Coordinate Invariance of the Mellin Calculus without Asymptotics for Manifolds with Conical Singularities

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Abstract

Let D be a manifold with conical singularities, and denote by $\mathbb{D}$ the smooth bounded manifold with cylindrical ends obtained by blowing up near the singularities.

B.-W. Schulze has developed a framework for a pseudodifferential calculus on D by defining various classes of distribution spaces and operator algebras, working in fixed coordinates on the manifold $\mathbb{D}$. I am showing here that the Mellin Sobolev spaces without asymptotics, the cone algebra without asymptotics, and its ideal of smoothing operators are independent of the choice of coordinates and therefore may be considered intrinsic objects for manifolds with conical singularities.

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Introduction

The Mellin calculus in the form developed by B.-W. Schulze [11], [9] provides a general framework for the analysis on manifolds with conical singularities.

A manifold with conical singularities is a topological space D with a finite set of exceptional points, $v_1, \ldots, v_r$ such that $D \setminus \{v_1, \ldots, v_r\}$ is smooth manifold, while, in a neighborhood of each v_j, the manifold is homeomorphic to a cone $X_j \times \overline{\mathbf{R}}_+ / X_j \times \{0\}$, where each X_j is a compact manifold without boundary.

The basic idea now is the following: Outside the singular set, one uses the standard pseudodifferential calculus and the standard Sobolev spaces. Near a singularity v_j, however, one identifies the manifold with the cylinder $X_j \times \mathbf{R}_+$. For simplicity let us assume we are dealing with a single singularity v with associated cross-section X of dimension n. The analysis then relies on Mellin type operators and Mellin Sobolev spaces. More precisely, one considers Mellin operators on $\mathbf{R}_+$ taking values in the algebra of all pseudodifferential operators on X; the Mellin Sobolev spaces without asymptotics, $\mathcal{H}^{s,\gamma}(X^\wedge)$, are easiest to describe for $s \in \mathbf{N}$ when they consist of all functions u on $X \times \mathbf{R}_+$ such that $t^{n/2-\gamma}(t\partial_t)^k D_x^\alpha u(x,t) \in L^2(X \times \mathbf{R}_+)$ whenever $k + |\alpha| \leq s$.

Basic contributions to the theory of differential problems on spaces with singularities have been made by Kondrat'ev [3], Melrose [5], Plamenevskij [6], Schulze, and others. The novelty of Schulze's approach is that he introduces a complete pseudodifferential calculus with a very small ideal of residual elements, the so-called Green operators, leading to a Fredholm theory and results on the regularity and asymptotics of solutions to elliptic equations.

The Mellin calculus without asymptotics provides the analytic frame within which the main operations can be performed. Its components are (i) the Mellin Sobolev spaces, (ii) the cone algebra without asymptotics, and (iii) the corresponding space of residual operators, the smoothing Mellin operators.

In addition, the Mellin calculus on manifolds with conical singularities lays the analytical foundations for a variety of more elaborate constructions such as pseudodifferential calculi for manifolds with edges and and corners, boundary value problems [11], [7], [8], manifolds with components of different dimensions [12], etc.

One of the drawbacks of the theory so far has been that all constructions are performed in a fixed set of coordinates. While it could, of course, be conjectured that the actual choice of coordinates was irrelevant, this has never been shown. The present paper finally settles this question.

The paper starts with a short review of all important objects involved. Next, I derive some general properties of the coordinate transforms and analyze the expressions that naturally come up in connection with the invariance proof. Then I show that, under a change of coordinates, the Mellin Sobolev spaces are preserved, cf. Theorem 2.10. This immediately implies the invariance of the classes of residual elements, see Corollary 2.11. The final part studies the pullbacks of Mellin operators of arbitrary order under diffeomorphisms.

The point of main interest is the analysis on $X \times \overline{\mathbf{R}}_+$. In this text, the diffeomorphisms will depend on both the variables $x \in X$ and $t \in \overline{\mathbf{R}}_+$. Of course, one might argue that the manifold $I\!D$ can be endowed with a system of normal coordinates in a neighborhood of the boundary $X \times \{0\}$ of the cylinder $X \times \mathbf{R}_+$. The variable t would then be given invariantly as the geodesic distance to the boundary with respect to an arbitrarily chosen Riemannian metric, and all changes of coordinates were of the form $(x,t) \mapsto (\chi(x), t)$, χ

being some change of coordinates on X. An important special case, however, is the case where X is a single point only. Within this framework there would be only one possible choice of coordinates on $\mathbf{R}_+ = X \times \mathbf{R}_+$, yet it is one of the points here to show that also in this case the calculus is invariant under diffeomorphisms.

A topic of equal if not higher interest is the coordinate invariance of the Mellin calculus *with* asymptotics. This will be the subject of a forthcoming publication.

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1 A Short Review of the Mellin Calculus

1.1 Manifolds with Conical Singularities and Mellin Sobolev Spaces

1.1 Definition. A *manifold with conical singularities D of dimension* $n+1$ is a topological (second countable) Hausdorff space with a finite subset $\Sigma \subset D$ ('singularities') such that $D \backslash \Sigma$ is an $n+1$-dimensional manifold, and for every $v \in \Sigma$ there is an open neighborhood U_v of v, a compact manifold without boundary X_v of dimension n, and a system $\mathcal{F}_v \neq \emptyset$ of mappings with the following properties

(1) For all $\phi \in \mathcal{F}_v$, $\phi : U_v \to X_v \times [0,1)/X_v \times \{0\}$ is a homeomorphism with $\phi(v) = X_v \times \{0\}/X_v \times \{0\}$.

(2) Given $\phi_1, \phi_2 \in \mathcal{F}_v$, the restriction $\phi_1\phi_2^{-1} : X_v \times (0,1) \to X_v \times (0,1)$ extends to a *diffeomorphism* $X_v \times (-1,1) \to X_v \times (-1,1)$.

(3) The charts $\phi \in \mathcal{F}_v$ are compatible with the charts for the manifold for $D\backslash\Sigma$: The restriction $\phi : U_v \backslash \{v\} \to X_v \times (0,1)$ is a diffeomorphism.

The system $\mathcal{F}_v$ is assumed to be maximal with respect to these properties. In this article, I shall also assume that D is compact. $\mathbb{D}$, the *stretched object associated with D* is the topological space constructed by replacing, for every singularity v, the neighborhood U_v by $X_v \times [0,1)$ via glueing with any one of the diffeomorphisms ϕ. $\mathbb{D}$ is a compact manifold with boundary; $\mathrm{int}\mathbb{D}$ is its interior.

Throughout this article the notation D and $\mathbb{D}$ will be kept fixed. For simplicity we assume that there is only one singularity with cross-section X. Write $X^\wedge = X \times \mathbf{R}_+$. Let X be endowed with a Riemannian metric and let $X^\wedge$ carry the canonical (cylindrical) metric.

We shall say that a function or distribution is supported *close to the boundary of $\mathbb{D}$* if it vanishes outside the part of $\mathbb{D}$ that is identified with $X \times [0,1)$.

1.2 Definition. Let $U = U_1 \times U_2 \subseteq \mathbf{R}^n \times \mathbf{R}^n$ be open. We say that $p \in S^\mu(U, \mathbf{R}^n)$ provided that, for all multi-indices α, β, γ, the estimate

$$|D_\eta^\alpha D_y^\beta D_{y'}^\gamma p(y,y',\eta)| \le C_{\alpha,\beta,\gamma} \langle\eta\rangle^{\mu-|\alpha|}$$

holds. Writing $đ\eta = (2\pi)^{-n} d\eta$, the pseudodifferential operator $\mathrm{op}\, p$ is defined by

$$[\mathrm{op}\, p(f)](y) = \int \int_{U_2} e^{i(y-y')\eta} p(y,y',\eta) f(y') dy' đ\eta$$

for $f \in C_0^\infty(U_2), y \in U_1$. This reduces to

$$[\operatorname{op} p(f)](y) = \int e^{iy\eta} p(y,\eta)\hat{f}(\eta)đ\eta,$$

for 'simple' symbols, i.e. those that are independent of y'. Here, $\hat{f}(\eta) = \int e^{-iy\eta} f(y)dy$ is the Fourier transform of f.

We may also consider the case where a part of the covariables serves as parameters: For $U \subseteq \mathbf{R}^n$ open, $p \in S^\mu(U_y, \mathbf{R}^n_\eta \times \mathbf{R}^l_\lambda)$ then defines a parameter-dependent operator $\operatorname{op} p(\lambda)$ by

$$[\operatorname{op} p(\lambda)f](y) = \int e^{iy\eta} p(y,\eta,\lambda)\hat{f}(\eta)đ\eta,$$

$f \in C_0^\infty(U)$, similarly for 'double' symbols $p(y, y', \eta, \lambda)$. Of course, all symbols can take values in vector bundles, and all results carry over to this case. For the sake of simplicity, however, I shall consider scalar symbols only.

1.3 The Manifold Case. Let Ω be a smooth manifold and $P : C_0^\infty(\Omega) \to C^\infty(\Omega)$ a continuous operator. We say that $P \in L^\mu(\Omega)$ if the following holds:

(i) For all C_0^∞ functions ϕ, ψ, supported in the same coordinate neighborhood, the operator $(\phi P \psi)_* : C_0^\infty(U) \to C^\infty(U)$ induced on $U \subseteq \mathbf{R}^n$ by $\phi P \psi$ and the coordinate maps has the form $(\phi P \psi)_* = \operatorname{op} p$ for some $p \in S^\mu(U, \mathbf{R}^n)$.

(ii) For all C_0^∞ functions ϕ, ψ, with disjoint supports, the operator $\phi P \psi$ is given as an integral operator with a kernel in $C^\infty(\Omega \times \Omega)$ (more precisely a kernel section, see [2, Section 23.4]).

The fact that the pseudodifferential symbol classes on $\mathbf{R}^n$ are invariant under diffeomorphisms implies that property (i) is independent of the particular choice of the chart.

If P depends on a parameter $\lambda \in \mathbf{R}^l$, then (i) carries over, while in (ii) we ask that the integral kernel belongs to $\mathcal{S}(\mathbf{R}^l, C^\infty(\Omega \times \Omega))$. I shall then write $P \in L^\mu(\Omega; \mathbf{R}^l)$.

Suppose we are given a locally finite covering of the manifold by relatively compact coordinate neighborhoods $\{\Omega_j\}$ with associated coordinate maps $\chi_j : \Omega_j \to U_j$. Then we can find $p_j \in S^\mu(U_j, \mathbf{R}^n)$ and an integral operator with C^∞-kernel, K_j, such that $P(f \circ \chi_j)(\chi^{-1}(x)) = \operatorname{op} p_j(f)(x) + K_j f(x)$ for all $f \in C_0^\infty(U_j)$. We shall call the tuple $\{p_j\}$ the symbol of P.

1.4 The Mellin Transform. For $\beta \in \mathbf{R}$, Γ_β denotes the vertical line $\{z \in \mathbf{C} : \operatorname{Re} z = \beta\}$. The Mellin transform Mu of a complex-valued $C_0^\infty(\mathbf{R}_+)$-function u is given by

$$(Mu)(z) = \int_0^\infty t^{z-1} u(t)\, dt. \tag{1}$$

M is closely related to the Fourier transform and extends to an isomorphism $M : L^2(\mathbf{R}_+) \to L^2(\Gamma_{\frac{1}{2}})$. (1) also makes sense for functions with values in a Fréchet space E. The fact that $Mu|_{\Gamma_{\frac{1}{2}-\gamma}}(z) = M_{t\to z}(t^{-\gamma}u)(z+\gamma)$ motivates the following definition of the *weighted Mellin transform* M_γ:

$$M_\gamma u(z) = M_{t\to z}(t^{-\gamma}u)(z+\gamma), \quad u \in C_0^\infty(\mathbf{R}_+, E).$$

The inverse of M_γ is given by

$$[M_\gamma^{-1}h](t) = \frac{1}{2\pi i}\int_{\Gamma_{1/2-\gamma}} t^{-z}h(z)dz.$$

1.5 Sobolev Spaces and Weighted Mellin Sobolev Spaces. (a) $H^s(\Omega), s \in \mathbf{R}$, is the usual Sobolev space over a smooth compact manifold Ω with or without boundary. For non-compact Ω we will have to specify additionally a density on Ω.

(b) For $s \in \mathbf{N}$ and $\gamma \in \mathbf{R}$, the space $\mathcal{H}^{s,\gamma}(X^\wedge)$ is the set of all $u \in \mathcal{D}'(X^\wedge)$ such that $t^{\frac{n}{2}-\gamma}(t\partial_t)^k Du(x,t) \in L^2(X^\wedge)$ for all $k \le s$ and all differential operators D of order $\le s-k$ on X. Here we use the canonical cylindrical metric on $X^\wedge$ for the definition of $L^2(X^\wedge)$. Next we define $\mathcal{H}^{s,\gamma}(X^\wedge)$ for $s \ge 0$ by interpolation, then for $s \le 0$ by duality: $\mathcal{H}^{s,\gamma}(X^\wedge) = [\mathcal{H}^{-s,-\gamma}(X^\wedge)]'$ with respect to the pairing

$$(u,v) = \frac{1}{2\pi i}\int_{\Gamma_{\frac{n+1}{2}}} (Mu(z), Mv(z))_{L^2(X)}\, dz.$$

Finally, $\mathcal{H}^{\infty,\gamma}(X^\wedge) = \bigcap_{s\ge 0}\mathcal{H}^{s,\gamma}(X^\wedge)$. This is not quite the standard definition, which relies on parameter-elliptic pseudodifferential operators, but it is equivalent to it, cf. [9, Section 2.1.1, Proposition 2], and better adapted to our purposes.

(c) We can apply the same definition for $X = \mathbf{R}^n$. It is then easily seen that

$$\|u\|_{\mathcal{H}^{s,\gamma}(\mathbf{R}^n\times\mathbf{R}_+)} = \|\Phi_{n,\gamma}u\|_{H^s(\mathbf{R}^n\times\mathbf{R})}$$

with $\Phi_{n,\gamma}v(r) = \exp(r(\frac{n+1}{2}-\gamma))v(e^r) = \left(t^{\frac{n+1}{2}-\gamma}v(t)\right)\Big|_{t=e^r}$, cf. [9, 2.1.6(4)]. Via a partition of unity on X we obtain a relation between the standard Sobolev spaces on $X^\wedge$ and the Mellin Sobolev spaces on $X^\wedge$.

(d) The following relations hold: $\mathcal{H}^{s,\gamma}(X^\wedge) \subseteq H^s_{loc}(X^\wedge)$; $\mathcal{H}^{s,\gamma}(X^\wedge) = t^\gamma\mathcal{H}^{s,0}(X^\wedge)$; $\mathcal{H}^{0,0}(X^\wedge) = t^{-n/2}L^2(X^\wedge)$.

(e) Fix a smooth function ω on $\mathbb{D}$, equal to 1 close to the boundary and supported close to the boundary, cf. Definition 1.1. Given a distribution $u \in \mathcal{D}'(\mathrm{int}\mathbb{D})$ we can write $u = u_1 + u_2$ with $u_1 = \omega u$ supported close to the boundary and $u_2 = (1-\omega)u$ supported away from the boundary. We shall say that $u \in \mathcal{H}^{s,\gamma}(\mathbb{D})$, provided that $u_1 \in \mathcal{H}^{s,\gamma}(X^\wedge)$ and $u_2 \in H^s(\mathbb{D})$. According to (d), the definition is independent of the choice of ω. We can topologize $\mathcal{H}^{s,\gamma}(\mathbb{D})$ as a Hilbert space, using the Hilbert space structures on $\mathcal{H}^{s,\gamma}(X^\wedge)$ and $H^s(\mathbb{D})$. We then have interpolation and duality as above.

1.6 Remark. An easy fact which is useful to recall: For every $0 < k \in \mathbf{N}$ there are universal constants c_{kj} and d_{kj} such that

$$(t\partial_t)^k = \sum_{j=1}^k c_{kj}t^j\partial_t^j \quad\text{and}\quad t^k\partial_t^k = \sum_{j=1}^k d_{kj}(t\partial_t)^j.$$

1.2 The Cone Algebra without Asymptotics

1.7 Notation. In the following let $\mu, \gamma \in \mathbf{R}$ be fixed. Given $f \in C^\infty(\mathbf{R}_+ \times \mathbf{R}_+, L^\mu(X; \Gamma_{\frac{1}{2}-\gamma}))$ we shall write $f = f(t, t', z)$, where z indicates the variable in $\Gamma_{\frac{1}{2}-\gamma}$. For t, t', z fixed, $f(t, t', z)$ is a pseudodifferential operator on X. As before, I am assuming that all symbols are scalar.

1.8 Definition. Let $f \in C^\infty(\mathbf{R}_+ \times \mathbf{R}_+, L^\mu(X; \Gamma_{\frac{1}{2}-\gamma}))$. We define the *Mellin operator* $\mathrm{op}_M^\gamma f$ with the *(Mellin) symbol* f on $C_0^\infty(X^\wedge) = C_0^\infty(\mathbf{R}_+, C^\infty(X))$ by

$$[\mathrm{op}_M^\gamma(f)u](t) = \frac{1}{2\pi i} \int_{\Gamma_{\frac{1}{2}-\gamma}} \int_0^\infty (t/t')^{-z} f(t, t', z) u(t') \frac{dt'}{t'} dz \quad . \tag{1}$$

The right hand side of (1) is to be understood as an iterated integral. If f is independent of t' or, equivalently, $f \in C^\infty(\mathbf{R}_+, L^\mu(X; \Gamma_{\frac{1}{2}-\gamma}))$, then (1) reduces to

$$[\mathrm{op}_M^\gamma(f)u](t) = \frac{1}{2\pi i} \int_{\Gamma_{\frac{1}{2}-\gamma}} t^{-z} f(t, z) [M_\gamma u](z) dz. \tag{2}$$

We did not specify the variable x in (1) or (2), understanding that, for fixed $t', u(t') = u(\cdot, t')$ is in $C^\infty(X)$ and that $f(t, t', z)$ acts as a pseudodifferential operator with respect to the x-variables.

Like pseudodifferential double symbols, Mellin double symbols are not uniquely determined. It is immediate from integration by parts in (1) that

$$\mathrm{op}_M^\gamma[\ln^k(t/t') f(t, t', z)] = \mathrm{op}_M^\gamma[\partial_z^k f(t, t', z)]. \tag{3}$$

Similarly, it follows from a consideration of the integral kernels that

$$\mathrm{op}_M^\gamma[\phi(t/t') f(t, t', z)] = \mathrm{op}_M^\gamma[M_{\rho \to z}\{\phi(\rho) M_{\gamma, \zeta \to \rho}^{-1} f(t, t', \zeta)\}] \tag{4}$$

for every $\phi \in C_0^\infty(\mathbf{R}_+)$. For $f \in C^\infty(\mathbf{R}_+ \times \mathbf{R}_+, L^\mu(X; \Gamma_{\frac{1}{2}-\gamma}))$ or $f \in C^\infty(\mathbf{R}_+, L^\mu(X; \Gamma_{\frac{1}{2}-\gamma}))$ we will have a continuous map

$$\mathrm{op}_M^\gamma f : C_0^\infty(X^\wedge) \to C^\infty(X^\wedge).$$

Smoothness of f up to zero yields continuity of $\mathrm{op}_M^\gamma f$ on the weighted Mellin-Sobolev spaces, cf. Theorem 1.9; the preceding relation (3), however, shows that smoothness is not necessary.

1.9 Theorem. *Let $f \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, L^\mu(X; \Gamma_{\frac{1}{2}-\gamma}))$. Given $s \in \mathbf{R}$ and $\omega_1, \omega_2 \in C_0^\infty(\overline{\mathbf{R}}_+)$, there is a bounded extension*

$$\omega_1 [\mathrm{op}_M^\gamma f] \omega_2 : \mathcal{H}^{s, \gamma + \frac{n}{2}}(X^\wedge) \to \mathcal{H}^{s-\mu, \gamma + \frac{n}{2}}(X^\wedge).$$

We will also need the following results. They show that, just as in the case of pseudodifferential operators, one has asymptotic summation of symbols. Moreover, one obtains smoothing operators by a special analytic procedure that Schulze calls 'kernel excision'.

1.10 Asymptotic Summation. Let $\mu_1, \mu_2, \ldots$ be a sequence in $\mathbf{R}$ tending to $-\infty$, $f_j \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, L^{\mu_j}(X; \Gamma_{\frac{1}{2}-\gamma}))$, and $\mu = \max \mu_j$. Then there is an

$$f \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, L^\mu(X; \Gamma_{\frac{1}{2}-\gamma}))$$

with $f \sim \sum_{j=1}^\infty f_j$, i.e, for any $N \in \mathbf{N}$ there is a J with

$$f - \sum_{j=1}^{J} f_j \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, L^{\mu-N}(X; \Gamma_{\frac{1}{2}-\gamma})). \tag{1}$$

This f is unique modulo $C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, L^{-\infty}(X; \Gamma_{\frac{1}{2}-\gamma}))$.

1.11 Theorem. *Let $\phi \in C_0^\infty(\mathbf{R}_+)$ and suppose that $\phi(t) \equiv 1$ near $t = 1$. For $f \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, L^\mu(X; \Gamma_{\frac{1}{2}-\gamma}))$ let*

$$\begin{aligned} f_1(t, t', z) &= M_{\rho \to z}[\phi(\rho) M^{-1}_{\gamma, \zeta \to \rho} f(t, t', z)] \\ f_2(t, t', z) &= M_{\rho \to z}[(1 - \phi(\rho)) M^{-1}_{\gamma, \zeta \to \rho} f(t, t', z)]. \end{aligned}$$

Then $f_1 \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, L^\mu(X; \Gamma_{1/2-\gamma}))$ and $f_2(t, t', z) \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, L^{-\infty}(X; \Gamma_{\frac{1}{2}-\gamma}))$.

1.12 The Cone Algebra without Asymptotics.

(a) $ML_\gamma^{-\infty}(\mathbb{D})$ is the set of all operators $G : C_0^\infty(\mathrm{int}\mathbb{D}) \to \mathcal{D}'(\mathrm{int}\mathbb{D})$ such that, for all $s \in \mathbf{R}$, there is a continuous extension $G : \mathcal{H}^{s,\gamma}(\mathbb{D}) \to \mathcal{H}^{\infty,\gamma}(\mathbb{D})$.

(b) $ML_\gamma^\mu(\mathbb{D})$ is the space of all operators $A : C_0^\infty(\mathrm{int}\mathbb{D}) \to \mathcal{D}'(\mathrm{int}\mathbb{D})$ that can be written $A = A_M + A_\psi + t^{-\mu} G$, where

- A_M is a Mellin operator supported close to the boundary, i.e., there are functions $\omega_1, \omega_2 \in C^\infty(\mathbb{D})$, supported close to the boundary of $\mathbb{D}$, cf. Definition 1.1, and there is a Mellin symbol $f \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, L^\mu(X; \Gamma_{1/2-\gamma}))$ such that $A_M = t^{-\mu} \omega_1 \mathrm{op}_M^\gamma f \omega_2$;
- A_ψ is a pseudodifferential operator supported away from the boundary, i.e., there are functions ϕ_1, ϕ_2 vanishing in a neighborhood of the boundary of $\mathbb{D}$, and there is a symbol $p \in S^\mu(\mathrm{int}\mathbb{D})$ such that $A_\phi = \phi_1 \,\mathrm{op}\, p\, \phi_2$. Finally,
- G is an operator in $ML_\gamma^{-\infty}(\mathbb{D})$.

The collection of all the spaces $ML_\gamma^\mu(\mathbb{D})$, for $\mu, \gamma \in \mathbf{R}$ is the cone algebra without asymptotics.

1.13 Remark. (a) It is obvious that, for fixed μ, γ, the operators in $ML_\gamma^\mu(\mathbb{D})$ form a vector space and that an operator $A \in ML_\gamma^\mu(\mathbb{D})$ induces a continuous mapping

$$A : \mathcal{H}^{s,\gamma+n/2}(\mathbb{D}) \to \mathcal{H}^{s-\mu,\gamma-\mu+n/2}(\mathbb{D}).$$

In particular: If the Mellin symbol f of A_M is an element of $C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, L^{-\infty}(X; \Gamma_{1/2-\gamma}))$, then $A_M \in t^{-\mu} ML_\gamma^{-\infty}(\mathbb{D})$.

It is not so trivial that the cone algebra without asymptotics is an 'algebra' in the sense that, for all $\mu, \tilde{\mu}, \gamma \in \mathbf{R}$, the composition of operators induces a continuous multiplication

$$ML^{\tilde{\mu}}_{\gamma-\mu}(\mathbb{D}) \times ML^\mu_\gamma(\mathbb{D}) \to ML^{\mu+\tilde{\mu}}_\gamma(\mathbb{D}).$$

(b) It follows from the mapping properties that the operators in $ML_\gamma^{-\infty}(\mathbb{D})$ form an ideal in the sense that the above multiplication restricts to continuous maps

$$\begin{aligned} t^{-\tilde{\mu}}ML^{-\infty}_{\gamma-\mu}(\mathbb{D}) \times ML^{\mu}_{\gamma}(\mathbb{D}) &\to t^{-\mu-\tilde{\mu}}ML^{-\infty}_{\gamma}(\mathbb{D}); \\ ML^{\mu}_{\gamma-\mu}(\mathbb{D}) \times t^{-\mu}ML^{-\infty}_{\gamma}(\mathbb{D}) &\to t^{-\mu-\tilde{\mu}}ML^{-\infty}_{\gamma}(\mathbb{D}). \end{aligned}$$

(c) In particular, for $\mu = 0$ and arbitrary γ, $ML^0_\gamma(\mathbb{D})$ is an algebra in the usual sense, and $ML_\gamma^{-\infty}(\mathbb{D})$ is an ideal.

We shall need the following relation between Mellin and pseudodifferential operators.

1.14 Theorem. (a) *Let $\phi, \psi \in C_0^\infty(\mathrm{int}\mathbb{D})$, and let A_M be a Mellin operator as in Definition* 1.12(b). *Denote, for the moment, by M_ϕ and M_ψ the operators of multiplication by ϕ and ψ, respectively. Then there is a pseudodifferential operator $B \in L^\mu(\mathrm{int}\mathbb{D})$, supported in the interior of $\mathbb{D}$, with $M_\phi A_M M_\psi = B$.*

(b) *If $\phi, \psi \in C^\infty(\mathbb{D})$ and $\mathrm{supp}\phi \cap \mathrm{supp}\psi = \emptyset$, then $M_\phi A_M M_\psi \in t^{-\mu}ML_\gamma^{-\infty}(\mathbb{D})$.*

2 Coordinate Invariance

2.1 Outline. Assumptions on the Coordinate Transforms and Their Properties

2.1 Outline. In order to show the coordinate invariance of the Mellin calculus, we will first establish the coordinate invariance of the spaces $\mathcal{H}^{s,\gamma}(\mathbb{D})$, cf. Theorem 2.10. It entails the coordinate invariance of the residual classes $ML_\gamma^{-\infty}(\mathbb{D})$, cf. Corollary 2.11. The pseudodifferential operators in the calculus are supported away from the singular set. Their invariance is a consequence of the well-known fact that the pseudodifferential calculus on smooth manifolds is well-defined. The only subtle point therefore will be the behavior of the Mellin operators: If $\kappa : U \to V$ is a diffeomorphism of bounded open sets $U, V \subseteq \mathbf{R}^n \times \overline{\mathbf{R}}_+$, and

$$A = \mathrm{op}_M^\gamma f : C_0^\infty(V) \to C^\infty(V)$$

is a Mellin operator, what can we say about the pullback

$$A_* : C_0^\infty(U) \to C^\infty(U)$$

defined by

$$(A_*u)(\tilde{x}) = [A(u \circ \kappa^{-1})](\kappa(\tilde{x})), \quad u \in C_0^\infty(U), \tilde{x} \in U?$$

I shall show that A_* again is a Mellin operator by computing a Mellin symbol simply by substitution in the oscillatory integrals, then deriving an asymptotic expansion of this symbol and showing that it makes sense. In order to make this more precise, we need some notation.

2.2 Notation and Elementary Properties of the Changes of Coordinates. We want the change of coordinates to preserve the cylinder $X \times \overline{\mathbf{R}}_+$. So let U, V be bounded open subsets of $\mathbf{R}^n \times \overline{\mathbf{R}}_+$, and let

$$\kappa : U \to V \qquad \text{diffeomorphically.}$$

Write

$$(\underline{x},\underline{t}) = \kappa(x,t) = (\chi(x,t),\sigma(x,t)) = (\chi_1(x,t),...,\chi_n(x,t),\sigma(x,t)).$$

Since the boundary is preserved, $\kappa(x,0) = (\chi(x,0),0)$; hence, for all x with $(x,0) \in U$,

$$\partial_{x_j}\sigma(x,0) = 0, \qquad j = 1,...,n.$$

Furthermore, the total derivative $\partial\kappa$ is regular, so we necessarily have $\partial_t\sigma(x,0) \neq 0$ for all x, even

$$\partial_t\sigma(x,0) > 0,$$

since the $\mathbf{R}_+$-direction is preserved. In the total derivative $\partial\kappa$, written as an $(n+1)\times(n+1)$ matrix

$$\partial\kappa = \begin{pmatrix} \partial_x\chi & \partial_t\chi \\ \partial_x\sigma & \partial_t\sigma \end{pmatrix},$$

$\partial_x\chi(x,0)$ will be regular for all x. We may therefore find an $\epsilon_1 > 0$ such that, for all $(x,t) \in U$,

$$\epsilon_1 \leq \partial_t\sigma(x,t) \leq 1/\epsilon_1; \tag{1}$$

$$\epsilon_1 \leq |\det \partial_x\chi(x,t)| \leq 1/\epsilon_1. \tag{2}$$

Moreover, since we are only interested in changes of coordinates in a neighborhood of $X \times \{0\}$, we may assume that there is an $\epsilon_2 \ll \epsilon_1$, with

$$0 \leq |\partial_x\sigma(x,t)| < \epsilon_2 \tag{3}$$

for all $(x,t) \in U$, and that

$$U = U' \times [0,\epsilon_3) \tag{4}$$

for some convex open subset $U' \subseteq \mathbf{R}^n$, where $\epsilon_3 > 0$ is small.

2.3 Outline (continued). I would like to first explain the concept without specifying a particular symbol class. Assume that $f = f(\underline{t},\underline{t}',\underline{z},\underline{x},\underline{y},\underline{\xi})$ is a smooth function on $\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+ \times \Gamma_{\frac{1}{2}-\gamma} \times U' \times U' \times \mathbf{R}^n$, where U' is as in 2.2(4), that f vanishes unless $\underline{t}$ and $\underline{t}'$ are both small, and that f is subject to reasonable growth conditions. In order to simplify even more, let $\gamma = 1/2$, so that $\Gamma_{1/2-\gamma}$ is the imaginary axis, and use the variable $\tau \in \mathbf{R}$ instead of $z \in i\mathbf{R}$. We want to compute the pull-back of the Mellin operator $\mathrm{op}_M^{1/2} f$ induced by f. So we choose $u \in C_0^\infty(V), (x,t) \in U$, and let $\underline{u} = u \circ \kappa^{-1}$. By definition,

$$\begin{aligned}
&[\mathrm{op}_M^{\frac{1}{2}} f]_* u(x,t) = [\mathrm{op}_M^{\frac{1}{2}} f]\underline{u}(\underline{x},\underline{t}) \\
= &\int_{\mathbf{R}}\int_0^\infty \iint e^{i(\underline{x}-\underline{y})\underline{\xi}} \left(\frac{\underline{t}}{\underline{t}'}\right)^{-i\underline{\tau}} f(\underline{t},\underline{t}',\underline{\tau},\underline{x},\underline{y},\underline{\xi})\, \underline{u}(\underline{y},\underline{t}')\, d\underline{y} đ\underline{\xi} \frac{d\underline{t}'}{\underline{t}} đ\underline{\tau}.
\end{aligned} \tag{1}$$

Ignoring problems about the existence of the integrals and writing $\underline{y} = \chi(y,t'), \underline{t}' = \sigma(y,t')$,

$$\begin{aligned}
[\mathrm{op}_M^{\frac{1}{2}} f]_* u(x,t) \;=\; &\int_{\mathbf{R}}\iint\int_0^\infty e^{i(\chi(x,t)-\chi(y,t'))\underline{\xi}} \left(\frac{\sigma(x,t)}{\sigma(y,t')}\right)^{-i\underline{\tau}} \\
&f(\sigma(x,t),\sigma(y,t'),\underline{\tau},\chi(x,t),\chi(y,t'),\underline{\xi})\, u(y,t') \frac{J(y,t')}{\sigma(y,t')} t' \frac{dt'}{t'} dy đ\underline{\xi} đ\underline{\tau},
\end{aligned} \tag{2}$$

where $J(y,t') = |\det \partial\kappa(y,t')|$. We therefore will be interested in the behavior of $J(y,t')\frac{t'}{\sigma(y,t')}$. Assuming that everything works well, we then write

$$e^{i(\chi(x,t)-\chi(y,t'))\underline{\xi}}\left(\frac{\sigma(x,t)}{\sigma(y,t')}\right)^{-i\underline{\tau}} = e^{i(x-y)B_1(x,t,y,t')(\frac{\xi}{\tau})}\left(\frac{t}{t'}\right)^{-iB_2(x,t,y,t')(\frac{\xi}{\tau})}, \tag{3}$$

where B_1 is a suitable $n\times(n+1)$ matrix and B_2 an $1\times(n+1)$ matrix. Our above assumptions on the change of coordinates will imply that the $(n+1)\times(n+1)$ matrix $B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ is invertible. Let $A = B^{-1}$ be of the form $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ with an $n\times(n+1)$ and an $1\times(n+1)$ matrix, and let

$$\begin{bmatrix} \xi \\ \tau \end{bmatrix} = B\begin{bmatrix} \underline{\xi} \\ \underline{\tau} \end{bmatrix}, \quad \text{i.e.} \quad \begin{bmatrix} \underline{\xi} \\ \underline{\tau} \end{bmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}\begin{bmatrix} \xi \\ \tau \end{bmatrix} \tag{4}$$

Then we can change variables in (2) and obtain

$$\begin{aligned}
[\mathrm{op}_M^{\frac{1}{2}} f]_* u(x,t) &= \int_{\mathbb{R}}\iint\int_0^\infty e^{i(x-y)\xi}\left(\frac{t}{t'}\right)^{-i\tau}\cdot \\
&\quad \cdot f(\sigma(x,t),\sigma(y,t'),A_2\begin{bmatrix} \xi \\ \tau \end{bmatrix},\chi(x,t),\chi(y,t'),A_1\begin{bmatrix} \xi \\ \tau \end{bmatrix})\cdot \\
&\quad \cdot u(y,t')\,\frac{J(y,t')}{\sigma(y,t')}t'\,|\det A(x,t,y,t')|\,\frac{dt'}{t'}dy\,d\xi\,d\tau.
\end{aligned} \tag{5}$$

In other words, this purely formal computation shows that $[\mathrm{op}_M^{\frac{1}{2}} f]_* = \mathrm{op}_M^{\frac{1}{2}}\tilde{g}$ with

$$\begin{aligned}
&\tilde{g}(t,t',\tau,x,y,\xi) \\
= \;& f(\sigma(x,t),\sigma(y,t'),A_2\begin{bmatrix} \xi \\ \tau \end{bmatrix},\chi(x,t),\chi(y,t'),A_1\begin{bmatrix} \xi \\ \tau \end{bmatrix})\frac{J(y,t')}{\sigma(y,t')}\;t'\;|\det A(x,t,y,t')|.
\end{aligned} \tag{6}$$

Of course, $\tilde{g}$ will in general not be an element of the "right" symbol class, and we will have to replace $\tilde{g}$ by a symbol g such that $[\mathrm{op}_M^{\frac{1}{2}} f]_* - \mathrm{op}_M^{\frac{1}{2}} g$ belongs to the corresponding class of residual operators. Our first task now will be the analysis of the term $\frac{J(y,t')}{\sigma(y,t')}t'\;|\det A(x,t,y,t')|$.

2.4 Lemma. *There is a function $\psi \in C_b^\infty(U)$ with*

$$\sigma(y,t') = t'e^{\psi(y,t')}, \qquad (y,t') \in U.$$

We have $J(y,t')\frac{t'}{\sigma(y,t')} \in C_b^\infty(U)$, moreover, this function is bounded away from zero, provided that the constant ϵ_2 in 2.2(3) is sufficiently small.

Proof. We have $\sigma(y,t') = \sigma(y,t') - \sigma(y,0) = \int_0^1 \partial_t\sigma(y,\vartheta t')d\vartheta\cdot t'$. Since $0 < \epsilon_1 \le \partial_t\sigma \le 1/\epsilon$ on U by assumption 2.2(1), the integral is a smooth function of y and t', both bounded and bounded away from zero. Thus

$$\psi(y,t') = \ln\int_0^1 \partial_t\sigma(y,\vartheta t')d\vartheta \in C_b^\infty(U),$$

and the first assertion is proven.
By definition, $J(y,t') = |\det \partial\kappa(y,t')|$. Suppose we had $\partial_x\sigma(y,t') \equiv 0$ on U. Then estimates 2.2(1) and 2.2(2) would imply $J \geq \epsilon_1^2$. Hence the continuity of the determinant shows that $J \geq \epsilon_1^2/2$, if ϵ_2 is small. Moreover, all entries of the matrix for $\partial\kappa$ are C_b^∞ functions on U, so $\det\partial\kappa$ is C_b^∞, and so is $|\det\partial\kappa|$. Since $\frac{t'}{\sigma(y,t')} = e^{-\psi(y,t')}$ we get the desired result. $\triangleleft$

Now let us let have a look at identity 2.3(3). As a preparation we shall need the following lemma.

2.5 Lemma. *For $t, t' \in \mathbf{R}_+$ let $T(t,t') = \frac{t-t'}{\ln t - \ln t'}$. Then T is a smooth positive function on $\mathbf{R}_+ \times \mathbf{R}_+$, $T(t,t) = t$. Moreover,*

(a) $T(t,t') \leq \max\{t,t'\}$.

(b) *Let $0 < \delta < 1$. Then on the set $\{(t,t') \in (0,1)\times(0,1) : |t/t'-1| < \delta\}$ the functions $t'^{-1}(t'\partial_{t'})^k\, T(t,t')$ are bounded, $k = 0,1,\ldots$;*

(c) $t'^{k-1}\partial_{t'}^k T(t,t')|_{t'=t}$ *is smooth up to $t=0$; $k = 0,1,\ldots$.*

Note that T cannot be continued to a function in $C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+)$.

Proof. T is smooth and ≥ 0, for ln is smooth and monotonely increasing on $\mathbf{R}_+$. Moreover, T has no zero, since $T(t,t) = t > 0$.
(a): $|\ln t - \ln t'| = \left|\int_0^1 \frac{d\vartheta}{t'+\vartheta(t-t')}\right| |t-t'| \geq \min\{1/t, 1/t'\}|t-t'|$.
(b) Let $x = t/t'$. Then $1-\delta < x < 1+\delta$, and $T(t,t') = t'\frac{t/t'-1}{\ln t/t'} = t'\frac{x-1}{\ln x}$. So T is of the form $T(t,t') = t'\varphi(x)|_{x=t/t'}$ with a C_b^∞-function φ. But then $t'\partial_t'[t'\varphi(x)|_{x=t/t'}] = t'\varphi(x)|_{x=t/t'} - t'(x\partial_x)\varphi(x)|_{x=t/t'}$ is of the same form. Hence we get the assertion.
(c) Consider first the function $M(t,t') = T(t,t')^{-1}$ and show that $\partial_{t'}^k M(t,t')\,|_{t'=t} = c_k t^{-k-1}$ for suitable $c_k \in \mathbf{R}$, $k = 0,1,\ldots$. By induction, $\partial_{t'}^k[M(t,t')^{-1}]$ is a linear combination of terms of the form

$$M(t,t')^{-r-1}\prod_{l=1}^{r}\partial_{t'}^{j_l}M(t,t'),$$

where $r \leq k$ and $\sum_{l=1}^r j_l = k$. This implies that $\partial_{t'}^k T(t,t')\,|_{t'=t} = \partial_{t'}^k[M(t,t')^{-1}]\,|_{t'=t}$ is a linear combination of terms $t^{r+1}t^{-r-k}$, $0 \leq r \leq k$. $\triangleleft$

2.6 Lemma. *We have, in the notation of 2.2 and 2.3,*

$$e^{i(\chi(x,t)-\chi(y,t'))\underline{\xi}}\left(\frac{\sigma(x,t)}{\sigma(y,t')}\right)^{-i\underline{\tau}} = e^{i(x-y)(\tilde{D}_1^T\underline{\xi}+\tilde{D}_2^T\underline{\tau})}\left(\frac{t}{t'}\right)^{-i(\tilde{D}_3^T T(t,t')\underline{\xi}+\tilde{D}_4^T T(t,t')\underline{\tau})} \tag{1}$$

with

$$\begin{aligned}
\tilde{D}_1 = \quad \tilde{D}_1(x,t,y,t') \quad &= \int_0^1 \partial_x\chi(y+\vartheta(x-y), t'+\vartheta(t-t'))d\vartheta;\\
\tilde{D}_2 = \quad \tilde{D}_2(x,t,y,t') \quad &= -\int_0^1 [\partial_x \ln\sigma](y+\vartheta(x-y), t'+\vartheta(t-t'))d\vartheta;\\
\tilde{D}_3 = \quad \tilde{D}_3(x,t,y,t') \quad &= -\int_0^1 \partial_t\chi(y+\vartheta(x-y), t'+\vartheta(t-t'))d\vartheta;\\
\tilde{D}_4 = \quad \tilde{D}_4(x,t,y,t') \quad &= \int_0^1 [\partial_t \ln\sigma](y+\vartheta(x-y), t'+\vartheta(t-t'))d\vartheta.
\end{aligned}$$

Note that $\tilde{D}_1, ..., \tilde{D}_4$ are matrices of functions of sizes $n \times n, 1 \times n, n \times 1, 1 \times 1$, respectively, $(\cdot)^T$ denotes the transposed matrix. The matrices $\tilde{D}_1^T, \tilde{D}_2^T$ form the matrix B_1 of 2.3(3), while $\tilde{D}_3^T T(t,t'), \tilde{D}_4^T T(t,t')$ form B_2.

Proof. The left hand side of (1) equals $\exp\{i([\chi(x,t)-\chi(y,t')]\underline{\xi}-[\ln\sigma(x,t)-\ln\sigma(y,t')]\underline{\tau})\}$ $= \exp\{i([\int_0^1 \partial_x\chi d\vartheta(x-y)+\int_0^1 \partial_t\chi d\vartheta(t-t')]\underline{\xi}-[\int_0^1 \partial_x \ln\sigma d\vartheta(x-y)+\int_0^1 \partial_t \ln\sigma d\vartheta(t-t')]\underline{\tau}\}$, where the argument $y+\vartheta(x-y), t'+\vartheta(t-t'))$ has been omitted under the integrals. ◁

2.7 Lemma. *The matrices $\tilde{D}_1, ..., \tilde{D}_4$ of 2.6 are smooth functions on $U \times U \cap \{t, t' > 0\}$. Moreover, we have the following properties.*
(a) $\det \tilde{D}_1 \in C_b^\infty(U \times U)$ *is bounded away from zero on $U \times U$ provided that $|x-y|$ and $|t-t'|$ both are sufficiently small.*
(b) $\tilde{D}_2 \in C_b^\infty(U \times U)$.
(c) $\tilde{D}_3 \in C_b^\infty(U \times U)$.
(d) $\tilde{D}_4(x,t,y,t') \cdot T(t,t') = 1 + r(x,t,y,t')T(t,t')$ *with a function $r \in C_b^\infty(U \times U)$.*

Proof. According to 2.4, $\ln\sigma$ is a smooth function on $U \cap \{t > 0\}$, hence all functions are smooth on $U \times U \cap \{t, t' > 0\}$. Now (a) is immediate from 2.2(2). In order to see (b) note that, in the notation of 2.4,

$$\partial_x \ln\sigma(x,t) = \partial_x\psi(x,t) \in C_b^\infty(U).$$

(c) is trivial. For (d), observe that $\partial_t \ln\sigma(x,t) = \frac{1}{t} + \partial_t\psi(x,t)$; therefore

$$\begin{aligned} & \int_0^1 (\partial_t \ln\sigma)(y+\vartheta(x-y), t'+\vartheta(t-t'))d\vartheta \\ = \; & \frac{\ln t - \ln t'}{t-t'} + \int_0^1 (\partial_t\psi)(y+\vartheta(x-y), t'+\vartheta(t-t'))d\vartheta. \end{aligned}$$

This leads to the desired form. ◁

2.8 Corollary. Let, as it has been outlined in 2.3 and 2.6,

$$B(x,t,y,t') = \begin{pmatrix} \tilde{D}_1^T(x,t,y,t') & \tilde{D}_2^T(x,t,y,t') \\ \tilde{D}_3^T(x,t,y,t')T(t,t') & \tilde{D}_4(x,t,y,t')T(t,t') \end{pmatrix} \tag{1}$$

with the matrices $\tilde{D}_1, \tilde{D}_2, \tilde{D}_3, \tilde{D}_4$ of Lemma 2.6. As before, $(\cdot)^T$ denotes the transposed matrix. Then $\det B$ is smooth on $U \times U \cap \{t, t' > 0\}$; it is bounded and bounded away from zero on $U \times U$, provided that

(i) $|x-y|$ is sufficiently small, and

(ii) t and t' both are small.

Notice that (ii) will be automatically fulfilled, if the constant ϵ_3 in 2.2(4) is sufficiently small. Furthermore,

$$\det B(x,t,y,t') = T(t,t') \det \begin{pmatrix} \tilde{D}_1^T & \tilde{D}_2^T \\ \tilde{D}_3^T & \tilde{D}_4^T \end{pmatrix} (x,t,y,t'), \tag{2}$$

hence $\begin{pmatrix} \tilde{D}_1^T & \tilde{D}_2^T \\ \tilde{D}_3^T & \tilde{D}_4^T \end{pmatrix}$ is regular on $U \times U \cap \{t, t' > 0\}$.

Proof. This follows from Lemma 2.7 in connection with Lemma 2.5(a). $\lhd$

The principal result of this section now is the following proposition.

2.9 Proposition. *Let $A(x,t,y,t') = B(x,t,y,t')^{-1}$ with the matrix B of Corollary 2.8: in order for this definition to make sense we assume that* (i) *and* (ii) *of* 2.8 *are fulfilled. Fix $0 < \delta < 1$. Then, for all $k \in \mathbf{N}$, all multi-indices α, β, and all t, t' with $|t/t' - 1| < \delta$, we have*

$$\|t'^k \partial_{t'}^k D_x^\alpha D_y^\beta A(x,t,y,t')\|_{\mathcal{L}(\mathbf{C}^{n+1})} \le C_{k\alpha\beta\delta}. \tag{1}$$

Moreover, the matrix function

$$t'^k \partial_{t'}^k D_x^\alpha D_y^\beta A(x,t,y,t')|_{t'=t} \tag{2}$$

is smooth up to $t = 0$.

Proof. We have $A = B^{-1}$; hence $\partial_{t'}^k D_x^\alpha D_y^\beta A$ by induction is a linear combination of terms of the form

$$B^{-1}[\partial_{t'}^{k_1} D_x^{\alpha_1} D_y^{\beta_1} B] B^{-1} [\partial_{t'}^{k_2} D_x^{\alpha_2} D_y^{\beta_2}] B^{-1} \ldots [\partial_{t'}^{k_l} D_x^{\alpha_l} D_y^{\beta_l} B] B^{-1}$$

with $k_1 + \ldots + k_l = k, \alpha_1 + \ldots + \alpha_l = \alpha$, $\beta_1 + \ldots + \beta_l = \beta$, and $l \le |\alpha + \beta| + k$. It is therefore sufficient to show that

(i) $\|A\|_{\mathcal{L}(\mathbf{C}^{n+1})} = \|B^{-1}\|_{\mathcal{L}(\mathbf{C}^{n+1})}$ is bounded, and

(ii) $\|t'^k \partial_{t'}^k D_x^\alpha D_y^\beta B\|_{\mathcal{L}(\mathbf{C}^{n+1})}$ is bounded.

By Lemma 2.7, all entries of B are bounded functions on $U \times U$; moreover, the determinant is bounded away from zero for small $|x - y|$ and small $|t - t'|$. Hence (i) follows from Cramer's rule.
For (ii) we recall that $t'^k \partial_{t'}^k = \sum_{j=0}^k e_{jk} (t'\partial_{t'})^j$ for suitable e_{jk}. Then we use Lemma 2.7, Lemma 2.5(b), and Leibniz' rule. In order to show that the function (2) is smooth up to $t = 0$, it is sufficient to prove

(iii) $A(x,t,y,t) = B(x,t,y,t)^{-1}$ is smooth up to $t = 0$, and

(iv) $t'^k \partial_{t'}^k D_x^\alpha D_y^\beta B(x,t,y,t')|_{t'=t}$ is smooth up to $t = 0$.

Since $T(t,t')|_{t'=t} = t$, relation (iii) follows from Lemma 2.7. Also (iv) is immediate from Lemma 2.7 in connection with Lemma 2.5(c). $\lhd$

2.2 Invariance of the Cone Algebra without Asymptotics

Let us start with the Mellin Sobolev spaces.

2.10 Theorem. *The spaces $\mathcal{H}^{s,\gamma}(\mathbb{D})$, $s, \gamma \in \mathbf{R}$, are invariant under changes of coordinates.*

Proof. In view of interpolation and duality we may assume that $s \in \mathbf{N}$. Moreover, since $\mathcal{H}^{s,\gamma}(\mathbb{D}) \hookrightarrow H^s_{loc}(\mathrm{int}\,\mathbb{D})$ and the invariance of the usual Sobolev spaces is well-known, it is sufficient to consider functions with support close to the boundary of $\mathbb{D}$. Let therefore $\kappa : U \to V$ be a diffeomorphism of bounded open subsets of $\overline{\mathbf{R}}^{n+1}_+$ as in 2.2, and suppose that $u : V \to \mathbf{C}$ is a function with compact support in V, satisfying

$$\underline{t}^{\gamma-\frac{n}{2}}(\underline{t}\partial_{\underline{t}})^k\partial^\alpha_{\underline{x}}u(\underline{x},\underline{t}) \in L^2(V) \tag{1}$$

whenever $k + |\alpha| \le s$. Our task is to show that, for the pullback $u \circ \kappa$,

$$t^{t-\frac{n}{2}}(t\partial_t)^k\partial^\alpha_x(u \circ \kappa)(x,t) \in L^2(U). \tag{2}$$

Writing, as in 2.2 and 2.4, $\kappa(x,t) = (\chi(x,t),\sigma(x,t)) = (\chi(x,t),te^{\psi(x,t)}) = (\underline{x},\underline{t})$, we have

$$\begin{aligned}\partial_x[u(\chi(x,t),te^{\psi(x,t)})] &= (\partial_{\underline{x}}u)(\chi(x,t),te^{\psi(x,t)})(\partial_x\chi)(x,t) + \\ &\quad +(\partial_{\underline{t}}u)(\chi(x,t),te^{\psi(x,t)})te^{\psi(x,t)}(\partial_x\psi)(x,t)\end{aligned}$$

and

$$\begin{aligned}(t\partial_t)[u(\chi(x,t),te^{\psi(x,t)})] &= t(\partial_{\underline{x}}u)(\chi(x,t),te^{\psi(x,t)})(\partial_t\chi)(x,t) + \\ &\quad +t(\partial_{\underline{t}}u)(\chi(x,t),te^{\psi(x,t)})[e^{\psi(x,t)} + te^{\psi(x,t)}\partial_t\psi(x,t)] \\ &= (\partial_{\underline{x}}u)(\underline{x},\underline{t})(t\partial_t\chi)(x,t) + (\underline{t}\partial_{\underline{t}}u)(\underline{x},\underline{t})[1 + (t\partial_t\psi)(x,t)].\end{aligned}$$

By induction, $(t\partial_t)^k\partial^\alpha_x(u \circ \kappa)(x,t)$ is a linear combination of expressions of the form $[(\underline{t}\partial_{\underline{t}})^j\partial^\beta_{\underline{x}}u]\big|_{(\underline{x},\underline{t})=\kappa(x,t)} b_{j\beta}(x,t)$ with smooth bounded functions $b_{j\beta}$ and $j + |\beta| \le s$. In order to see this, note that t is bounded on supp u and that all derivatives of κ are bounded. Therefore, $\int |t^{\gamma-\frac{n}{2}}(t\partial_t)^k\partial^\alpha_x(u\circ\kappa)(x,t)|^2dxdt$ can be estimated by a finite linear combination of integrals of the form

$$\begin{aligned}&\int |t^{\gamma-\frac{n}{2}}[(\underline{t}\partial_{\underline{t}})^j\partial^\beta_{\underline{x}}u](\kappa(x,t))b_{j\beta}(x,t)|^2dxdt \\ =\ &\int |[\kappa^{-1}(\underline{x},\underline{t})]^{\gamma-\frac{n}{2}}_{n+1}(\underline{t}\partial_{\underline{t}})^j\partial^\beta_{\underline{x}}u(\underline{x},\underline{t})b_{j\beta}(\kappa^{-1}(\underline{x},\underline{t}))|^2\tilde{J}(\underline{x},\underline{t})d\underline{x}d\underline{t}\end{aligned} \tag{3}$$

Here $\tilde{J} = \partial\kappa^{-1}$ denotes the Jacobian determinant, which is bounded, and $[\kappa^{-1}(\underline{x},\underline{t})]_{n+1}$ is the $(n+1)$–st component of the vector $\kappa^{-1}(\underline{x},\underline{t})$, i.e., t in the new coordinates.

Since $\underline{t} = t\exp\psi(x,t)$ and ψ is a bounded function, there are constants c_1, c_2, with $c_1 \le t/\underline{t} = [\kappa^{-1}(\underline{x},\underline{t})]_{n+1}/\underline{t} \le c_2$, hence (1) implies that all integrals in (3) are bounded, and the proof is complete. $\lhd$

2.11 Corollary. The class $ML^{-\infty}_\gamma(\mathbb{D})$ is invariant under changes of coordinates, since all spaces $\mathcal{H}^{s,\gamma}(\mathbb{D})$, $s \in \mathbf{R} \cup \{\infty\}, \gamma \in \mathbf{R}$, are.

2.12 Outline and Reduction of the Task. Let us now have a look at the operators in the cone algebra without asymptotics. According to Definition 1.12, an element of $A \in ML^\mu_\gamma(\mathbb{D})$ is a sum of three operators: $A = A_M + A_\psi + t^{-\mu}G$ where A_M is a Mellin operator supported close to the boundary of $\mathbb{D}$, A_ψ is a pseudodifferential operator in the interior, and G is an operator in $ML^{-\infty}_\gamma(\mathbb{D})$. Our task now is to show that such a representation is independent of the choice of coordinates.

Step 1. We have seen in Corollary 2.11 that the operators in $ML_\gamma^{-\infty}(\mathbb{D})$ are invariantly defined. In the notation of Lemma 2.4, $\sigma(x,t)^{-\mu} = t^{-\mu}\exp(-\mu\psi(x,t))$ with the C_b^∞ function ψ. Noting that the function $\exp(-\mu\psi(x,t))$ has a smooth extension up to $t=0$, the factor $t^{-\mu}$ can be ignored. Moreover, it is known that the pseudodifferential calculus is coordinate free. Hence, it remains to consider the Mellin part

$$A_M = t^{-\mu}\omega_1 \mathrm{op}_M^\gamma f\,\omega_2$$

of A. Here, ω_1, ω_2 are functions that are supported close to the boundary of $\mathbb{D}$ and equal to 1 close to the boundary, and $f \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, L^\mu(X;\Gamma_{1/2-\gamma}))$. As before, the factor $t^{-\mu}$ can be ignored.

Step 2. We know from Remark 1.13(a) that, for $f \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, L^{-\infty}(X;\Gamma_{1/2-\gamma}))$, the operator A_M will be an element of $ML_\gamma^{-\infty}(\mathbb{D})$. So let $f \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, L^\mu(X;\Gamma_{1/2-\gamma}))$, and suppose for the moment that ϕ, ψ are smooth functions, supported in a single coordinate neighborhood U' for X and satisfying $\phi\psi = \phi$; by $M_\phi, M_\psi, M_{1-\psi}$ denote the operators of multipliction by ϕ, ψ, and $1-\psi$ respectively. Then $(t,t',z) \mapsto M_\phi f(t,t',z) M_{1-\psi}$ is an element of $C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, L^{-\infty}(X;\Gamma_{1/2-\gamma}))$, and the corresponding Mellin operator can be ignored in our considerations.

On the other hand, the operator-valued function $M_\phi f(t,t',z)M_\psi$ is given by a local parameter-dependent symbol, depending on the variables $t,t' \in \overline{\mathbf{R}}_+$ and the covariable $z \in \Gamma_{1/2-\gamma}$. The operator A_M therefore can be localized to a coordinate neighborhood for $\mathbb{D}$ of the form $U = U' \times [0,\epsilon)$, with suitable $\epsilon > 0$.

Step 3. The constant ϵ can be chosen arbitrarily small. In order to see this, choose smooth functions $\omega_3, \omega_4, \omega_5$ supported in an arbitrarily small neighborhood of the boundary of $\mathbb{D}$ and satisfying $\omega_3(t) = \omega_4(t) = \omega_5(t) \equiv 1$ close to the boundary, while $\omega_3\omega_4 = \omega_4$ and $\omega_4\omega_5 = \omega_5$. Write

$$A_M = \omega_3 A_M \omega_4 + (1-\omega_3)A_M\omega_4 + \omega_5 A_M(1-\omega_4) + (1-\omega_5)A_M(1-\omega_4).$$

According to Theorem 1.14(b), the operators $(1-\omega_3)A_M\omega_4$ and $\omega_5 A_M(1-\omega_4)$ are elements of $ML_\gamma^{-\infty}(\mathbb{D})$. In view of Theorem 1.14(a), the operators $(1-\omega_5)A_M(1-\omega_4)$ are pseudodifferential operators supported in the interior of $\mathbb{D}$. We know that both these classes are preserved, so we focus on $\omega_3 A_M \omega_4$.

Step 4. Hence we are reduced to the case where $\omega_1 \mathrm{op}_M^\gamma(f)\omega_2$ is the operator defined on $C_0^\infty(U' \times (0,\epsilon))$ by

$$\begin{aligned}&[\omega_1 \mathrm{op}_M^\gamma(f)\omega_2](u)(x,t)\\ =\;& \frac{1}{2\pi i}\int_{\Gamma_{1/2-\gamma}}\int_0^\infty \iint e^{i(x-y)\xi}\left(\frac{t}{t'}\right)^{-z}\tilde{f}(t,t',z,x,y,\xi)u(y,t')\,dy\,đ\xi\,\frac{dt'}{t'}\,dz. \qquad (1)\end{aligned}$$

with a function $\tilde{f} \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, S^\mu(U' \times U', \mathbf{R}^n \times \Gamma_{1/2-\gamma}))$. Obviously, the choice of the line $\Gamma_{1/2-\gamma}$ is irrelevant, and it is no restriction to assume $\gamma = 1/2$. In the integral (1) we may then replace the line $\Gamma_{1/2-\gamma}$ by $\Gamma_0 = i\mathbf{R}$ and the variable z by $i\tau$. Writing $đ\tau = 1/(2\pi)d\tau = 1/(2\pi i)dz$ and $f(t,t',\tau,x,y,\xi) = \tilde{f}(t,t',i\tau,x,y,\xi)$, we will then precisely have the situation of 2.3(1).

Step 5. We may assume that $f(t,t',\tau,x,y,\xi)$ vanishes unless $|x-y|$ is small: Otherwise, we might choose a function $\Phi \in C^\infty(U' \times U')$ supported in a small neighborhood of the diagonal $\{x = y\}$ and equal to 1 in a smaller one. Replacing $f(t,t',\tau,x,y,\xi)$ by

$f(t,t',\tau,x,y,\xi)\Phi(x,y)$ results in an error which is an element of $C^\infty(\overline{\mathbf{R}}_+\times\overline{\mathbf{R}}_+, S^{-\infty}(U'\times U',\mathbf{R}^n\times\mathbf{R}))$, hence induces an operator in $ML^{-\infty}_{1/2}(I\!\!D)$.

Step 6. We finally make one further simplification. We choose a function $\tilde\psi\in C_0^\infty(\mathbf{R}_+)$ supported in a small neighborhood of 1 with $\tilde\psi(\rho)\equiv 1$ for ρ close to 1. Then, according to Theorem 1.11 and 1.8(4), we have

$$\mathrm{op}_M^{\frac12} f-\mathrm{op}_M^{\frac12}[\tilde\psi(t/t')f]\in ML^{-\infty}_{1/2}(I\!\!D).$$

In other words, we may assume that $f(t,t',\tau,x,y,\xi)$ vanishes unless $|t/t'-1|$ is small.

Step 7. The simplifications of Steps 3, 5, and 6 show that the Mellin symbol $f=f(t,t',\tau,x,y,\xi)$ satisfies the assumptions necessary for Lemma 2.7, Corollary 2.8, and Proposition 2.9: $f(t,t',\tau,x,y,\xi)$ vanishes, unless

(i) $|x-y|$ is small,

(ii) t,t' are small, and

(iii) $|t/t'-1|$ is small.

Step 8. The idea now is the following: We have seen that the pullback of $\mathrm{op}_M^{\frac12}f$ can be written in the form used in 2.3. Considering the integral an oscillatory integral we may indeed change the order of integration, perform the substitutions and conclude as in 2.3(6) that the pullback is the Mellin operator with the symbol

$$\tilde g(t,t',\tau,x,y,\xi)=f(\sigma(x,t),\sigma(y,t'),A_2\begin{bmatrix}\xi\\ \tau\end{bmatrix},\chi(x,t),\chi(y,t'),A_1\begin{bmatrix}\xi\\ \tau\end{bmatrix})F(x,t,y,t')$$

with smooth matrix-valued functions A_1,A_2, and F, analyzed in Section 2.1. A_1 and A_2 also depend on (x,t,y,t'). Here I have written $F(x,t,y,t')$ instead of the expression

$$\frac{J(y,t')}{\sigma(y,t')}t'\,|\det A(x,t,y,t')|$$

employed in 2.3(6); recall that A is the $(n+1)\times(n+1)$ matrix formed by the $n\times(n+1)$ matrix A_1 and the $1\times(n+1)$ matrix A_2.

We will next use a Taylor expansion of order $N\in\mathbf{N}$ for the function $\tilde g$ at $t'=t$. We will show that the terms of the expansion furnish Mellin symbols in $C^\infty(\overline{\mathbf{R}}_+,S^\mu(U'\times U',\mathbf{R}^n\times\mathbf{R}))$, while, for given $M>0$, the remainder term will induce a bounded linear operator between the Mellin Sobolev spaces $\mathcal{H}^{-M,(n+1)/2}$ and $\mathcal{H}^{M,(n+1)/2}$, provided N is sufficiently large.

Using asymptotic summation, cf. Theorem 1.10, we conclude that the pullback of $\mathrm{op}_M^{1/2}f$ is a sum of a Mellin operator and an operator in $ML^{-\infty}_{1/2}(I\!\!D)$. This will take some time. The remainder of the proof is therefore split up into a series of lemmata; we will, however, keep the notation we have introduced so far.

2.13 Lemma. *Write, with arbitrary $N\in\mathbf{N}$,*

$$\tilde g(t,t',\tau,x,y,\xi)=\sum_{j=0}^{N-1}\frac{(t'-t)^j}{j!}\partial_{t'}^j\tilde g(t,t',\tau,x,y,\xi)|_{t'=t}+(t'-t)^N r_N(t,t',\tau,x,y,\xi),\qquad \textit{with}$$

$$r_N(t,t',\tau,x,y,\xi)=\frac{1}{(N-1)!}\int_0^1(1-\vartheta)^{N-1}\partial_{t'}^N\tilde g(t,t+\vartheta(t'-t),\tau,x,y,\xi)d\vartheta.$$

Then, for each j, $\partial_{t'}^{j}\tilde g(t,t',\tau,x,y,\xi)$ is a linear combination (with universal coefficients) of terms of the form

$$\begin{aligned}
&(\partial_{t'}^{j_1}\partial_{\tau}^{j_2}\partial_y^{\alpha}\partial_\xi^{\beta}f)(\sigma(x,t),\sigma(y,t'),A_2\begin{bmatrix}\xi\\ \tau\end{bmatrix},\chi(x,t),\chi(y,t'),A_1\begin{bmatrix}\xi\\ \tau\end{bmatrix})\ \cdot\\
&\cdot\ \prod_{p=1}^{j_1}\partial_{t'}^{k_p+1}\sigma(y,t')\prod_{q=1}^{j_2}[\partial_{t'}^{l_q+1}a_{2,\ell_2}(x,t,y,t')\xi_{\ell_2}]\ \cdot \qquad (1)\\
&\cdot\ \prod_{\nu=1}^{n}\prod_{r_\nu=1}^{\alpha_\nu}\partial_{t'}^{m_{r_\nu}+1}\chi_\nu(y,t')\prod_{\nu=1}^{n}\prod_{s_\nu=1}^{\beta_\nu}[\partial_{t'}^{n_{s_\nu}+1}a_{1,\nu\ell_1}(x,t,y,t')\xi_{\ell_1}]\ \cdot \qquad (2)\\
&\cdot\ \partial_{t'}^{j_3}F(x,t,y,t').
\end{aligned}$$

Here, $j_1,j_2,j_3,k_p,l_q,m_{r_\nu},n_{s_\nu}\in\mathbf{N}$,

$$j_1+j_2+j_3+|\alpha|+|\beta|+\sum_{p=1}^{j_1}k_p+\sum_{q=1}^{j_2}l_q+\sum_{\nu=1}^{n}\sum_{r_\nu=1}^{\alpha_\nu}m_{r_\nu}+\sum_{\nu=1}^{n}\sum_{s_\nu=1}^{\alpha_\nu}n_{s_\nu}=j;$$

ℓ_1 and ℓ_2 stand for any elements of $\{1,\dots,n+1\}$; $a_{2,\ell},\ell=1,\dots,n+1$ are the entries of the $1\times(n+1)$ matrix A_2; $a_{1,\nu\ell},\nu=1,\dots,n,\ell=1,\dots,n+1$ are the entries of the $n\times(n+1)$ matrix A_1; and, to avoid additional notational complications $(\xi_1,\dots,\xi_{n+1})=(\xi,\tau)$.

Proof. The formula is proven by induction, and it is much easier than it looks. It is obviously true for $j=0$. Assume it holds for some j. Then take an additional derivative with respect to t'. According to Leibniz rule, it will result in a derivative of one of the factors.

(i) In case we have to take a derivative of the first factor, we will get a derivative of f with respect to either t',τ,y_ν, or ξ_ν and a corresponding factor $\partial_{t'}\sigma(y,t')$, $\partial_{t'}a_{2,\ell}(x,t,y,t')\xi_\ell$, $\partial_{t'}\chi_\nu(y,t')$, or $\partial_{t'}a_{1,\nu\ell}(x,t,y,t')\xi_\ell$; moreover, one of $j_1,j_2,|\alpha|$, and $|\beta|$ will increase by 1. Otherwise the form above is preserved.

(ii) In case we have to take a derivative of one of the products in (1) or (2), we again apply Leibniz' rule. One of k_p,l_q,m_{r_ν}, and n_{s_ν} will increase by 1; the others remain unchanged.

(iii) Finally, we might have to take a derivative of $\partial_{t'}^{j_3}F(x,t,y,t')$, which only increases j_3 by 1. ◁

2.14 Lemma. *There is a function $\phi\in C_0^\infty(\mathbf{R}_+)$ with $\phi(\rho)\equiv 1$ near $\rho=1$ such that*

$$\phi(t/t')\tilde g(t,t',\tau,x,y,\xi)=\tilde g(t,t',\tau,x,y,\xi).$$

In particular, we will have

$$\begin{aligned}
\mathrm{op}_M^{\frac12}\tilde g &= \mathrm{op}_M^{\frac12}[\phi(t/t')\tilde g]\\
&= \sum_{j=0}\mathrm{op}_M^{\frac12}[\phi(t/t')\frac{(t'-t)^j}{j!}g_j(t,\tau,x,y,\xi)]+\mathrm{op}_M^{\frac12}[\phi(t/t')r_N]
\end{aligned}$$

with $g_j(t,\tau,x,y,\xi)=[\partial_{t'}^{j}\tilde g](t,t',\tau,x,y,\xi)|_{t'=t}$.

Proof. In view of the fact that $f(t,t',\tau,x,y,\xi)$ vanishes unless $|t/t'-1|$ is small, we know that $\tilde{g}(t,t',\tau,x,y,\xi)$ vanishes unless $|\sigma(x,t)/\sigma(y,t')-1|$ is small. Next we note that $\sigma(x,t)/\sigma(y,t') = t/t'\cdot\exp[\psi(x,t)-\psi(y,t')]$ with the function $\psi\in C_b^\infty$ introduced in Lemma 2.4. Hence, $|\sigma(x,t)/\sigma(y,t')-1|$ cannot stay small as t/t' tends to 0 or $+\infty$, and $\tilde{g}(t,t',\tau,x,y,\xi)$ will vanish for t/t' outside a compact set in $\mathbf{R}_+$.

The remaining statement follows immediately. ◁

2.15 Lemma. *Let g_j denote the functions introduced in Lemma 2.14. Then $t^j g_j \in C^\infty(\overline{\mathbf{R}}_+, S^\mu(U'\times U', \mathbf{R}^n\times\mathbf{R}))$.*

Proof. We use the form for g_j implied by Lemma 2.13. By Proposition 2.9 we have $[t'^k\partial_{t'}^k A](x,t,y,t')|_{t'=t}$ and $[t'^k\partial_{t'}^k F](x,t,y,t')|_{t'=t}$ bounded for all k; moreover, $\partial_{t'}^l\sigma(y,t')$ and $\partial_{t'}^l\chi(y,t')$ are clearly bounded for arbitrary l. The assertion will therefore be proven if we show that

$$\begin{aligned}&\partial_t^l\partial_\tau^k D_\xi^{\tilde\alpha} D_x^{\tilde\beta} D_y^{\tilde\gamma}\\ &(\partial_{t'}^{j_1}\partial_\tau^{j_2}\partial_y^\alpha\partial_\xi^\beta f)(\sigma(x,t),\sigma(y,t'),A_2\begin{bmatrix}\xi\\ \tau\end{bmatrix},\chi(x,t),\chi(y,t'),A_1\begin{bmatrix}\xi\\ \tau\end{bmatrix})\Big|_{t'=t}\\ =\;& O(\langle\xi,\tau\rangle^{\mu-j_2-|\beta|-|\tilde\alpha|-k}).\end{aligned}$$

Obviously, derivatives with respect to x,y, and t produce terms just like those we have analyzed. So we may assume that $|\tilde\beta|=|\tilde\gamma|=l=0$, and the only point to clarify is the behavior of derivatives $\partial_\tau^k D_\xi^{\tilde\alpha}$. This, however, is easy: These derivatives are linear combinations of terms of the form

$$\begin{aligned}&(\partial_{t'}^{j_1}\partial_\tau^{j_2+n_1}\partial_y^\alpha\partial_\xi^{\beta+\delta} f)(\sigma(x,t),\sigma(y,t'),A_2\begin{bmatrix}\xi\\ \tau\end{bmatrix},\chi(x,t),\chi(y,t'),A_1\begin{bmatrix}\xi\\ \tau\end{bmatrix})\;\cdot\\ &\cdot\;\prod_{l=1}^{n+1} a_{2,l}(x,t,y,t')^{r_l}\prod_{\nu=1}^{n}\prod_{l=1}^{n+1} a_{1,\nu l}(x,t,y,t')^{\delta_{\nu,l}}|_{t'=t}\end{aligned}$$

with $r_l,\delta_{\nu,l}\in\mathbf{N}$, $n_1+|\delta|=k+|\tilde\alpha|$. In view of the properties of f and the fact that the functions $a_{2,l}$ as well as $a_{1,\nu l}$ are bounded, the expression is

$$O(\langle A_1\begin{bmatrix}\xi\\ \tau\end{bmatrix}, A_2\begin{bmatrix}\xi\\ \tau\end{bmatrix}\rangle^{\mu-j_2-|\beta|-n_1-|\delta|}) = O(\langle A\begin{bmatrix}\xi\\ \tau\end{bmatrix}\rangle^{\mu-j_2-|\beta|-k-|\tilde\alpha|}).$$

Now we know from Corollary 2.8 and Proposition 2.9 that $\det A$ is both bounded and bounded away from zero. Hence Cramer's rule shows that

$$c_1|(\xi,\tau)|\le |A\begin{bmatrix}\xi\\ \tau\end{bmatrix}|\le c_2|(\xi,\tau)|$$

for suitable positive constants c_1 and c_2, and we obtain the desired estimate. ◁

2.16 Corollary. We have

$$\begin{aligned}
&\operatorname{op}_M^{\frac{1}{2}}[\phi(t/t')\frac{(t'-t)^j}{j!}g_j(t,\tau,x,y,\xi)] \\
= \; & \frac{1}{j!}\operatorname{op}_M^{\frac{1}{2}}[\phi(t/t')(t'/t-1)^j t^j g_j(t,\tau,x,y,\xi)] \\
= \; & \frac{1}{j!}\operatorname{op}_M^{\frac{1}{2}}[\phi(t/t')(t'/t-1)^j \ln^{-j}(t'/t)t^j(-i)^j\partial_\tau^j g_j(t,\tau,x,y,\xi)] \\
= \; & \frac{1}{j!}\operatorname{op}_M^{\frac{1}{2}}M[\phi(\rho)(\rho^{-1}-1)^j\ln^{-j}(\rho^{-1})M^{-1}_{1/2,z\to\rho}(t^j\partial_z^j g_j(t,-iz,x,y,\xi))].
\end{aligned}$$

For the last identities, we have used 1.8(3) and 1.8(4). Notice that, by Lemma 2.15 in connection with Theorem 1.11,

$$\begin{aligned}
& M_{\rho\to\zeta}[\phi(\rho)(\rho^{-1}-1)^j\ln^{-j}(\rho^{-1})M^{-1}_{1/2,z\to\rho}(t^j\partial_z^j g_j(t,-iz,x,y,\xi))] \\
\in \; & C^\infty(\overline{\mathbf{R}}_+, S^{\mu-j}(U'\times U', \mathbf{R}^n\times\mathbf{R}_\zeta)).
\end{aligned}$$

2.17 Outline (continued). We shall now consider the remainder term $\operatorname{op}_M^{\frac{1}{2}}[\phi(t/t')(t'-t)^N r_N(t,t',\tau,x,y,\xi)]$ and show that, given an arbitrary $M>0$,

$$\operatorname{op}_M^{\frac{1}{2}}[\phi(t/t')(t'-t)^N r_N]:\mathcal{H}^{-M,(n+1)/2}(\mathbf{R}^n\times\mathbf{R}_+)\to\mathcal{H}^{M,(n+1)/2}(\mathbf{R}^n\times\mathbf{R}_+)$$

is bounded, provided N is sufficently large. In order to prove the boundedness we will consider the distributional kernel of the operator, i.e., the function $k_N = k_N(x,t,y,t')$ that satisfies

$$\operatorname{op}_M^{\frac{1}{2}}[\phi(t/t')(t'-t)^N r_N]u(x,t) = \int_0^\infty\int k_N(x,t,y,t')u(y,t')dydt'/t'.$$

An argument based on a modified Hausdorff-Young inequality, see Lemmata 2.19 and 2.21, below, will then conclude the proof.

2.18 Remark. Note that $u = u(x,t)\in\mathcal{H}^{s,(n+1)/2}(\mathbf{R}^n_x\times\mathbf{R}_{+,t}), s\in\mathbf{N}$, if and only if $(t\partial_t)^l\partial_x^\alpha u\in L^2(\mathbf{R}^n\times\mathbf{R}_+, dxdt/t)$ for all $l+|\alpha|\le s$.

2.19 Lemma. (Modified Hausdorff-Young Inequality) *Let $\lambda\in L^1(\mathbf{R}^n\times\mathbf{R}_+, dxdt/t)$ and $u\in L^2(\mathbf{R}^n\times\mathbf{R}_+, dxdt/t)$. By* exp *denote for the moment the mapping $\mathbf{R}^n\times\mathbf{R}\to\mathbf{R}^n\times\mathbf{R}_+$ given by $(x,t)\mapsto(x,e^t)$. Then*

$$\begin{aligned}
& \left\|\int_0^\infty\int\lambda(x-y,t/t')u(y,t')dydt'/t'\right\|_{L^2(\mathbf{R}^n\times\mathbf{R}_+,dxdt/t)} \\
= \; & \left\|\iint\lambda(x-y,e^s/e^{s'})u(y,e^{s'})dyds'\right\|_{L^2(\mathbf{R}^n\times\mathbf{R},dxds)} \\
= \; & \|[\lambda\circ\exp] * [u\circ\exp]\|_{L^2(\mathbf{R}^n\times\mathbf{R})} \\
\le \; & \|[\lambda\circ\exp]\|_{L^1(\mathbf{R}^n\times\mathbf{R})}\,\|[u\circ\exp]\|_{L^2(\mathbf{R}^n\times\mathbf{R})} \\
= \; & \|\lambda\|_{L^1(\mathbf{R}^n\times\mathbf{R}_+,dxdt/t)}\,\|u\|_{L^2(\mathbf{R}^n\times\mathbf{R}_+,dxdt/t)},
\end{aligned}$$

where the only inequality is Hausdorff-Young's.

2.20 Remark. Let $M \in \mathbf{N}$. Then $\mathcal{H}^{-M,(n+1)/2}(\mathbf{R}^n \times \mathbf{R}_+)$ is the space of all distributions $u \in \mathcal{D}'(\mathbf{R}^n \times \mathbf{R}_+)$ that can be written

$$u(x,t) = \sum_{l+|\alpha|\le M} (t\partial_t)^l \partial_x^\alpha u_{l\alpha}(x,t) \text{ with suitable } u_{k\alpha} \in \mathcal{H}^{0,(n+1)/2}(\mathbf{R}^n \times \mathbf{R}_+).$$

2.21 Lemma. (a) *Let $k = k(x,t,y,t')$ be a continuous function on $\mathbf{R}^n \times \mathbf{R}_+ \times \mathbf{R}^n \times \mathbf{R}_+$ and suppose that there is a function $g_0 \in L^1(\mathbf{R}^n)$ and a function $\psi_0 \in L^1(\mathbf{R}_+)$ with $|k(x,t,y,t')| \le \psi_0(t/t')g_0(x-y)$. Then the operator K, defined by*

$$Ku(x,t) = \int_0^\infty \int k(x,t,y,t')u(y,t')dydt'/t' \tag{1}$$

for $u \in C_0^\infty(\mathbf{R}^n \times \mathbf{R}_+)$, has a continuous extension

$$K : \mathcal{H}^{0,(n+1)/2}(\mathbf{R}^n \times \mathbf{R}_+) \to \mathcal{H}^{0,(n+1)/2}(\mathbf{R}^n \times \mathbf{R}_+).$$

(b) *Let $M \in \mathbf{N}$, and suppose that, for some $g_M \in L^1(\mathbf{R}^n)$ and $\psi_M \in L^1(\mathbf{R}_+)$, we have $|(t'\partial_{t'})^{j_1}(t\partial_t)^{j_2}\partial_x^\alpha\partial_y^\beta k(x,t,y,t')| \le \psi_M(t/t')g_M(x-y)$ whenever $j_1 + j_2 + |\alpha| + |\beta| \le M$. Then the above operator K extends to a continous operator*

$$K : \mathcal{H}^{-M,(n+1)/2}(\mathbf{R}^n \times \mathbf{R}_+) \to \mathcal{H}^{M,(n+1)/2}(\mathbf{R}^n \times \mathbf{R}_+).$$

Proof. (a) We apply Remark 2.18 as well as the modified Hausdorff-Young inequality and obtain

$$\begin{aligned}
\|Ku\|_{\mathcal{H}^{0,(n+1)/2}(\mathbf{R}^n\times\mathbf{R}_+)} &= \|Ku\|_{L^2(\mathbf{R}^n\times\mathbf{R}_+,dxdt/t)} \\
&= \|\int_0^\infty \int k(x,t,y,t')u(y,t')dydt'/t'\|_{L^2(\mathbf{R}^n\times\mathbf{R}_+,dxdt/t)} \\
&\le \|\int_0^\infty \int \psi_0(t/t')g_0(x-y)|u(y,t')|dydt'/t'\|_{L^2(\mathbf{R}^n\times\mathbf{R}_+,dxdt/t)} \\
&\le \|\psi_0\|_{L^1(\mathbf{R}_+,dt/t)}\|g_0\|_{L^1(\mathbf{R}^n)}\|u\|_{L^2(\mathbf{R}^n\times\mathbf{R}_+,dxdt/t)} \\
&= \|\psi_0\|_{L^1(\mathbf{R}_+,dt/t)}\|g_0\|_{L^1(\mathbf{R}^n)}\|u\|_{\mathcal{H}^{0,(n+1)/2}(\mathbf{R}^n\times\mathbf{R}_+)}.
\end{aligned}$$

(b) It follows from the definition of $\mathcal{H}^{M,(n+1)/2}(\mathbf{R}^n \times \mathbf{R}_+)$ and Remark 2.20 that K has the asserted continuity property if and only if, for all j_1, j_2, α, β with $j_1 + j_2 + |\alpha| + |\beta| \le M$, the operators with the kernels $k_{j_1j_2\alpha\beta}(x,t,y,t') = (t'\partial_{t'})^{j_1}(t\partial_t)^{j_2}\partial_x^\alpha\partial_y^\beta k(x,t,y,t')$ are bounded on $\mathcal{H}^{0,(n+1)/2}(\mathbf{R}^n \times \mathbf{R}_+)$. In view of (a), the estimate guarantees precisely this. $\lhd$

2.22 Outline (continued). It remains to show that the kernel k_N for $\mathrm{op}_M^{\frac{1}{2}}[\phi(t/t')(t'-t)^N r_N$ satisfies the estimates in Lemma 2.21. This is essentially very easy, since we already have the function $\phi(t/t')$ implying compact support with respect to t/t', and we know that the kernel has compact support with respect to (x,y). So we only have to establish the existence of the kernel and to check that the derivatives $(t'\partial_{t'})^{j_1}(t\partial_t)^{j_2}\partial_x^\alpha\partial_y^\beta k_N(x,t,y,t'), j_1 + j_2 + |\alpha| + |\beta| \le M$ are *bounded*, provided N is large. So let us have a look at the kernel:

$$
\begin{aligned}
&k_N(x,t,y,t') \\
&= \iint e^{i(x-y)\xi}\left(\frac{t}{t'}\right)^{-i\tau}\phi(t/t')(t'-t)^N r_N(t,t',\tau,x,y,\xi)d\tau d\xi \\
&= \iint e^{i(x-y)\xi}\left(\frac{t}{t'}\right)^{-i\tau}\phi(t/t')(1-t/t')^N \ln^{-N}(t/t')(-it')^N\partial_\tau^N r_N(t,t',\tau,x,y,\xi)d\tau d\xi \\
&= \frac{(-i)^N}{(N-1)!}\phi(t/t')(1-t/t')^N\ln^{-N}(t/t') \\
&\quad\cdot\iint e^{i(x-y)\xi}\left(\frac{t}{t'}\right)^{-i\tau}\int_0^1(1-\vartheta)^{N-1}\tilde r_{N\vartheta}(t,t',\tau,x,y,\xi)d\vartheta d\tau d\xi,
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde r_{N\vartheta}(t,t',\tau,x,y,\xi) &= \partial_\tau^N t'^N(\partial_{t'}^N\tilde g)(t,t+\vartheta(t'-t),\tau,x,y,\xi) \\
&= (t'/u)^N\partial_\tau^N u^N\partial_u^N\tilde g(t,u,\tau,x,y,\xi)|_{u=t+\vartheta(t'-t)}. \qquad (1)
\end{aligned}
$$

Note that $u/t'=t/t'+\vartheta(1-t/t')$ is both bounded and bounded away from zero on supp ϕ. As far as the derivatives are concerned, we use the following lemma.

2.23 Lemma. *For arbitrary* $N,j\in\mathbf{N}$,

$$\partial_\tau^N t'^j\partial_{t'}^j\tilde g(t,t',\tau,x,y,\xi)=O(\langle(\xi,\tau)\rangle^{\mu-N}).$$

Proof. We already have computed these derivatives in the proof of Lemma 2.15. Instead of the smoothness of $(t'\partial_{t'})^kF(x,t,y,t')|_{t'=t}$ and $(t'\partial_{t'})^kA(x,t,y,t')|_{t'=t}$, we now use the boundedness of $(t'\partial_{t'})^kF(x,t,y,t')$ and $(t'\partial_{t'})^kA(x,t,y,t')$, which was established in Lemma 2.4 and Proposition 2.9. $\lhd$

2.24 Remark. Let $\tilde\phi\in C_0^\infty(\mathbf{R}_+)$. Then, for all $j,j'\in\mathbf{N}$,

$$(t'\partial_{t'})^{j'}(t\partial_t)^j\tilde\phi(t/t')\quad\text{is bounded on}\quad\mathbf{R}_+\times\mathbf{R}_+.$$

Moreover, it again is a C_0^∞ function of t/t', since

$$t\partial_t\tilde\phi(t/t')=(x\partial_x)\tilde\phi(x)|_{x=t/t'}\ \text{ and }\ t'\partial_{t'}\tilde\phi(t/t')=(-x\partial_x)\tilde\phi(x)|_{x=t/t'}.$$

2.25 Conclusion. According to 2.22 we only have to check the boundedness of the derivatives $(t'\partial_{t'})^{j_1}(t\partial_t)^{j_2}D_x^\alpha D_y^\beta k_N(x,t,y,t')$ for all $j_1+j_2+|\alpha+\beta|\le M$.

By Lemma 2.23 the integral for k_N in Lemma 2.22 will converge whenever $\mu-N<-n-1$ and furnish a bounded continuous function on $\mathbf{R}^n\times\mathbf{R}_+\times\mathbf{R}^n\times\mathbf{R}_+$. Moreover, differentiating under the integral sign, we see that $D_x^\alpha D_y^\beta k_N(x,t,y,t')$ will be bounded if $\mu-N+|\alpha+\beta|<-n-1$.

What about the totally characteristic derivatives $(t'\partial_{t'})^{j_1}(t\partial_t)^{j_2}D_x^\alpha D_y^\beta k_N(x,t,y,t')$? According to Remark 2.24, we need not worry about the terms $\phi(t/t')(1-t/t')^N\ln^{-N}(t/t')$; furthermore, we have

$$(t'\partial_{t'})^{j_1}(t\partial_t)^{j_2}(t/t')^{-i\tau}=(-1)^{j_1}(-i\tau)^{j_1+j_2}(t/t')^{-i\tau},$$

so we may focus on the question how the derivatives act on $\tilde{r}_{N\vartheta}$.

From the observations that (i) $u^N\partial_u^N$ is a linear combination of totally characteristic derivatives $(u\partial_u)^j, j \le N$, and that (ii) $u/t' = t/t' + \vartheta(1 - t/t')$ is a function of t/t', we conclude that it is sufficient to show that $(t'\partial_{t'})^{j_1}(t\partial_t)^{j_2}D_x^\alpha D_y^\beta \partial_\tau^N \tilde{g}(t,t',\tau,x,y,\xi) = O(\langle(\xi,\tau)\rangle^{\mu-N}$. This, however, is (essentially) what we have done already in Lemmata 2.15 and 2.23, so the proof is complete. $\lhd$

References

[1] Agranovič, M.S., and Višik, M.I.: Elliptic problems with a parameter and parabolic problems of general type (Russ.), *Usp. Mat. Nauk*, **19** (1963), 53-161 (Engl transl. *Russ. Math. Surveys*, **19** (1963), 53 - 159).

[2] Dieudonné, J.: *Grundzüge der modernen Analysis*, vol. 7. VEB Deutscher Verlag der Wissenschaften, Berlin 1982.

[3] Kondrat'ev, V.A.: Boundary value problems in domains with conical or angular points, Transactions Moscow Math. Soc. **16** (1967) 227-313.

[4] Lewis, J.E., and Parenti, C.: Pseudodifferential operators of Mellin type, *Comm. in Partial Diff. Equ.* **8** (1983), 477 - 544.

[5] Melrose, R.: Transformation of boundary problems, *Acta Math.* **147** (1981), 149 - 236.

[6] Plamenevskij, B. A.: *Algebras of Pseudodifferential Operators* (Russ.), Nauka, Moscow 1986.

[7] Schrohe, E., and Schulze, B.-W.: Boundary value problems in Boutet de Monvel's algebra for manifolds with conical singularities I. In: *Advances in Partial Differential Equations* **1**: *Pseudodifferential Operators and Mathematical Physics.* Akademie Verlag, Berlin, 1994, 97 - 209.

[8] Schrohe, E., and Schulze, B.-W.: Boundary value problems in Boutet de Monvel's algebra for manifolds with conical singularities II. in preparation

[9] Schulze, B.-W.: *Pseudo-Differential Operators on Manifolds with Singularities,* North-Holland, Amsterdam 1991.

[10] Schulze, B.-W.: Mellin representations of pseudo-differential operators on manifolds with corners. *Ann. Global Anal. and Geometry* **8** (1990), 261–297.

[11] Schulze, B.-W.: *Pseudo-Differential Boundary Value Problems, Conical Singularities and Asymptotics*, Akademie Verlag, Berlin 1994.

[12] Schulze, B.-W.: Transmission algebras on singular spaces with components of different dimensions, to appear in Proceedings "Partial Differential Equations, Holzhau, 1994", Birkhäuser, Boston, Basel 1995

[13] Višik, M.I., and Eskin, G.I.: Convolution equations in bounded domains in spaces with weighted norms, *Math. USSR Sb.* **69** (1966), 65 - 110.