and a permutation π of $\{1, \ldots, t\}$ such that for all $i \in \{1, \ldots, t\}$ we have $\tau(e_i) = p^{n_i} e_{\pi(i)}$, where $n_i \in \{0,1\}$ is 1 if and only if we have $e_i \in (\oplus_{i=1}^l F^1_{K_2}) \otimes_{K_2} \bar{K}_2$.*

Proof: For $i \in \{1, 2, \ldots, dl\}$ and $j \in \{0,1\}$ let ${}^iF^j$ be the K_0-vector subspace of O_0 on which $\tau^i(\mu_2)$ acts trivially if $j = 0$ and via the inverse of the identical character of $\mathbb{G}_m$ if $j = 1$. For a function $\bar{f} : \{1, 2, \ldots, dl\} \to \{0,1\}$ let

$$F_{\bar{f}} := \bigcap_{i \in \{1,2,\ldots,dl\}} {}^iF^{\bar{f}(i)}.$$

Let $\mathcal{Q}$ be the set of such functions $\bar{f}$ with $F_{\bar{f}} \neq 0$. As $\tau^i(\mu_2)$'s commute (being cocharacters of $\mathcal{T}^l_{1K_0}$) we have a direct sum decomposition $O_0 = \oplus_{\bar{f} \in \mathcal{Q}} F_{\bar{f}}$. Let

$$\bar{\tau} : \mathcal{Q} \to \mathcal{Q}$$

be the bijection defined by the rule: $\bar{\tau}(\bar{f})(i) = \bar{f}(i-1)$, where $\bar{f}(0) := \bar{f}(dl)$.

Let $I^0_{\bar{\tau}} := \{\bar{f} \in \mathcal{Q} | \bar{\tau}(\bar{f}) = \bar{f}\}$. Let $\bar{\tau} = \prod_{j \in I_{\bar{\tau}}} \bar{\tau}_j$ be written as a product of disjoint cyclic permutations. We allow trivial cyclic permutations i.e., we have a disjoint union

$$I_{\bar{\tau}} = I^1_{\bar{\tau}} \cup I^0_{\bar{\tau}}$$

with the property that $j \in I_{\bar{\tau}}$ belongs to $I^1_{\bar{\tau}}$ if and only if $\bar{\tau}_j$ is a non-trivial permutation. If $j \in I^1_{\bar{\tau}}$, then each function $\bar{f} \in \mathcal{Q}$ such that we have $\bar{\tau}_j(\bar{f}) \neq \bar{f}$ is said to be associated to $\bar{\tau}_j$. Also $\bar{f} \in I^0_{\bar{\tau}}$ is said to be associated to $\bar{\tau}_{\bar{f}}$. As $\bar{\tau}^{dl} = 1_{\mathcal{Q}}$, the order d_j of the cyclic permutation $\bar{\tau}_j$ divides dl. For each $j \in I_{\bar{\tau}}$ we choose arbitrarily an element $\bar{f}_j$ of $\mathcal{Q}$ which is associated to $\bar{\tau}_j$. We have $\tau^{d_j}(F_{\bar{f}_j}) = \sigma_2^{d_j}(F_{\bar{f}_j}) = F_{\bar{f}_j}$. Let ${}_pF_{\bar{f}_j} := \{x \in F_{\bar{f}_j} | \sigma_2^{d_j}(x) = x\}$. Let $K_0(\bar{f}_j)$ be the maximal subfield of K_0 such that ${}_pF_{\bar{f}_j}$ is a $K_0(\bar{f}_j)$-vector space. It contains K_{00}. By reasons of dimensions we have $F_{\bar{f}_j} = {}_pF_{\bar{f}_j} \otimes_{K_0(\bar{f}_j)} K_0$.

For each $j \in I_{\bar{\tau}}$ we choose a $K_0(\bar{f}_j)$-basis $\{e_s | s \in \mathcal{B}_j\}$ for ${}_pF_{\bar{f}_j}$; we also view it as a K_0-basis for $F_{\bar{f}_j}$. For each cyclic permutation $\bar{\tau}_j$ of length ≥ 2 (i.e., for when we deal with a $j \in I_{\bar{\tau}}^1$) and for every element $\bar{f} \in \mathcal{Q}$ associated to $\bar{\tau}_j$ but different from $\bar{f}_j$, let $u(\bar{f}) \in \mathbb{N}$ be the smallest number such that $\bar{f} = \bar{\tau}^{u(\bar{f})}(\bar{f}_j)$ and let $n_{u(\bar{f}),j} := \sum_{i=1}^{u(\bar{f})} \bar{f}_j(i)$. We get a K_0-basis $\{\frac{1}{p^{n_{u(\bar{f}),j}}}\tau^{u(\bar{f})}(e_s) | s \in \mathcal{B}_j\}$ for $F_{\bar{f}}$. The expressions of $n_{u(\bar{f}),j}$'s are a consequence of the following iteration formula $\tau^{u(\bar{f})} = [(\prod_{m=1}^{u(\bar{f})} \sigma_2^m(\mu_2))(\frac{1}{p})]\sigma_2^{u(\bar{f})}$.

Let $\mathcal{B} = \{e_1, \ldots, e_t\}$ be the K_0-basis for O_0 obtained by putting together the chosen K_0-bases for $F_{\bar{f}}$'s with $\bar{f} \in \mathcal{Q}$. Let π be the unique permutation of $\{1, \ldots, t\}$ such that for all $i \in \{1, \ldots, t\}$ we have $\tau(e_i) \in K_0 e_{\pi(i)}$. From constructions we get that $\tau(e_i) = p^{n_i} e_{\pi(i)}$, where n_i is as mentioned in the Proposition. $\square$

5.2.2. End of the proof of 4.1 (b). Let V_0 and $\bar{V}_2$ be the ring of integers of K_0 and $\bar{K}_2$ (respectively). Let W_0 be the V_0-lattice of O_0 generated by elements of $\mathcal{B}$. Let F_0^1 be its direct summand generated by elements of $\mathcal{B} \cap ((\oplus_{i=1}^l F_{K_2}^1) \otimes_{K_2} \bar{K}_2)$. Let $\bar{W}_2 := W_0 \otimes_{V_0} \bar{V}_2$. We consider an arbitrary $B(k)$-submodule $\bigstar$ of $M[\frac{1}{p}]$ which is normalized by ϕ. Let $\bar{W}_2^0 := \bar{W}_2 \cap (\oplus_{i=1}^l \bigstar \otimes_{B(k)} \bar{K}_2)$; it is a $\bar{V}_2$-module which is a direct summand of $\bar{W}_2$ left invariant by τ. As in Mazur theorem of [Ka, Thm. 1.4.1] we get that the Newton polygon of l copies of $(\bigstar, \phi)$ is below the Hodge polygon of $(\bar{W}_2^0, \bar{W}_2^0 \cap (F_0^1 \otimes_{V_0} \bar{V}_2))$. But this Hodge polygon coincides with the Hodge polygon of l copies of $(\bigstar, \phi, (\bigstar \otimes_{B(k)} K_2) \cap F_{K_2}^1)$. Thus the Newton polygon of $(\bigstar, \phi)$ is below the Hodge polygon of $(\bigstar, \phi, (\bigstar \otimes_{B(k)} K_2) \cap F_{K_2}^1)$. Therefore the filtered module $(M[\frac{1}{p}], \phi, F_{K_2}^1)$ over K_2 is weakly admissible. Thus Theorem 4.1 (b) holds. $\square$

5.3. Proof of 4.1 (c). We begin the proof of Theorem 4.1 (c) with some étale considerations. To prove Theorem 4.1 (c) we can assume that there exists an E-pair $(\mathcal{T}_{1B(k)}, \mu_1)$ of $\mathcal{C}$ which is plus admissible. Let $\mathcal{T}_{1\mathbb{Q}_p}$, K_2 and $F_{K_2}^1$ be as in Definitions 2.3 (b), (d), and (h). The torus $Z^0(\mathcal{G}_{\mathbb{Q}_p})$ is naturally a subtorus of $\mathcal{T}_{1\mathbb{Q}_p}$. The triple $(M[\frac{1}{p}], \phi, F_{K_2}^1)$ is an admissible filtered module. Let $M_{\mathbb{Q}_p}$, $CG_{\mathbb{Q}_p}$, ρ, $\mathcal{W}$, $\mathfrak{L}$, and $(v_\alpha)_{\alpha \in \mathcal{J}}$ be as in Subsubsections 2.4.5 and 2.4.6. As $p \geq 3$, from [Br, Cor. 5.3.3] we get that the Galois representation $\rho : \mathrm{Gal}(K_2) \to \mathbf{GL}_{\mathcal{W}}$ is associated to an isogeny class of p-divisible groups over the ring of integers V_2 of K_2. Let $\mathcal{J}_1 := \mathrm{Lie}(\mathcal{T}_{1\mathbb{Q}_p})$. For $\alpha \in \mathcal{J}_1$ let $t_\alpha := \alpha$. To prove Theorem 4.1 (c) we can assume that

$$\mathcal{J} \cap \mathcal{J}_1 = \mathrm{Lie}(Z^0(\mathcal{G}_{\mathbb{Q}_p})) \subseteq \mathrm{Lie}(\mathcal{T}_{1\mathbb{Q}_p}) \tag{7}$$

and that for each $\alpha \in \mathcal{J} \cap \mathcal{J}_1$ the two definitions of t_α define the same tensor of $\mathcal{T}(M[\frac{1}{p}])$. Let $\mathcal{J}_2 := \mathcal{J} \cup \mathcal{J}_1$. Let $\mathcal{G}'_{\mathbb{Q}_p}$ be the subgroup of $\mathbf{GL}_{\mathcal{W}}$ that fixes v_α for all $\alpha \in \mathcal{J}$; it is an inner form of $\mathcal{G}_{\mathbb{Q}_p}$. As ρ factors through $\mathcal{G}'_{\mathbb{Q}_p}(\mathbb{Q}_p)$, by enlarging $\mathcal{J}$ we can assume that there exists a subset $\mathcal{J}_3$ of $\mathcal{J}$ such that $CG_{\mathbb{Q}_p}$ is the subgroup of $\mathbf{GL}_{M_{\mathbb{Q}_p}}$ that fixes t_α for all $\alpha \in \mathcal{J}_3$.

The image of $\mathfrak{L}$ in $H^1(\mathbb{Q}_p, CG_{\mathbb{Q}_p})$ is the trivial class, cf. Subsubsection 2.4.5. Thus there exists a $\mathbb{Q}_p$-linear isomorphism

$$\mathfrak{J} : M_{\mathbb{Q}_p} \tilde{\to} \mathcal{W}$$

that takes t_α to v_α for all $\alpha \in \mathcal{J}_3$. We use it to identify naturally $\mathcal{G}_{\mathbb{Q}_p} = \mathcal{G}'_{\mathbb{Q}_p}$ (this is possible as $CG_{\mathbb{Q}_p}$ normalizes $\mathcal{G}_{\mathbb{Q}_p}$). Let $\mathcal{G}'_{\mathbb{Z}_p}$ be the Zariski closure of $\mathcal{G}'_{\mathbb{Q}_p}$ in $\mathcal{L} := \mathfrak{J}(M_{\mathbb{Z}_p})$; we identify it with $\mathcal{G}_{\mathbb{Z}_p}$.

5.3.1. Crystalline considerations. We now apply the crystalline machinery of [Fa] and [Va1, Subsection 5.2] to show first that $[\mathcal{C}] \in PSZ^{\mathrm{ram}}(\mathcal{Y}(\mathcal{F}))$. Let K_3 be a finite field extension of K_2 such that $\rho(\mathrm{Gal}(K_3))$ normalizes $\mathcal{L}$ and its ramification index e is at least 2. Let D_{K_3} be the p-divisible group over K_3 defined by the representation

$$\rho_1 : \mathrm{Gal}(K_3) \to \mathbf{GL}_{\mathcal{L}}$$

induced naturally by ρ. It extends to a p-divisible group D_{V_3} over the ring of integers V_3 of K_3 (cf. the isogeny class part of the first paragraph of Subsection 5.3) and this extension is unique (cf. [Ta1]).

We use the notations of Subsection 2.7. To avoid extra notations, (by performing the operation $\mathfrak{D}_1$) we can assume that the residue field of K_3 is k. We fix a uniformizer π_3 of V_3. Let $\widetilde{Re} \twoheadrightarrow V_3$ be a $W(k)$-epimorphism defined by π_3, cf. Fact 2.7.1. Let

$$(M_{\tilde{R}e}, \phi_{M_{\tilde{R}e}}, \nabla)$$

be the Dieudonné F-crystal over $\tilde{R}e/p\tilde{R}e$ of $D_{V_3} \times_{\mathrm{Spec}(V_3)} \mathrm{Spec}(V_3/pV_3)$ (see the proof of Theorem 3.5).

Let $B^+(V_3)$ be the crystalline Fontaine ring of V_3 as defined in [Fa]. We recall that $B^+(V_3)$ is an integral, local $W(k)$-algebra which is endowed with a decreasing, exhaustive, and separated filtration $(F^i(B^+(W(k)))_{i\in\mathbb{N}\cup\{0\}}$, with a Frobenius lift $\mathfrak{F}$, and with a natural Galois action by $\mathrm{Gal}(K_3)$. Moreover we have a natural $W(k)$-epimorphism compatible with the natural Galois actions by $\mathrm{Gal}(K_3)$

$$s_{V_3} : B^+(V_3) \twoheadrightarrow \overline{V_3}^\wedge,$$

where $\overline{V_3}^\wedge$ is the p-adic completion of the normalization $\overline{V_3}$ of V_3 in $\overline{B(k)}$. We refer to loc. cit. for the natural $W(k)$-monomorphism $\tilde{R}e \hookrightarrow B^+(V_3)$ which respects the Frobenius lifts (and which is associated to the uniformizer π_3). We apply Fontaine comparison theory to D_{V_3} (see loc. cit. and [Va1, Subsection 5.2]). We get a $B^+(V_3)$-monomorphism

$$i_{D_{V_3}} : M_{\tilde{R}e} \otimes_{\tilde{R}e} B^+(V_3) \hookrightarrow \mathcal{L} \otimes_{\mathbb{Z}_p} B^+(V_3)$$

which has the following two properties:

(a) It respects the tensor product filtrations (the filtration of $\mathcal{L}$ is defined by: $F^1(\mathcal{L}) = 0$ and $F^0(\mathcal{L}) = \mathcal{L}$).

(b) It respects the Galois actions (the Galois action on $M_{\tilde{R}e} \otimes_{\tilde{R}e} B^+(V_3)$ is defined naturally via s_{V_3} and the fact that $\mathrm{Ker}(s_{V_3})$ has a natural divided power structure).

The existence of $\mathfrak{J}$ and our hypothesis on the subset $\mathcal{J}_0$ of $\mathcal{J}$ implies that the family of tensors $(\mathfrak{J}(t_\alpha))_{\alpha\in\mathcal{J}_0}$ of $\mathcal{T}(\mathcal{L})$ is $\mathbb{Z}_p$-very well position for $\mathcal{G}'_{\mathbb{Z}_p}$. For $\alpha \in \mathcal{J}_0$ (resp. $\alpha \in \mathcal{J}_2 \setminus \mathcal{J}_0$) let $u_\alpha \in \mathcal{T}(M_{\tilde{R}e})$ (resp. $u_\alpha \in \mathcal{T}(M_{\tilde{R}e}[\frac{1}{p}])$) be the tensor that corresponds to $\mathfrak{J}(t_\alpha)$ via $i_{D_{V_3}}$, cf. [Fa, Cor. 9] and the fact that the family of tensors $(\mathfrak{J}(t_\alpha)_{\alpha\in\mathcal{J}_0})$ is of partial degrees at most $p-2$. As in [Va1, Subsubsections 5.2.12 to 5.2.17] we argue that the Zariski closure $\tilde{\mathcal{G}}_{\tilde{R}e}$ in $\mathbf{GL}_{M_{\tilde{R}e}}$ of the closed subgroup scheme of $\mathbf{GL}_{M_{\tilde{R}e}[\frac{1}{p}]}$ that fixes u_α for all $\alpha \in \mathcal{J}$, is a reductive subgroup scheme.

The tensorization of $(M_{\tilde{R}e}[\frac{1}{p}], \phi_{M_{\tilde{R}e}})$ with the natural epimorphism $\tilde{R}e[\frac{1}{p}] \twoheadrightarrow B(k)$ that takes X to 0, is the F-isocrystal of D (cf. the very definition of D_{V_3}) and thus it is is canonically isomorphic to $(M[\frac{1}{p}], \phi)$. Under the resulting identification $(M_{\tilde{R}e}[\frac{1}{p}], \phi_{M_{\tilde{R}e}}) \otimes_{\tilde{R}e} B(k) = (M[\frac{1}{p}], \phi)$, u_α is identified with t_α for all $\alpha \in \mathcal{J}_3 \cup \mathcal{J}_1$. This implies that under the identification (6), u_α gets identified with t_α for all $\alpha \in \mathcal{J}_3 \cup \mathcal{J}_1$. In particular, we get that there exists a maximal torus of $\tilde{\mathcal{G}}_{\tilde{R}e[\frac{1}{p}]}$ whose Lie algebra is $\tilde{R}e[\frac{1}{p}]$-generated by those u_α with $\alpha \in \mathcal{J}_1$ (i.e., which corresponds to the maximal torus $\mathcal{T}_{1B(k)}$ of $\mathcal{G}_{1B(k)}$ via the identification (6)).

Due to the existence of the cocharacter $\mu_{1K_2} : \mathbb{G}_m \to \mathcal{T}_{1K_2}$ that acts on $F^1_{K_2}$ via the inverse of the identical character of $\mathbb{G}_m$, as in [Va1, Subsubsection 5.3.1 and Lem. 5.3.2] we argue that there exists a cocharacter $\tilde{\mu}_{\tilde{R}e} : \mathbb{G}_m \to \tilde{\mathcal{G}}_{\tilde{R}e}$ such that the following two properties hold:

(c) there exists a direct sum decomposition $M_{\tilde{R}e} = F^1_{\tilde{R}e} \oplus F^0_{\tilde{R}e}$ such that $F^1_{\tilde{R}e}$ lifts the Hodge filtration $F^1_{V_3}$ of $M_{\tilde{R}e} \otimes_{\tilde{R}e} V_3$ defined by D_{V_3} and for each $i \in \{0, 1\}$, every element $\beta \in \mathbb{G}_m(\tilde{R}e)$ acts on $F^i_{\tilde{R}e}$ through $\tilde{\mu}_{\tilde{R}e}$ as the multiplication with β^{-i};

(d) μ_{1K_3} and the pull back of $\tilde{\mu}_{\tilde{R}e}$ to a cocharacter of $\tilde{\mathcal{G}}_{K_3} = \mathcal{G}_{K_3}$ are $\mathcal{G}_{K_3}(K_3)$-conjugate (the identification $\tilde{\mathcal{G}}_{K_3} = \mathcal{G}_{K_3}$ used here is the one defined naturally by the tensorization of (6) with K_3 over $\tilde{R}e[\frac{1}{p}]$).

Let $(M_1, F^1_1, \mathcal{G}_1, \tilde{\mu}) := (M_{\tilde{R}e}, F^1_{\tilde{R}e}, \tilde{\mathcal{G}}_{\tilde{R}e}, \tilde{\mu}_{\tilde{R}e}) \otimes_{\tilde{R}e} W(k)$. We have two identifications $M_1[\frac{1}{p}] = M[\frac{1}{p}]$ and $\mathcal{G}_{1B(k)} = \mathcal{G}_{B(k)}$ and moreover the pair (M_1, ϕ) is a Dieudonné module. As $\phi_{M_{\tilde{R}e}}(M_{\tilde{R}e} + \frac{1}{p}F^1_{\tilde{R}e}) = M_{\tilde{R}e}$, the cocharacter $\tilde{\mu}$ is a Hodge cocharacter of $(M_1, \phi, \mathcal{G}_1)$. Thus $(M_1, F^1_1, \phi, \mathcal{G}_1)$ is a Shimura filtered F-crystal over k. Moreover the triple

$$(M_{\tilde{R}e}, F^1_{V_3}, \phi_{M_{\tilde{R}e}}, \tilde{\mathcal{G}}_{\tilde{R}e})$$

is a ramified lift of $(M_1, \phi, \mathcal{G}_1)$ to V_3. It is of CM type, cf. the existence of the maximal torus $\mathcal{T}_{1\tilde{R}e[\frac{1}{p}]}$ of $\tilde{\mathcal{G}}_{\tilde{R}e[\frac{1}{p}]}$. Let $h \in \mathbf{GL}_M(B(k))$ be such that we have $h(M) = M_1$. Due to the property (d), it is easy to see that there exists an element $\tilde{h} \in \mathcal{G}(B(k))$ such that we have $F^1_1 = \tilde{h}(F^1[\frac{1}{p}]) \cap M_1$. Therefore we have $h \in \mathfrak{P}(\mathcal{C})$ as well as an identity $(M_1, \phi, \mathcal{G}_1) = (h(M), \phi, \mathcal{G}(h))$. As $h \in \mathfrak{P}(\mathcal{C})$ and as $(M_{\tilde{R}e}, F^1_{V_3}, \phi_{M_{\tilde{R}e}}, \tilde{\mathcal{G}}_{\tilde{R}e})$ is a ramified lift of $(M_1, \phi, \mathcal{G}_1) = (h(M), \phi, \mathcal{G}(h))$ to V_3 of CM type, we have $[\mathcal{C}] \in PSZ^{\mathrm{ram}}(\mathcal{Y}(\mathcal{F}))$. Thus the first part of Theorem 4.1 (c) holds.

5.3.2. End of the proof. To end the proof of Theorem 4.1 (c) we have to show that if $\mathcal{G} = N_{C_1}(\mathcal{G})$ (resp. if $Z^0(\mathcal{G}) = Z^0(C_1)$ and $\mathfrak{L}$ is the trivial class), then we can can choose $\mathfrak{J} : M_{\mathbb{Q}_p} \tilde{\to} \mathcal{W}$ such that there exists an element $h \in \mathcal{G}(B(k))$ with the property that $h(M) = M_1$. The below arguments will not rely on the way we constructed M_1; they will only use the fact that the Zariski closure $\mathcal{G}_1$ of $\mathcal{G}_{B(k)}$ in $\mathbf{GL}_{M_1}$ is a reductive group scheme over $W(k)$. Under the identification (6) we have $t_\alpha = u_\alpha$ for all $\alpha \in \mathcal{J}_3$ (resp. we can assume that we have $t_\alpha = u_\alpha$ for all $\alpha \in \mathcal{J}$, as we can choose $\mathfrak{J}$ such that we have $\mathfrak{J}(t_\alpha) = v_\alpha$ for all $\alpha \in \mathcal{J}$). The Lie algebra $\mathrm{Lie}(C)$ is the Lie algebra defined by a semisimple $W(k)$-subalgebra S of $\mathrm{End}(M)$. Let $S_{\mathbb{Z}_p} := \{s \in S | \phi(s) = s\}$. We have $S = S_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} W(k)$, cf. Theorem 2.4.2 (a). We can assume that there exists a subset $\mathcal{J}_4$ of $\mathcal{J}$ such that $\{t_\alpha | \alpha \in \mathcal{J}_4\} = S_{\mathbb{Z}_p}$. Let $S_{1\mathbb{Z}_p}$ be the semisimple $\mathbb{Z}_p$-subalgebra of $\mathrm{End}(M_1)$ that corresponds to $\mathfrak{J}(S_{\mathbb{Z}_p})$ via Fontaine comparison theory. The group $CG_{\mathbb{Q}_p}$ normalizes the group scheme $C_{\mathbb{Q}_p}$ of invertible elements of $S_{\mathbb{Q}_p}$ (the notations match i.e., the extension of $C_{\mathbb{Q}_p}$ to $B(k)$ is the generic fibre of C). As the image of $\mathfrak{L}$ in $H^1(\mathbb{Q}_p, CG_{\mathbb{Q}_p})$ is the trivial class, from the previous sentence we get that we can identify naturally $S_{\mathbb{Z}_p}[\frac{1}{p}] = S_{1\mathbb{Z}_p}[\frac{1}{p}] =: S_{\mathbb{Q}_p}$. The abstract groups H and H_1 of invertible elements of $S_{\mathbb{Z}_p}$ and respectively $S_{1\mathbb{Z}_p}$ are two (resp. are the same) hyperspecial subgroups (resp. subgroup) of the group $C_{\mathbb{Q}_p}$. Thus there exists an element $c \in C_{\mathbb{Q}_p}(\mathbb{Q}_p)$ such that $cHc^{-1} = H_1$, cf. [Ti2, Subsection 1.10] and the fact that the group $C_{\mathbb{Q}_p}(\mathbb{Q}_p)$ surjects onto $C^{\mathrm{ad}}_{\mathbb{Q}_p}(\mathbb{Q}_p)$ (resp. for $c := 1_{M[\frac{1}{p}]} \in C_{\mathbb{Q}_p}(\mathbb{Q}_p)$ we have $cHc^{-1} = H_1$).

By replacing (M_1, ϕ) with $(c^{-1}(M_1), c^{-1}\phi c) = (c^{-1}(M_1), \phi)$ we can assume that $S_{\mathbb{Z}_p} = S_{1\mathbb{Z}_p}$. The map $S_{\mathbb{Z}_p} \to S_{\mathbb{Z}_p}$ which takes t_α to u_α for $\alpha \in \mathcal{J}_4$ is an automorphism of $S_{\mathbb{Z}_p}$ and thus by performing a similar replacement defined this time by an element $c_1 \in H(\mathbb{Z}_p)$ (resp. by $c_1 = 1_M$) we can assume that we have $t_\alpha = u_\alpha$ for all $\alpha \in \mathcal{J}_4$. But the subgroup of $\mathbf{GL}_{M[\frac{1}{p}]}$ that fixes t_α for all $\alpha \in \mathcal{J}_3 \cup \mathcal{J}_4$ (resp. for all $\alpha \in \mathcal{J}$) is $\mathcal{G}_{B(k)}$, cf. the identity $\mathcal{G} = N_{C_1}(\mathcal{G})$ (resp. cf. the definition of $\mathcal{J}$). Thus we can assume that we have $t_\alpha = u_\alpha$ for all $\alpha \in \mathcal{J}$. In particular, we get that M_1 is a C-module.

The existence of the element $h \in \mathcal{G}(B(k))$ is expressed in terms of a right torsor of $\mathcal{G}$ being trivial. As $\mathcal{G}$ is smooth, we can work with the flat topology instead of the étale topology of $\mathrm{Spec}(W(k))$. Thus to show the existence of the element $h \in \mathcal{G}(B(k))$ we can tensor M and M_1 over $W(k)$ with $V(\bar{k})$, where $V(\bar{k})$ is an arbitrary finite, discrete valuation ring extension of $W(\bar{k})$. Thus as the hyperspecial subgroups $\mathcal{G}(W(\bar{k}))$ and $\{g \in \mathcal{G}(B(\bar{k})) | g(M_1 \otimes_{W(k)} W(\bar{k})) = M_1 \otimes_{W(k)} W(\bar{k})\}$ of $\mathcal{G}(B(\bar{k}))$ are $\mathcal{G}^{\mathrm{ad}}(B(\bar{k}))$-conjugate (see [Ti2, p. 47]) and as each element of $\mathcal{G}^{\mathrm{ad}}(B(\bar{k}))$ is the image in $\mathcal{G}^{\mathrm{ad}}(V(\bar{k})[\frac{1}{p}])$ of some element of $\mathcal{G}(V(\bar{k})[\frac{1}{p}])$ for a suitable choice of $V(\bar{k})$, by replacing $W(k)$ with $V(\bar{k})$ we can assume that these two hyperspecial subgroups are equal. The reason we deal with a discrete valuation ring $V(\bar{k})$ whose residue field is $\bar{k}$ and not k is that we want to pass from inclusions of hyperspecial subgroups to closed monomorphisms between reductive group schemes and this is possible in general only if we have residue fields which are infinite (see [Va1, Prop. 3.1.2.1 a) and b)]). Thus to show the existence of the element $h \in \mathcal{G}(B(k))$ we can assume that $\mathcal{G}_{V(\bar{k})}$ is a closed subgroup scheme of $\mathbf{GL}_{M_1 \otimes_{W(k)} V(\bar{k})}$.

Let TG be the reductive, closed subgroup scheme of $\mathbf{GL}_M$ generated by $\mathcal{G}$ and by a maximal torus of C. By performing $\mathfrak{D}_1$ we can assume that TG is split. Thus we can write $M = \oplus_{i \in I_{\mathcal{G}}} O^i$ as a direct sum of absolutely irreducible TG-modules. For $i \in I_{\mathcal{G}}$

the representation ρ_{ki} of CG_k on O^i/pO^i is absolutely irreducible, cf. proof of Theorem 2.4.2 (b). Moreover, if $i_1,\ i_2 \in I_{\mathcal{G}}$ are two distinct elements, then ρ_{ki_1} and ρ_{ki_2} are unequivalent representations of TG_k. Thus as $TG(V(\bar{k}))$ normalizes both $M \otimes_{W(k)} V(\bar{k})$ and $M_1 \otimes_{W(k)} V(\bar{k})$, we have

$$M_1 \otimes_{W(k)} V(\bar{k}) = c(M \otimes_{W(k)} V(\bar{k})),$$

where $c \in \prod_{i\in I_{\mathcal{G}}} Z(\mathbf{GL}_{O^i})(V(\bar{k})[\frac{1}{p}])$ acts on $O^i \otimes_{W(k)} V(\bar{k})$ by multiplication with $\pi_{V(\bar{k})}^{n_i}$; here $n_i \in \mathbb{Z}$ and $\pi_{V(\bar{k})}$ is a fixed uniformizer of $V(\bar{k})$. As $C_{V(\bar{k})}$ normalizes both $M \otimes_{W(k)} V(\bar{k})$ and $M_1 \otimes_{W(k)} V(\bar{k})$, we have $n_{i_1} = n_{i_2}$ for all elements $i_1, i_2 \in I_{\mathcal{G}}$ such that the representations of $\mathcal{G}$ on O^{i_1} and O^{i_2} are isomorphic. Thus $c \in Z^0(C_1)(V(\bar{k})[\frac{1}{p}]) \subseteq Z^0(\mathcal{G})(V(\bar{k})[\frac{1}{p}])$. Thus the desired element $h \in \mathcal{G}(B(k))$ exists. This ends the proof of Theorem 4.1 (c). $\square$

5.3.3. Simple facts and variants. (a) We assume that we have a principal bilinear quasi-polarization $\lambda_M : M \otimes_{W(k)} M \to W(k)$ of $\mathcal{C}$. Then $\mathfrak{J}(\lambda_M) : \mathcal{L} \otimes_{\mathbb{Z}_p} \mathcal{L} \to \mathbb{Z}_p$ gives birth via Fontaine comparison theory to a principal bilinear quasi-polarization λ_{M_1} of (M_1, ϕ).

Let $\mathcal{G}^0_{\mathbb{Q}_p}$, $DG_{\mathbb{Q}_p}$, $DG^0_{\mathbb{Q}_p}$, and $\mathfrak{L}^0$ be as in Subsubsection 2.4.7. If the image of $\mathfrak{L}$ in $H^1(\mathbb{Q}_p, DG_{\mathbb{Q}_p})$ is the trivial class, then in Subsubsection 5.3.1 we can choose $\mathfrak{J}$ such that we have $\lambda_{M_1} \in \mathbb{G}_m(\mathbb{Q}_p)\lambda_M$. If the image of $\mathfrak{L}^0$ in $H^1(\mathbb{Q}_p, DG^0_{\mathbb{Q}_p})$ is the trivial class, then we can choose $\mathfrak{J}$ such that in fact we have $\lambda_M = \lambda_{M_1}$.

Moreover, we have $[(\mathcal{C}, \lambda_M)] \in Z(\mathcal{Y}(\mathcal{F}, \lambda_M))$, provided we also assume that $\mathfrak{L}^0$ is the trivial class and $Z^0(\mathcal{G}) = Z^0(C_1(\lambda_M)^0)$. Argument: with the notations of Theorem 2.4.2 (c) we only have to add that as λ_M defines perfect bilinear forms on both M and M_1, in the end of Subsubsection 5.3.2 we have $c \in Z^0(C_1(\lambda_M)^0)(V(\bar{k})[\frac{1}{p}]) \subseteq Z^0(\mathcal{G})(V(\bar{k})[\frac{1}{p}])$ and in fact c fixes λ_M.

(b) We refer to Subsubsection 5.3.1. Each element of $\mathrm{End}(M[\frac{1}{p}])$ fixed by ϕ and $\mathcal{T}_{1B(k)}$ defines an endomorphism of $(M[\frac{1}{p}], \phi, F^1_{K_2})$ and therefore a $\mathbb{Q}_p$-endomorphism of D_{V_3}. Thus the homomorphism ρ_3 of Subsubsection 5.3.1 factors through the group of $\mathbb{Z}_p$-valued points of the Zariski closure in $\mathbf{GL}_{\mathcal{L}}$ of the subtorus of $\tilde{\mathcal{G}}_{\mathbb{Q}_p}$ that fixes the $\mathbb{Q}_p$-étale realizations of the $\mathbb{Q}_p$-endomorphisms of D_{V_3} that correspond to those t_α with $\alpha \in \mathcal{J}_1$. Therefore D_{V_3} is with complex multiplication.

6. Proof of Basic Theorem 4.2

In this Section we prove Theorem 4.2. Let $(M_{\mathbb{Z}_p}, \mathcal{G}_{\mathbb{Z}_p}, (t_\alpha)_{\alpha\in\mathcal{J}})$ be the $\mathbb{Z}_p$ structure of $(M, \phi, \mathcal{G}, (t_\alpha)_{\alpha\in\mathcal{J}})$, cf. Subsection 2.4. Each simple factor of $\mathcal{G}^{\mathrm{ad}}_{\mathbb{F}_p}$ is of the form $\mathrm{Res}_{k_0/\mathbb{F}_p}\mathcal{G}^0_{k_0}$, where k_0 is a finite field and $\mathcal{G}^0_{k_0}$ is an absolutely simple, adjoint group over k_0 (cf. [Ti1, Prop. 3.1.2]). Thus each simple factor of $\mathcal{G}^{\mathrm{ad}}_{\mathbb{Z}_p}$ is of the form $\mathrm{Res}_{W(k_0)/\mathbb{Z}_p}\mathcal{G}^0$, where $\mathcal{G}^0$ is an absolutely simple, adjoint group over $W(k_0)$ whose special fibre is $\mathcal{G}^0_{k_0}$ (cf. [DG, Vol. III, Exp. XXIII, Prop. 1.21]). Until Section 8 we will assume that each such group scheme $\mathcal{G}^0$ is of B_n, C_n, or D_n Dynkin type.

By performing the operation $\mathfrak{D}_1$ we can assume that $\mathcal{G}$ is split. Thus the field k contains each such field k_0. We assume that there exists a maximal torus $\mathcal{T}_{1B(k)}$ of $\mathcal{G}_{B(k)}$

of $\mathbb{Q}_p$-endomorphisms of $\mathcal{C}$. Let $\mathcal{T}_{1\mathbb{Q}_p}$, K, K_1, and K_2 be as in Definition 2.3 (c). Until Section 8 we will also assume that $L^0_{\mathcal{G}}(\phi)$ is a Levi subgroup scheme of $P^+_{\mathcal{G}}(\phi)$ and that $\mu : \mathbb{G}_m \to \mathcal{G}$ factors through a maximal torus $\mathcal{T}$ of $L^0_{\mathcal{G}}(\phi)$ contained in a Borel subgroup scheme B of $\mathcal{P}$, cf. Subsection 2.6. Let $L^0_{\mathcal{G}}(\phi)_{\mathbb{Z}_p}$ be the $\mathbb{Z}_p$ structure of $L^0_{\mathcal{G}}(\phi)$ obtained as in Subsection 2.4. It is a reductive, closed subgroup scheme of $\mathcal{G}_{\mathbb{Z}_p}$ which is the centralizer of the rank 1 split torus of $\mathcal{G}_{\mathbb{Z}_p}$ whose extension to $B(k)$ is the image of the Newton cocharacter of $\mathcal{C}$, cf. Fact 2.6.1. In Subsection 6.1 we include some reduction steps. In Subsection 6.2 we include few simple properties. In Subsection 6.3 (resp. Subsection 6.4) we deal with the cases related to Shimura varieties of B_n and $D_n^{\mathbb{R}}$ (resp. C_n and $D_n^{\mathbb{H}}$) type. The proof of Theorem 4.2 ends in Subsection 6.5.

6.1. Some reductions and notations. Let $\mu^{\mathrm{ad}} : \mathbb{G}_m \to \mathcal{G}^{\mathrm{ad}}$ be the composite of the cocharacter $\mu : \mathbb{G}_m \to \mathcal{G}$ with the natural epimorphism $\mathcal{G} \twoheadrightarrow \mathcal{G}^{\mathrm{ad}}$. As $\mathcal{G}$ is split, each cocharacter of $\mathcal{G}^{\mathrm{ad}}_{K_2}$ which is $\mathcal{G}^{\mathrm{ad}}(K_2)$-conjugate to $\mu^{\mathrm{ad}}_{K_2}$, lifts uniquely to a cocharacter of $\mathcal{G}_{K_2}$ which is $\mathcal{G}_{K_2}(K_2)$-conjugate to μ_{K_2}. Below we will consider only E-pairs of $\mathcal{C}$ which are as in Example 2.3.1. Thus based on the last two sentences, on Subsubsection 2.4.8, and on the fact that the statements 4.2 (a) and (b) pertain only to images in $\mathcal{G}^{\mathrm{ad}}_{K_2}$ of suitable products of cocharacters of $\mathcal{G}_{K_2}$ that factor through $\mathcal{T}_{1K_2}$, to prove Theorem 4.2 we can assume that the adjoint group scheme $\mathcal{G}^{\mathrm{ad}}_{\mathbb{Z}_p}$ is $\mathbb{Z}_p$-simple and that the cocharacter $\mu^{\mathrm{ad}}_{K_2}$ is non-trivial. Thus $\mathcal{G}^{\mathrm{ad}}_{\mathbb{Z}_p} = \mathrm{Res}_{W(k_0)/\mathbb{Z}_p}\mathcal{G}^0$. If $\mathcal{G}^0$ is of D_n Dynkin type, then it splits over $W(k_{02})$, where k_{02} is the quadratic extension of k_0 (cf. [Se1, Cor. 2 of p. 182]).

Let $m \in \mathbb{N}$ be such that $k_0 := \mathbb{F}_{p^m}$. We write

$$\mathcal{G}^{\mathrm{ad}} = \prod_{i=1}^{m} \mathcal{G}_i,$$

where $\mathcal{G}_i$ is a split, absolutely simple, adjoint group scheme over $W(k)$ and the numbering of $\mathcal{G}_i$'s is such that we have $\phi(\mathrm{Lie}(\mathcal{G}_i[\frac{1}{p}])) = \mathrm{Lie}(\mathcal{G}_{i+1}[\frac{1}{p}])$ for all $i \in \{1, \ldots, m\}$. Here and in all that follows the left lower or upper index $m+1$ has the same role as 1 (thus $\mathcal{G}_{m+1} := \mathcal{G}_1$, etc.). Let $\mathcal{G}^i$ be the semisimple, normal, closed subgroup scheme of $\mathcal{G}^{\mathrm{der}}$ which is naturally isogenous to $\mathcal{G}_i$. We view the isomorphism (2) of Subsubsection 2.4.1 as an identification and therefore we can write $\phi = g(1_{M_{\mathbb{Z}_p}} \otimes \sigma)\mu(\frac{1}{p})$, where $g \in \mathcal{G}(W(k))$ (to be compared with Subsubsection 2.4.8). Let $g^{\mathrm{ad}} \in \mathcal{G}^{\mathrm{ad}}(W(k))$ be the image of g. For $\alpha \in \mathbb{Q}$ let $\mathcal{D}_\alpha$ be the central division algebra over $B(k_0)$ of invariant α.

6.1.1. Fact. *To prove the Theorem 4.2 we can also assume that we have a direct sum decomposition*

$$M = \oplus_{i=1}^m M_i$$

into $\mathcal{G}$-modules such that the following two conditions hold:

(i) *if $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, i-1, i+1, \ldots, m\}$, then M_i has no trivial $\mathcal{G}_i$-submodule and M_j is a trivial $\mathcal{G}^i$-module;*

(ii) *we have an identity $Z(\mathcal{G}) = \prod_{i=1}^m Z^i$, where each Z^i is a torus of $\boldsymbol{GL}_M$ that acts trivially on $\oplus_{j\in\{1,\ldots,i-1,i+1,\ldots,m\}} M_j$.*

Proof: The arguments for this Fact are the same as the ones of the proof of [Va1, Thm. 6.5.1.1 or Subsubsection 6.6.5] but much simpler as we are over $\mathbb{Z}_p$ and not over $\mathbb{Z}_{(p)}$ and as we do not have to bother about quasi-polarizations or Hodge $\mathbb{Q}$–structures. We recall the essence of loc. cit.

We first assume that $\mathcal{G}^0$ is of B_n Lie type. Thus $\mathcal{G}^0$ is split. We consider the spin faithful representation $\mathcal{G}^{0\mathrm{sc}} \hookrightarrow \mathbf{GL}_{M_0}$ over $W(k_0)$. Let **GSpin** be the closed subgroup scheme of $\mathbf{GL}_{M_0}$ generated by $\mathcal{G}^{0\mathrm{sc}}$ and $Z(\mathbf{GL}_{M_0})$. Let $\mathcal{G}'_{\mathbb{Z}_p} := \mathrm{Res}_{W(k_0)/\mathbb{Z}_p}\mathbf{GSpin}$. We consider its faithful representation on $M'_{\mathbb{Z}_p}$, where $M'_{\mathbb{Z}_p}$ is M_0 but viewed as a $\mathbb{Z}_p$-module. We identify naturally $\mathcal{G}^{\mathrm{ad}}_{\mathbb{Z}_p} = \mathcal{G}'^{\mathrm{ad}}_{\mathbb{Z}_p}$. Let $M' := M'_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} W(k)$. Let $\mathcal{G}' := \mathcal{G}'_{W(k)}$. We have a unique direct sum decomposition $M' = \oplus_{i=1}^m M'_i$ of $\mathcal{G}'$-modules which are also $W(k_0) \otimes_{\mathbb{Z}_p} W(k)$-modules. Let $g' \in \mathcal{G}'(W(k))$ be such that its image in $\mathcal{G}'^{\mathrm{ad}}(W(k))$ is g^{ad}. Let μ' be a cocharacter of $\mathcal{G}'$ such that the cocharacter of $\mathcal{G}'^{\mathrm{ad}} = \mathcal{G}^{\mathrm{ad}}$ it defines naturally is μ^{ad} and the triple $\mathcal{C}' := (M', g'(1'_{M_{\mathbb{Z}_p}} \otimes \sigma)\mu'(\frac{1}{p}), \mathcal{G}')$ is a Shimura F-crystal over k. Let $\mathcal{T}'_{1B(k)}$ be the maximal torus of $\mathcal{G}'_{B(k)}$ whose image in $\mathcal{G}'^{\mathrm{ad}}_{B(k)}$ is the same as of $\mathcal{T}_{1B(k)}$; it is a maximal torus of $\mathcal{G}'_{B(k)}$ of $\mathbb{Q}_p$-endomorphisms of $\mathcal{C}'$. Thus $\mathcal{C}'$ is semisimple, cf. Fact 3.1 (b). Let μ'_1 be the cocharacter of $\mathcal{T}'_{K_1}$ which over K_2 is $\mathcal{G}'(K_2)$-conjugate to μ'_{K_2} and such that it defines the same cocharacter of $\mathcal{G}'^{\mathrm{ad}}_{K_1} = \mathcal{G}^{\mathrm{ad}}_{K_1}$ as μ_1. The E-pair $(\mathcal{T}'_{1B(k)}, \mu'_1)$ of $\mathcal{C}'$ satisfies the $\mathfrak{C}$ condition if and only if the E-pair $(\mathcal{T}_{1B(k)}, \mu_1)$ of $\mathcal{C}$ satisfies the $\mathfrak{C}$ condition. Similarly, $TT\mathfrak{R}$ holds for $\mathcal{C}'$ if and only if it holds for $\mathcal{C}$. Thus to prove Theorem 4.2 for the case when $\mathcal{G}^0$ is of B_n Lie type, we can replace $\mathcal{C}$ by $\mathcal{C}'$. As the two conditions (i) and (ii) obviously hold if $\mathcal{C}$ is $\mathcal{C}'$, the Fact holds if $\mathcal{G}^0$ is of B_n Lie type.

If $\mathcal{G}^0$ is of either C_n or D_n Dynkin type, we will only list the modifications required to be performed to the previous paragraph. If $\mathcal{G}^0$ is of C_n Lie type, then the spin representation has to be replaced by the standard rank $2n$ faithful representation $\mathcal{G}^{0\mathrm{sc}} \hookrightarrow \mathbf{GL}_{M_0}$ over $W(k_0)$. If $\mathcal{G}^0$ is of D_n Dynkin type, then we have two disjoint subcases (related to Shimura varieties of $D_n^{\mathbb{H}}$ and respectively of $D_n^{\mathbb{R}}$ type). The second case can be defined rigurously by the following two properties:

(iii.a) the adjoint group scheme of the centralizer of μ^{ad} in $\mathcal{G}^{\mathrm{ad}}$ is a product of split, simple groups of either D_n or D_{n-1} Lie type;

(iii.b) if $n = 4$, then for each $i \in \{1, \ldots, m\}$ the non-trivial images of the cocharacters $\phi^s(\mu) : \mathbb{G}_m \to \mathcal{G}$ with $s \in \mathbb{Z}$ in $\mathcal{G}_i$ are $\mathcal{G}_i(W(k))$-conjugate.

In the first subcase the spin representation has to be replaced by the standard rank $2n$ faithful representation $\mathcal{G}^1 \hookrightarrow \mathbf{GL}_{M_0}$ over $W(k_0)$. Here $\mathcal{G}^1$ is an isogeny cover of $\mathcal{G}^0$ for which such a representation is possible; its existence is implied by the fact that $\mathcal{G}^0$ splits over $W(k_{02})$. If $n > 4$, then $\mathcal{G}^1$ is unique. If $n = 4$, then we choose $\mathcal{G}^1$ such that the construction of μ' is possible (we have only one choice for $\mathcal{G}^1$, due to the fact that the two subcases are disjoint). The second subcase is in essence the same as the previous paragraph (the only difference being that $\mathcal{G}^{0\mathrm{sc}}$ is not necessarily split; however, as it splits over $W(k_{02})$, its splin representation is well defined over $W(k_0)$). $\square$

6.2. Simple properties. We first consider the case when $L^0_{\mathcal{G}}(\phi)$ is a torus (i.e., we have $L^0_{\mathcal{G}}(\phi) = \mathcal{T}$). Thus we have $\mathcal{T}_{1B(k)} = \mathcal{T}_{B(k)}$ and $K_1 \subseteq B(k)$. Let $\tau_1 \in \mathrm{Gal}(K_1/\mathbb{Q}_p)$ be the restriction of σ to K_1. The E-triple $(\mathcal{T}_{1B(k)}, \mu_{B(k)}, \tau_1)$ satisfies the condition 2.3 (e1) and is obviously admissible. Thus Theorem 4.2 holds if $L^0_{\mathcal{G}}(\phi)$ is a torus. From now on until Section 7 we will assume that $L^0_{\mathcal{G}}(\phi)$ is not a torus (i.e., we have $L^0_{\mathcal{G}}(\phi) \neq \mathcal{T}$). Let L_0 be the $\mathbb{Q}_p$-form of $L^0_{\mathcal{G}}(\phi)_{B(\bar{k})}$ with respect to $(M[\frac{1}{p}], \phi)$. The tori $Z^0(\mathcal{G}_{\mathbb{Q}_p})$ and $\mathcal{T}_{1\mathbb{Q}_p}$ are subtori of L_0. We have a direct sum decomposition

$$\mathrm{Lie}(L_0) \otimes_{\mathbb{Q}_p} B(k_0) = \mathrm{Lie}(Z^0(L_0)) \otimes_{\mathbb{Q}_p} B(k_0) \bigoplus_{i=1}^{m} \mathcal{L}^i_0, \tag{8}$$

where $\mathcal{L}^i_0 := (\mathrm{Lie}(L^{\mathrm{der}}_0) \otimes_{\mathbb{Q}_p} B(k_0)) \cap \mathrm{Lie}(\mathcal{G}_{iB(\bar{k})})$. Each $\mathcal{L}^i_0$ is a semisimple Lie algebra and thus it is also the Lie algebra of a semisimple group L^i_0 over $B(k_0)$. Moreover, we have $\phi(\mathcal{L}^i_0) = \mathcal{L}^{i+1}_0$. Based on this and (8) we get that each p-adic field over which L_0 splits must contain $B(k_0)$. Therefore $B(k_0) \subseteq K_1$.

6.2.1. Lemma. *We assume that $p \geq 3$. Let $H_1 := Gal(K_1/\mathbb{Q}_p)$. Let H_0 be a subgroup of $H^0_1 := Gal(K_1/B(k_0))$ of even index. If m is odd, then there exists an element $\tau_1 \in H_1$ such that the following two conditions hold:*

(i) *all orbits under τ_1^m of the left translation action of H_1 on H_1/H_0 have an even number of elements;*

(ii) *the action of τ_1 on the residue field l_1 of K_1 is the Frobenius automorphism of l_1 whose fixed field is $\mathbb{F}_p$.*

Proof: For $s \in \mathbb{N} \cup \{0\}$ let H_{1s} be the s-th ramification group of H_1. Thus $H_1 = H_{10}$, H_1/H_{11} is cyclic, and the subgroup H_{12} of H_1 is normal and (as $p \geq 3$) has odd order. By replacing H_1 with H_1/H_{12}, we can assume that $H_{12} = \{1_{K_1}\}$. Thus H_{11} is a subgroup of $\mathbb{G}_m(l_1)$ and therefore it is cyclic. By replacing H_1 with its quotient through a normal subgroup of H_{11} of odd order, we can assume that H_{11} is of order 2^t for some $t \in \mathbb{N} \cup \{0\}$. The case $t = 0$ is trivial and therefore we can assume that $t \geq 1$. Let H_{01} be the image of H_0 in H^0_1/H_{11} and let a be its index in H^0_1/H_{11}. If a is even, then the condition (i) is implied by (ii) and therefore we can choose any element $\tau_1 \in H_1$ for which the condition (ii) holds. If a is odd, then by replacing H_1 with its quotient through the subgroup of H_{11} of order 2^{t-1} we can assume that $t = 1$. Thus H_{11} has order 2. As H_{11} is a normal subgroup of H_1 of order 2, it is included in the center of H_1. Thus H_1 is either cyclic or isomorphic to $H_{11} \times H_1/H_{11}$. If H_1 is cyclic, then we can take $\tau_1 \in H_1$ such that it generates H_1 and the condition (ii) holds. If H_1 is isomorphic to $H_{11} \times H_1/H_{11}$, then we can take $\tau_1 = (\tau_{11}, \tau_{12})$ such that $\tau_{11} \in H_{11}$ and $\tau_{12} \in H_1/H_{11}$ generate these groups and the condition (ii) holds. In both cases the condition (i) also holds. $\square$

6.2.2. Factors. Let $\mathfrak{N}$ be the set of those elements $i \in \{1, \ldots, m\}$ for which the image of $\mu^{\mathrm{ad}} : \mathbb{G}_m \to \mathcal{G}^{\mathrm{ad}}$ in $\mathcal{G}_i$ is non-trivial. Let $\mathfrak{M} := \{1, \ldots, m\} \setminus \mathfrak{N}$. If $i \in \mathfrak{N}$ (resp. $i \in \mathfrak{N}$), then $\mathcal{G}_i$ is called a non-compact (resp. compact) factor of $\mathcal{G}^{\mathrm{ad}}$ with respect to μ^{ad}. As μ^{ad} is non-trivial, the set $\mathfrak{N}$ is non-empty. To simplify notations we will assume that $1 \in \mathfrak{N}$. Let v be the number of elements of $\mathfrak{N}$. We will choose the cocharacter $\mu : \mathbb{G}_m \to \mathcal{G}$ such that $\mathbb{G}_m$ acts via μ trivially on M_i for all $i \in \mathfrak{M}$.

6.3. Case 1. Until Subsection 6.4 we will assume that $\mathcal{G}^0$ is of either B_n or D_n Dynkin type and that $\mathcal{G}^{\mathrm{der}}$ is simply connected (under these assumptions, one can assume that the representation of $\mathcal{G}^i$ on M_i is the spin representation). Let $\mathcal{G}^{01} \to \mathcal{G}^0$ be an isogeny such that $\mathcal{G}^{01}$ is the **SO** group scheme of a quadratic form on a free $W(k_0)$-module O_0 of rank r. Here r is either $2n+1$ or $2n$ depending on the fact that $\mathcal{G}^0$ is of B_n or D_n Dynkin type. Let $\mathbf{S} := (\mathrm{Res}_{W(k_0)/\mathbb{Z}_p}\mathcal{G}^{01}) \times_{\mathbb{Z}_p} W(k)$; it is a semisimple group scheme over $W(k)$ whose adjoint group scheme is $\mathcal{G}^{\mathrm{ad}}$. Let $\mu_0 : \mathbb{G}_m \to \mathbf{S}$ be the unique cocharacter that lifts μ^{ad}. Let $\mathbf{SO}(O_i, b_i) := \mathcal{G}^{01} \times_{W(k_0)} W(k)$, where the $\mathbb{Z}_p$-embedding $W(k_0) \hookrightarrow W(k)$ is the same as the one that defines $\mathcal{G}_i = \mathcal{G}^0 \times_{W(k_0)} W(k)$. We have $\mathbf{S} = \prod_{i=1}^m \mathbf{SO}(O_i, b_i)$. We have an identification $O_0 \otimes_{\mathbb{Z}_p} W(k) = \oplus_{i=1}^m O_i$ of $W(k_0) \otimes_{\mathbb{Z}_p} W(k)$-modules. For $i \in \{1, \ldots, m\}$, let $W_i := O_i \otimes_{W(k)} K_2$.

Let $\mathcal{B}_i := \{e_1^i, \ldots, e_r^i\}$ be a K_2-basis for W_i such that the following two conditions hold:

(i) if $a, b \in \{1, \ldots, r\}$ with $a < b$, then the value of $b_i(e_a^i, e_b^i)$ is 0 or 1 depending on the fact that the pair (a, b) belongs or not to the set $\{(1, 2), \ldots, (2n-1, 2n)\}$;

(ii) the torus $\mathcal{T}_{1K_2}$ normalizes each $K_2 e_a^i$.

The natural action of $\mathrm{Gal}(K_2/\mathbb{Q}_p)$ on cocharacters of $\mathcal{T}_{1K_2}$ defines naturally an action of $\mathrm{Gal}(K_2/\mathbb{Q}_p)$ on $\mathcal{B} := \cup_{i=1}^m \mathcal{B}_i$. For $\star \in \mathrm{Gal}(K_2/\mathbb{Q}_p)$, let $\pi_\star$ be the permutation of $\mathcal{B}$ defined by $\star$. For each $i \in \{1, \ldots, m\}$, the set $\mathcal{B}_i$ is normalized by $\mathrm{Gal}(K_2/B(k_0))$.

If there exists an element $g_0 \in \mathbf{S}(W(k))$ whose image in $\mathcal{G}^{\mathrm{ad}}(W(k))$ is g^{ad}, then $\phi_0 := g_0(1_{O_0} \otimes \sigma)\mu(\frac{1}{p})$ is a σ-linear automorphism of $O_0 \otimes_{\mathbb{Z}_p} B(k)$. By a natural passage to $\bar{k}$ we can always assume that such an element g_0 exists, cf. [Va3, Fact 2.6.3].

Let $\mathcal{T}_{0B(k)}$ be the maximal subtorus of $\mathbf{S}_{B(k)}$ whose image in $\mathbf{S}_{B(k)}^{\mathrm{ad}} = \mathcal{G}_{B(k)}^{\mathrm{ad}}$ is $\mathcal{T}'_{0B(k)} := \mathrm{Im}(\mathcal{T}_{1B(k)} \to \mathcal{G}_{B(k)}^{\mathrm{ad}})$. Let $\mathcal{T}_{0\mathbb{Q}_p}$ be the $\mathbb{Q}_p$-form of $\mathcal{T}_{0B(\bar{k})}$ with respect to $(O_0 \otimes_{\mathbb{Z}_p} B(\bar{k}), \phi_0 \otimes \sigma_{\bar{k}})$; it is a form of $\mathcal{T}_{0B(k)}$ whose Lie algebra is $\{x \in \mathrm{Lie}(\mathcal{T}_{0B(k)}) = \mathrm{Lie}(\mathcal{T}'_{0B(k)}) | \phi(x) = x\}$. Let $\mathcal{T}'_{0\mathbb{Q}_p}$ be the $\mathbb{Q}_p$-form of $\mathcal{T}'_{0B(k)}$ which is the quotient of $\mathcal{T}_{0\mathbb{Q}_p}$ by its finite subgroup whose extension to $B(k)$ is $\mathcal{T}_{0B(k)} \cap Z(\mathbf{S}_{B(k)})$.

Until Subsubsection 6.3.4 we will assume that $\mathcal{C}$ is basic. Thus $L^0_{\mathcal{G}}(\phi) = \mathcal{G}$ and therefore L_0 is a $\mathbb{Q}_p$-form of $\mathcal{G}_{B(\bar{k})}$. To show that there exists an E-pair $(\mathcal{T}_{1B(k)}, \mu_1)$ of $\mathcal{C}$ as in Example 2.3.1, we first prove the following Lemma.

6.3.1. Lemma. *We recall that $\mathcal{C}$ is basic. The action of Gal$(K_2/B(k_0))$ on $\mathcal{B}_1$ has an orbit that contains $\{e^1_{2a_1-1}, e^1_{2a_1}\}$ for some element $a_1 \in \{1, \ldots, n\}$.*

Proof: In this proof by orbit we mean an orbit of the action of $\mathrm{Gal}(K_2/B(k_0))$ on $\mathcal{B}_1$. Suppose there exists an orbit $\tilde{o}_1$ whose elements are pairwise perpendicular with respect to b_1. To fix the notations, we can assume that there exists $a \in \{1, ..., n\}$ such that $e^1_{2a-1} \in \tilde{o}_1$. Let $\tilde{o}_2$ be the orbit that contains e^1_{2a}. The orbit decomposition of $\mathcal{B}_1$ corresponds to a direct sum decomposition of $O_1[\frac{1}{p}]$ in minimal $B(k)$-vector subspaces normalized by $\mathcal{T}_{1B(k)}$ and, in the case when the element g_0 exists, by ϕ_0^m. Let $O_{1,1}$ and $O_{1,2}$ be the $B(k)$-vector subspaces of $O_1[\frac{1}{p}]$ that correspond to $\tilde{o}_1$ and $\tilde{o}_2$ (respectively). The intersection $\mathcal{L}^1_0 \cap (\mathrm{End}(O_{1,1} \oplus O_{1,2})) \otimes_{B(k)} B(\bar{k}))$ is the Lie algebra of a split semisimple group of D_s Lie type over $B(k_0)$, where s is the number of elements of $\tilde{o}_1$. Argument: we can assume that the element $g_0 \in \mathbf{S}(W(k))$ exists and thus the statement is an easy consequence of the fact that (as $\mathcal{C}$ is basic) all Newton polygon slopes of $(O_{1,s}[\frac{1}{p}]\otimes_{B(k_0)} B(k), (\phi_0\otimes\sigma_k)^m)$ are 0 (here $s \in \{1,2\}$).

Thus if the Lemma does not hold, then L_0^1 split torus of rank n and therefore it is a split group over $B(k_0)$. Thus to prove the Lemma we only have to show that the semisimple group L_0^1 is non-split. This is a rational statement. Thus to check it, based on [Va3, Thm. 1.3.3 and Subsection 2.5] we can assume that $\phi(\text{Lie}(\mathcal{T})) = \text{Lie}(\mathcal{T})$ and (in order to use the above notations on $\mathcal{B}_i$'s) that $\mathcal{T}_{1B(k)} = \mathcal{T}_{B(k)}$. Thus $K_2 = B(k)$ and the actions of $\pi_{\sigma_{\mathbb{F}_p}}$ and ϕ on cocharacters of $\mathcal{T}$ coincide. We can also assume that $\mu_0 : \mathbb{G}_m \to \mathbf{S}$ fixes e_a^i if either $i \in \mathfrak{M}$ or $a \geq 3$. Let $g_1 \in N_{\mathcal{G}}(\mathcal{T})(W(k))$ be such that $g_1\phi(\text{Lie}(B)) \subseteq \text{Lie}(B)$, cf. [Va3, Subsection 2.5]. Let $w_1 \in N_{\mathcal{G}}(\mathcal{T})(W(k))$ be such that its image in $\mathcal{G}^{\text{ad}}(W(k))$ belongs to $\mathcal{G}_1(W(k))$ and takes the image of B in $\mathcal{G}_1$ to its opposite with respect to the image of $\mathcal{T}$ in $\mathcal{G}_1$. As $\mathcal{G}^{\text{der}}$ is simply connected of either B_n or D_n Dynkin type, w_1 takes the cocharacter of $\mathcal{G}_1$ defined by μ^{ad} to its inverse. Thus the Shimura F-crystal $(M, w_1g_1\phi, \mathcal{G})$ over k is basic (to be compared with [Va3, Cases 1 and 2 of Subsubsection 4.2.2]). Thus based on [Va3, Prop. 2.7.1] we can assume that $w_1g_1 = 1_M$ and therefore that:

(iii) $\pi_{\sigma_{\mathbb{F}_p}}^m$ restricted to $\mathcal{B}_1$ fixes e_a^1 for $a \geq 3$ and permutes e_1^1 and e_2^1.

Thus the group L_0^1 has a split torus $T_{1;n-1}^1$ of rank $n-1$: it is the torus of $\mathbf{S}$ that fixes e_1^1 and e_2^1 and that normalizes $B(k_0)e_a^1$ for each $a \in \{3,\ldots,r\}$. The centralizer of $T_{1;n-1}^1$ in L_0^1 is a non-split torus, cf. property (iii). Thus the group L_0^1 is non-split. $\square$

6.3.2. The choice of μ_1 for the basic context. Let a_1 be as in Lemma 6.3.1. Let o be the orbit of $e_{2a_1-1}^1$ under $\text{Gal}(K_2/\mathbb{Q}_p)$. For $i \in \mathfrak{N}\setminus\{1\}$ let $a_i \in \{1,\ldots,n\}$ be such that $\{e_{2a_i-1}^i, e_{2a_i}^i\} \subseteq o$. Let $\mu_2 : \mathbb{G}_m \to \mathbf{S}_{K_2}$ be the cocharacter that fixes all e_a^i's except those of the form $e_{2a_i-1+u}^i$, where $i \in \mathfrak{N}$ and $u \in \{0,1\}$, and that acts as the identical (resp. as the inverse of the identical) character of $\mathbb{G}_m$ on each $K_2e_{2a_i}^i$ (resp. $K_2e_{2a_i-1}^i$) with $i \in \mathfrak{N}$. To define the cocharacter $\mu_1 : \mathbb{G}_m \to \mathcal{T}_{1K_1}$ it is enough to define the cocharacter $\mu_{1K_2} : \mathbb{G}_m \to \mathcal{T}_{1K_2}$. Let $\mu_{1K_2} : \mathbb{G}_m \to \mathcal{T}_{1K_2}$ be the unique cocharacter such that the cocharacter of $\mathcal{G}_{K_2}^{\text{ad}}$ (resp. of $\mathcal{G}_{K_2}^{\text{ab}}$) it defines naturally is the composite of μ_2 with the isogeny $\mathbf{S}_{K_2} \to \mathcal{G}_{K_2}^{\text{ad}}$ (resp. is the one defined by μ_{K_2}). As $\{e_{2a_i-1}^i, e_{2a_i}^i\} \subseteq o$, the product of the cocharacters of $\mathcal{T}_{1K_2}$ that belong to the $\text{Gal}(K_2/\mathbb{Q}_p)$-orbit of μ_{1K_2} factors through $Z^0(\mathcal{G}_{K_2})$. Thus the E-pair $(\mathcal{T}_{1B(k)}, \mu_1)$ of $\mathcal{C}$ satisfies the $\mathfrak{C}$ condition, cf. Example 2.3.1.

6.3.3. Remark. If m is odd and the action of $\text{Gal}(K_2/B(k_0))$ on $\mathcal{B}_1$ has only one orbit, then it is easy to check based on Lemma 6.2.1 that there exists an E-pair $(\mathcal{T}_{1B(k)}, \mu_1)$ of $\mathcal{C}$ that satisfies the cyclic $\mathfrak{C}$ condition.

6.3.4. The non-basic context. Until the end of Case 1 we will assume that $\mathcal{C}$ is non-basic. We use the previous notations of Subsection 6.3. Let $\mathcal{T}_{0B(k)}^0$ be the subtorus of $\mathcal{T}_{0B(k)}$ whose image in $\mathcal{G}_{B(k)}^{\text{ad}}$ is the same as the image of $Z^0(L_{\mathcal{G}}^0(\phi))_{B(k)}$. As $L_{\mathcal{G}}^0(\phi)_{\mathbb{Z}_p}$ is the centralizer in $\mathcal{G}_{\mathbb{Z}_p}$ of a rank 1 split torus (see beginning of Section 6), the group $C_{\mathbf{S}_{B(k)}}(\mathcal{T}_{0B(k)}^0)$ is a product

$$\prod_{i=1}^{m}\prod_{j=1}^{m_f}\mathbf{SO}(O_{i,j}, b_{i,j}), \tag{9}$$

where $O_i[\frac{1}{p}] = \oplus_{j=1}^{m_f} O_{i,j}$ is the minimal direct sum decomposition normalized by $\mathcal{T}^0_{0B(k)}$ and $b_{i,j}$ is the restriction of b_i to $O_{i,j}$. We emphasize that $m_f \in \mathbb{N}$ does not depend on i and that, in the case when $b_{i,j} = 0$, we define $\mathbf{SO}(O_{i,j}, b_{i,j}) := \mathbf{GL}_{O_{i,j}}$. We choose the indices such that we have $\phi(\mathrm{End}(O_{i,j})) = \mathrm{End}(O_{i+1,j})$.

We define a cocharacter $\mu_2 : \mathbb{G}_m \to \mathbf{S}_{K_2}$ that factors through $\mathcal{T}_{0K_2}$ as follows. Let $j \in \{1, \ldots, m_f\}$. We first assume that $b_{1,j} \neq 0$. If $\mathbb{G}_m$ acts via μ_0 trivially (resp. non-trivially) on $\oplus_{i=1}^{m} O_{i,j}$, then we define the action of $\mathbb{G}_m$ via μ_2 on $O_{i,j} \otimes_{B(k)} K_2$ to be trivial (resp. to be obtained as in Subsubsection 6.3.2 but working with $\oplus_{i=1}^{m} O_{i,j}$ instead of with O_i).

Until Case 2 we assume that $b_{1,j} = 0$. Let $\tilde{j} \in \{1, \ldots, m_f\} \backslash \{j\}$ be the unique element such that $O_{i,\tilde{j}}$ is not perpendicular on $O_{i,j}$ with respect to b_i. Let $o_1, \ldots, o_s$ be the orbits of the action of $\mathrm{Gal}(K_2/\mathbb{Q}_p)$ on $\mathcal{B} \cap (\oplus_{i=1}^{m} O_{i,j}) \otimes_{B(k)} K_2$. Let $d_j := \dim_{B(k)}(O_{1,j})$. Let $S_{+,j}$ (resp. $S_{-,j}$) be the set of those elements $i \in \mathfrak{N}$ with the property that μ_0 acts via the inverse of the identical (resp. via the identical) character of $\mathbb{G}_m$ on a non-zero element of $O_{i,j}$. Let $c_{+,j}$ (resp. $c_{-,j}$) be the number of elements of $S_{+,j}$ (resp. of $S_{-,j}$). We have $S_{+,j} \cap S_{-,j} = \emptyset$ as otherwise $b_{1,j} \neq 0$. Thus $c_{+,j} + c_{-,j} \leq v$ (see Subsubsection 6.2.2 for v).

Let $p_j \in \mathbb{Z}$ and $q_j \in \mathbb{N}$ be such that $g.c.d.(p_j, q_j) = 1$ and $(c_{+,j} - c_{-,j})q_j = d_j p_j$. If the element $g_0 \in \mathbf{S}(W(k))$ exits, then the only Newton polygon slope of $(O_{i,j}, \phi_0^m)$ is $\frac{p_j}{q_j}$ and therefore q_j divides d_j. Thus the $B(k_0)$-subalgebra $\{x \in \mathrm{End}(O_{i,j}) \otimes_{B(k)} B(\bar{k}) | (\phi \otimes \sigma_{\bar{k}}^m(x) = x\}$ is (isomorphic to) $M_{d_j/q_j}(\mathcal{D}_{\frac{p_j}{q_j}})$, cf. Dieudonné classification of F-crystals over $\bar{k}$ (see [Ma, Section 2]; we recall that after a passage to $\bar{k}$ we can assume that the element $g_0 \in \mathbf{S}(W(k))$ exists). Thus for each $l \in \{1, \ldots, s\}$ there exists $e_l \in \mathbb{N}$ such that the number of elements of $o_l \cap \mathcal{B}_1$ is $q_j e_l$. We have $q_j(\sum_{l=1}^{s} e_l) = d_j$. For each $l \in \{1, \ldots, s\}$ we choose numbers $c_{l,j}^+$, $c_{l,j}^- \in \mathbb{N} \cup \{0\}$ such that the following three relations hold:

(i) $c_{l,j}^+ - c_{l,j}^- = p_j e_l$;

(ii) $\sum_{l=1}^{s} c_{l,j}^+ = c_{+,j}$;

(iii) $c_{l,j}^+ + c_{l,j}^- \leq v$.

For instance, if $p_j \geq 0$ we can choose $c_{1,j}^- = \cdots = c_{s-1,j}^- = 0$ and $c_{s,j}^- = c_{-,j}$, the numbers $c_{l,j}^+$'s being now determined uniquely by the relation (i). We define the action of $\mathbb{G}_m$ via μ_2 on $(O_{i,j} \oplus O_{i,\tilde{j}}) \otimes_{B(k)} K_2$ as follows. Let $l \in \{1, \ldots, s\}$. Let $i \in \{1, \ldots, m\}$ and $a \in \{1, \ldots, r\}$ be such that $e_a^i \in o_l$. The action of $\mathbb{G}_m$ via μ_2 on $K_2 e_a^i$ is:

(iv) via the identical character of $\mathbb{G}_m$ if a is the smallest number in $\{1, \ldots, r\}$ such that $e_a^i \in o_l$ and i is the s_i^{th} number in $S_{-,j}$, where $s_i \in \{1 + \sum_{x=1}^{l-1} c_{x,j}^-, \ldots, \sum_{x=1}^{l} c_{x,j}^-\}$;

(v) via the inverse of the identical character of $\mathbb{G}_m$ if a is the smallest number in $\{1, \ldots, r\}$ such that $e_a^i \in o_l$ and i is the s_i^{th} number in $S_{+,j}$, where $s_i \in \{1 + \sum_{x=1}^{l-1} c_{x,j}^+, \ldots, \sum_{x=1}^{l} c_{x,j}^+\}$;

(vi) trivial otherwise.

The action of $\mathbb{G}_m$ via μ_2 on a K_2-vector subspace $K_2 e_a^i$ of $O_{i,\tilde{j}}$ is defined uniquely by the requirement that μ_2 factors through the image of $\mathcal{T}_{0K_2}$ in $\mathbf{GL}_{(O_{i,j} \oplus O_{i,\tilde{j}}) \otimes_{B(k)} K_2}$.

Due to the relation (i) the product of the cocharacters of $\mathcal{T}_{0K_2}$ of the orbit under $\mathrm{Gal}(K_2/\mathbb{Q}_p)$ of the factorization of μ_2 through $\mathcal{T}_{0K_2}$ gives birth to a cocharacter of $\mathbf{GL}_{(O_{i,j}\oplus O_{i,\tilde{j}})\otimes_{B(k)}K_2}$ that factors through $Z(\mathbf{GL}_{O_{i,j}\otimes_{B(k)}K_2})\times_{K_2} Z(\mathbf{GL}_{O_{i,\tilde{j}}\otimes_{B(k)}K_2})$. Thus choosing μ_1 as in Subsubsection 6.3.2 we get that the product of the cocharacters of $\mathcal{T}_{1K_2}$ which belong to the orbit under $\mathrm{Gal}(K_2/\mathbb{Q}_p)$ of μ_{1K_2} factors through $Z^0(L^0_{\mathcal{G}}(\phi)_{K_2})$, cf. (9). Thus the E-pair $(\mathcal{T}_{1B(k)},\mu_1)$ of $(M,\phi,L^0_{\mathcal{G}}(\phi))$ satisfies the $\mathfrak{C}$ condition, cf. Example 2.3.1.

6.4. Case 2. Until Subsection 6.5 we assume that $\mathcal{G}^0$ is of C_n or D_n Dynkin type, that M_i has rank $2n$, and that $\mathcal{C}$ is basic. This Case 2 is very much the same as the Case 1 for $\mathcal{C}$ basic. We mention only the differences. The first differences are:

(i) we can assume that $O_i = M_i$ and thus that $\mathbb{G}_m$ acts via μ trivially on O_i for all $i\in\mathfrak{M}$;

(ii) we have $r = 2n$ and for the C_n Dynkin type the form b_i is alternating and not symmetric.

As $M_i = O_i$ let $\mathcal{B}$, $\mathcal{B}_i$, e^i_a, and $\pi_\star$ with $\star\in\mathrm{Gal}(K_2/\mathbb{Q}_p)$ be as in Subsection 6.3. Let $o_1,\dots,o_s$ be the orbits of the action of $\mathrm{Gal}(K_2/\mathbb{Q}_p)$ on $\mathcal{B}$ numbered in such a way that there exists $s_0\in\{0,\dots,s\}$ such that for an element $l\in\{1,\dots,s\}$ the orbit o_l contains the set $\{e^1_{2a_1-1},e^1_{2a_1}\}$ for some number $a_1\in\{1,\dots,n\}$ if and only if $l\le s_0$. The difference $s-s_0$ is an even number. We can also assume that if $s_1\in\{1,\dots,\frac{s-s_0}{2}\}$, then the union $o_{s_0+2s_1-1}\cup o_{s_0+2s_1}$ contains the set $\{e^1_{2a_1-1},e^1_{2a_1}\}$ for some number $a_1\in\{1,\dots,n\}$. If $l\le s_0$ (resp. $l>s_0$) let $u_l\in\mathbb{N}$ be such that the number of elements of the set $\tilde{o}_l := o_l\cap\mathcal{B}_1$ is $2u_l$ (resp. u_l). Lemma 6.3.1 gets replaced by the following weaker one.

6.4.1. Lemma. *We assume that v is odd. Then u_l is even for $l>s_0$.*

Proof: As $\mathcal{C}$ is basic, all Newton polygon slopes of (O_1,ϕ^m) are $\frac{v}{2}$. From this the Lemma follows. $\square$

To define μ_1 it is enough to define μ_{1K_2}. We consider two Subcases:

6.4.2. Choice of μ_1 for v odd. We know that u_l is even for $l>s_0$, cf. Lemma 6.4.1. Thus if $l>s_0$ we write $o_l = o_{l,1}\cup o_{l,2}$, where $o_{l,1}$ and $o_{l,2}$ have $\frac{u_l}{2}$ elements. Not to introduce extra notations we will assume that if $l-s_0\in 1+2\mathbb{N}$, then the sets $o_{l,1}\cap\mathcal{B}_i$ and $o_{l+1,2}\cap\mathcal{B}_i$ are perpendicular with respect to b_i for all $i\in\{1,\dots,m\}$. We choose μ_{1K_2} such that $\mathbb{G}_m$ acts through it:

(i) trivially on e^i_a if $i\in\mathfrak{M}$;

(ii) trivially on $e^i_a\in o_{l,j}$ if $i\in\mathfrak{N}$, $l\in\{s_0+1,\dots,s\}$, and $j\in\{1,2\}$ with $l-s_0-j$ even;

(iii) trivially on $e^i_a\in o_l$ if $i\in\mathfrak{N}$, $l\le s_0$, and a is odd;

(iv) via the inverse of the identical character on all other elements of $\mathcal{B}$.

Thus μ_{1K_2} acts non-trivially on precisely half of the elements of the set $\{o_l|l\in\{1,\dots,s\}\}$. Therefore the product of the cocharacters of $\mathcal{T}_{1K_2}$ which belong to the

$\mathrm{Gal}(K_2/\mathbb{Q}_p)$-orbit of μ_{1K_2} factors through $\prod_{i=1}^m Z(\mathbf{GL}_{M_i}) = Z^0(\mathcal{G}_{K_2})$. Thus $(\mathcal{T}_{1B(k)}, \mu_1)$ satisfies the $\mathfrak{C}$ condition, cf. Example 2.3.1.

6.4.3. Choice of μ_1 for v even. Let $\mathfrak{N}_0$ be a subset of $\mathfrak{N}$ that has $\frac{v}{2}$ elements. Let $\mathfrak{N}_1 := \mathfrak{N} \setminus \mathfrak{N}_0$. Let $e_a^i \in o_l$. If $l \leq s_0$, then we define the action of $\mathbb{G}_m$ via μ_{1K_2} on $K_2 e_a^i$ as in Subsubsection 6.4.2. If $l > s_0$, then we define the action of $\mathbb{G}_m$ via μ_{1K_2} on $K_2 e_a^i$ to be via the inverse of the identical character of $\mathbb{G}_m$ (resp. trivial) if and only if $i \in \mathfrak{N}_j$, where $j \in \{0, 1\}$ is congruent to $l - s_0$ modulo 2. Thus the number of elements of o_l on whose K_0-spans $\mathbb{G}_m$ acts via μ_{1K_2} as the inverse of the identical character of $\mathbb{G}_m$ is u_l (resp. is $\frac{u_l}{2}$) if $l \leq s_0$ (resp. if $l > s_0$). Therefore the $\mathrm{Gal}(K_2/\mathbb{Q}_p)$-orbit of μ_{1K_2} factors through $\prod_{i=1}^m Z(\mathbf{GL}_{M_i}) = Z^0(\mathcal{G}_{K_2})$. Thus again the E-pair $(\mathcal{T}_{1B(k)}, \mu_1)$ of $\mathcal{C}$ satisfies the $\mathfrak{C}$ condition.

6.4.4. Remark. We assume that $\mathcal{G}^0$ is of D_4 Dynkin type. Then Subsubsection 6.3.4 extends automatically to the present context $\mathrm{rk}_{W(k)}(M_i) = 8$ of Case 2. The only difference: we have $M_i = O_i$.

6.5. End of the proof of 4.2. We recall that Subsection 6.1 achieves the reduction to Cases 1 and 2 of Subsections 6.3 and 6.4. Thus Theorem 4.2 (a) (resp. Theorem 4.2. (b)) follows from Subsubsections 6.3.2, 6.4.2, and 6.4.3 (resp. from Subsubsection 6.3.4). $\square$

6.5.1. Remark. The approach of Subsubsection 6.3.4 extends in many cases to the case when $\mathcal{C}$ is basic and $\mathcal{G}^0$ is of A_n Dynkin type. However, one has to deal not with only two sets $S_{+,j}$ and $S_{-,j}$ but with n analogue sets and therefore in general it is much harder to show the existence of corresponding numbers $c^+_{l,u,j}$, where $u \in \{1, \ldots, n\}$ is a third index. This is the reason why in Subsection 6.4 we dealt only with the basic context (and why in Theorem 9.6 below we will rely as well on [Zi1, Thm. 4.4]).

7. Proofs of Corollaries 4.3, 4.4, and 4.5

In this Section we will assume that $p > 3$ and that the semisimple group scheme $\mathcal{G}^{\mathrm{der}}$ is simply connected. In Subsections 7.1, 7.2, and 7.3 we prove the Corollaries 4.3, 4.4, and 4.5 (respectively). If $[\mathcal{C}_g] \in Z^{\mathrm{ram}}(\mathcal{Y}(\mathcal{F}))$, then by performing $\mathfrak{D}_1$ we can assume that there exists a maximal torus of $\mathcal{G}_{B(k)}$ of $\mathbb{Q}_p$-endomorphisms of $\mathcal{C}_g$ (see Remark 3.3.2 (a) and Corollary 3.6.1). Thus as in Subsection 5.1 we get that there exists $m \in \mathbb{N}$ such that $(g\phi)^{rm}$ is a semisimple element of $\mathcal{G}(B(k))$. Therefore the element $(g\phi)^r \in \mathcal{G}(B(k))$ is also semisimple i.e., $\mathcal{C}_g$ is semisimple. Thus to end the proofs of Corollaries 4.3 to 4.5, we only have to show that under the assumptions of either Corollary 4.3 or Corollary 4.4 (resp. Corollary 4.5) we have $[\mathcal{C}] \in Z^{\mathrm{ram}}(\mathcal{Y}(\mathcal{F}))$ (resp. we have $[(\mathcal{C}, \lambda_M)] \in Z^{\mathrm{ram}}(\mathcal{Y}(\mathcal{F}, \lambda_M)))$ provided $\mathcal{C}$ is semisimple.

7.1. Proof of 4.3. We can assume that there exists a subset $\mathcal{J}_0$ of $\mathcal{J}$ such that $(t_\alpha)_{\alpha \in \mathcal{J}_0}$ is the family of tensors of $\mathcal{T}(M_{\mathbb{Z}_p})$ formed by the following three types of tensors:

(i) all elements of $\mathrm{Lie}(C_{\mathbb{Z}_p}) := \{e \in \mathrm{Lie}(C) | \phi(e) = e\}$;

(ii) the projector Π of Subsubsection 2.4.4 (it is also a projector of $\mathrm{End}(M_{\mathbb{Z}_p})$ as the trace form $\mathfrak{T}$ is perfect);

(iii) the endomorphism $\mathrm{End}(M_{\mathbb{Z}_p}) \to \mathrm{End}(M_{\mathbb{Z}_p})^*$ whose restriction to $\mathrm{Ker}(\Pi)$ is 0 and which induces the isomorphism $\mathrm{Lie}(\mathcal{G}^{\mathrm{der}}_{\mathbb{Z}_p}) \tilde{\to} \mathrm{Lie}(\mathcal{G}^{\mathrm{der}}_{\mathbb{Z}_p})^*$ defined naturally by $\mathfrak{K}$.

The family of tensors formed by the tensors of (i) (resp. of (ii) and (iii)) is $\mathbb{Z}_p$-very well positioned for $Z^0(\mathcal{G}_{\mathbb{Z}_p})$ (resp. for $\mathcal{G}^{\mathrm{der}}_{\mathbb{Z}_p}$), cf. [Va1, Subsubsection 4.3.13] (resp. [Va1, Prop. 4.3.10 b) and Rm. 4.3.10.1 1)]). Thus from [Va1, Rm. 4.3.6 2)] we get that the family $(t_\alpha)_{\alpha\in\mathcal{J}_0}$ of tensors of $\mathcal{T}(M_{\mathbb{Z}_p})$ is $\mathbb{Z}_p$-very well positioned for $\mathcal{G}_{\mathbb{Z}_p}$. As the tori $Z^0(\mathcal{G}_{\mathbb{Q}_p})$ and $\mathcal{G}^{\mathrm{ab}}_{\mathbb{Q}_p}$ are isomorphic (cf. our hypothesis on the isogeny $Z^0(\mathcal{G}) \to \mathcal{G}^{\mathrm{ab}}$), the set $H^1(\mathbb{Q}_p, \mathcal{G}_{\mathbb{Q}_p})$ has only one class. This follows from Lemma 2.4.6 (with the notations of Lemma 2.4.6 we have $\mathcal{G}_{\mathbb{Q}_p} = ZG_{\mathbb{Q}_p}$, as $Z^0(\mathcal{G}) = Z^0(C_1)$).

To prove Corollary 4.3, we can assume that $L^0_{\mathcal{G}}(\phi)$ is a reductive group scheme (cf. Subsection 2.6). We can also assume that there exists a maximal torus $\mathcal{T}_{1B(k)}$ of $\mathbb{Q}_p$-endomorphisms of $(M, \phi, L^0_{\mathcal{G}}(\phi))$. Let $(\mathcal{T}_{1B(k)}, \mu_1)$ be an E-pair of $(M, \phi, L^0_{\mathcal{G}}(\phi))$ that satisfies the $\mathfrak{C}$ condition, cf. Theorem 4.2 (b). Thus the E-pair $(\mathcal{T}_{1B(k)}, \mu_1)$ of $\mathcal{C}$ is admissible, cf. Theorem 4.1 (b). It is plus plus admissible, cf. previous paragraph. Thus as $Z^0(\mathcal{G}) = Z^0(C_1)$, Corollary 4.3 follows from Theorem 4.1 (c). □

7.2. Proof of 4.4. The proof of Corollary 4.4 is very much the same as the proof of Corollary 4.3. Only the argument for the plus plus admissibility part has to be changed slightly. Let $\mathcal{H}$ be a split, simply connected semisimple group scheme of D_n Lie type over a field of characteristic 0.

If n is even, then $Z(\mathcal{H})$ is $\mu_2 \times \mu_2$. Moreover we can assume that the kernel of the first (resp. second) half spin representation of $\mathcal{H}$ is the first (resp. the second) factor of this product. Thus the isogeny $Z^0(\mathcal{G}_{\mathbb{Q}_p}) \to \mathcal{G}^{\mathrm{ab}}_{\mathbb{Q}_p}$ is the square isogeny $2 : Z^0(\mathcal{G}_{\mathbb{Q}_p}) \to Z^0(\mathcal{G}_{\mathbb{Q}_p})$. Thus as in Subsection 7.1 we argue that the set $H^1(\mathbb{Q}_p, \mathcal{G}_{\mathbb{Q}_p})$ has only one class.

If n is odd, then the half spin representations of $\mathcal{H}$ have trivial kernels and are dual to each other (see [Bou2, p. 210]). Thus based on our hypothesis on $Z^{00}(\mathcal{G})$, the isogeny $Z^{00}(\mathcal{G}_{\mathbb{Q}_p}) \to \tilde{\mathcal{G}}^{\mathrm{ab}}_{\mathbb{Q}_p}$ is the square isogeny $2 : Z^{00}(\mathcal{G}_{\mathbb{Q}_p}) \to Z^{00}(\mathcal{G}_{\mathbb{Q}_p})$. Here $\tilde{\mathcal{G}}_{\mathbb{Q}_p}$ and $Z^{00}_{\mathbb{Q}_p}$ are the subgroups of $\mathcal{G}_{\mathbb{Q}_p}$ which are the $\mathbb{Q}_p$ forms of $\tilde{\mathcal{G}}_{B(k)}$ and $Z^{00}(\mathcal{G}_{B(k)})$ (respectively) obtained as in Subsection 2.4. The torus $Z^{00}(\mathcal{G}_{\mathbb{Q}_p})$ is the group scheme of invertible elements of an étale $\mathbb{Q}_p$-algebra. Thus as in Subsection 7.1 we argue that the set $H^1(\mathbb{Q}_p, \tilde{\mathcal{G}}_{\mathbb{Q}_p})$ has only one class. But the class $\mathfrak{L}$ of Subsubsection 2.4.5 is the image of a class $\mathfrak{L}_1 \in H^1(\mathbb{Q}_p, \tilde{\mathcal{G}}_{\mathbb{Q}_p})$. This is a consequence of the fact that $\tilde{\mathcal{G}}_{1\mathbb{Q}_p}$ is the subgroup of $\mathbf{GL}_{M_{\mathbb{Q}_p}}$ that fixes a family of tensors $(t_\alpha)_{\alpha\in\tilde{\mathcal{J}}}$ of $\mathcal{T}(M_{\mathbb{Q}_p})$, cf. [De3, Prop. 3.1 c)]. Therefore $\mathfrak{L}$ is the trivial class.

Regardless of the parity of n, as in Subsection 7.1 we argue that there exists an E-pair $(\mathcal{T}_{1B(k)}, \mu_1)$ of $\mathcal{C}$ which is plus plus admissible. Thus as $Z^0(\mathcal{G}) = Z^0(C_1)$, Corollary 4.4 follows from Theorem 4.1 (c). □

7.3. Proof of 4.5. The proof of Corollary 4.5 is the same as the proof of Corollary 4.3. As the group $H^1(\mathbb{Q}_p, \mathcal{G}^{0\mathrm{ad}}_{\mathbb{Q}_p})$ is trivial and as $\mathcal{G}^{0\mathrm{der}}_{\mathbb{Q}_p}$ is simply connected, the set $H^1(\mathbb{Q}_p, \mathcal{G}^0_{\mathbb{Q}_p})$ has only one class. Therefore each E-pair $(\mathcal{T}_{1B(k)}, \mu_1)$ of $\mathcal{C}$ which is admissible, is in fact plus plus admissible with respect to λ_M. The rest is the same, only the reference to Theorem 4.1 (c) has to be supplemented by the variant 5.3.3 (a). □

8. First applications to abelian varieties

Pairs of the form $(\ddagger, \lambda_\ddagger)$ will denote polarized abelian schemes. By abuse of notations, we also denote by $\lambda_\ddagger$ the different forms on the cohomologies (or homologies) of $\ddagger$ induced by $\lambda_\ddagger$. We now apply the results 4.1, 4.2, and 5.3.3 to the geometric context of Subsubsections 1.1.1 and 1.2. Applications to Conjecture 1.2.2 (i) and to Subproblems 1.2.3 and 1.2.4 are included in Corollary 8.3 and Remark 8.4. If $\triangle$ is an algebra, let $\triangle^{\mathrm{opp}}$ be its opposite algebra.

8.1. Geometric setting. Until the end we assume that D is the p-divisible group of an abelian variety A over k, that $\mathcal{C} = (M, \phi, \mathcal{G})$ is a Shimura filtered F-crystal over k such that axioms 2.4.1 (i) and (ii) hold, and that there exists a polarization λ_A of A whose crystalline realization (denoted in the same way) $\lambda_A : M \otimes_{W(k)} M \to W(k)$ has a $W(k)$-span normalized by $\mathcal{G}$. Let F^1 and μ be as in Subsection 2.1. Let $\mathcal{G}_{\mathbb{Z}_p}$ be as in Subsection 2.4. By performing the operation $\mathfrak{O}_1$ we can assume that the Zariski closure $\mathcal{T}(\phi)$ of the group $\{\phi^{rm} | m \in \mathbb{Z}\}$ in $\mathcal{G}_{B(k)}$ is a torus over $B(k)$. This implies that we have an identity $\mathrm{End}(A) = \mathrm{End}(A_{\bar{k}})$. We identify

$$E_A := \mathrm{End}(A)^{\mathrm{opp}} \otimes_{\mathbb{Z}} \mathbb{Q}$$

with a $\mathbb{Q}$–subalgebra of $\{x \in \mathrm{End}(M[\frac{1}{p}]) | \phi(x) = x\}$.

Let Π be as in Subsection 2.4.4. Let $\mathfrak{e}_{\mathbb{Q}_p} := (E_A \otimes_{\mathbb{Q}} \mathbb{Q}_p) \cap \mathrm{Im}(\Pi)$. Let $\mathfrak{e}^{\perp}_{\mathbb{Q}_p} := (E_A \otimes_{\mathbb{Q}} \mathbb{Q}_p) \cap \mathrm{Ker}(\Pi)$. As Π is fixed by ϕ, we have a direct sum decomposition

$$E_A \otimes_{\mathbb{Q}} \mathbb{Q}_p = \mathfrak{e}_{\mathbb{Q}_p} \oplus \mathfrak{e}^{\perp}_{\mathbb{Q}_p}$$

of $\mathbb{Q}_p$-vector spaces. Let $C(\phi)_{\mathbb{Q}}$ be the reductive group over $\mathbb{Q}$ of invertible elements of E_A; thus $\mathrm{Lie}(C(\phi)_{\mathbb{Q}})$ is the Lie algebra associated to E_A. A classical theorem of Tate says that $C(\phi) := C(\phi)_{B(k)}$ is the centralizer of ϕ^r in $\mathrm{End}(M[\frac{1}{p}])$. Thus $\mathfrak{e} := \mathfrak{e}_{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} B(k)$ is the Lie algebra of the centralizer $C_{\mathcal{G}_{B(k)}}(\phi)$ of ϕ^r in $\mathcal{G}_{B(k)}$. Let E^1_A be a semisimple $\mathbb{Q}$–subalgebra of E_A which (inside $\mathrm{End}(M[\frac{1}{p}])$) is formed by elements fixed by $\mathcal{G}_{B(k)}$ and which is stable under the involution of E_A defined naturally by λ_A.

8.2. Lemma. *Let $\mathcal{T}_{1B(k)}$ be a maximal torus of $\mathcal{G}_{B(k)}$ of $\mathbb{Q}_p$-endomorphisms of $\mathcal{C}$. Then there exists a maximal torus $\mathcal{T}^{\mathrm{big}}_{1B(k)}$ of $\boldsymbol{GL}_{M[\frac{1}{p}]}$ of $\mathbb{Q}_p$-endomorphisms of $(M, \phi, \boldsymbol{GL}_M)$ and there exists an element $u \in C(\phi)_{\mathbb{Q}}(\mathbb{Q}_p)$ such that the following four conditions hold:*

(i) *the element u normalizes M (i.e., we have $u(M) = M$) as well as any a priori fixed $W(k)$-lattice of $M[\frac{1}{p}]$;*

(ii) *the torus $u\mathcal{T}^{\mathrm{big}}_{1B(k)}u^{-1}$ is the extension to $B(k)$ of a maximal torus of $C(\phi)_{\mathbb{Q}}$;*

(iii) *the element u fixes λ_A and each element of E^1_A;*

(iv) *we have $\mathcal{T}_{1B(k)} = Z^0(\mathcal{T}^{\mathrm{big}}_{1B(k)} \cap \mathcal{G}_{B(k)})$.*

Proof: Let $C(\phi)^1_{\mathbb{Q}}$ be the identity component of the subgroup of $C(\phi)_{\mathbb{Q}}$ that normalizes the $\mathbb{Q}$–span of λ_A and that centralizes E^1_A. It is a reductive group over $\mathbb{Q}$. Let $\mathcal{T}^1_{\mathbb{Q}_p}$ be a maximal torus of $C(\phi)^1_{\mathbb{Q}_p}$ that contains the $\mathbb{Q}_p$-form $\mathcal{T}_{1\mathbb{Q}_p}$ of $\mathcal{T}_{1B(k)}$ with respect to $(M[\frac{1}{p}], \phi)$. From [Ha, Lem. 5.5.3] we deduce the existence of an element $u \in C(\phi)^1_{\mathbb{Q}}(\mathbb{Q}_p)$ such that the condition (i) holds and $u\mathcal{T}^1_{\mathbb{Q}_p}u^{-1}$ is the extension to $\mathbb{Q}_p$ of a maximal torus $\mathcal{T}^1_{\mathbb{Q}}$ of $C(\phi)^1_{\mathbb{Q}}$. We choose such an element u which also fixes λ_A. Thus the condition (iii) holds. Let $\mathcal{T}^{1\text{big}}_{\mathbb{Q}}$ be a maximal torus of $C(\phi)_{\mathbb{Q}}$ that contains $\mathcal{T}^1_{\mathbb{Q}}$. The conditions (ii) and (iv) hold for $\mathcal{T}^{\text{big}}_{1B(k)} := u^{-1}\mathcal{T}^{1\text{big}}_{B(k)}u$. $\square$

We have the following geometric consequences of Theorem 4.1 (c) and variant 5.3.3 (a).

8.3. Corollary. *We assume that $p \geq 3$ and that $Q++\mathfrak{A}$ holds for $\mathcal{C}$. We also assume that there exists a subset $\mathcal{J}_0$ of $\mathcal{J}$ such that the family $(t_\alpha)_{\alpha\in\mathcal{J}_0}$ of tensors of $\mathcal{T}(M_{\mathbb{Z}_p})$ is of partial degrees at most $p-2$ and is $\mathbb{Z}_p$-very well position for $\mathcal{G}_{\mathbb{Z}_p}$.*

(a) *Then by performing the operation $\mathfrak{O}_1$ we can assume there exist an element $h \in \mathfrak{P}(\mathcal{C})$ and an abelian variety $A(h)$ over k such that the following two conditions hold:*

(i) *the abelian variety $A(h)$ is $\mathbb{Z}[\frac{1}{p}]$-isogenous to A and, under this $\mathbb{Z}[\frac{1}{p}]$-isogeny, the Dieudonné module of its p-divisible group is $(h(M), \phi)$ and is a direct sum of F-crystals over k that have only one Newton polygon slope;*

(ii) *there exists an abelian scheme $A(h)_{V_3}$ with complex multiplication over a finite, totally ramified discrete valuation ring extension V_3 of $W(k)$ which is a ramified lift of $A(h)$ to V_3 with respect to the Zariski closure $\tilde{\mathcal{G}}(h)$ of $\tilde{\mathcal{G}}_{B(k)}$ in $\boldsymbol{GL}_{h(M)}$, where $\tilde{\mathcal{G}}$ is a $\boldsymbol{GL}_M(W(k))$-conjugate of $\mathcal{G}$ such that the triple $(M, \phi, \tilde{\mathcal{G}})$ is a Shimura F-crystal over k.*

(b) *We also assume that the polarization λ_A is of degree prime to p, that $Z^0(\mathcal{G}) = Z^0(C_1(\lambda_A)^0)$, and that $Q++\mathfrak{A}$ holds for $(\mathcal{C}, \lambda_A)$. Then by performing the operation $\mathfrak{O}_1$ we can assume that there exists an element $h \in \mathfrak{I}(\mathcal{C}, \lambda_A)$ and an abelian variety $A(h)$ over k such that the condition (i) and the following new condition (iii) hold:*

(iii) *there exists an abelian scheme $A(h)_{V_3}$ over a finite, totally ramified discrete valuation ring extension V_3 of $W(k)$ which lifts $A(h)$ in such a way that the Frobenius endomorphism of $A(h)$ also lifts to it, which is a ramified lift of $A(h)$ to V_3 with respect to $\mathcal{G}(h)$, and whose p-divisible group $D(h)_{V_3}$ is with complex multiplication.*

Proof: We can assume that $\mathcal{C}$ is basic, cf. Corollary 2.6.2 and Fact 2.6.3. Let $(\mathcal{T}_{1B(k)}, \mu_1)$ be an E-pair of $\mathcal{C}$ which is plus plus admissible. We first proof (a). Let $\mathcal{T}^{\text{big}}_{1B(k)}$ and u be as in Lemma 8.2; thus u is fixed by ϕ. Let $\tilde{\mathcal{G}}$, $\tilde{\mathcal{T}}^{\text{big}}_{1B(k)}$, $\tilde{\mathcal{T}}_{1B(k)}$, $\tilde{\mu}_1$, and $(\tilde{t}_\alpha)_{\alpha\in\mathcal{J}}$ be the inner conjugates of $\mathcal{G}$, $\mathcal{T}^{\text{big}}_{1B(k)}$, $\mathcal{T}_{1B(k)}$, μ_1, and $(t_\alpha)_{\alpha\in\mathcal{J}}$ (respectively) through the element $u \in \mathbf{GL}_M(B(k))$. Let $\tilde{\mathcal{C}} := (M, \phi, \tilde{\mathcal{G}})$. The Lie algebra $\text{Lie}(\tilde{\mathcal{T}}^{\text{big}}_{1B(k)})$ is $B(k)$-generated by elements of E_A and $(\tilde{\mathcal{T}}_{1B(k)}, \tilde{\mu}_1)$ is an E-pair of $\tilde{\mathcal{C}}$ which is plus plus admissible.

We apply the proof of Theorem 4.1 (c) to $\tilde{\mathcal{C}}$ and $(\tilde{\mathcal{T}}_{1B(k)}, \tilde{\mu}_1)$ (see Subsection 5.3). We deduce the existence of an element $h \in \mathfrak{P}(\tilde{\mathcal{C}})$ such that the p-divisible group $D(h)$ over k

whose Dieudonné module is $(h(M), \phi)$ has a lift $D(h)_{V_3}$ to a finite, discrete valuation ring extension V_3 of $W(k)$ such that each endomorphism of $D(h)$ whose crystalline realization is an element of $\mathrm{Lie}(\tilde{\mathcal{T}}^{\mathrm{big}}_{1B(k)})$ fixed by ϕ lifts to an endomorphism of $D(h)_{V_3}$ (cf. also property 5.3.3 (b)). Let $A(h)$ be the abelian variety over k defined by the condition (i). Let $A(h)_{V_3}$ be the abelian scheme over V_3 defined by $D(h)_{V_3}$, cf. Serre–Tate deformation theory. The fact that $A(h)_{V_3}$ is indeed an abelian scheme (and not only a formal abelian scheme over $\mathrm{Spf}(V_3)$) is implied by the fact that we are in a polarized context, cf. variant 5.3.3 (a) and property 8.2 (iii). By performing the operation $\mathfrak{O}_1$ we can assume that V_3 is a totally ramified extension of $W(k)$. We can also assume that $A(h)_{V_3}$ is a ramified lift of $A(h)$ to V_3 with respect to $\tilde{\mathcal{G}}(h)$, cf. Corollary 3.7.1 and Subsubsection 5.3.1. As $\tilde{\mathcal{C}}$ is basic, the part of the condition (i) on F-crystals over k holds (to be compared with [Va3, Subsection 4.1]). Thus the condition (i) holds. This proves (a).

To prove (b) we first remark that $\lambda_A : M \otimes_{W(k)} M \to W(k)$ is a principal quasi-polarization of $\mathcal{C}$. Part (b) follows from the proof of Theorem 4.1 (c) applied in the context of an E-pair $(\mathcal{T}_{1B(k)}, \mu_1)$ of $\mathcal{C}$ which is plus plus admissible with respect to λ_A. We get the existence of an element $h \in \mathfrak{I}(\mathcal{C}, \lambda_A)$, of a finite, discrete valuation ring extension V_3 of $W(k)$, and of a p-divisible group $D(h)_{V_3}$ over V_3 that is a ramified lift of $D(h)$ to V_3 with respect to $(h(M), \phi, \mathcal{G}(h))$ and that has the property that each endomorphism of $D(h)$ whose crystalline realization is an element of $\mathrm{Lie}(\mathcal{T}_{1B(k)}) \cap \mathrm{End}(M)$ fixed by ϕ lifts to an endomorphism of $D(h)_{V_3}$ (cf. Subsubsections 5.3.1, 5.3.2, and 5.3.3 (b)). By performing the operation $\mathfrak{O}_1$ we can assume that V_3 is a totally ramified extension of $W(k)$. Let $A(h)$ and $A(h)_{V_3}$ be obtained as above. From the property 5.3.3 (b) we get that the p-divisible group $D(h)_{V_3}$ is with complex multiplication. As $\phi^r \in \mathcal{T}_{1B(k)}(B(k))$ leaves invariant $h(M)$, the Frobenius endomorphism of $A(h)$ lifts to $A(h)_{V_3}$ (cf. also property 5.3.3 (b) and Serre–Tate deformation theory). As above we argue that the condition (i) holds. As $D(h)_{V_3}$ is a ramified lift of $D(h)$ to V_3 with respect to $(h(M), \phi, \mathcal{G}(h))$, the abelian scheme $A(h)_{V_3}$ is a ramified lift of $A(h)$ to V_3 with respect to $\mathcal{G}(h)$. $\square$

8.4. Remark. Corollary 8.3 (b) is our partial solution to Subproblems 1.2.3 and 1.2.4. We refer to Corollary 8.3 (a). If $\mathrm{Lie}(\mathcal{T}_{1B(k)})$ is $B(k)$-generated by elements of $\mathfrak{e}_{\mathbb{Q}}$, then we can take u to be 1_M. If moreover the assumptions of Corollary 8.3 (b) hold, then the condition (ii) holds with $h \in \mathfrak{I}(\mathcal{C}, \lambda_A)$ and thus with $\tilde{\mathcal{G}}(h) = \mathcal{G}(h)$. This solves Conjecture 1.2.2 (i) under all assumptions of Corollary 8.3.

9. The context of standard Hodge situations

If $(G, \mathcal{X})$ is a Shimura pair, let $E(G, \mathcal{X})$ be the subfield of $\mathbb{C}$ which is its reflex field, let $(G^{\mathrm{ad}}, \mathcal{X}^{\mathrm{ad}})$ be its adjoint Shimura pair, and let $\mathrm{Sh}(G, \mathcal{X})$ be the canonical model over $E(G, \mathcal{X})$ of $\mathrm{Sh}(G, \mathcal{X})$ (see [De1], [De2], [Mi3, Subsections 1.1 to 1.8], and [Va1, Subsections 2.2 to 2.8]). Let $\mathrm{Sh}(G, \mathcal{X})/\mathcal{K}$ be the quotient of $\mathrm{Sh}(G, \mathcal{X})$ by a compact subgroup $\mathcal{K}$ of $G(\mathbb{A}_f)$. See [Va1, Subsection 2.4] for injective maps between Shimura pairs. For general properties of Shimura varieties of PEL type we refer to [Zi1], [LR], [Ko2, Ch. 5], [Mi3, p. 161], and [RaZ] (we emphasize that in [Mi3, p. 161] one has to add that the axiom [De2, 2.1.1.3] holds). The injective maps in Siegel modular varieties that define Shimura varieties of PEL type as used in these references, will be referred as *PEL type embeddings*. Let $O_{(w)}$ be the localization of the ring of integers of a number field with respect to a finite prime w of it.

In Subsection 9.1 we mainly introduce notations and a setting. Different "properties" pertaining to the setting of Subsection 9.1 are introduced in Subsection 9.2. In Subsections 9.3 to 9.6 we prove results which fully support the point of view that the two things 1.4 (i) and (ii) are indeed the last ingredients required to complete the proof of the Langlands–Rapoport conjecture for $p \geq 5$ and for Shimura varieties of A_n, B_n, C_n, and $D_n^{\mathbb{R}}$ type (cf. also Remark 9.8 (b)). The main results are Theorems 9.4, 9.5.1, and 9.6. Not to make this paper too long, we include only one example (see Example 9.7) of how the new techniques of Subsections 9.2 to 9.6 apply to Shimura varieties of Hodge type which are not of PEL type (and thus to which the techniques of [Zi1] and [Ko2] do not apply).

9.1. Standard Hodge situation. We recall part of the setting of [Va3, Section 5] pertaining to good reduction cases of Shimura varieties of Hodge type. We start with an injective map

$$f : \mathrm{Sh}(G, \mathcal{X}) \hookrightarrow \mathrm{Sh}(\mathbf{GSp}(W, \psi), \mathcal{S})$$

of Shimura pairs. Here the Shimura pair $(\mathbf{GSp}(W, \psi), \mathcal{S})$ defines a Siegel modular variety, cf. Subsubsection 1.2.1. We consider a $\mathbb{Z}$-lattice L of W such that ψ induces a perfect form $\psi : L \otimes_{\mathbb{Z}} L \to \mathbb{Z}$. Let $L_{(p)} := L \otimes \mathbb{Z}_{(p)}$. Until the end we will assume that:

the Zariski closure $G_{\mathbb{Z}_{(p)}}$ of G in $\boldsymbol{GSp}(L_{(p)}, \psi)$ is a reductive group scheme over $\mathbb{Z}_{(p)}$.

It is easy to see that the group scheme $G^0_{\mathbb{Z}_{(p)}} := G_{\mathbb{Z}_{(p)}} \cap \mathbf{Sp}(L_{(p)}, \psi)$ is reductive (cf. [Va3, Subsection 5.1, Formula (11)]). Let $\mathcal{K}_p := \mathbf{GSp}(L_{(p)}, \psi)(\mathbb{Z}_p)$; it is a hyperspecial subgroup of $\mathbf{GSp}(W, \psi)_{\mathbb{Q}_p}(\mathbb{Q}_p)$. As $G_{\mathbb{Z}_{(p)}}$ is a reductive group scheme over $\mathbb{Z}_{(p)}$, the intersection $H := G_{\mathbb{Q}_p}(\mathbb{Q}_p) \cap \mathcal{K}_p$ is a hyperspecial subgroup of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$. Let v be a prime of the reflex field $E(G, \mathcal{X})$ that divides p; it is unramified over p (cf. [Mi4, Prop. 4.6 and Cor. 4.7]). Let $k(v)$ be the residue field of v. Let $r := \frac{\dim_{\mathbb{Q}}(W)}{2} \in \mathbb{N}$. Let $\mathbb{A}_f$ (resp. $\mathbb{A}_f^{(p)}$) be the $\mathbb{Q}$–algebra of finite adèles (resp. of finite adèles with the p-component omitted). We have an identity $\mathbb{A}_f = \mathbb{A}_f^{(p)} \times \mathbb{Q}_p$.

For integral canonical models of (suitable quotients of) Shimura varieties we refer to [Va1, Subsubsections 3.2.3 to 3.2.6]. It is well known that the $\mathbb{Z}_{(p)}$-scheme $\mathcal{M}$ that parameterizing isomorphism classes of principally polarized abelian schemes of relative dimension r over $\mathbb{Z}_{(p)}$-schemes which have compatible level-N symplectic similitude structures for all natural numbers N relatively prime to p, together with the natural action of $\mathbf{GSp}(W, \psi)(\mathbb{A}_f^{(p)})$ on it, is an integral canonical model of $\mathrm{Sh}(\mathbf{GSp}(W, \psi), \mathcal{S})/\mathcal{K}_p$ (for instance, see [De1, Thm. 4.21] and [Va1, Ex. 3.2.9 and Subsection 4.1]). These structures and this action are defined naturally via the $\mathbb{Z}$-lattice L of W (see [Va1, Subsection 4.1]). It is known that $\mathrm{Sh}(G, \mathcal{X})/H$ is a closed subscheme of $\mathcal{M}_{E(G,\mathcal{X})} = \mathrm{Sh}(\mathbf{GSp}(W, \psi), \mathcal{S})_{E(G,\mathcal{X})}/\mathcal{K}_p$, cf. [Va1, Rm. 3.2.14].

Let $\mathcal{N}$ be the normalization of the Zariski closure of $\mathrm{Sh}(G, \mathcal{X})/H$ in $\mathcal{M}_{O_{(v)}}$. Let $(\mathcal{A}, \Lambda_{\mathcal{A}})$ be the pull back to $\mathcal{N}$ of the universal principally polarized abelian scheme over $\mathcal{M}$. Let $(v_\alpha)_{\alpha \in \mathcal{J}}$ be a family of tensors of $\mathcal{T}(L^*_{(p)})$ such that G is the subgroup of $\mathbf{GL}_W$

that fixes v_α for all $\alpha \in \mathcal{J}$ (cf. [De3, Prop. 3.1 c)]). As G contains $Z(\mathbf{GL}_W)$, we have $v_\alpha \in \oplus_{n=0}^{\infty} L_{(p)}^{*\otimes n} \otimes_{\mathbb{Z}_{(p)}} L_{(p)}^{\otimes n}$ for all $\alpha \in \mathcal{J}$, The choice of L and $(v_\alpha)_{\alpha\in\mathcal{J}}$ allows a moduli interpretation of $\mathrm{Sh}(G,\mathcal{X})$ (see [De1], [De2], [Mi4], and [Va1, Subsection 4.1 and Lem. 4.1.3]). For instance, the set $\mathrm{Sh}(G,\mathcal{X})/H(\mathbb{C})$ is naturally identified with $G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)})\backslash(\mathcal{X}\times G(\mathbb{A}_f^{(p)}))$ (see [Mi4, Prop. 4.11 and Cor. 4.12]) and therefore it is the set of isomorphism classes of complex principally polarized abelian varieties of dimension r that carry a family of Hodge cycles indexed by the set $\mathcal{J}$, that have compatible level-N symplectic similitude structures for all natural numbers N relatively prime to p, and that satisfy certain axioms (see [Va1, Subsection 4.1]). This moduli interpretation endows naturally the abelian scheme $\mathcal{A}_{E(G,\mathcal{X})}$ with a family $(w_\alpha^{\mathcal{A}})_{\alpha\in\mathcal{J}}$ of Hodge cycles (the Betti realizations of pull backs of $w_\alpha^{\mathcal{A}}$ via $\mathbb{C}$-valued points of $\mathcal{N}_{E(G,\mathcal{X})}$ correspond naturally to v_α).

Let H_0 be a compact, open subgroup of $G(\mathbb{A}_f^{(p)})$ that has the following three properties:

(a) there exists $N_0 \in \mathbb{N}$ such that $(N_0,p)=1$, $N_0 \geq 3$, and we have an inclusion

$$H_0 \times H \subseteq \mathcal{K}(N_0) := \{g \in \mathbf{GSp}(L,\psi)(\hat{\mathbb{Z}}) | g \equiv 1_{L\otimes_{\mathbb{Z}}\hat{\mathbb{Z}}} \,\mathrm{modulo}\, N_0\};$$

(b) the triple $\mathcal{R} := (\mathcal{A}, \Lambda_{\mathcal{A}}, (w_\alpha^{\mathcal{A}})_{\alpha\in\mathcal{J}})$ is the pull back of an analogue triple $\mathcal{R}(H_0) = (\mathcal{A}_{H_0}, \Lambda_{\mathcal{A}_{H_0}}, (w_\alpha^{\mathcal{A}_{H_0}})_{\alpha\in\mathcal{J}})$ over $\mathcal{N}/H_0$, where $(\mathcal{A}_{H_0}, \Lambda_{\mathcal{A}_{H_0}})$ is the pull back via the natural morphism $\mathcal{N}/H_0 \to \mathcal{M}/\mathcal{K}^p(N_0)$ of the universal principally polarized abelian scheme over $\mathcal{M}/\mathcal{K}^p(N_0)$ (here $\mathcal{K}^p(N_0)$ is the unique subgroup of $\mathbf{GSp}(W,\psi)(\mathbb{A}_f^{(p)})$ such that we have $\mathcal{K}(N_0) = \mathcal{K}^p(N_0) \times \mathcal{K}_p$);

(c) the scheme $\mathcal{N}$ is a pro-étale cover of $\mathcal{N}/H_0$ (cf. [Va1, Prop. 3.4.1]).

9.1.1. Some notations. Let $k=\mathbb{F}_q$ be a finite field that contains $k(v)$. We consider a $W(k)$-morphism $z : \mathrm{Spec}(W(k)) \to \mathcal{N}/H_0$. Let

$$(A_{W(k)}, \lambda_{A_{W(k)}}, (w_\alpha)_{\alpha\in\mathcal{J}}) = z^*(\mathcal{R}(H_0)).$$

Let $y : \mathrm{Spec}(k) \to \mathcal{N}_{k(v)}/H_0$ and (A,λ_A) be the special fibres of z and $(A_{W(k)}, \lambda_{A_{W(k)}})$ (respectively). Let (M,ϕ,λ_A) be the principally quasi-polarized Dieudonné module of (A,λ_A). Let F^1 be the Hodge filtration of M defined by $A_{W(k)}$. For $\alpha\in\mathcal{J}$ let $t_\alpha \in \mathcal{T}(M[\frac{1}{p}])$ be the de Rham component of the Hodge cycle w_α on $A_{W(k)}$. Let $\tilde{\mathcal{G}}$ be the Zariski closure in $\mathbf{GL}_M$ of the subgroup of $\mathbf{GL}_{M[\frac{1}{p}]}$ that fixes t_α for all $\alpha \in \mathcal{J}$. Until the end we will assume that the triple (f,L,v) is a *standard Hodge situation* in the sense of [Va3, Def. 5.1.2]. Therefore the following two properties hold:

(a) the $O_{(v)}$-scheme $\mathcal{N}/H_0$ is smooth;

(b) for each point $z \in \mathcal{N}/H_0(W(k))$, $\tilde{\mathcal{G}}$ is a reductive, closed subgroup scheme of $\mathbf{GL}_M$ and the triple $(M,\phi,\tilde{\mathcal{G}})$ is a Shimura F-crystal over k.

Let $\mathcal{G} := G_{W(k)}$ and $\mathcal{G}^0 := G^0_{W(k)}$. Each tensor t_α is fixed under the natural action of ϕ on $\mathcal{T}(M[\frac{1}{p}])$, cf. [Va3, Cor. 5.1.6]. By performing the operation $\mathfrak{D}_1$ we can assume that $\tilde{\mathcal{G}}$ is isomorphic to $\mathcal{G}$. By multiplying each v_α by a fixed integral power of p we can assume that for all points $z \in \mathcal{N}/H_0(W(k))$ we have $t_\alpha \in \mathcal{T}(M)$ for all $\alpha \in \mathcal{J}$. To match the notations with those of Sections 1 to 8, we will identify (non-canonically) $\tilde{\mathcal{G}} = \mathcal{G}$. Thus let $\mathcal{C} := (M, \phi, \mathcal{G})$. Obviously the axiom 2.4.1 (i) holds for $\mathcal{C}$. The fact that the axiom 2.4.1 (ii) holds for $\mathcal{C}$ is implied by [De2, axiom 2.1.1.3]. The triple $\mathcal{C}$ depends only on y and not on z (cf. [Va3, paragraph before Subsubsection 5.1.7] and therefore we call it the *Shimura F-crystal attached* to the point $y \in \mathcal{N}_{k(v)}/H_0(k)$. Let $z_\infty : \mathrm{Spec}(W(\bar{k})) \to \mathcal{N}$ be such that the resulting $W(\bar{k})$-valued point of $\mathcal{N}/H_0$ factors through z. We refer to $\mathcal{C} \otimes \bar{k}$ as the Shimura F-crystal attached to the special fibre $y_\infty : \mathrm{Spec}(\bar{k}) \to \mathcal{N}_{k(v)}$ of z_∞. We also refer to y_∞ (resp. z_∞) as an infinite lift of y (resp. of z). We also refer to F^1 as the lift of $\mathcal{C}$ defined by the point $z \in \mathcal{N}/H_0(W(k))$ that lifts $y \in \mathcal{N}_{k(v)}/H_0(k)$.

If we have another point $y_j \in \mathcal{N}/H_0(k)$, then (A_j, λ_{A_j}), $\mathcal{C}_j = (M_j, \phi_j, \mathcal{G}_j, \lambda_{A_j})$, $y_{j\infty}$, and $(t_{j\alpha})_{\alpha \in \mathcal{J}}$ will be the analogues of (A, λ_A), $\mathcal{C}$, y_∞, and $(t_\alpha)_{\alpha\in\mathcal{J}}$ obtained by replacing y with y_j.

9.1.2. PEL type embeddings. Let $C_{\mathbb{Q}} := C_{\mathbf{GL}_W}(G)$. Let G_1 be the identity component of $C_{1\mathbb{Q}} := \mathbf{GSp}(W, \psi) \cap C_{\mathbf{GL}_W}(C_{\mathbb{Q}})$; it contains G. Let $\mathcal{X}_1$ be the $G_1(\mathbb{R})$-conjugacy class of homomorphisms $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_{1\mathbb{R}}$ that contain the composites of elements of $\mathcal{X}$ with the monomorphism $G_{\mathbb{R}} \hookrightarrow G_{1\mathbb{R}}$. We get a PEL type embedding $f_1 : (G_1, \mathcal{X}_1) \hookrightarrow (\mathbf{GSp}(W, \psi), \mathcal{S})$ through which f factors. We call it the *PEL-envelope* of f, cf. [Va1, Rm. 4.3.12].

Let $G_{2\mathbb{Z}_{(p)}} := C_{\mathbf{GSp}(L_{(p)}, \psi)}(Z^0(G_{\mathbb{Z}_{(p)}}))$; it is a reductive group scheme over $\mathbb{Z}_{(p)}$ (cf. [DG Vol. III, Exp. XIX, Subsection 2.8]). Let G_2 be the generic fibre of $G_{2\mathbb{Z}_{(p)}}$; it contains G_1 and moreover we have $Z^0(G_1) = Z^0(G_2)$. As in the previous paragraph we get an injective map $f_2 : (G_2, \mathcal{X}_2) \hookrightarrow (\mathbf{GSp}(W, \psi), \mathcal{S})$ through which both f and f_1 factor naturally.

Let $i \in \{1, 2\}$. Let $H_i := G_{i\mathbb{Q}_p}(\mathbb{Q}_p) \cap \mathcal{K}_p$. Let v_i be the prime of the subfield $E(G_i, \mathcal{X}_i)$ of $E(G, \mathcal{X})$ which is divided by v. Let $\mathcal{N}_i$ be the normalization of the Zariski closure of $\mathrm{Sh}(G_i, \mathcal{X}_i)/H_i$ in $\mathcal{M}_{O_{(v_i)}}$. By replacing H_0 with a compact, open subgroup of it we can assume that:

(a) there exists a compact, open subgroup H_{0i} of $G_i(\mathbb{A}_f^{(p)})$ which is contained in $\mathcal{K}^p(N_0)$, which contains H_0, and for which the quotient morphism $\mathcal{N}_i \to \mathcal{N}_i/H_{0i}$ is a pro-étale cover (see proof of [Va1, Prop. 3.4.1]).

We can also assume that H_{01} is a subgroup of H_{02}. The injective map f_i is a PEL type embedding. The triple $(f_2, L_{(p)}, v_2)$ is a standard Hodge situation (this well known fact follows from either [Zi1, Subsection 3.5] or [LR]). Let $(\mathcal{A}_i, \Lambda_{\mathcal{A}_i})$ be the pull back to $\mathcal{N}_i$ of the universal abelian scheme over $\mathcal{M}$. Let $\mathcal{G}_i$ be the integral, closed subgroup scheme of $\mathbf{GL}_M$ which has the analogue meaning of $\mathcal{G} = \tilde{\mathcal{G}}$ but obtained working with the k-valued point y_i of $\mathcal{N}_i/H_{0i}$ defined by y. The group scheme $\mathcal{G}_2$ is reductive.[1]

We use the notations of Subsection 8.1 and (by performing the operation $\mathfrak{O}_1$) until the end we will assume that $\mathcal{T}(\phi)$ is a torus. Let $E_{1A} := E_A \cap \mathrm{Lie}(\mathcal{G}_{1B(k)})$. Identifying the opposite of the $\mathbb{Q}$–algebra that defines $\mathrm{Lie}(C_{\mathbb{Q}})$ with a semisimple $\mathbb{Q}$–subalgebra of E_A, we

1 If either $p > 3$ or $p = 2$ and $C_{1\mathbb{Q}}$ is connected, then it is easy to see that Theorem 2.4.2 (b) implies that $\mathcal{G}_1$ is also a reductive group scheme (see [LR] and [Ko2]).

get that E_{1A} is the maximal $\mathbb{Q}$–vector subspace of E_A that centralizes $\mathrm{Lie}(C_{\mathbb{Q}})$ and that leaves invariant the $\mathbb{Q}$–span of λ_A. We can assume that $Z_{1A} := \mathrm{Lie}(Z^0(G_{1\mathbb{Z}_{(p)}}))$, when viewed as a set, is included in $\{v_\alpha|\alpha \in \mathcal{J}\}$.

9.1.3. Rational stratification. Let $\mathfrak{S}_{\mathrm{rat}}$ be the rational stratification of $\mathcal{N}_{k(v)}$ defined in [Va3, Subsection 5.3]. We recall that if $y_1 \in \mathcal{N}/H_0(k)$, then $y_{1\infty}$ and y_∞ are $\bar{k}$-valued points of the same (reduced) stratum of $\mathfrak{S}_{\mathrm{rat}}$ if and only if there exists an isomorphism $(M_1 \otimes_{W(k)} B(\bar{k}), \phi_1 \otimes \sigma_{\bar{k}}) \xrightarrow{\sim} (M \otimes_{W(k)} B(\bar{k}), \phi \otimes \sigma_{\bar{k}})$ that takes $t_{1\alpha}$ to t_α for all $\alpha \in \mathcal{J}$. The number of strata of $\mathfrak{S}_{\mathrm{rat}}$ is finite, cf. [Va3, Rm. 5.3.2 (b)]. Let $\mathfrak{s}_0$ be the $G(\mathbb{A}_f^{(p)})$-invariant reduced, closed subscheme of $\mathcal{N}_{k(v)}$ defined by the following property: the point y_∞ factors through $\mathfrak{s}_0$ if and only if $\mathcal{C}$ is basic. Obviously $\mathfrak{s}_0$ is a union of strata of $\mathcal{N}_{k(v)}$.

9.2. Some properties. Let $h \in \mathfrak{I}(\mathcal{C}, \lambda_A) \subseteq \mathcal{G}^0(B(k))$. Let $A(h)$ be as in Subsubsection 1.1.1. We denote also by λ_A the principal polarization of $A(h)$ defined naturally by λ_A. Let

$$y(h) : \mathrm{Spec}(k) \to \mathcal{M}_{\mathbb{F}_p}/\mathcal{K}^p(N_0)$$

be the morphism defined by $(A(h), \lambda_A)$ and its level-N_0 symplectic similitude structure induced naturally from the one of (A, λ_A) defined by the point $y \in \mathcal{N}_{k(v)}/H_0(k)$. Let $y(h)_\infty : \mathrm{Spec}(\bar{k}) \to \mathcal{M}_{\mathbb{F}_p}$ be an infinite lift of $y(h)$.

(a) For $p \geq 3$ (resp. $p = 2$) we say the *isogeny property* holds for (f, L, v) if for each point $y \in \mathcal{N}_{k(v)}/H_0(k)$ and for every element $h \in \mathfrak{I}(\mathcal{C}, \lambda_A)$, the (resp. up to the operation $\mathfrak{O}_1$ the) morphism $y(h)$ factors through $\mathcal{N}_{k(v)}/H_0$ and there exists a point $z(h) \in \mathcal{N}/H_0(W(k))$ which lifts this factorization (denoted in the same way) $y(h) : \mathrm{Spec}(k) \to \mathcal{N}_{k(v)}/H_0$ and for which t_α is the de Rham realization of $z(h)^*(w_\alpha^{\mathcal{A}_{H_0}})$ for all $\alpha \in \mathcal{J}$.

(b) We say the *weak isogeny property* holds for (f, L, v) if $\mathfrak{S}_{\mathrm{rat}}$ has only one closed stratum which is $\mathfrak{s}_0$ itself.

(c) We say the *Milne conjecture* holds for (f, L, v) if for each point $y \in \mathcal{N}_{k(v)}/H_0(k)$ there exists a symplectic isomorphism $(M, \lambda_A) \xrightarrow{\sim} (L^* \otimes_{\mathbb{Z}} W(k), \psi^*)$ that takes t_α to v_α for all $\alpha \in \mathcal{J}$. Here ψ^* is the alternating form on L^* defined naturally by ψ.

(d) We say the *ST property* holds for (f, L, v) if there exists an open, dense, $G(\mathbb{A}_f^{(p)})$-invariant subset $\mathcal{O}$ of $\mathcal{N}_{k(v)}$ such that whenever we have $y \in \mathcal{O}/H_0(k)$, there exists a unique Hodge cocharacter μ_{can} of $\mathcal{C}$ whose generic fibre factors through the torus of $\mathcal{G}_{B(k)}$ generated by $Z^0(\mathcal{G}_{B(k)})$ and $Z^0(C_{\mathcal{G}_{B(k)}}(\mathcal{T}(\phi)))$ and moreover the lift F^1_{can} of $\mathcal{C}$ defined by μ_{can} is the lift defined by a unique point $z \in \mathcal{N}/H_0(W(k))$ that lifts y. Here ST stands for Serre–Tate.

(e) Suppose that $p \geq 3$. We say the *GFT property* holds for (f, L, v) if there exists a subset $\mathcal{J}_0$ of $\mathcal{J}$ such that $(v_\alpha)_{\alpha \in \mathcal{J}_0}$ is a family of tensors of $\mathcal{T}(L^*_{(p)})$ of partial degrees at most $p-2$ and, when viewed as a family of tensors of $\mathcal{T}(L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p)$, it is also $\mathbb{Z}_p$-very well position for $G_{\mathbb{Z}_p}$. Here GFT stands for a good family of tensors.

9.2.1. Remarks. **(a)** In [Va6] it is proved that for $p \geq 3$ the Milne conjecture holds for (f, L, v). In [Va4] it is proved that for $p \geq 3$ the ST property holds for (f, L, v). The GFT property holds for (f, L, v) in most cases (like if $p \geq \max\{5, r\}$, cf. [Va1, Cor. 5.8.6]).

(b) The isogeny property was announced in [Va1, Subsubsection 1.7.1]. We outline the very essence of one way to prove it for $p \geq 3$. We assume that the weak isogeny property holds for (f, L, v) at least if $p \geq 3$. Due to this, standard specialization arguments show that to prove that the isogeny property holds for (f, L, v) it suffices to prove it only for those points $y \in \mathcal{N}_{k(v)}/H_0(k)$ for which $\mathcal{C}$ is basic. If G^{der} is simply connected, then the motivic theory of [Mi5] when combined with (a), Proposition 9.3, and Subsubsection 9.4.1 will imply that the isogeny property holds for such a point y (see also Remark 9.4.2). But the part of the Main Problem that pertains to $D_n^{\mathbb{H}}$ types and to relative PEL situations will allow us to remove the assumption that G^{der} is simply connected (to be compared with the paragraph before Subproblem 1.2.3).

(c) For each $\beta \in \mathbb{G}_m(W(k))$ there exists an element $g \in \mathcal{G}(W(k))$ that acts on the $W(k)$-span of λ_A via multiplication with β. Thus if there exists an isomorphism $(M, (t_\alpha)_{\alpha \in \mathcal{J}}) \xrightarrow{\sim} (L^* \otimes_{\mathbb{Z}} W(k), (t_\alpha)_{\alpha \in \mathcal{J}})$, then there exists also an isomorphism of the form $(M, (t_\alpha)_{\alpha \in \mathcal{J}}, \lambda_A) \xrightarrow{\sim} (L^* \otimes_{\mathbb{Z}} W(k), (t_\alpha)_{\alpha \in \mathcal{J}}, \psi^*)$. Moreover, as $\mathcal{G}^0$ is smooth and has a connected special fibre, such isomorphisms $(M, (t_\alpha)_{\alpha \in \mathcal{J}}, \lambda_A) \xrightarrow{\sim} (L^* \otimes_{\mathbb{Z}} W(k), (t_\alpha)_{\alpha \in \mathcal{J}}, \psi^*)$ exist if and only if they exist in the flat topology of $W(k)$.

9.2.2. Theorem. *If $Z^0(G) = Z^0(G_1)$, then the Milne conjecture holds for (f, L, v).*

Proof: It is known that we can identify $(H^1_{\acute{e}t}(A_{\overline{B(k)}}, \mathbb{Z}_p), \lambda_A) = (L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p, \psi^*)$ in such a way that the p-component of the étale component of $w_\alpha := z^*_{B(k)}(w^{\mathcal{A}}_\alpha)$ is v_α for all $\alpha \in \mathcal{J}$ (see [Va1, top of p. 473]). Strictly speaking loc. cit. mentions a $\mathbb{G}_m(\mathbb{Z}_p)$-multiple $\beta_{(p)}$ of ψ^*; as the complex $0 \to G^0_{\mathbb{Z}_p}(\mathbb{Z}_p) \to G_{\mathbb{Z}_p}(\mathbb{Z}_p) \to \mathbb{G}_m(\mathbb{Z}_p) \to 0$ is exact, we can assume that $\beta_{(p)} = 1$. Thus as $B(\bar{k})$ is a field of dimension ≤ 1 (see [Se2]) and due to Fontaine comparison theory, there exists an isomorphism $j_A : (M \otimes_{W(k)} B(\bar{k}), \lambda_A) \xrightarrow{\sim} (L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} B(\bar{k}), \psi^*)$ that takes t_α to v_α for all $\alpha \in \mathcal{J}$. As $Z^0(G) = Z^0(G_1)$, we have $Z^0(\mathcal{G}) = Z^0(C_1(\lambda_A))$ (cf. Theorem 2.4.2 (c)); here $Z^0(C_1(\lambda_A))$ is defined as in Theorem 2.4.2 (c) but for the pair $(\mathcal{C}, \lambda_A)$. As in Subsubsections 5.3.2 and 5.3.3 (a) we argue that we can assume that $j_A(M \otimes_{W(k)} W(\bar{k})) = L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} W(\bar{k})$. Thus the Milne conjecture holds for (f, L, v), cf. last sentence of Remark 9.2.1 (c). $\square$

9.3. Proposition. *We assume that the Milne conjecture holds for (f, L, v).*

(a) *Then $\mathfrak{s}_0$ is a stratum of $\mathfrak{S}_{\text{rat}}$ which is closed.*

(b) *We also assume that G^{der} is simply connected and that y factors through $\mathfrak{s}_0/H_0$. Let $y_0 \in \mathfrak{s}_0/H_0(k)$. Let $y_{0\infty} : \text{Spec}(\bar{k}) \to \mathcal{N}_{k(v)}$ be an infinite lift of y_0. Then up to the operation $\mathfrak{O}_1$, there exist elements $t \in G_2(\mathbb{A}_f^{(p)})$ and $h \in \mathfrak{I}(\mathcal{C}, \lambda_A)$ such that we have an identity $y_{0\infty} t = y(h)_\infty$ of $\bar{k}$-valued points of $\mathcal{M}_{k(v)}$.*

Proof: The connected components of $\mathcal{N}$ are permuted transitively by $G(\mathbb{A}_f^{(p)})$, cf. [Va1, Lem. 3.3.2]. Thus to prove the Proposition, we can assume that $y \in \mathfrak{s}_0/H_0(k)$ and that both y_∞ and $y_{0\infty}$ factor through the special fibre of the same connected component $\mathcal{N}^0$ of $\mathcal{N}$. Let π (resp. π_0) be the Frobenius endomorphism of A (resp. of A_0). See [Ch, Subsection 3.a] for the Frobenius tori $\mathcal{T}_\pi$ and $\mathcal{T}_{\pi_0}$ over $\mathbb{Q}$ of π and π_0 (respectively). The crystalline realization of π is $\phi^r \in \mathcal{G}(B(k))$ and therefore we have an identity $\mathcal{T}(\phi) = \mathcal{T}_{\pi B(k)}$. Each element $b \in Z_{1A}$ defines naturally a $\mathbb{Z}_{(p)}$-endomorphism of any pull back of $\mathcal{A}$, $\mathcal{A}_1$, or $\mathcal{A}_2$, to be denoted also by b. Thus we view $Z^0(G)$ and $\mathcal{T}_\pi$ as subtori of $C(\phi)_{\mathbb{Q}}$.

We prove (a). As $y \in \mathfrak{s}_0/H_0(k)$, the Newton quasi-cocharacter of $(M, \phi, \mathcal{G})$ factors through $Z^0(\mathcal{G}_{B(k)})$ (see [Va3, Cor. 2.3.2]) and thus it can be identified with a quasi-cocharacter μ_0 of $Z^0(\mathcal{G}_{B(k)})$. This quasi-cocharacter depends only on the $\mathrm{Gal}(\mathbb{Q}_p)$-orbit of the composite $\mu^{\mathrm{ab}} : \mathbb{G}_m \to \mathcal{G}^{\mathrm{ab}}$ of μ with the canonical epimorphism $\mathcal{G} \twoheadrightarrow \mathcal{G}^{\mathrm{ab}}$. Moreover μ^{ab} is uniquely attached to $\mathcal{X}$, cf. [Va3, Subsubsections 5.1.1 and 5.1.8]. We conclude that, as the notation suggests, μ_0 does not depend on the point $y \in \mathcal{N}/H_0(k)$.

The torus $\mathcal{T}_\pi$ is the smallest subtorus of $C(\phi)_{\mathbb{Q}}$ with the property that μ_0 is a quasi-cocharacter of $\mathcal{T}_{\pi B(k)}$, cf. Serre's result of [Pi, Prop. 3.5]. Thus $\mathcal{T}_\pi$ is naturally identified with a subtorus of $Z^0(G)$ uniquely determined by $\mathcal{X}$. Applying this also to y_0 we get $\mathcal{T}_\pi = \mathcal{T}_{\pi_0}$. Thus $\pi_0 \in \mathbb{Q}[\pi]$ is such that its image in each number field factor F_0 of $\mathbb{Q}[\pi]$ is non-trivial. Therefore from [Ta2] we get that the images of π and π_0 in F_0 are both Weil q-integers. Thus $\frac{\pi}{\pi_0}$ is a root of unity and therefore by performing the operation $\mathfrak{O}_1$ we can assume that $\pi = \pi_0 \in \mathcal{T}_\pi(\mathbb{Q}) = \mathcal{T}_{\pi_0}(\mathbb{Q}) \leqslant Z^0(G)(\mathbb{Q})$. Let $i : A \to A_0$ be the $\mathbb{Q}$–isogeny defined by this equality, cf. [Ta2]. Let $(M_0[\frac{1}{p}], \phi) \xrightarrow{\sim} (M[\frac{1}{p}], \phi_0)$ be the isomorphism defined by i; we will view it as a natural identification.

Let $\mathcal{G}^{0\prime}_{\mathbb{Q}_p}$ be the $\mathbb{Q}_p$-form of $\mathcal{G}^0_{B(k)}$ with respect to $(M[\frac{1}{p}], \phi)$. We have $\mathcal{G}^{0\prime\mathrm{ab}}_{\mathbb{Q}_p} = \mathcal{G}^{0\mathrm{ab}}_{\mathbb{Q}_p}$ and therefore let $\mathcal{G}^{0\prime\mathrm{ab}}_{\mathbb{Z}_p} := \mathcal{G}^{0\mathrm{ab}}_{\mathbb{Z}_p}$.

As the Milne conjecture holds for (f, L, v), there exists an element $j \in \mathbf{GL}_M(B(k))$ such that $j(M) = M_0$ and j takes λ_A to λ_{A_0} and takes t_α to $t_{0\alpha}$ for all $\alpha \in \mathcal{J}$. Thus j commutes with $\phi^r = \phi_0^r \in Z^0(\mathcal{G}_{B(k)})(B(k))$. We can also assume that j takes a Hodge cocharacter of $\mathcal{C}$ to a Hodge cocharacter of $\mathcal{C}_0$ (to be compared with [Va3, Lem. 5.1.8]). Thus we can identify $j^{-1}\phi j = g\phi$, where $g \in \mathcal{G}^0(W(k))$. From [Va3, Prop. 2.7.1 and Subsection 4.7] we deduce the existence of an element $h \in \mathcal{G}^0(B(\bar{k}))$ such that we have $g(\phi\otimes\sigma_{\bar{k}}) = h^{-1}(\phi\otimes\sigma_{\bar{k}})h$. In other words, there exists an isomorphism $(M_0\otimes_{W(k)}B(\bar{k}), \phi_0\otimes\sigma_{\bar{k}}, (t_{0\alpha})_{\alpha\in\mathcal{J}}) \xrightarrow{\sim} (M\otimes_{W(k)} B(\bar{k}), \phi\otimes\sigma_{\bar{k}}, (t_\alpha)_{\alpha\in\mathcal{J}})$. Thus $\mathfrak{s}_0$ is a stratum of $\mathfrak{S}_{\mathrm{rat}}$; it is closed by its very definition. Thus (a) holds.

We probe (b); thus G^{der} is simply connected. As j commutes with ϕ^r we get that $\phi^r = (j^{-1}\phi j)^r = (g\phi)^r$. Thus g defines a class $\gamma_g \in H^1(\mathrm{Gal}(B(k)/\mathbb{Q}_p), \mathcal{G}^{0\prime}_{\mathbb{Q}_p})$ whose image in $H^1(\mathrm{Gal}(B(k)/\mathbb{Q}_p), \mathcal{G}^{0\prime\mathrm{ab}}_{\mathbb{Q}_p})$ factors through $H^1(\mathrm{Gal}(W(k)/\mathbb{Z}_p), \mathcal{G}^{0\prime\mathrm{ab}}_{\mathbb{Z}_p}) = 0$. As the group $H^1(\mathrm{Gal}(W(k)/\mathbb{Z}_p), \mathcal{G}^{0\prime\mathrm{ab}}_{\mathbb{Z}_p})$ is trivial (cf. Lang theorem) and as the homomorphism $\mathcal{G}^0(W(k)) \to \mathcal{G}^{0\mathrm{ab}}(W(k))$ is surjective, we can assume that we have $g \in \mathcal{G}^{\mathrm{der}}(W(k))$. Thus γ_g is the image of some class $\gamma_g^{\mathrm{der}} \in H^1(\mathrm{Gal}(B(k)/\mathbb{Q}_p), \mathcal{G}^{\prime\mathrm{der}}_{\mathbb{Q}_p})$. As G^{der} is simply connected, the class γ_g^{der} is trivial (cf. [Kn, Thm. 1]). Thus we can assume that $h \in \mathcal{G}^{\mathrm{der}}(B(k))$. Therefore $g\phi = h^{-1}\phi h$ and thus we have $h \in \mathfrak{I}(\mathcal{C}, \lambda_A)$ and $j^{-1}\phi j = h^{-1}\phi h$. Let $\tilde{h} := hj^{-1} \in \mathbf{GL}_M(B(k))$; it is fixed by ϕ and thus it is a $\mathbb{Q}_p$-valued point of $C(\phi)_{\mathbb{Q}_p}$.

Let $ZC(\phi)_{\mathbb{Q}}$ be the reductive subgroup of $C(\phi)_{\mathbb{Q}}$ that fixes λ_A and $Z_{1A}[\frac{1}{p}]$. We now check that we can assume that i takes b to b for all $b \in Z_{1A}[\frac{1}{p}]$ and takes λ_A to λ_{A_0}. Let $\gamma \in H^1(\mathbb{Q}, ZC(\phi)_{\mathbb{Q}})$ be the class that "measures" the existence of such a choice of i. Let l be a rational prime. We check that the image of γ in $H^1(\mathbb{Q}_l, ZC(\phi)_{\mathbb{Q}_l})$ is the trivial class. If $l = p$ (resp. if $l \neq p$), then this is so due to the previous paragraph (resp. due to the existence of all level-l^m symplectic similitude structures of $\mathcal{A}$ with $m \in \mathbb{N}$ and on the fact that $\pi = \pi_0 \in T_\pi(\mathbb{Q}) = T_{\pi_0}(\mathbb{Q}) \leqslant Z^0(G)(\mathbb{Q})$). The triples (A, λ_A, Z_{1A}) and $(A_0, \lambda_{A_0}, Z_{1A})$ lift to characteristic 0. But all pull backs of $(\mathcal{A}, \Lambda_{\mathcal{A}}, Z_{1A})$ via complex valued points of $\mathcal{N}^0$ are $\mathbb{R}$-isogenous (as each connected component of $\mathcal{X}$ is a $G^0(\mathbb{R})$-conjugacy class). Thus (A, λ_A, Z_{1A}) and $(A_0, \lambda_{A_0}, Z_{1A})$ are $\mathbb{R}$-isogenous. Thus the image of γ in $H^1(\mathbb{R}, ZC(\phi)_{\mathbb{R}})$ is also the trivial class.

The group $ZC(\phi)_{\mathbb{C}}$ is isomorphic to the centralizer of a torus of $\mathbf{Sp}(W, \psi)_{\mathbb{C}}$. Thus it is the product of some $\mathbf{GL}_n$ groups with either a trivial group or with a $\mathbf{Sp}_{2n}$ group (the ranks n do depend on the factors of such a product). Therefore we have a product decomposition $ZC(\phi)_{\mathbb{Q}} = Z_1 \times_{\mathbb{Q}} Z_2$, where:

(i) there exists a semisimple $\mathbb{Q}$–algebra Z_{11} with involution ι_{11} such that Z_1 is the group scheme of invertible elements of Z_{11} fixed by $*$;

(ii) Z_2 is either trivial or a simple connected semisimple group of C_n Dynkin type.

The pair (Z_{11}, ι_{11}) is a product of semisimple $\mathbb{Q}$–algebras endowed with involutions which are either trivial or of second type. Thus Z_1 is a product of Weil restrictions of reductive groups whose derived groups are forms of $\mathbf{SL}_n$ groups ($n \in \mathbb{N}$) and whose abelianizations are of rank 1. This implies that the Hasse principle holds for Z_1 (even if some n's are even). It is well known that the Hasse principle holds for Z_2. We conclude that:

(iii) the Hasse principle holds for $ZC(\phi)_{\mathbb{Q}}$ and therefore the class γ is trivial (cf. previous paragraph).

It is well known that $Z_1(\mathbb{Q})$ is dense in $Z_1(\mathbb{Q}_p)$. As $ZC(\phi)_{B(k)}$ is $C_{\mathbf{Sp}(M[\frac{1}{p}], \lambda_A)}(\mathcal{T}(\phi))$ and as $Z^0(\mathcal{G})$ splits over a finite unramified extension of $W(k)$, the group $Z_{2B(\bar{k})}$ is split. Thus $Z_2(\mathbb{Q})$ is dense in $Z_2(\mathbb{Q}_p)$, cf. [Mi4, Lem. 4.10]. Thus we get:

(iv) the group $ZC(\phi)_{\mathbb{Q}}(\mathbb{Q})$ is dense in $ZC(\phi)_{\mathbb{Q}}(\mathbb{Q}_p)$.

Due to the property (iii), we can assume that $j \in \mathcal{G}^0(B(k))$. Thus $\tilde{h} \in \mathcal{G}(B(k))$ is a $\mathbb{Z}_p$-isomorphism between the principally quasi-polarized Dieudonné modules with endomorphisms associated to $(A(h), \lambda_A, Z_{1A})$ and $(A_0, \lambda_{A_0}, Z_{1A})$. Let $s \in \mathbb{N}$. A Theorem of Tate says that $\mathrm{Hom}_k(A_0, A(h)) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is the set of $\mathbb{Z}_p$-homomorphisms between the Dieudonné modules of the p-divisible groups of $A(h)$ and A_0 (see [Ta2, p. 99]; the passage from $\mathbb{Q}_p$ coefficients to $\mathbb{Z}_p$ coefficients is trivial). Based on this and the property (iv) we get that there exists a $\mathbb{Z}_{(p)}$-isomorphism $\tilde{h}_{(p)}$ between $(A_0, \lambda_{A_0}, Z_{1A})$ and $(A(h), \lambda_A, Z_{1A})$ whose crystalline realization is congruent modulo p^s to $\tilde{h}$.

Due to existence of the $\mathbb{Z}_{(p)}$-isomorphism $\tilde{h}_{(p)}$, there exists an element $t \in \mathbf{GSp}(W, \psi)(\mathbb{A}_f^{(p)})$ such that we have an identity $y_{0\infty}t = y(h)_\infty$ of $\bar{k}$-valued points of $\mathcal{M}_{k(v)}$ (cf. also [Mi2, Section 3]). The fact that we can take $t \in G_2(\mathbb{A}_f^{(p)})$ is checked easily by considering the level-l^m symplectic similitude structures of $y_{0\infty}^*(\mathcal{A}, \Lambda_{\mathcal{A}})$ and $(y(h)_\infty)^*(\mathcal{A}, \Lambda_{\mathcal{A}})$ (here l is a prime different from p and $m \in \mathbb{N}$). □

9.4. Theorem. *We assume that the Milne conjecture, the isogeny property, and the ST property hold for (f, L, v). We also assume that $p \geq 3$, that G^{der} is simply connected, and that y factors through $\mathfrak{s}_0/H_0$. Then in Proposition 9.3 (b) we can assume that in fact we have $t \in G(\mathbb{A}_f^{(p)})$.*

Proof: Let $\mathcal{N}^0_t$ be the right translation of $\mathcal{N}^0$ through t i.e., the normalization of $\mathcal{M}_{O_{(v)}}$ in the right translation of $\mathcal{N}^0_{E(G,\mathcal{X})}$ through t. It is a finite scheme over $\mathcal{M}_{O_{(v)}}$, cf. proof of [Va1, Prop. 3.4.1]. As the isogeny property holds for (f,L,v), the point $u:=y(h)_\infty$ is a $\bar{k}$-valued point of $\mathcal{N}_{k(v)}$ that factors also through $\mathcal{N}^0_{tk(v)}$. We can identify the principally quasi-polarized Dieudonné module associated to $u^*(\mathcal{A},\Lambda_{\mathcal{A}})$ with $\mathcal{D}(h):=(h(M)\otimes_{W(k)} W(\bar{k}),\phi\otimes\sigma_{\bar{k}},\lambda_A)$. Let F^1_h be the lift of $(h(M)\otimes_{W(k)} W(\bar{k}),\phi\otimes\sigma_{\bar{k}},\mathcal{G}(h)_{W(\bar{k})})$ defined by a $W(\bar{k})$-valued point of $\mathcal{N}$ that lifts both u and a point $z(h)\in\mathcal{N}/H_0(W(k))$ as in the property 9.2 (a).

As $y_{0\infty}t=u$ we can identify naturally $M_0\otimes_{W(k)} W(\bar{k})=h(M)\otimes_{W(k)} W(\bar{k})$. Thus $\mathcal{G}_{0W(\bar{k})}$ is the reductive, closed subgroup scheme of $\mathbf{GL}_{h(M)\otimes_{W(k)}W(\bar{k})}$ that fixes $t_{0\alpha}$ for all $\alpha\in\mathcal{J}$. Let F^1_0 be a lift of $(h(M)\otimes_{W(k)} W(\bar{k}),\phi\otimes\sigma_{\bar{k}},\mathcal{G}_{0W(\bar{k})})$ defined by a $W(\bar{k})$-valued point of $\mathcal{N}^0_t$ that lifts u.

The closed subgroup schemes $\mathcal{G}(h)_{W(\bar{k})}$ and $\mathcal{G}_{0W(\bar{k})}$ of $\mathbf{GL}_{h(M)\otimes_{W(k)}W(\bar{k})}$ are conjugate under an element $h_0\in\mathbf{Sp}(h(M),\lambda_A)(W(\bar{k}))$ which is fixed by $\phi\otimes\sigma_{\bar{k}}$ and which is congruent modulo p^s to $1_{h(M)\otimes_{W(k)}W(\bar{k})}$ (see the last part of the proof of Proposition 9.3 (b)). Let $d:=\dim_{W(k)}(\mathcal{G}^{\mathrm{der}})$. Let $R_d:=W(\bar{k})[[x_1,\ldots,x_d]]$. Let Φ_{R_d} be the Frobenius lift of R_d that is compatible with $\sigma_{\bar{k}}$ and that takes each x_i to x_i^p for all $i\in\{1,\ldots,d\}$.

The local deformation space $\mathcal{D}$ (resp. $\mathcal{D}_t$) of $u^*(\mathcal{A},\Lambda_{\mathcal{A}})$ defined by $\mathcal{N}_{W(\bar{k})}$ (resp. $\mathcal{N}^0_{tW(\bar{k})}$) depends only on $\mathcal{G}(h)^{\mathrm{der}}_{W(\bar{k})}$ (resp. $\mathcal{G}^{\mathrm{der}}_{0W(\bar{k})}$), cf. [Va1, Subsubsections 5.4.4 to 5.4.8 and Subsection 5.5]. More precisely the principally quasi-polarized filtered F-crystal over R_d/pR_d defined by the pull back of $(\mathcal{A},\Lambda_{\mathcal{A}})$ via a formally smooth $W(\bar{k})$-morphism $\mathrm{Spec}(R_d)\to\mathcal{N}$ (resp. $\mathrm{Spec}(R_d)\to\mathcal{N}^0_t$) that lifts u is isomorphic to

$$(h(M)\otimes_{W(k)} R_d, F^1_h\otimes_{W(\bar{k})} R_d, g^{\mathrm{der}}_{\mathrm{univ}}(\phi\otimes\Phi_{R_d}),\lambda_A,\nabla_h) \tag{10}$$

(resp. $(h(M)\otimes_{W(k)} R_d, F^1_0\otimes_{W(\bar{k})} R_d, g^{\mathrm{der}}_{0\mathrm{univ}}(\phi\otimes\Phi_{R_d}),\lambda_A,\nabla_{0h})$). Here $g^{\mathrm{der}}_{\mathrm{univ}}:R_d\to\mathcal{G}(h)^{\mathrm{der}}_{W(\bar{k})}$ and $g^{\mathrm{der}}_{0\mathrm{univ}}:R_d\to\mathcal{G}^{\mathrm{der}}_{0W(\bar{k})}$ are formally étale morphisms which (due to the previous paragraph) coincide modulo p^s. Thus the special fibres of $\mathcal{D}$ (resp. $\mathcal{D}_t$) coincide, cf. Lemma 3.4 and Serre–Tate deformation theory. Thus the images in $\mathcal{M}_{k(v)}$ of the connected components $\mathcal{U}_t$ and $\mathcal{U}$ of $\mathcal{N}^0_{tk(v)}$ and $\mathcal{N}_{k(v)}$ (respectively) through which u factors, are the same.

Let $\mathcal{O}$ be as in the property 9.2 (d). Let $\mathcal{O}\mathcal{U}$ be the image of $\mathcal{O}\cap\mathcal{U}$ in $\mathcal{M}_{k(v)}$. Let $(M_o,\phi_o,\lambda_{A_o})$ be the principally quasi-polarized Dieudonné module associated naturally to a point $y_{o\infty}\in\mathcal{O}\mathcal{U}(\bar{k})$. A $\bar{k}$-valued point of either $\mathcal{U}$ or $\mathcal{U}_t$ that factors through $y_{o\infty}$, will be also denoted by $y_{o\infty}$.

As the ST property holds for (f,L,v), each such $\bar{k}$-valued point $y_{o\infty}$ of $\mathcal{U}$ (resp. of $\mathcal{U}_t$) has a unique $W(\bar{k})$-lift $z_{o\infty}$ (resp. $z_{ot\infty}$) to $\mathcal{N}$ (resp. to $\mathcal{N}^0_t$) such that there exists a Hodge cocharacter μ_o (resp. μ_{ot}) of the Shimura F-crystal $(M_o,\phi_o,\mathcal{G}_o)$ (resp. $(M_o,\phi_o,\mathcal{G}_{o0})$) attached to $y_{o\infty}\in\mathcal{U}(\bar{k})$ (resp. to $y_{o\infty}t^{-1}\in\mathcal{N}^0(\bar{k})$) whose generic fibre factors through the center of the subgroup of $\mathbf{GL}_{M_o[\frac{1}{p}]}$ that commutes with $Z^0(\mathcal{G}_o)_{B(\bar{k})}=Z^0(\mathcal{G}_{o0})_{B(\bar{k})}$ and with the integral powers of the Frobenius endomorphism of ${y_o}^*(\mathcal{A}_{H_0})$; here y_o is an $\mathbb{F}_{p^m}$-valued point of $\mathcal{O}/H_0$ that has $y_{o\infty}$ as an infinite lift, for some $m\in\mathbb{N}$ which is big enough. As μ_o and μ_{ot} commute, the two lifts of $(M_o,\phi_o,\mathbf{GL}_{M_o})$ they define coincide. Thus as $p\geq 3$, we have an identity $z_{o\infty} = z_{ot\infty}$ of $W(\bar{k})$-valued points of $\mathcal{M}_{O_{(v)}}$. The normalization of the Zariski closure of these $z_{o\infty} = z_{ot\infty}$ points in $\mathcal{M}_{O_{(v)}}$ is on one hand a connected component of $\mathcal{N}$ and on the other hand it is $\mathcal{N}_t^0$.

But $\mathcal{N}(\mathbb{C}) = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)})\backslash(\mathcal{X} \times G(\mathbb{A}_f^{(p)}))$ and its analogue holds for $\mathcal{N}_2(\mathbb{C})$, cf. [Mi4, Prop. 4.11 and Cor. 4.12]. Thus as $\mathcal{N}_t^0$ is a connected component of $\mathcal{N}$ we easily get that $t \in G_{2\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)})(\mathbb{Z}_{(p)})G(\mathbb{A}_f^{(p)})$. Thus to prove the Theorem, we can assume that $t \in G_{2\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)})$. By considering level-$l^m$ symplectic similitude structures with $l \neq p$ fixed and with $m \in \mathbb{N}$ varying, we get that $t \in G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)})$. $\square$

9.4.1. An interpretation. As Milne conjecture is assumed to hold for (f, L, v), the $\mathbb{Z}_p$ structure of $\mathcal{C}$ is isomorphic to $(L^* \otimes_{\mathbb{Z}} \mathbb{Z}_p, G_{\mathbb{Z}_p}, (v_\alpha)_{\alpha \in \mathcal{J}})$. Thus the condition [Va3, 5.2 (b)] holds for (f, L, v). Thus from [Va3, Subsubsection 4.2.2 and the proof of Thm. 5.2.3] we get the existence of a point $z_{0\infty} \in \mathcal{N}(W(\bar{k}))$ that lifts a $\bar{k}$-valued point of $\mathfrak{s}_0$ and such that the Mumford–Tate group of each complex extension of $z_{0\infty}^*(\mathcal{A})$ is a torus. By enlarging k we can assume that the $W(\bar{k})$-valued point of $\mathcal{N}/H$ defined by $z_{0\infty}$ factors through a point $z_0 \in \mathcal{N}/H(W(k))$. Thus Theorem 9.4 can be interpreted as follows. If $y \in \mathfrak{s}_0/H_0(k)$ and if we work under the assumptions of Theorem 9.4, then up to the operations $\mathfrak{O}_1$ and $\mathfrak{O}_2$ we can assume that the lift $z \in \mathcal{N}/H_0(W(k))$ of y is such that $A_{W(k)}$ is with complex multiplication.

9.4.2. Remark. We assume that the hypotheses of Proposition 9.3 (b) hold. Let $t \in G_2(\mathbb{A}_f^{(p)})$ and $h \in \mathfrak{I}(\mathcal{C}, \lambda_A)$ be such that we have an identity $y_{0\infty}t = y(h)_\infty$ of $\bar{k}$-valued points of $\mathcal{M}_{k(v)}$, cf. Proposition 8.3 (b). We also assume that $\mathfrak{e}$ (see Subsection 8.1) is the extension to $B(k)$ of the Lie algebra of a subgroup $E(\phi)_{\mathbb{Q}}$ of $C(\phi)_{\mathbb{Q}}$ (for instance, this holds if the cycle Π of Subsection 8.1 is the crystalline realization of an algebraic cycle of $A_{W(k)}$). Let $E(\phi)_{\mathbb{Q}}^0$ be the subgroup of $E(\phi)_{\mathbb{Q}}$ that fixes λ_A; it is a $\mathbb{Q}$–form of $\mathcal{G}_{B(k)}^0$ and thus it is connected. The group $E(\phi)_{\mathbb{Q}}^0(\mathbb{Q})$ is dense in $E(\phi)_{\mathbb{Q}}^0(\mathbb{Q}_p)$, cf. [Mi4, Lem. 4.10]. Thus as in the proof of Proposition 9.3 we get the existence of an element $h_1 \in E(\phi)_{\mathbb{Q}}^0(\mathbb{Q})$ such that by denoting also by h_1 its crystalline realization, we have $h_1(h(M)) = M$. This implies that we can choose $y(h)_\infty$ to be the translation of y_∞ by an element of $G(\mathbb{A}_f^{(p)})$ defined naturally by h_1. Thus $y(h)$ factors through $\mathcal{N}_{k(v)}/H_0$. This implies directly that $t \in G(\mathbb{A}_f^{(p)})$.

In future work we will show independently of [Mi5] i.e., based mainly on the weak isogeny property and on [Va1], that for $p \geq 3$ the point $y(h)$ factors through $\mathcal{N}/H_0$ if either $k(v) = \mathbb{F}_p$ or if all simple factors of $(G^{\mathrm{ad}}, \mathcal{X}^{\mathrm{ad}})$ are of A_n, B_n, C_n, or $D_n^{\mathbb{R}}$ type.

9.5. The non-basic context. In this Subsection we assume that the isogeny property and the GFT property hold for (f, L, v), that $p \geq 3$, and that $\mathcal{C}$ is not basic. We also assume that $Q + +\mathfrak{A}$ holds for $(\mathcal{C}, \lambda_A)$ and that $Z^0(\mathcal{G}) = Z^0(G_1)$. Thus the Milne conjecture holds for (f, L, v), cf. Theorem 9.2.2. All assumptions of Corollary 8.3 hold (cf. also Subsubsection 9.1.1) and therefore we will use the notations of Corollary 8.3, with $h \in \mathfrak{I}(\mathcal{C}, \lambda_A) \subseteq \mathcal{G}^0(B(k))$. Let $(\mathcal{T}_{1B(k)}, \mu_1)$ be the E-pair of $\mathcal{C}$ which is plus plus admissible and which was used in the proof of Corollary 8.3 (b). Let $\mathcal{T}_{1\mathbb{Q}_p}$ and K_2 be as in Definition 2.3 (c). We know that $L^0_{\mathbf{GL}_{h(M)}}(\phi)$ is a reductive, closed subgroup scheme of $\mathbf{GL}_{h(M)}$, cf.

property (i) of Corollary 8.3 (a). Let $Z_h(\phi)$ be the split rank 1 subtorus of $Z(L^0_{\mathbf{GL}_{h(M)}}(\phi))$ such that the Newton quasi-cocharacter of $\mathcal{C}$ factors through $Z_h(\phi)_{B(k)}$; it is a torus of $\mathcal{G}(h)$.

As the property 9.2 (a) holds for y and h, we can assume that $y(h) \in \mathcal{N}_{k(v)}/H_0(k)$. Let $\bar{V}_3 := V_3 \otimes_{W(k)} W(\bar{k})$. Let $\lambda_{A(h)_{V_3}}$ be the principal polarization of $A(h)_{V_3}$ that lifts $\lambda_{A(h)} = \lambda_A$. Let $\tilde{z}_\infty : \mathrm{Spec}(\bar{V}_3) \to \mathcal{M}_{O_{(v)}}$ be the morphism that lifts the composite of $y(h)_\infty$ with the morphism $\mathcal{N}_{k(v)} \to \mathcal{M}_{k(v)}$ and that is defined by the pull back of $(A(h)_{V_3}, \lambda_{A(h)_{V_3}})$ to $\bar{V}_3$. Let $z(h)_\infty \in \mathcal{N}(W(\bar{k}))$ be a point that lifts $y(h)_\infty$. The part of the proof of Theorem 9.4 that pertains to (10) holds even if $\mathcal{C}$ is not basic, cf. [Va1, Subsubsections 5.4.4 to 5.4.8 and Section 5.5]. Thus as $p \geq 3$, from Theorem 3.6 and Corollary 3.7.3 we get that $\tilde{z}_\infty$ factors through $\mathcal{N}$. We fix an $E(G,\mathcal{X})$-embedding $i_3 : \bar{V}_3[\frac{1}{p}] \hookrightarrow \mathbb{C}$ and we use it to naturally identify $\overline{B(k)}$ with a subfield of $\mathbb{C}$.

We identify $H^1_{\acute{e}t}(A(h)_{\overline{B(k)}}, \mathbb{Z}_p) = H^1(A(h)_{\mathbb{C}}, \mathbb{Z}_{(p)}) \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$ (cf. [AGV, Exp. XI]) and $H^1(A(h)_{\mathbb{C}}, \mathbb{Z}_{(p)}) = L^*_{(p)}$ in such a way that v_α becomes the p-component of the étale component of $\tilde{z}^*_\infty(w^{\mathcal{A}}_\alpha)$ for all $\alpha \in \mathcal{J}$ (cf. [Va1, pp. 472–473]). Thus we can identify the Mumford–Tate group of $A(h)_{\mathbb{C}}$ with a reductive subgroup $\tilde{G}_3$ of G. Let G_3 be a reductive subgroup of G which is maximal under the properties that it contains $\tilde{G}_3$ and we have $G_3^{\mathrm{der}} = \tilde{G}_3^{\mathrm{der}}$. As the Frobenius endomorphism of $A(h)$ lifts to A_{V_3}, it defines naturally a $\mathbb{Q}$–valued point π of $Z^0(G_3)$. Let $\mathcal{X}_3$ be the $G_3(\mathbb{R})$-conjugacy class of the homomorphism $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_{3\mathbb{R}}$ that defines the Hodge $\mathbb{Q}$–structure on W defined by $A(h)_{\mathbb{C}}$. The pair $(G_3, \mathcal{X}_3)$ is a Shimura pair.

Let Z_3 be the subtorus of $\mathbf{GL}_{H^1_{\acute{e}t}(A(h)_{\overline{B(k)}}, \mathbb{Z}_p)}$ that corresponds to $Z^0(L_{\mathbf{GL}_{h(M)}}(\phi))$ via Fontaine comparison theory for $A(h)_{V_3}$. It exists as $p \geq 3$ and as we can identify it naturally with the group scheme of invertible elements of the semisimple $\mathbb{Z}_p$-subalgebra of $\mathrm{End}(h(M))$ formed by elements of $\mathrm{Lie}(Z^0(L_{\mathbf{GL}_{h(M)}}(\phi)))$ fixed by ϕ. Its generic fibre commutes with $G_{3\mathbb{Q}_p}$. The subtorus $Z_h(\phi)_{\acute{e}t}$ of Z_3 that corresponds to $Z_h(\phi)$ via Fontaine comparison theory for $A(h)_{V_3}$, is a $\mathbb{G}_m$ subgroup scheme of the Zariski closure of $Z^0(G_{3\mathbb{Q}_p})$ in $\mathbf{GL}_{H^1_{\acute{e}t}(A(h)_{\overline{B(k)}}, \mathbb{Z}_p)} = \mathbf{GL}_{L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p}$. Let T_3 be a maximal torus of G_3 such that the following two things hold (cf. [Ha, Lem. 5.5.3]):

(i) the torus $T_{3\mathbb{R}}$ is the extension of a compact torus by $Z(\mathbf{GL}_{W \otimes_{\mathbb{Q}} \mathbb{R}})$;

(ii) there exists an element $g \in G_3^{\mathrm{ad}}(\mathbb{Q}_p)$ such that $gT_{3\mathbb{Q}_p}g^{-1}$ is the subtorus of $G_{3\mathbb{Q}_p}$ which is isomorphic to $\mathcal{T}_{1\mathbb{Q}_p}$ and which corresponds to $\mathcal{T}_{1V_3[\frac{1}{p}]}$ via Fontaine comparison theory for $A(h)_{V_3}$.

Let $\mu_3 : \mathbb{G}_m \to T_{3\mathbb{C}}$ be the cocharacter such that $g\mu_3g^{-1}$ is obtained from μ_1 by extension of scalars under the restriction $i_{K_2} : K_2 \hookrightarrow \mathbb{C}$ of i_3 to K_2. From the property (ii) we get:

(iii) the cocharacter μ_3 is $G_3(\mathbb{C})$-conjugate to the Hodge cocharacters of $\mathbf{GL}_{W \otimes_{\mathbb{Q}} \mathbb{C}}$ that define Hodge $\mathbb{Q}$–structures on W associated to points $x_3 \in \mathcal{X}_3$.

Let $S_{3\mathbb{C}}$ be the subtorus of $T_{3\mathbb{C}}$ generated by $Z(\mathbf{GL}_{W \otimes_{\mathbb{Q}} \mathbb{C}})$ and $\mathrm{Im}(\mu_3)$. As the torus $T_{3\mathbb{R}}/Z(\mathbf{GL}_{W \otimes_{\mathbb{Q}} \mathbb{R}})$ is compact, $S_{3\mathbb{C}}$ is the extension to $\mathbb{C}$ of a subtorus $S_{3\mathbb{R}}$ of $T_{3\mathbb{R}}$. From the property (iii) we get that we can identify $S_{w\mathbb{R}} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ and thus we get a natural monomorphism

$$h_3 : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \hookrightarrow T_{3\mathbb{R}}.$$

Let $\mathcal{X}_3'$ be the $G_3(\mathbb{R})$-conjugacy class of h_3. As in the proof of [Va3, Thm. 5.2.3] one checks that we can choose T_3 such that the pair $(G_3, \mathcal{X}_3')$ is a Shimura pair (this time it is irrelevant what the Zariski closures of T_3 and G_3 in $\mathbf{GL}_{L_{(p)}}$ are and therefore loc. cit. can be adapted to our present context).

The adjoint Shimura pairs $(G_3^{\mathrm{ad}}, \mathcal{X}_3'^{\mathrm{ad}})$ and $(G_3^{\mathrm{ad}}, \mathcal{X}_3^{\mathrm{ad}})$ of $(G_3, \mathcal{X}_3')$ and $(G_3, \mathcal{X}_3)$ (respectively) coincide i.e., we have an identity $\mathcal{X}_3^{\mathrm{ad}} = \mathcal{X}_3'^{\mathrm{ad}}$ (cf. the property (iii) and [De2, Prop. 1.2.7 and Cor. 1.2.8]). As the group $G_3^{\mathrm{ad}}(\mathbb{Q})$ is dense in $G_3^{\mathrm{ad}}(\mathbb{R})$, it permutes the connected components of $\mathcal{X}_3^{\mathrm{ad}}$. Thus by replacing the injective map

$$i_3 : (T_3, \{h_3\}) \hookrightarrow (G_3, \mathcal{X}_3')$$

by its composite with an isomorphism

$$(G_3, \mathcal{X}_3') \tilde{\to} (G_3, \mathcal{X}_3)$$

defined by an element of $G_3^{\mathrm{ad}}(\mathbb{Q})$, we can assume that $\mathcal{X}_3' = \mathcal{X}_3$.

As the cocharacter μ_1 is defined over K_1, the reflex field $E(T_3, \{h_3\})$ is a subfield of K_1. Let v_3 be the prime of $E(T_3, \{h_3\})$ such that the local ring $O_{(v_3)}$ of it is dominated by the ring of integers of K_1. Let $H_3 := T_3(\mathbb{Q}_p) \cap H$ and let $H_{03} := T_3(\mathbb{A}_f^{(p)}) \cap H_0$. It is easy to see that the natural morphism $\mathrm{Sh}(T_3, \{h_3\})/H_3 \to \mathrm{Sh}(G, \mathcal{X})_{E(T_3,\{h_3\})}/H$ (see [De1, Cor. 5.4]) is a closed embedding. Let $\mathcal{T}_3$ be the Zariski closure of $\mathrm{Sh}(T_3, \{h_3\})/H_3$ in $\mathcal{N}_{O_{(v_3)}}$. Let O_0 be a finite, discrete valuation ring extension of the completion O_{v_3} of $O_{(v_3)}$ such that we have a morphism $z_0 : \mathrm{Spec}(O_0) \to \mathcal{T}_3/H_{03}$ and $K_0 := O_0[\frac{1}{p}]$ is a Galois extension of $\mathbb{Q}_p$. Let $A_{0O_0} := z_0^*(\mathcal{A}_{H_0})$, where we denote also by z_0 the O_0-valued point of $\mathcal{N}/H_0$ defined naturally by z_0. By performing the operation $\mathfrak{O}_1$ to $\mathcal{C}$, we can assume that the residue field of O_0 is k and that the abelian scheme A_0 has complex multiplication. We can assume that z_0 and $\tilde{z}_\infty$ give birth to complex valued points of the same connected component of $\mathrm{Sh}(G_3, \mathcal{X}_3)/(G_3(\mathbb{A}_f) \cap (H_0 \times H))$. Let y_0 be the special fibre of z_0 identified as well with a k-valued point of $\mathcal{N}_{k(v)}/H_0$. We recall that $(A_0, \lambda_{A_0}) = y_0^*(\mathcal{A}_{H_0}, \Lambda_{\mathcal{A}_{H_0}})$ and that $(M_0, \phi_0, \mathcal{G}_0)$ is the Shimura F-crystal attached to y_0. By performing the operation $\mathfrak{O}_1$, we can also assume that all endomorphisms of $A_{0\bar{k}}$ are pull backs of endomorphisms of A_0. Let $Z(\phi_0)$ be the subtorus of $\mathcal{G}_0$ that corresponds to $Z_h(\phi)_{\acute{e}t}$ via Fontaine comparison theory for the abelian scheme A_{0O_0}. Let $\mathcal{G}_{3B(k)}$ be the reductive subgroup of $\mathcal{G}_{0B(k)}$ that corresponds to $G_{3\mathbb{Q}_p}$ via Fontaine comparison theory for A_{0K_0}.

We denote also by μ_3 the cocharacter $\mathbb{G}_m \to T_{3O_{v_3}[\frac{1}{p}]}$ whose extension to $\mathbb{C}$ is μ_3 (i.e., it is the Hodge cocharacter defined by h_3, cf. the very definition of reflex fields). The Newton quasi-cocharacter of $(M_0, \phi_0, \mathcal{G}_0)$ is the quasi-cocharacter of $T_{3B(k)}$ (viewed as a maximal torus of $\mathcal{G}_{0B(k)}$) which is the mean-average of the $\mathrm{Gal}(K_0/\mathbb{Q}_p)$-orbit of μ_3, cf. [Ko1, Subsections 2.8 and 4.2] (see also either [RR, Thm. 1.15] or [RaZ, Prop. 1.21]). Thus it factors through $Z^0(\mathcal{G}_{3B(k)})$, cf. the $G_3^{\mathrm{ad}}(\mathbb{Q}_p)$-conjugacy of the property (ii) and the fact that the Newton quasi-cocharacter of $(M, \phi, \mathcal{G})$ factors through the subtorus $\mathcal{T}(\phi)$ of

$Z^0(\mathcal{G}_{3B(k)})$. Thus as in the proof of Proposition 9.3 (a) we argue that up to performing the operation $\mathfrak{O}_1$ to $\mathcal{C}$ we can assume that the Betti realization of the Frobenius endomorphism of A_0 obtained via A_{0O_0} is the element $\pi \in Z^0(G_3)$.

9.5.1. Theorem. *We assume that the isogeny property, the ST property, and the GFT property hold for (f, L, v), that $p \geq 3$, that $\mathcal{C}$ is not basic, and that G^{der} is simply connected. We also assume that $Q + +\mathfrak{A}$ holds for $(\mathcal{C}, \lambda_A)$ and that $Z^0(\mathcal{G}) = Z^0(G_1)$. Let $y_0 \in \mathcal{N}_{k(v)}/H_0(k)$ be as above (i.e., the special fibre of a composite morphism $z_0 : Spec(O_0) \to \mathcal{T}_3/H_{03} \to \mathcal{N}/H_0$). Then up to performing $\mathfrak{O}_1$, there exists $h_1 \in \mathfrak{I}(\mathcal{C}, \lambda_A)$, $t_1 \in G(\mathbb{A}_f^{(p)})$, and an infinite lift $y_{0\infty} \in \mathcal{N}_{k(v)}(\bar{k})$ of $y_0 \in \mathcal{N}_{k(v)}/H_0(k)$ such that $y_{0\infty}t_1$ is an infinite lift of $y(h_1) \in \mathcal{N}_{k(v)}/H_0(k)$.*

Proof: The proof of this is entirely the same as the proofs of Proposition 9.3 (b) and Theorem 9.4. Only the part of the proof of Proposition 9.3 (b) that pertains to an element $g \in \mathcal{G}^0(W(k))$ has to be slightly modified. The role of Z_{1A} is now replaced by the maximal $\mathbb{Z}_{(p)}$-subalgebra Z_{3A} of $\mathrm{End}(L_{(p)})$ fixed by $Z^0(G_{\mathbb{Z}_{(p)}})$ and by $\pi \in Z^0(G_3)(\mathbb{Q})$. Up to the operation $\mathfrak{O}_1$, each element $b \in Z_{3A}$ defines a $\mathbb{Z}_{(p)}$-endomorphism of either A_0 or A. As we can assume that $\pi_0 = \pi \in Z^0(G_3)(\mathbb{Q})$, as in the proof of Proposition 9.3 we argue that we have a $\mathbb{Q}$–isogeny $i : A(h) \to A_0$ that defines naturally an identification $(M_0[\frac{1}{p}], \phi_0) = (M[\frac{1}{p}], \phi)$, an element $j \in \boldsymbol{GL}_M(B(k))$ such that $j(h(M)) = M_0$, and an element $g \in \mathcal{G}(h)^0(W(k))$ such that $j^{-1}\phi j = g\phi$.

Both $Z_h(\phi)$ and $j^{-1}Z(\phi_0)j$ are $\mathbb{G}_m$ subgroup schemes of $\mathcal{G}(h)$ and their Lie algebras have natural generators fixed by ϕ. As these generators are identified naturally with elements of $Z^0(G_{3\mathbb{Q}_p})$ and as the generic fibres of both points $\tilde{z}_\infty$ and z_0 factor through $\mathrm{Sh}(G_3, \mathcal{X}_3)/(G_3(\mathbb{A}_f) \cap (H_0 \times H))$, over $B(\bar{k})$ these generators are $\mathcal{G}^0(B(\bar{k}))$-conjugate and therefore they are also $\mathcal{G}(h)^0(W(k))$-conjugate. Thus we can assume that $j^{-1}Z(\phi_0)j = Z_h(\phi)$ and therefore that $g \in (L^0_{\mathcal{G}(h)}(\phi) \cap \mathcal{G}(h)^0)(W(k))$. As in the proof of Proposition 9.3 we argue that we can assume that $g \in L^{0\mathrm{der}}_{\mathcal{G}(h)}(\phi)(W(k))$. As G^{der} is simply connected, from Fact 2.6.1 we get that $L^{0\mathrm{der}}_{\mathcal{G}(h)}(\phi)$ is simply connected. Thus as in the proof of Proposition 9.3 we argue that we can assume that there exists an element $h' \in L^{0\mathrm{der}}_{\mathcal{G}(h)}(\phi)(B(k))$ such that we have $g\phi = h_1'^{-1}\phi h'$. If $h_1 := h'h \in \mathcal{G}^0(B(k))$, then we have an isomorphism $h'j^{-1} : (M_0, \phi_0) \xrightarrow{\sim} (h_1(M), \phi)$ and therefore $h_1 \in \mathfrak{I}(\mathcal{C}, \lambda_A)$. The rest of the proof is as the last part of the proof of Proposition 9.3 (b). $\square$

9.6. Theorem. *Let $n \in \mathbb{N} \setminus \{1\}$. We assume that all simple factors of $(G^{\mathrm{ad}}, \mathcal{X}^{\mathrm{ad}})$ are of either C_n or $D_n^{\mathbb{H}}$ type, that all simple factors of $(G_1^{\mathrm{ad}}, \mathcal{X}_1^{\mathrm{ad}})$ are of A_{2n-1} type, and that $C_{\mathbb{Q}}$ is indecomposable (equivalently, and that $C_{\mathbb{Q}}$ is the group scheme of invertible elements of a simple $\mathbb{Q}$–algebra). We also assume that $Z^0(G) = Z^0(G_1)$ and that the monomorphism $G^{\mathrm{der}}_{\mathbb{C}} \hookrightarrow G^{\mathrm{der}}_{1\mathbb{C}}$ is a product of monomorphisms of one of the forms: $\boldsymbol{Sp}_{2m} \hookrightarrow \boldsymbol{SL}_{2m}$, $\boldsymbol{SO}_{2m} \hookrightarrow \boldsymbol{SL}_{2m}$, and $\boldsymbol{SL}_m \hookrightarrow \boldsymbol{SL}_m$. Then $TT\mathfrak{A}$ holds for $\mathcal{C}$.*

Proof: We have $C_{1\mathbb{Q}} = G_1$ (i.e., the group $C_{1\mathbb{Q}}$ is connected), cf. the hypothesis that pertains to the A_{2n-1} type. The Zariski closure $C_{\mathbb{Z}_{(p)}}$ of $C_{\mathbb{Q}}$ in $\mathbf{GL}_{L_{(p)}}$ is a reductive group scheme, cf. Theorem 2.4.2 (b) applied over $\mathbb{Z}_p$. Thus the Zariski closure of G_1 in $\mathbf{GL}_{L_{(p)}}$ is a reductive group scheme, the triple (f_1, L, v_1) is a standard Hodge situation, and the $O_{(v_1)}$-scheme $\mathcal{N}_1/H_{01}$ is smooth (see [LR] and [Ko2]). Let y_1 and $\mathcal{G}_1$ be as in Subsubsection 9.1.2. Thus $\mathcal{C}_1 := (M, \phi, \mathcal{G}_1)$ is the Shimura F-crystal attached to the point $y_1 \in \mathcal{N}_{1k(v_1)}/H_{01}(k)$. The statement of the Theorem depends only on $\mathcal{C}$ up to the operations $\mathfrak{O}_1$ and $\mathfrak{O}_2$ and therefore from now we will forget about f and $\mathcal{N}$ and we will only keep in mind that the quadruple $(M, \phi, \mathrm{Lie}(C_{\mathbb{Z}_{(p)}}), \lambda_A)$ is the crystalline realization of a principally polarized abelian variety endowed with $\mathbb{Z}_{(p)}$-endomorphisms $(A, \lambda_A, \mathrm{Lie}(C_{\mathbb{Z}_{(p)}}))$ over k and that our hypotheses get translated into properties of the group schemes $\mathcal{G}$, $\mathcal{G}_1$, etc. By performing the operation $\mathfrak{O}_1$, we can assume that $\mathcal{G}$ and $\mathcal{G}_1$ are split. The main property required below is the following one (cf. hypotheses):

(i) We have $Z^0(\mathcal{G}) = Z^0(\mathcal{G}_1)$ and the monomorphism $\mathcal{G}^{\mathrm{der}} \hookrightarrow \mathcal{G}_1^{\mathrm{der}}$ is a product of monomorphisms of one of the forms: $\mathbf{Sp}_{2m} \hookrightarrow \mathbf{SL}_{2m}$, $\mathbf{SO}_{2m}^{\mathrm{split}} \hookrightarrow \mathbf{SL}_{2m}$, and $\mathbf{SL}_m \hookrightarrow \mathbf{SL}_m$.

By performing the operation $\mathfrak{O}_2$, we can assume that $L^0_{\mathcal{G}}(\phi)$ is a reductive group scheme, cf. Subsection 2.6. Thus we can replace $\mathcal{C}$ by $(M, \phi, L^0_{\mathcal{G}}(\phi))$. From Fact 2.6.1 and the property (i) we get the existence of a direct sum decomposition

$$(M, \phi) = \oplus_{j \in J} (M_j, \phi)$$

into F-crystals over k that have only one Newton polygon slope such that for each $j \in J$ the following two properties hold:

(ii) the adjoint of the image $L^0(j)$ of $L^0_{\mathcal{G}}(\phi)$ in $\mathbf{GL}_{M_j}$ via the projection $\prod_{j \in J} \mathbf{GL}_{M_j} \to \mathbf{GL}_{M_j}$, has all simple factors of the same Lie type $\theta(j) \in \{C_m, D_m, A_m | m \in \{1, \ldots, n\}\}$;

(iii) the image $L^0_1(j)$ of $L^0_{\mathcal{G}_1}(\phi)$ in $\mathbf{GL}_{M_j}$ via the same projection is either $L^0(j)$ or its adjoint has all simple factors of the same Lie type A_{2m-1} and $\theta(j) \in \{C_m, D_m\}$.

Let $\mathcal{T}_{1B(k)}$ be a maximal torus of $\mathcal{G}_{B(k)}$ of $\mathbb{Q}_p$-endomorphisms of $\mathcal{C}$. Its centralizer $\mathcal{T}'_{1B(k)}$ in $\mathcal{G}_{1B(k)}$ is a maximal torus of $\mathcal{G}_{1B(k)}$ of $\mathbb{Q}_p$-endomorphisms of $\mathcal{C}_1$, cf. property (i) and the fact that in Subsubsection 9.1.2 we assumed that $\mathcal{T}(\phi)$ is a torus. Let $\mathcal{T}_{11B(k)}$ be the centralizer of $\mathcal{T}_{1B(k)}$ in $C_{\mathbf{GL}_{M[\frac{1}{p}]}}(C_{B(k)})$; it is a torus over $B(k)$. We use the notations of Definition 2.3 (c), an upper index $'$ being used in connection to $\mathcal{T}'_{1B(k)}$. We apply [Ha, Lem. 5.5.3] to the reductive subgroup of $C(\phi)_{\mathbb{Q}}$ that normalizes the $\mathbb{Q}$–span of λ_A and that fixes each $\mathbb{Q}$–endomorphism of A defined by an element of $\mathrm{Lie}(C_{\mathbb{Q}})$. Thus up to a replacement of $(M, \phi, L^0_{\mathcal{G}}(\phi))$ by $(M, \phi, hL^0_{\mathcal{G}}(\phi)h^{-1})$, where $h \in \mathcal{G}_1(W(k))$ commutes with ϕ, we can assume that $\mathrm{Lie}(\mathcal{T}'_{1B(k)})$ is $B(k)$-generated by elements of E_A. From [Zi1, Thm. 4.4] we get that up to the operations $\mathfrak{O}_1$ and $\mathfrak{O}_2$, the isogeny class of abelian varieties endowed with endomorphisms whose crystalline realization is $(M[\frac{1}{p}], \phi, (\mathrm{Lie}(\mathcal{T}_{11B(k)}) + \mathrm{Lie}(C_{B(k)})) \cap \mathrm{End}(A))$ has a lift to the ring of fractions of a finite field extension K'_3 of K'_2. Thus the Hodge cocharacter of this lift, when viewed in the crystalline context, is the extension to K'_3 of a cocharacter $\mu'_1 : \mathbb{G}_m \to \mathcal{T}'_{1K'_1}$ such the E-pair $(\mathcal{T}'_{1B(k)}, \mu'_1)$ of $\mathcal{C}_1$ is admissible. But the product $\mathfrak{Q}$ of the simple factors of $L^0_{\mathcal{G}}(\phi)^{\mathrm{ad}}$ which are of some A_m Lie type, $m \geq 2$, is the same as the similar product for $L^0_{\mathcal{G}_1}(\phi)^{\mathrm{ad}}$, cf. property (i). Thus we choose a cocharacter $\mu_1 : \mathbb{G}_m \to \mathcal{T}_{1K_1}$ such that the following three properties hold (to be compared with Subsection 6.1):

(iv.a) the cocharacters of $\mathfrak{Q}_{\overline{B(k)}}$ defined by μ_1 and μ'_1 coincide;

(iv.b) the cocharacter of the product of the simple factors of $L^0_{\mathcal{G}}(\phi)^{\mathrm{ad}}_{K_1}$ which are not subgroups of $\mathfrak{Q}_{K_2}$ defined by μ_1 is constructed based on Subsection 6.4;

(iv.c) a $\mathcal{G}(K_2)$-conjugate of μ_{1K_2} is the extension to K_2 of a Hodge cocharacter of $\mathcal{C}$.

Let $F^1_{K_2}$ be as in Definition 2.3 (h). The admissible filtered modules over K_2 are stable under direct sums. Thus to check that $(\mathcal{T}_{1B(k)}, \mu_1)$ is admissible (i.e., to end the proof of the Theorem), we can work with a fixed $j_0 \in J$ and we have to show that the filtered module $(N_{j_0}[\frac{1}{p}], \phi, F^1_{K_2} \cap (N_{j_0} \otimes_{W(k)} K_2))$ over K_2 is admissible. Let $J_A := \{j \in J | \theta(j) = A_m,\ m \in \mathbb{N}\backslash\{1\}\}$. If $j_0 \in J_A$, then $(N_{j_0}[\frac{1}{p}], \phi, F^1_{K_2} \cap N_{j_0} \otimes_{W(k)} K_2)$ is admissible as $(\mathcal{T}'_{1B(k)}, \mu_2)$ is admissible. If $j_0 \in J \setminus J_A$, then the fact that $(N_{j_0}[\frac{1}{p}], \phi, F^1_{K_2} \cap N_{j_0} \otimes_{W(k)} K_2)$ is admissible follows from Theorems 4.1 (b) and 4.2 (b) (cf. Subsection 6.4). $\square$

9.7. Example. There exists a second approach (besides the one mentioned in Remark 9.2.1 (b)) toward the proof that the isogeny property holds for (f, L, v). We exemplify it as well as Subsections 9.2 to 9.6 in the following concrete context.

9.7.1. Assumptions. Let $n \in \mathbb{N} \setminus \{1\}$. Let $m \in \mathbb{N}$. We assume that p does not divide $n-1$, that $p \geq 5$, that $2r = \dim_{\mathbb{Q}}(W) = 4nm$, and that $G^{\mathrm{der}}_{\mathbb{Z}_p} = \prod_{i=1}^m G^i$, where each G^i is an $\mathbf{Sp}_{2n}$ group scheme over $\mathbb{Z}_p$. We also assume that we have a direct sum decomposition

$$L_p := L \otimes_{\mathbb{Z}} \mathbb{Z}_p = \oplus_{i=1}^{2m} L^i_p \tag{11}$$

into free $\mathbb{Z}_p$-modules of rank $2n$ which is normalized by $G^{\mathrm{der}}_{\mathbb{Z}_p}$ and for which the following property holds:

(i) the representation of G^i on L^j_p is trivial if $j \notin \{2i-1, 2i\}$ and it is the standard rank $2n$ representation if $j \in \{2i-1, 2i\}$.

Let T_i be the $\mathbb{G}_m$ subgroup scheme of $\mathbf{GL}_{L_p}$ that acts trivially on L^j_p if $j \notin \{2i-1, 2i\}$ and as the inverse of the identical (resp. as the identical) character of $\mathbb{G}_m$ on L^{2i-1}_p (resp. on L^{2i}_p). We also assume that $G_{\mathbb{Z}_p}$ is generated by $G^{\mathrm{der}}_{\mathbb{Z}_p}$, by $Z(\mathbf{GL}_{L_p})$, and by the tori T_i with $i \in \{1, \ldots, m\}$; thus $G_{\mathbb{Z}_p}$ is split. This last assumption implies that for two elements $i_1,\ i_2 \in \{1, \ldots, 2m\}$, the $\mathbb{Z}_p$-modules $L^{i_1}_p$ and $L^{i_2}_p$ are perpendicular with respect to ψ if and only if $(i_1, i_2) \notin \{(1,2), (2,1), (3,4), (4,3), \ldots, (2m-1, 2m), (2m, 2m-1)\}$. Finally, we also assume that $G^{\mathrm{ad}}_{\mathbb{Q}}$ is $\mathbb{Q}$–simple and that the group $G^{\mathrm{ad}}_{\mathbb{R}}$ has compact, simple factors.

9.7.2. First properties of (f, L, v). We list some simple properties.

(a) We have $C_{\mathbb{Z}_p} := C_{\mathbf{GL}_{L_p}}(G_{\mathbb{Z}_p}) = \prod_{i=1}^{2m} Z(\mathbf{GL}_{L^i_p})$. Moreover $C_{\mathbf{GL}_{L_p}}(C_{\mathbb{Z}_p}) = \prod_{i=1}^{2m} \mathbf{GL}_{L^i_p}$. From the perpendicular aspects of Subsubsection 9.7.1 we get that $Z(G_{1\mathbb{Z}_p}) = Z(G_{\mathbb{Z}_p})$ and that $G^{\mathrm{der}}_{1\mathbb{Z}_p} = \prod_{i=1}^m G^i_1$, where each G^i_1 is an $\mathbf{SL}_{2n}$ group scheme that contains G^i. Thus G^i_1 acts trivially on L^j_p if $j \notin \{2i-1, 2i\}$ and we can assume that the representation of G^i_1 on L^{2i-1}_p is the standard rank $2n$ representation. Thus we have $G^i = N_{G^i_1}(G^i)$ for all $i \in \{1, \ldots, m\}$ and therefore $G_{\mathbb{Z}_p} = N_{G_{1\mathbb{Z}_p}}(G_{\mathbb{Z}_p})$.

(b) As p does not divide $2(n-1)$ the Killing and the trace forms on $\mathrm{Lie}(G^{\mathrm{der}}_{\mathbb{Z}_p})$ are perfect (see [Va1, Lem. 5.7.2.1]). Thus as in Subsection 7.1 we can assume that there exists a subset $\mathcal{J}_0$ of $\mathcal{J}$ such that $(v_\alpha)_{\alpha\in\mathcal{J}}$ is of partial degrees at most 2 and, when viewed as a family of tensors of $\mathcal{T}(L^*_{(p)}\otimes_{\mathbb{Z}_{(p)}}\mathbb{Z}_p)$, is $\mathbb{Z}_p$-very well positioned for $G_{\mathbb{Z}_p}$ (cf. also [Va1, Thm. 5.7.1]). Thus (f,L,v) is a standard Hodge situation (cf. [Va1, Thm. 5.1 and Rm. 5.6.5]) for which the GFT property holds. As $Z^0(G_{\mathbb{Z}_p})=Z^0(G_{1\mathbb{Z}_p})$, the Milne conjecture holds for (f,L,v) (cf. Theorem 9.2.2).

(c) The projector of L_p on L_p^i associated to (11) is fixed by $G_{\mathbb{Z}_p}$ and thus it is a $\mathbb{Z}_p$-linear combination of tensors of $\mathrm{End}(\mathrm{End}(L_{(p)}))$ fixed by $G_{\mathbb{Z}_{(p)}}$. Thus all of part (a) transfers automatically to the crystalline context of the Shimura F-crystal $\mathcal{C}$ of Subsubsection 9.1.1. Therefore we have a direct sum decomposition $M=\oplus_{i=1}^{2m}M^i$ normalized by $\mathcal{G}$ and formed by free $W(k)$-modules of rank $2n$. We have $\lambda_A(M^{i_1}\otimes M^{i_2})=0$ if and only if $(i_1,i_2)\notin\{(1,2),(2,1),\ldots,(2m-1,2m),(2m,2m-1)\}$. As in the part (a) we argue that $\mathcal{G}=N_{\mathcal{G}_1}(\mathcal{G})$ (the group scheme $\mathcal{G}_1$ being identified with the intersection of $\mathbf{GSp}(M,\lambda_A)$ with the double centralizer of $\mathcal{G}$ in $\mathbf{GL}_M$). Moreover, we have a direct sum decomposition $\mathcal{G}^{\mathrm{der}}=\prod_{i=1}^m\mathcal{G}^i$ with the property that each $\mathcal{G}^i$ acts trivially on M^j if $j\neq i$. As $G^{\mathrm{der}}_{\mathbb{Q}_p}$ is simply connected and $G^{0\mathrm{ab}}_{\mathbb{Q}_p}$ is split, the set $H^1(\mathbb{Q}_p,G^0_{\mathbb{Q}_p})$ has only one class.

(d) We show that the assumption that $C_{\mathbb{Q}}$ is decomposable leads to a contradiction. As G^{ad} is $\mathbb{Q}$–simple, any subrepresentation of the representation of G^{der} on W has dimension at least $2nm$. Thus if $C_{\mathbb{Q}}$ is decomposable, then there exists a direct sum decomposition $W=W_1\oplus W_2$ in G-modules of dimension $2nm$. Let F be a totally real number field such that G^{ad} is the $\mathrm{Res}_{F/\mathbb{Q}}$ of an absolutely simple F-group $\tilde{G}^{\mathrm{ad}}_F$ (cf. [De2, Subsubsection 2.3.4 (a)]); we have $[F:\mathbb{Q}]=2m$. The maximal subgroup of $\mathbf{GL}_{W_i}$ that commutes with G is $\mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$. This implies that $Z^0(G_{\mathbb{R}})$ is a split torus of dimension $m+1$ (more precisely, we have a short exact sequence $0\to\mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m\to Z^0(G)\to\mathbb{G}_m\to 0$). But the maximal split torus of $Z^0(G_{\mathbb{R}})$ is $Z(\mathbf{GL}_{W\otimes_{\mathbb{Q}}\mathbb{R}})$ and thus it has dimension 1. As $m+1>1$, we reached a contradiction. Thus $C_{\mathbb{Q}}$ is indecomposable. Thus $TT\mathfrak{A}$ holds for $\mathcal{C}$, cf. Theorem 9.6. From this and the end of part (c) we get that $TT++\mathfrak{A}$ holds for $(\mathcal{C},\lambda_A)$.

9.7.3. The isogeny property. We check that the isogeny property holds for (f,L,v). Let $h\in\mathfrak{I}(\mathcal{C},\lambda_A)$. Let D and $D(h)$ be the p-divisible groups of A and $A(h)$ (respectively). Let $D=\oplus_{i=1}^{2m}D^i$ and $D(h)=\oplus_{i=1}^{2m}D(h)^i$ be the direct sum decompositions that correspond naturally to (11). For each $i\in\{1,\ldots,m\}$, the duals of D^{2i-1} and $D(h)^{2i-1}$ are D^{2i} and $D(h)^{2i}$ (respectively). Let $T(i)$ be the image of μ in $G^{i\mathrm{ad}}$. If the torus $T(i)$ is trivial, then $(M^{2i-1}\oplus M^{2i},F^1\cap(M^{2i-1}\oplus M^{2i}),\phi)$ is a canonical lift (for all points $z\in\mathcal{N}/H_0(W(k))$). For the remaining part of this paragraph we assume that $T(i)$ is non-trivial. Let ψ_{2i-1} be a perfect alternating form on L_p^{2i-1} normalized by the image of $G_{\mathbb{Z}_p}$ in $\mathbf{GL}_{L_p^{2i-1}}$ via the projection $\prod_{i=1}^{2m}\mathbf{GL}_{L_p^i}\to\mathbf{GL}_{L_p^{2i-1}}$; it is unique up to a $\mathbb{G}_m(\mathbb{Z}_p)$-multiple. Let $\lambda_{D^{2i-1}}$ and $\lambda_{D(h)^{2i-1}}$ be the principal quasi-polarizations of D^{2i-1} and $D(h)^{2i-1}$ that correspond naturally to ψ_{2i-1}. Let $j_k:(D^{2i-1},\lambda_{D^{2i-1}})\to(D(h)^{2i-1},\lambda_{D(h)^{2i-1}})$ be the $\mathbb{Q}_p$-isogeny defined by the equality $M^{2i-1}[\frac{1}{p}]=h(M^{2i-1})[\frac{1}{p}]$. From the proof of [FC, Ch. VII, Prop. 4.3] we get that j_k lifts to a $\mathbb{Q}_p$-isogeny

$$j_{k[[X]]}:(E^{2i-1}_{k[[X]]},\lambda_{E^{2i-1}_{k[[X]]}})\to(E(h)^{2i-1}_{k[[X]]},\lambda_{E(h)^{2i-1}_{k[[X]]}})$$

between principally quasi-polarized p-divisible groups over $k[[X]]$ such that the fibre of $E^{2i-1}_{k[[X]]}$ over $k((X))$ is ordinary. Let $R_1 := W(k)[[X]]$ and Φ_{R_1} be as in Subsection 3.3. We can identify the principally quasi-polarized F-crystal of $(E^{2i-1}_{k[[X]]}, \lambda_{E^{2i-1}_{k[[X]]}})$ over $k[[X]]$ with

$$(M^{2i-1} \otimes_{W(k)} R_1, g_{2i-1}(\phi \otimes \Phi_{R_1}), \nabla_{2i-1}, \lambda_{M^{2i-1}})$$

where $\lambda_{M^{2i-1}}$ is the perfect alternating form on M^{2i-1} defined by $\lambda_{D^{2i-1}}$ and $g_{2i-1} \in \mathbf{Sp}(M^{2i-1}, \lambda_{M^{2i-1}})(R_1) = \mathcal{G}^i(R_1)$.

Let $g \in \mathcal{G}^{\mathrm{der}}(R_1)$ be such that for each $i \in \{1, \ldots, m\}$ its component in $\mathcal{G}^{2i-1}(R_1)$ is the identity element (resp. is g_{2i-1}) if $T(2i-1)$ is trivial (resp. is non-trivial). Let $\mathcal{C}^{\bar{R}_1}$ be the extension to $\bar{R}_1 := W(\bar{k})[[X]]$ of $(M \otimes_{W(k)} R_1, g(\phi \otimes \Phi_{R_1}), \nabla, \lambda_A, (t_\alpha)_{\alpha \in \mathcal{J}})$, where the connection ∇ on $M \otimes_{W(k)} R_1$ is obtained as in Lemma 3.4. Let d, R_d, and Φ_{R_d} be as in the proof of Theorem 9.4. Let $J : \mathrm{Spec}(R_d) \to \mathcal{N}_{W(\bar{k})}$ be a formally smooth morphism through which y_∞ factors. The principally quasi-polarized filtered F-crystal with tensors over R_d/pR_d defined by the pull back of the triple $\mathcal{R}$ via J is of the form

$$\mathcal{C}^{R_d} = (M \otimes_{W(k)} R_d, g_d(\phi \otimes \Phi_{R_d}), \nabla_d, \lambda_A, (t_\alpha)_{\alpha \in \mathcal{J}}),$$

where $g_d \in \mathcal{G}^{\mathrm{der}}(R_d)$ and ∇_d is a connection on $M \otimes_{W(k)} R_d$ whose Kodaira–Spencer map has as image the image of $\mathrm{Lie}(\mathcal{G}) \otimes_{W(k)} R_d$ in $\mathrm{Hom}_{W(k)}(M/F^1, F^1) \otimes_{W(k)} R_d$ (cf. [Va1, Subsubsections 5.4.4 to 5.4.8 and Subsection 5.5]). Moreover, $\mathcal{C}^{R_1}$ is induced from $\mathcal{C}^{R_d}$ via a $W(\bar{k})$-morphism $j_d : R_d \to R_1$ that maps the ideal $(X_1, \ldots, X_d)$ to the ideal (X) (cf. [Fa, Thm. 10 and Rm. (iii) of p. 136]).

We now check that $y(h)$ up to the operation $\mathfrak{O}_1$ factors through $\mathcal{N}/H_0$. Let M_1 be a $W(k)$-lattice of $M[\frac{1}{p}]$ such that we have a direct sum decomposition $M_1 = \oplus_{i=1}^m M_1 \cap M^i[\frac{1}{p}]$ and each ψ_{2i-1} induces a perfect alternating form on $M_1 \cap M^{2i-1}[\frac{1}{p}]$. If moreover (M_1, ϕ, λ_A) is a principally quasi-polarized Dieudonné module, then it is easy to see that there exists an element $h \in \mathcal{G}^0(B(k))$ such that we have $h(M) = M_1$ (cf. also Remark 9.2.1 (c)). We have a natural variant of this for $\overline{k((X))}$-valued points of $\mathcal{N}/H_0$. Thus due to existence of j_d and $j_{k[[X]]}$, using a standard specialization argument to check that $y(h)$ up to the operation $\mathfrak{O}_1$ factors through $\mathcal{N}/H_0$ we can assume that (M, ϕ) is ordinary. Let F^1_0 be the canonical lift of (M, ϕ). As $p \geq 3$ and as the $W(\bar{k})$-morphism $\mathcal{N}_{W(\bar{k})} \to \mathcal{M}_{W(\bar{k})}$ is a formally closed embedding at each $\bar{k}$-valued point of $\mathcal{N}_{W(\bar{k})}$ (cf. [Va1, Cor. 5.6.1]), there exists a unique point $z_0 \in \mathcal{N}/H_0(W(k))$ such that its attached Shimura filtered F-crystal is $(M, F^1_0, \phi, \mathcal{G})$. Not to introduce extra notations we will assume that $F^1 = F^1_0$ and $z = z_0$. As (M, ϕ) is ordinary, the direct summand $F^1[\frac{1}{p}] \cap h(M)$ of $h(M)$ is a lift of $(h(M), \phi, \mathcal{G})$. Let $z(h) : \mathrm{Spec}(W(k)) \to \mathcal{M}_{O_{(v)}}/\mathcal{K}^p(N_0)^p$ be the lift of $y(h)$ such that $F^1[\frac{1}{p}] \cap h(M)$ is the Hodge filtration of $h(M)$ defined by $A(h)_{W(k)}$, where $(A(h)_{W(k)}, \lambda_{A(h)_{W(k)}})$ is the pull back of the universal abelian scheme over $\mathcal{M}_{O_{(v)}}/\mathcal{K}^p(N_0)^p$ via $z(h)$. We have a $\mathbb{Z}[\frac{1}{p}]$-isogeny $j_{W(k)} : A_{W(k)} \to A(h)_{W(k)}$ that lifts the $\mathbb{Z}[\frac{1}{p}]$-isogeny $A \to A(h)$ and that is compatible with the principal polarizations. Let $j_{\mathbb{C}}$ be its extension to $\mathbb{C}$ via an $O_{(v)}$-embedding $W(k) \hookrightarrow \mathbb{C}$. Let $L_0 := H_1(A_{\mathbb{C}}, \mathbb{Z})$. Let $L_1 := H_1(A(h)_{\mathbb{C}}, \mathbb{Z})$. We can identify $W = L_0 \otimes_{\mathbb{Z}} \mathbb{Q} = L_1 \otimes_{\mathbb{Z}} \mathbb{Q}$ in such a way that ψ and each v_α with $\alpha \in \mathcal{J}$ are the Betti realizations of the principal polarizations of $A_{\mathbb{C}}$ and $A(h)_{\mathbb{C}}$ and respectively of the Hodge cycle w_α, cf. [Va1, Subsection 4.1]. Such an identification is unique up to an element of $G^0(\mathbb{Q})$. We have $L_0[\frac{1}{p}] = L_1[\frac{1}{p}]$. Also $G(\mathbb{Q}_p) = G(\mathbb{Q})H_0$, cf. [Mi4, Lem. 4.9]. Thus in order to get that, up to the operation $\mathfrak{O}_1$, the point $z(h)$ factors through $\mathcal{N}/H_0$, we only have to show that there exists an element $g_{\mathbb{Q}_p} \in G(\mathbb{Q}_p)$ such that $g_{\mathbb{Q}_p}(L_0 \otimes_{\mathbb{Z}} \mathbb{Z}_p) = L_1 \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

Let $M = F^1 \oplus F^0$ be the direct sum decomposition left invariant by ϕ. We have $h(M) = (h(M) \cap F^1[\frac{1}{p}]) \oplus (h(M) \cap F^0[\frac{1}{p}])$. Let $\mu_{\text{can}} : \mathbb{G}_m \to \mathbf{GL}_M$ be the canonical split cocharacter of (M, F^1, ϕ) as defined in [Wi, p. 512]. It normalizes F^1 and fixes F^0 and thus it fixes each element of $\mathcal{T}(M)$ fixed by ϕ. Thus μ_{can} factors through $\mathcal{G}$. Let $\mathfrak{F}$ be the Frobenius lift of the Fontaine ring $B^+(W(\bar{k}))$ of $W(\bar{k})$ (to be compared with Subsubsection 5.3.1). Let $\beta_0 \in F^1(B^+(W(\bar{k}))$ be as in [Fa, p. 125]. We have $\mathfrak{F}(\beta_0) = p\beta_0$ and $\text{Gal}(B(\bar{k}))$ acts on β_0 via the cyclotomic character. As $A(h)_{W(\bar{k})}$ is a canonical lift, its p-divisible group is a direct sum of nm copies of $\mathbb{Q}_p/\mathbb{Z}_p \oplus \mu_{p^\infty}$. Thus from Fontaine comparison theory for $A(h)_{W(k)}$ we deduce the existence of a $B^+(W(\bar{k}))$-isomorphism

$$(h(M) \cap F^1[\frac{1}{p}]) \otimes_{W(k)} B^+(W(\bar{k})) \oplus (h(M) \cap F^0[\frac{1}{p}]) \otimes \frac{1}{\beta_0} B^+(W(\bar{k})) \xrightarrow{\sim} L_0^* \otimes_{\mathbb{Z}} B^+(W(\bar{k}))$$

that takes λ_A to ψ^* and (cf. [Va4, Fact 8.1.3]) takes t_α to v_α for all $\alpha \in \mathcal{J}$. Thus as h fixes λ_A, the existence of $g_{\mathbb{Q}_p}$ is implied by the fact that each torsor of $G^0_{\mathbb{Z}_p}$ trivial with respect to the flat topology is trivial. Thus the isogeny property holds for (f, L, v).

9.7.4. The ST property. The pull back $\mathcal{O}$ to $\mathcal{N}_{k(v)}$ of the ordinary locus of $\mathcal{M}_{k(v)}$ is Zariski dense in $\mathcal{N}_{k(v)}$, cf. Subsubsection 9.7.3. We now assume that y factors through $\mathcal{O}/H_0$. We can assume that z is such that (M, F^1, ϕ) is a canonical lift. Let μ_{can} and $M = F^1 \oplus F^0$ be as in Subsubsection 9.7.3. The generic fibre of μ_{can} factors through the center of the centralizer of $\mathcal{T}(\phi)$ in $\mathbf{GL}_{M[\frac{1}{p}]}$. Also μ_{can} is the only Hodge cocharacter of $\mathcal{C}$ that commutes with μ_{can}. Thus the ST property holds for (f, L, v), cf. the uniqueness of z in Subsubsection 9.7.3.

9.7.5. Conclusion. All assumptions of Subsections 9.3, 9.4, 9.5, and 9.5.1 hold in the context described in Subsubsection 9.7.1, cf. Subsubsections 9.7.2 (c) to 9.7.4. Thus the results 9.4, 9.4.1, and 9.5.1 hold. These results can be interpreted as follows. Up to the operations $\mathfrak{O}_1$ and $\mathfrak{O}_2$ (i.e., up to replacements of k by a finite field extension of it and of y by an $y(h)$ with $h \in \mathfrak{I}(\mathcal{C}, \lambda_A)$), we can assume that there exists a finite, totally ramified, discrete valuation ring extension V of $W(k)$ and a point $z_y \in \mathcal{N}/H_0(V)$ such that the abelian scheme $z_y^*(\mathcal{A}_{H_0})$ is with complex multiplication. This represents the extension of [Zi1, Thm. 4.4] to the context of (f, L, v).

As $G^{\text{ad}}_{\mathbb{R}}$ has compact simple factors of C_n Dynkin type, the adjoint of $\text{Sh}(G, \mathcal{X})$ is not the adjoint of a Shimura variety of PEL type (see [Sh]).

9.8. Remarks. (a) Let $(\tilde{G}, \tilde{\mathcal{X}})$ be a simple, adjoint Shimura pair of C_n type. We assume that $\tilde{G}_{\mathbb{R}}$ is not split. Let $\tilde{F}$ be a totally real number field such that $\tilde{G}$ is $\text{Res}_{\tilde{F}/\mathbb{Q}} \tilde{G}'_{\tilde{F}}$, with $\tilde{G}'_{\tilde{F}}$ as an absolutely simple $\tilde{F}$-group. We assume that there exists a totally imaginary quadratic extension $\tilde{K}$ of $\tilde{F}$ such that $\tilde{G}'_{\tilde{K}}$ is split. We consider the standard $2n$ dimensional faithful representation $\tilde{f} : \tilde{G}'^{\mathrm{sc}}_{\tilde{K}} \hookrightarrow \mathbf{GL}_{\tilde{W}}$ over $\tilde{K}$. Let W be $\tilde{W}$ but viewed as a $\mathbb{Q}$–vector space. We view naturally $\tilde{G}^{\mathrm{sc}} = \mathrm{Res}_{\tilde{F}/\mathbb{Q}} \tilde{G}'^{\mathrm{sc}}_{\tilde{F}}$ as a subgroup of $\mathrm{Res}_{\tilde{K}/\mathbb{Q}} \tilde{G}'^{\mathrm{sc}}_{\tilde{K}}$ and thus of $\mathbf{GL}_W$. Let G be the subgroup of $\mathbf{GL}_W$ generated by $\tilde{G}^{\mathrm{sc}}$, by $Z(\mathbf{GL}_W)$, and by the maximal torus of $\mathrm{Res}_{\tilde{K}/\mathbb{Q}} \mathbb{G}_m$ which over $\mathbb{R}$ is compact. It is easy to see that there exists a $G(\mathbb{R})$-conjugacy class $\mathcal{X}$ of homomorphisms $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to G_{\mathbb{R}}$ that define Hodge $\mathbb{Q}$–structures on W of type $\{(0,-1),(-1,0)\}$. Let ψ be a non-degenerate alternating form on W such that we have an injective map $f : (G, \mathcal{X}) \hookrightarrow (\mathbf{GSp}(W,\psi), \mathcal{S})$ of Shimura pairs, cf. [De2, Cor. 2.3.3]. If the group $G_{\mathbb{Q}_p}$ is split and if (f, L, v) is a standard Hodge situation such that v divides a prime p which is at least 5 and does not divide $n-1$, then all assumptions of Subsubsection 9.7.1 hold. Thus Subsubsection 9.7.5 applies.

(b) In [Zi1, Thm. 4.4] and thus also in Theorem 9.6, the assumption that $C_{\mathbb{Q}}$ is indecomposable is not necessary (being inserted only to ease the notations). If $(G^{\mathrm{ad}}, \mathcal{X}^{\mathrm{ad}})$ is of A_n type and if the group $G^{\mathrm{ad}}_{\mathbb{Q}_p}$ is unramified, then we can choose (see [Va5]) the injective map $f : (G, \mathcal{X}) \hookrightarrow (\mathbf{GSp}(W,\psi), \mathcal{S})$ to be a PEL type embedding for which there exists a $\mathbb{Z}$-lattice L of W with the property that for each prime v of $E(G, \mathcal{X})$ that divides p, the triple (f, L, v) is a standard Hodge situation; thus the variant of [Zi1, Thm. 4.4] which does not assume that $C_{\mathbb{Q}}$ is indecomposable applies to it. This and Remark 6.5.1 are the main reasons why in Sections 4 to 9 we focused more on the B_n, C_n, and D_n Dynkin type (see Corollaries 4.3 to 4.5, Theorem 9.6, Example 9.7, etc.).

(c) The methods of (a) and Example 9.7 apply to any simple, adjoint Shimura pair $(\tilde{G}, \tilde{\mathcal{X}})$ of C_n type for which the groups $\tilde{G}_{\mathbb{R}}$ and $\tilde{G}_{\mathbb{Q}_p}$ are non-split and split (respectively).

(d) Theorems 9.4 and 9.5.1 are the very essence of the extension of [Zi1] one gets for $p \geq 5$ and for all Shimura varieties of Hodge type, once the program of Remark 9.2.1 (b) is accomplished.

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