Extended Moduli Spaces and the Kan Construction. II. Lattice Gauge Theory

Johannes Huebschmann

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
53225 Bonn
GERMANY

MPI/95-92

dg-ga/9506006

EXTENDED MODULI SPACES AND THE KAN CONSTRUCTION. II. LATTICE GAUGE THEORY

JOHANNES HUEBSCHMANN†

Max Planck Institut für Mathematik
Gottfried Claren Str. 26
D-53 225 BONN
huebschm@mpim-bonn.mpg.de

August 10, 1995

ABSTRACT. Let Y be a CW-complex with a single 0-cell, let K be its Kan group, a free simplicial group whose realization is a model for the space ΩY of based loops on Y, and let G be a compact, connected Lie group. We carry out an explicit purely finite dimensional construction of generators of the equivariant cohomology of the geometric realization of the cosimplicial manifold $\mathrm{Hom}(K, G)$ and hence, in view of earlier results, of the space $\mathrm{Map}^o(Y, BG)$ of based maps from Y to the classifying space BG of G where G acts on BG by conjugation. For a smooth manifold Y, this may be viewed as a rigorous approach to lattice gauge theory, and we show that it then yields, (i) when $\dim(Y) = 2$, equivariant de Rham representatives of generators of the equivariant cohomology of twisted representation spaces of the fundamental group of a closed surface including generators for moduli spaces of semi stable holomorphic vector bundles on complex curves so that, in particular, the known structure of a stratified symplectic space results; (ii) when $\dim(Y) = 3$, equivariant cohomology generators including the Chern-Simons function; (iii) when $\dim(Y) = 4$, the generators of the relevant equivariant cohomology from which for example Donaldson polynomials are obtained by evaluation against suitable fundamental classes corresponding to moduli spaces of ASD connections.

1991 *Mathematics Subject Classification.* 18G30, 18G55, 55R35, 55R40, 55U10, 57M05, 58D27, 58E15, 81T13.

Key words and phrases. Kan group, extended representation spaces, extended moduli spaces, geometry of moduli spaces, lattice gauge theory.

† The author carried out this work in the framework of the VBAC research group of EUROPROJ.

1. Introduction

The paper might as well have been entitled "A purely finite dimensional approach to gauge theory". We pursue further the approach in [60]: Let Y be a CW-complex with a single 0-cell, let K be its Kan group [30], a free simplicial group whose realization is a model for the space ΩY of based loops on Y, and let G be a compact and connected Lie group. In this paper we carry out an explicit construction of the generators of the G-equivariant de Rham cohomology of the realization $|\mathrm{Hom}(K,G)|$ of the cosimplicial manifold $\mathrm{Hom}(K,G)$ and hence, in view of the main result in our paper [60], of the space $\mathrm{Map}^o(Y,BG)$ of based maps from Y to the classifying space BG of G where G acts on BG by conjugation; we thereby exploit the fact that the chains on the simplicial nerve of K yield a model for the chains of Y. Since $\mathrm{Hom}(K,G)$ is a smooth finite dimensional manifold in each cosimplicial degree, cf. [60], the construction is *purely finite dimensional: every de Rham form will be constructed on a piece of finite dimension.* A concise statement is given in Theorem 7.1 below. The finite dimensional pieces belong to the cosimplicial manifold $\mathrm{Hom}(K,G)$. Integration then carries these forms to forms on the realization $|\mathrm{Hom}(K,G)|$. This requires a suitable interpretation of forms on mapping spaces. Using the theory of *differentiable space* [56], [57] or, what amounts to the same, that of *"diffeological" space* ("espace difféologique") [63], [64], forms on mapping spaces admit a purely finite dimensional interpretation in terms of what are called *plots* [56], [57] or *"plaques"* [63], [64] and do *not* require infinite dimensional techniques. For a smooth manifold Y, our construction of forms may be viewed as a rigorous approach to lattice gauge theory, whereby plots admit a natural interpretation as (equivariant) *families of principal bundles with connection*; see Section 5 below for details.

We offer three applications; they may be viewed as classical topological field theory constructions: At first we show that the construction yields, when $\dim(Y)=2$, explicit equivariant de Rham representatives of equivariant generators of the cohomology of moduli spaces of twisted representation spaces of the fundamental group of a closed surface; in particular, this yields the structure of a stratified symplectic space on such a moduli space already obtained by other means [21], [23], [25]. We expect that part of what is said in [65] can be understood within our framework. It is worthwhile pointing out, though, that even for the case of a bundle on a closed surface Σ, the present more general construction involving a model for the full loop space rather than merely a presentation of the fundamental group of the surface [21], [22], [25], [27], [28] goes beyond earlier constructions: The realization $|\mathcal{H}|$ of $\mathcal{H}=\mathrm{Hom}(K\Sigma,G)$ contains the spaces of based gauge equivalence classes of *all* central Yang-Mills connections [2], not just those which correspond to the absolute minimum or, equivalently, to projective representations of the fundamental group π of Σ, and hence the space $|\mathcal{H}|$ comes with a kind of Harder-Narasimhan filtration, cf. Section 2 of [60]. The latter cannot be obtained from the earlier extended moduli space constructions. Perhaps information about the multiplicative structure of the cohomology of moduli spaces can be derived from the models we shall construct below or from variants thereof. Secondly, when $\dim(Y)=3$, we obtain equivariant cohomology generators including an explicit expression for the Chern-Simons function on our model of the space of based gauge equivalence classes of connections, thereby answering a question raised by ATIYAH in [4] where he comments on a possible combinatorial approach to the path integral quantization of the Chern-Simons function. Thirdly, when $\dim(Y) = 4$, we obtain the generators of the equivariant cohomology of the appropriate space from which for example Donaldson polynomials are obtained by evaluation against suitable fundamental classes corresponding to moduli spaces of ASD connections.

Our construction is rigid in the sense that it gets away with various choices made in the earlier approaches; the theory, admittedly technically a bit complicated, will take care of itself, *no* choices of appropriate data must be made except that of various chains representing certain homology classes, and the occurrence of the homotopy operator on forms in the cited references will get its natural explanation in terms of a realization procedure involving integration of forms; see Section 5 below for details. Our approach is vastly more general than those in [21], [22], [25], [27], [28] since it applies to a bundle over an arbitrary smooth compact manifold via a cell decomposition or triangulation as explained above. Formally it is not even necessary to know that the simplicial group we are working with arises from a smooth manifold; we shall therefore expose the theory for an arbitrary simplicial group or groupoid. In this way we arrive at a kind of gauge theory over arbitrary CW-complexes. By means of the simplicial groupoid constructed in [17] for an arbitrary connected simplicial set the present approach can be extended to arbitrary connected simplicial complexes, in particular, to triangulated smooth manifolds.

Our models for the space of gauge equivalence classes of connections involve classical low dimensional topology notions such as *identity among relations* (Section 3 of [60]) and *universal quadratic group* (Section 4 of [60]); this somewhat establishes a link between classical algebraic topology and the more recent gauge theory developments in low dimensions. We expect that our models will also prove useful for various calculations recently done in quantum cohomology and that related finite dimensional constructions may be applied to other gauge theory situations.

Some historical comments about the origin of the present purely finite dimensional techniques may be in order: Extending an approach by KARSHON [33], A. Weinstein [48] constructed a closed equivariant 2-form on (the smooth part) of certain spaces of homomorphisms $\mathrm{Hom}(\pi, G)$ from the fundamental group π of a closed surface to a Lie group G with a biinvariant metric and showed by techniques from equivariant cohomology [53] that this 2-form descends to (the non-singular part of) $\mathrm{Rep}(\pi, G)$. In [21], [25], and [27], Weinstein's method has been refined so as to yield a smooth finite dimensional symplectic manifold with a hamiltonian action so that the space of representations and more general twisted versions thereof arise by symplectic reduction; this approach has been extended thereafter in [22] and [59] to more general planar groups than just surface groups so that for example moduli spaces of parabolic bundles can be successfully treated. Another generalization in [28] yields explicit representatives for NEWSTEAD'S generators [62] for various moduli spaces over a surface; in the algebro-geometric context, these arise as moduli spaces of certain semi stable holomorphic vector bundles on complex curves. The present paper gives such a construction for an arbitrary gauge theory situation. It may be viewed as the "grand unified theory" searched for by A. Weinstein in [48]. Our *principal innovation* is to replace the bar construction of a discrete group coming into play in [33], [48], and in the subsequent papers [21], [22], [25], [27], [28], [59], by the simplicial nerve of the Kan group K on Y so that we can handle the space of based gauge equivalence classes of connections on an arbitrary principal bundle with compact structure group on a *general* manifold Y.

Any unexplained notation is the same as that in our papers [21] and [60]. Details about cosimplicial spaces may be found in [8] and [10]. All spaces are assumed to be compactly generated, that is to say, a set that meets every compact set in a closed set is closed.

I am indebted to S. Bauer, H. J. Baues, and D. Puppe for discussions, and to J. Stasheff for a number of most helpful comments on a draft of the manuscript. The paper has been written during a stay at the Max Planck Institut at Bonn. I wish to express my gratitude to it and to its director Professor F. Hirzebruch for hospitality and support.

2. Forms on spaces of representations

Let Π be a finitely generated groupoid, for example a group, and write $(C_\natural(\Pi), \partial_\natural)$ and $(C^\natural(\Pi), \delta^\natural)$ for the complexes of normalized chains and cochains, respectively, on its *nerve* $N\Pi$ or *inhomogeneous reduced normalized bar construction*. We use the dummy symbol $\natural$ to distinguish bar resolution and hence group or groupoid (co)-homology degree from form degree which will be written $*$. Further, let G be a connected Lie group; the extension of the construction to be given below to general non-connected Lie groups will be studied elsewhere. View G as a groupoid with a single object, and consider the space $\mathrm{H} = \mathrm{Hom}(\Pi, G)$. This space is not necessarily smooth at every point, and the interpretation of de Rham forms will in general require some care. However in the present paper we shall only need the special case where Π is *free* so that H amounts to a product of finitely many copies of G.

Equivariant de Rham forms on H may be constructed in the following way: Given a k-tuple $[x_1|x_2|\ldots|x_k]$ of elements of Π, $k \geq 1$, and an equivariant de Rham form $\alpha \in \Omega_G^{i,j}(G^k)$, $i,j \geq 0$, the evaluation map

$$(2.1) \qquad E_{[x_1|x_2|\ldots|x_k]}\colon \mathrm{Hom}(\Pi, G) \to G^k, \quad \phi \mapsto (\phi(x_1), \ldots, \phi(x_k)),$$

yields the form

$$(2.2) \qquad E^*_{[x_1|x_2|\ldots|x_k]}(\alpha) \in \Omega_G^{i,j}(\mathrm{Hom}(\Pi, G)).$$

This construction can be formalized in the following way:

Let $k \geq 0$, and consider the differential graded algebra

$$(2.3) \qquad \Omega^*(\mathrm{Hom}(\Pi, G) \times \Pi^k) = \Omega^*(\mathrm{Hom}(\Pi, G)) \otimes C^k(\Pi).$$

The evaluation map E from $\mathrm{Hom}(\Pi, G) \times \Pi^k$ to G^k is compatible with the obvious G-actions and induces a morphism

$$(2.4) \qquad E^{*,*}\colon (\Omega_G^{*,*}(G^k), d, \delta_G) \to (\Omega_G^{*,*}(\mathrm{Hom}(\Pi, G)); d, \delta_G) \otimes C^k(\Pi)$$

of equivariant de Rham algebras. Moreover, as k varies, these maps assemble to a morphism

$$(2.5) \qquad (\Omega_G^{*,*}(G^\natural); d, \delta_G, \delta^\natural) \to (\Omega_G^{*,*}(\mathrm{Hom}(\Pi, G)); d, \delta_G) \otimes (C^\natural(\Pi), \delta^\natural)$$

of tricomplexes; in a given tridegree (i,j,k), it goes from $\Omega_G^{i,j}(G^k)$ to $\Omega_G^{i,j}(\mathrm{Hom}(\Pi,G)) \otimes C^k(\Pi)$. For each bar complex degree k, pairing with chains in $C_k(\Pi)$, we obtain the graded bilinear pairing

$$\langle \cdot,\cdot \rangle \colon \big((\Omega_G^{*,*}(\mathrm{H}); d, \delta_G) \otimes C^k(\Pi)\big) \otimes C_k(\Pi) \to \big(\Omega_G^{*,*}(\mathrm{H}); d, \delta_G\big) \tag{2.6}$$

which is compatible with the operators d and δ_G and, for every $u \in \Omega_G^{*,*}(\mathrm{H}) \otimes C^k(\Pi)$ and every $v \in C_{k+1}(\Pi)$, satisfies

$$\langle u, \partial_\natural v\rangle = (-1)^{k+1}\langle \delta^\natural u, v\rangle, \tag{2.7}$$

where the right-hand side refers to (2.6) for $k+1$ rather than k; here the sign $(-1)^{k+1}$ is forced by the Eilenberg-Koszul convention for the differential on a Hom-complex. Combining (2.6) with (2.5) and abusing the notation $\langle \cdot,\cdot \rangle$ slightly, we then obtain the pairing

$$\langle \cdot,\cdot \rangle \colon \big(\Omega_G^{*,*}(G^\natural); d, \delta_G, \delta^\natural\big) \otimes \big(C_\natural(\Pi), \partial_\natural\big) \to \big(\Omega_G^{*,*}(\mathrm{H}); d, \delta_G\big) \tag{2.8}$$

which is compatible with the operators d and δ_G and, moreover, satisfies

$$\langle Q, \partial_\natural c\rangle = (-1)^{k+1}\langle \delta^\natural Q, c\rangle, \quad Q \in \Omega_G^{*,*}(G^k), \ \ c \in C_{k+1}(\Pi), \tag{2.9}$$

whatever $k \geq 0$. Thus pairing a form Q in $\Omega_G^{*,*}(G^\natural)$ against a chain c in $C_\natural(\Pi)$, we obtain the form $\langle Q, c\rangle$ in $\Omega_G^{*,*}(\mathrm{H})$. We shall need an explicit expression for the value $D\langle Q, c\rangle$ in terms of Q and c of the total differential D on the right-hand side of (2.8). There is no real obstacle to calculating this value in terms of the pairing (2.8) and the operators $d, \delta_G, \delta^\sharp$, and $\partial^\sharp$, but since (2.8) does not behave as a pairing of chain complexes for the operators $\delta^\sharp$ and $\partial^\sharp$, cf. (2.9), this calculation is somewhat of a mess. The cure is provided by an extension of the construction which leads to the formula (2.15) below: Recall that, for an arbitrary differential graded coalgebra C with diagonal Δ and arbitrary ground ring R, — in fact, we could take an arbitrary differential graded algebra here — the cap pairing $\cap$ from $\mathrm{Hom}(C,R) \otimes C$ to C is given by the composite

$$\mathrm{Hom}(C,R) \otimes C \xrightarrow{\mathrm{Id}\otimes\Delta} \mathrm{Hom}(C,R) \otimes C \otimes C \xrightarrow{\mathrm{ev}\otimes\mathrm{Id}_C} R \otimes C \xrightarrow{\cong} C$$

where "ev" denotes the evaluation pairing. When we take for C the inhomogeneous reduced normalized bar construction of Π, we obtain the cap pairing from $(C^\natural(\Pi), \delta^\natural) \otimes (C_\natural(\Pi), \partial_\natural)$ to $(C_\natural(\Pi), \partial_\natural)$ inducing on homology the cap pairing $\cap$ from $\mathrm{H}^\natural(\Pi) \otimes \mathrm{H}_{\natural+\ell}(\Pi)$ to $\mathrm{H}_\ell(\Pi)$, for $\ell \geq 0$. Tensoring the identity morphism with the cap pairing yields an extension

$$\begin{aligned} \mathrm{Id} \otimes \cap \colon (\Omega_G^{*,*}(\mathrm{H}); d, \delta_G) \otimes (C^\natural(\Pi), \delta^\natural) \otimes (C_\natural(\Pi), \partial_\natural) \\ \to \big(\Omega_G^{*,*}(\mathrm{H}); d, \delta_G\big) \otimes (C_\natural(\Pi), \partial_\natural) \end{aligned} \tag{2.10}$$

of (2.6) above which is compatible with all the operators coming into play and hence induces a pairing

$$\big|(\Omega_G^{*,*}(\mathrm{H}); d, \delta_G)\big| \otimes (C^\natural(\Pi), \delta^\natural) \otimes (C_\natural(\Pi), \partial_\natural) \to \big|(\Omega_G^{*,*}(\mathrm{H}); d, \delta_G)\big| \otimes (C_\natural(\Pi), \partial_\natural) \tag{2.11}$$

of the chain complexes resulting from totalization, which we write $|\cdot|$. Recall that the total differential on $\left|(\Omega_G^{*;*}(\mathrm{H}); d, \delta_G)\right|$ is simply the sum $d + \delta_G$. Finally, when we combine (2.10) with (2.5), we obtain the pairing

$$\langle\cdot,\cdot\rangle\colon \left(\Omega_G^{*;*}(G^\natural); d, \delta_G, \delta^\natural\right) \otimes \left(C_\natural(\Pi), \partial_\natural\right) \to \left(\Omega_G^{*;*}(\mathrm{H}); d, \delta_G\right) \otimes \left(C_\natural(\Pi), \partial_\natural\right) \tag{2.12}$$

which is compatible with all the operators and induces a pairing

$$\langle\cdot,\cdot\rangle\colon \left|(\Omega_G^{*;*}(G^\natural); d, \delta_G, \delta^\natural)\right| \otimes \left(C_\natural(\Pi), \partial_\natural\right) \to \left|(\Omega_G^{*;*}(\mathrm{H}); d, \delta_G)\right| \otimes \left(C_\natural(\Pi), \partial_\natural\right) \tag{2.13}$$

of the chain complexes resulting from totalization. We remind the reader that, for every (i,j,k), on the homogeneous component $\Omega_G^{i,j}(G^k)$, the total differential d_G on $\left|(\Omega_G^{*;*}(G^\natural); d, \delta_G, \delta^\natural)\right|$ is given by

$$d_G = d + \delta_G + (-1)^{i+j}\delta^\natural. \tag{2.14}$$

The compatibility property of (2.13) means that, when D refers to the tensor product differential on the right-hand side $\left|(\Omega_G^{*;*}(\mathrm{H}); d, \delta_G)\right| \otimes \left(C_\natural(\Pi), \partial_\natural\right)$ of (2.13), for $Q \in |\Omega_G^{*;*}(G^\natural)|$ and $c \in C_\natural(\Pi)$,

$$D\langle Q, c\rangle = \langle d_G Q, c\rangle + (-1)^{|Q|}\langle Q, \partial_\natural c\rangle \tag{2.15}$$

where $|Q|$ denotes the total degree of Q. Notice in a given quadruple degree $(i,j,k,k+\ell)$, (2.12) goes from $\Omega_G^{i,j}(G^k) \otimes C_{k+\ell}(\Pi)$ to $\Omega_G^{i,j}(\mathrm{H}) \otimes C_\ell(\Pi)$.

The pairing (2.12) and hence (2.13) is natural, in fact *covariant*, in the variable Π but notice that Π also occurs in $\mathrm{H} = \mathrm{Hom}(\Pi, G)$ so that the forms $\Omega_G^{*;*}(\mathrm{H})$ are also covariant in Π.

The total complex $\left|(\Omega_G^{*;*}(G^\natural); d, \delta_G, \delta^\natural)\right| = \left(|\Omega_G^{*;*}(G^\natural)|, d_G\right)$ inherits a structure of differential graded algebra in the following way: For each pairs (i,j) and (i',j') of bidegrees and for each k, k', consider the canonical pairing

$$\Omega_G^{i,j}(G^k) \otimes \Omega_G^{i',j'}(G^{k'}) \to \Omega_G^{i+i',j+j'}(G^k \times G^{k'});$$

it amounts to the dual of the *Alexander-Whitney* map for the usual bar construction. These pairings induce the searched for structure of differential graded algebra. It is natural in terms of the data.

The differential graded algebra $\left(|\Omega_G^{*;*}(G^\natural)|, d_G\right)$ computes the equivariant real cohomology algebra of the classifying space BG for G where G acts on BG via conjugation. To recall what this cohomology looks like, we assume henceforth G compact; the general case may as usual be reduced to this one by taking a maximal compact subgroup. Let Ig be the graded algebra of invariant polynomials on g, where g is endowed with degree 2 as usual; it is well known to be itself a finitely generated polynomial algebra. Inspection of the Serre spectral sequence for the Borel construction $EG \times_G BG$ shows at once that the equivariant cohomology algebra $\mathrm{H}_G^*(BG)$ of BG is isomorphic to $Ig \otimes Ig$. In particular, every class in $\mathrm{H}^*(BG)$ has an equivariantly closed representative in the total complex $\left(|\Omega_G^{*;*}(G^\natural)|, d_G\right)$, that is, the restriction mapping from $(|\Omega_G^{*,*}(G^\natural)|, d_G)$ to $|(\Omega^*(G^\natural), d, \delta^\natural)|$ induces a surjection from $H_G^*(BG)$ to $H^*(BG)$ on cohomology.

Explicit generators arise as follows: Take the realization of the nerve NG of G as a model for the classifying space BG, and let Q be an invariant degree r polynomial on g. SHULMAN'S simplicial Chern-Weil construction [45], [7], [9], applied to the universal simplicial principal G-bundle $(\xi_0, \xi_1, \dots)$ over the simplicial space NG, yields forms

$$(2.16) \qquad Q^{r,r} \in \Omega^r(G^r), \quad Q^{r+1,r-1} \in \Omega^{r+1}(G^{r-1}), \quad \dots \quad , \quad Q^{2r-1,1} \in \Omega^{2r-1}(G),$$

and the sum $Q^{r,r} + \dots + Q^{2r-1,1}$ is a closed element of $|(\Omega^*(G^\natural); d, \delta^\natural)|$ which represents the class $[Q] \in H^{2r}(BG)(= H^{2r}(NG))$ arising from Q. More precisely, for each $q \geq 1$, the Maurer-Cartan forms yield a connection on the corresponding (trivial) principal G-bundle $\xi_q: G^{q+1} \times \Delta_q \to G^q \times \Delta_q$ having curvature $F_q \in \Omega^2(G^q \times \Delta_q, \mathrm{ad}(\xi_q))$, and, for $1 \leq q \leq r$,

$$Q^{2r-q,q} = \int_{\Delta_q} Q(F_q) \in \Omega^{2r-q}(G^q).$$

Note that $Q^{2r-1,1}$ is a closed form on G representing the generator of $H^{2r-1}(G)$ which transgresses to $[Q]$.

As observed in [28], the equivariant Chern-Weil construction [54] yields explicit equivariant extensions of these forms: Given a Lie group H and an arbitrary H-equivariant principal G-bundle $\xi: P \to M$, for an H-equivariant connection on ξ with connection from $\vartheta \in \Omega^1(P, g)^H$, define the *moment* $\mu = \mu_\vartheta \in \Omega^{2,0}(M, \mathrm{ad}(\xi))$ of the connection by

$$\mu: h \to \Omega^0(M, \mathrm{ad}(\xi)) = C^\infty(P, g)^G, \quad \mu(X) = \vartheta(X_P)$$

where X_P denotes the vector field on P induced by X. Then an invariant degree r polynomial Q on g determines the closed form

$$Q(F + \mu) = \widetilde{Q}^{0,2r} + \widetilde{Q}^{2,2r-2} + \dots + \widetilde{Q}^{2r,0} \in |\Omega_H^{*,*}(M)|^{2r}$$

where $\widetilde{Q}^{i,j} \in \Omega_H^{i,j}(M)$. When we apply this to the principal G-bundle ξ_q with $H = G$ acting by conjugation, with the notation $\mu_q \in \Omega^{2,0}(G^q \times \Delta_q, \mathrm{ad}(\xi_q))$ for the corresponding moment, we obtain the closed form

$$Q(F_q + \mu_q) = \widetilde{Q}^{0,2r} + \widetilde{Q}^{2,2r-2} + \dots + \widetilde{Q}^{2r,0} \in |\Omega_G^{*,*}(G^q \times \Delta_q)|^{2r}$$

where $Q^{i,j} \in \Omega_G^{i,j}(G^q \times \Delta_q)$, and integration yields the forms

$$Q^{0,2r-q,q} = \int_{\Delta_q} \widetilde{Q}^{0,2r} \in \Omega^{0,2r-q}(G^q),$$

$$Q^{2,2r-2-q,q} = \int_{\Delta_q} \widetilde{Q}^{2,2r-2} \in \Omega^{2,2r-2-q}(G^q),$$

$$\dots$$

$$Q^{2r-q,0,q} = \int_{\Delta_q} \widetilde{Q}^{2r-q,q} \in \Omega^{2r-q,0}(G^q), \qquad \text{if } q \text{ is even},$$

$$Q^{2r-q-1,1,q} = \int_{\Delta_q} \widetilde{Q}^{2r-q-1,q+1} \in \Omega^{2r-q-1,1}(G^q), \qquad \text{if } q \text{ is odd}.$$

For an invariant polynomial Q on g of degree r, write

$$\Omega_Q = \Sigma Q^{2i,j,q}, \quad 2i+j+q=2r, \quad q \leq 2i+j, \quad q \leq r;$$

this is a closed element of $|(\Omega^{*,*}(G^{\natural}); d, \delta_G \delta^{\natural})|$ representing a class $[\Omega_Q] \in \mathrm{H}^{2r}_G(BG)$.

Theorem 2.17. *For every invariant polynomial Q on g of degree r, the class $[\Omega_Q] \in \mathrm{H}^{2r}_G(BG)$ restricts to the class $[Q] \in \mathrm{H}^{2r}(BG)$ arising from Q. Furthermore, when Q runs through a set of polynomial generators of Ig, the classes $[\Omega_Q]$ together with the elements Q viewed as elements of $\Omega^{*,0}(G^0)$ constitute a set of polynomial generators of $\mathrm{H}^*_G(BG) = \mathrm{H}^* |(\Omega^{*,*}(G^{\natural}); d, \delta_G, \delta^{\natural})|$.*

Proof. The first statement is immediate. The "Furthermore" clause is an immediate formal consequence thereof. □

For example, let Q be an invariant symmetric bilinear form $\cdot$ on g, so that $r=2$. The above construction then yields

$$Q^{0,3,1} \in \Omega^{0,3}_G(G), \quad Q^{2,1,1} \in \Omega^{2,1}_G(G), \quad \text{for } q=1,$$
$$Q^{0,2,2} \in \Omega^{0,2}_G(G \times G), \quad Q^{2,0,2} \in \Omega^{2,0}_G(G \times G), \quad \text{for } q=2,$$

and their sum is a closed 4-form in the total complex $|(\Omega^{*,*}(G^{\natural}); d, \delta_G, \delta^{\natural})|$. Actually it may be shown that the term $Q^{2,0,2}$ is irrelevant and may be dropped. The element $Q^{0,3,1}$ is the fundamental 3-form on G constructed by E. Cartan.

The singular cochains $C^*(G)$ of G constitute a Hopf algebra, the requisite diagonal map being induced from the multiplication mapping on G by means of the shuffle map, and it is well known and classical that the cobar construction on $C^*(G)$ yields a model for the (singular) cochains on BG. The bar de Rham bicomplex $(\Omega^*(G^{\natural}); d, \delta^{\natural})$ serves as a replacement for the *cobar* construction on the differential graded algebra $\Omega^*(G)$ of forms on G which is not available in the strict sense; while the multiplication mapping of G induces a map from $\Omega^*(G)$ to $\Omega^*(G \times G)$ we cannot algebraically project down the latter to $\Omega^*(G) \otimes \Omega^*(G)$ in such a way that a coalgebra structure on $\Omega^*(G)$ results. The bar de Rham bicomplex may be viewed as a completed cobar construction.

3. Representations of free simplicial groupoids

Recall that any cosimplicial manifold $M = \{M_{\sharp}\}$ gives rise to a simplicial differential graded de Rham algebra $\Omega M = (\Omega^*(M_{\sharp}), d, \dots)$, cf. [8]. Here $\Omega^j(M_q)$ are the j-forms on M_q, for $q \geq 0$, the operator d is the usual de Rham operator on each M_q, and $\dots$ stands for the operators between usual de Rham algebras induced by the cosimplicial structure. In particular, let $\Omega\mathcal{H}$ be the simplicial differential graded algebra of de Rham forms on the cosimplicial manifold $\mathcal{H} = \mathrm{Hom}(K, G)$. The construction in the previous Section yields de Rham forms on $\mathcal{H}$. We shall explain this in Section 4 below. We need some preparation first.

Let $k \geq 0$. Since K is assumed free, the product $\mathrm{Hom}(K,G) \times K^k$ inherits a canonical structure of *cosimplicial-simplicial* manifold, and the *coend* $\mathrm{Hom}(K,G) \times_{\Delta} K^k$, cf. e. g. [36], is a (non-connected) smooth manifold; actually the coend will play no rôle in this paper. Moreover, the canonical evaluation map

$$E\colon \mathrm{Hom}(K,G) \times K^k \to G^k \tag{3.1}$$

is well defined and smooth in the sense that, for each (simplicial) degree q, the corresponding component

$$E_q\colon \mathrm{Hom}(K_q, G) \times K_q^k \to G^k \tag{3.2}$$

is smooth; we note that the evaluation map factors through the coend $\mathrm{Hom}(K, G) \times_\Delta K^k$ but this will not be important for us. We can now apply the construction in the previous Section separately for each simplicial degree q. However the naturality of the constructions provides mores structure.

Write $C_\natural(\Pi)$ and $C^\natural(\Pi)$ for the *normalized* Eilenberg-Mac Lane chains and cochains, respectively, of an ordinary groupoid Π so that $C_\natural(\Pi) = C_\natural(N\Pi)$ and $C^\natural(\Pi) = C^\natural(N\Pi)$. The *nerve* or *simplicial bar construction* NK of K inherits a structure of bisimplicial set, one simplicial structure coming from that of K and the other one from the nerve construction. Its bicomplex CNK of chains which are *normalized* in the $\natural$-direction looks like

$$(C_\natural(K_\sharp), \partial_\natural, \partial_\sharp). \tag{3.3}$$

Its vertical differentials $\partial_\sharp$ are induced by the alternating sums of the face operations induced by the simplicial structure of $K = \{K_\sharp\}$ and its horizontal ones $\partial_\natural$ by the alternating sums of the face operations induced by the nerve construction for K_q separately for each K_q. Normalizing in the $\sharp$-direction yields the bicomplex

$$(\overline{C}_\natural(K_\sharp), \partial_\natural, \partial_\sharp) \tag{3.4}$$

where the notation $\partial_\natural, \partial_\sharp$ is abused. Its total complex

$$|NK| = (|\overline{C}NK|, \partial) = |(\overline{C}_\natural(K_\sharp), \partial_\natural, \partial_\sharp)|$$

has $|NK|_0 = \mathbf{Z}$ and

$$|NK|_r = \overline{C}_r(K_0) \oplus \overline{C}_{r-1}(K_1) \oplus \cdots \oplus \overline{C}_1(K_{r-1}), \quad r \geq 1, \tag{3.5}$$

and the total differential ∂ is given by

$$\partial = \partial_\natural + \partial_\flat \tag{3.6}$$

where on elements of $\overline{C}_k(K_{r-k})$ the operator $\partial_\flat$ looks like

$$\partial_\flat = (-1)^k \partial_\sharp \colon \overline{C}_k(K_{r-k}) \to \overline{C}_k(K_{r-k-1}). \tag{3.7}$$

Note that, by normalization, $\overline{C}_0(K_r)$ is zero for $r \geq 1$. Thus for

$$c = c_{r,0} + c_{r-1,1} + \cdots + c_{1,r-1}, \quad c_{k,q} \in \overline{C}_k(K_q), \tag{3.8}$$

we have

$$\begin{aligned} \partial(c) = \partial_\natural(c) + \sum_{k+q=r} (-1)^k \partial_\sharp c_{k,q} \\ = \quad & \partial_\natural(c_{r,0}) + (-1)^{r-1}\partial_\sharp(c_{r-1,1}) \\ & + \partial_\natural(c_{r-1,1}) + (-1)^{r-2}\partial_\sharp(c_{r-2,2}) \\ & + \cdots \\ & + \partial_\natural(c_{2,r-2}) + \partial_\sharp(c_{1,r-1}). \end{aligned} \tag{3.9}$$

Let $K = KY$ for a CW-complex Y. Since K is a loop complex for Y, the map from Y to $B|K|$ is a homotopy equivalence, and the two spaces $B|K|$ and $|NK|$ are homeomorphic; as CW-complexes they are not the same, though, and the cell decomposition of $|NK|$ must be refined, in the same way as the canonical homeomorphism between the realization $|S_1 \times S_2|$ of the product of two simplicial sets S_1 and S_2 and the product $|S_1| \times |S_2|$ of the realizations will be a cellular isomorphism only after refinement of the decomposition of $|S_1 \times S_2|$; cf. [42] for details. It follows that the homology of $|NK|$ coincides with that of Y. However this may be seen directly. To this end we observe at first that, for $\sharp$ fixed, since each $K_\sharp$ is a free group, the chain complex (3.3) amounts to an exact sequence

$$0 \leftarrow \mathrm{H}_1(K_\sharp) \overset{\varepsilon_1}{\longleftarrow} C_1(K_\sharp) \leftarrow \cdots \leftarrow C_k(K_\sharp) \leftarrow \cdots \tag{3.10}$$

and (3.3), viewed as a simplicial chain complex, with simplicial structure in the $\sharp$-direction, induces a structure of simplicial abelian group $\mathrm{H}_1(K_\sharp) = \{\mathrm{H}_1(K_q)\}_{q\geq 0}$. Here we have written $\varepsilon_\sharp$ for the projection from 1-cycles to homology. For $q \geq 0$, denote by $\overline{K}_q$ the free group generated by the degree q generators, that is, by the non-degenerate basis elements of K_q. By construction, the normalized chain complex $|\mathrm{H}_1(K_\sharp)|$ of $\mathrm{H}_1(K_\sharp)$ has

$$|\mathrm{H}_1(K_\sharp)|_q = \mathrm{H}_1(\overline{K}_q) = (\overline{K}_q)^{\mathrm{Ab}} = C_{q+1}(Y), \quad q \geq 0, \tag{3.11}$$

where $C_*(Y)$ refers to the cellular chains on Y. Furthermore, $\mathrm{H}_0(K_\sharp) = \{\mathrm{H}_0(K_q)\}_{q\geq 0}$ amounts to the free simplicial abelian group generated by a single point.

Proposition 3.12. *The canonical projection map from* (3.3) *onto* $\mathrm{H}_1(K_\sharp)$ *induced by* $\varepsilon_\sharp$ *together with the canonical map from* $|NK|_0 = \mathbf{Z}$ *onto* $C_0(Y) = \mathbf{Z}$ *passes to a deformation retraction from* $|NK|$ *onto* $C_*(Y)$ *which is natural in* Y.

Proof. The canonical projection map from (3.3) onto $\mathrm{H}_1(K_\sharp)$ induced by $\varepsilon_\sharp$ yields a deformation retraction from the totalization $|(C_\natural(K_\sharp), \partial_\natural, \partial_\sharp)|$ onto the totalization of $\mathrm{H}_1(K_\sharp)$. A little thought reveals that this implies the claim. □

N. B. The normalization $\overline{C}_\natural(K_\sharp)$ contains $C_\natural(\overline{K}_\sharp)$ in an obvious fashion but does *not* coincide with it since products of degenerate free generators are in general non-degenerate.

4. Forms on representations of free simplicial groupoids

We can now extend the construction of forms in Section 2 to representations of the simplicial groupoid K. For each simplicial degree q, with $\Pi = K_q$, the pairing (2.12) looks like

$$(\Omega_G^{*;*}(G^\natural); d, \delta_G, \delta^\natural) \otimes (C_\natural(K_q), \partial_\natural) \to (\Omega_G^{*;*}(\mathrm{H}_q); d, \delta_G) \otimes (C_\natural(K_q), \partial_\natural), \tag{4.1}$$

and these assemble to the pairing

$$(\Omega_G^{*;*}(G^\natural); d, \delta_G, \delta^\natural) \otimes (C_\natural(K_\sharp), \partial_\natural) \to (\Omega_G^{*;*}(\mathrm{H}_\sharp); d, \delta_G) \otimes (C_\natural(K_\sharp), \partial_\natural). \tag{4.2}$$

The left- and right-hand side of (4.2) both inherit a simplicial structure from that of K; in fact, on the left-hand side we have such a structure on $C_\natural(K)$ and, on the right-hand side, the induced cosimplicial structure on $\mathcal{H} = \mathrm{Hom}(K, G)$ induces a simplicial structure on $\left(\Omega_G^{*,*}(\mathcal{H}); d, \delta_G\right)$. The naturality of the constructions implies that (4.2) is compatible with these structures, whence we arrive at the pairing

$$(4.3)\quad (\Omega_G^{*,*}(G^\natural); d, \delta_G, \delta^\natural) \otimes (C_\natural(K_\sharp); \partial_\natural, \partial_\sharp) \to ([(\Omega_G^{*,*}(\mathrm{H}_\sharp); d, \delta_G) \otimes (C_\natural(K_\sharp), \partial_\natural)], \partial_\sharp)$$

compatible with all the operators. In a given quintuple degree $(i, j, k, k+\ell, q)$, this pairing goes from $\Omega_G^{i,j}(G^k) \otimes C_{k+\ell}(K_q)$ to $\Omega_G^{i,j}(\mathrm{H}_q) \otimes C_\ell(K_q)$. We note that, with reference to the $\sharp$-grading, $(\Omega_G^{*,*}(\mathrm{H}_\sharp); d, \delta_G) \otimes (C_\natural(K_\sharp), \partial_\natural)$ is the graded object underlying the diagonal of a certain bisimplicial object; we have chosen the parentheses '[' and ']' on the right-hand side of (4.3), with the operator $\partial_\sharp$ outside these parentheses to indicate this.

The *normalization* $(\overline{\Omega}_G^{*,*}(\mathrm{H}_\sharp); d, \delta_G, \partial_\sharp) = \left(\Omega_G^{*,*}(\mathrm{H}_\sharp)/\Omega_G^{*,*}(\mathrm{H}_\sharp)^{\mathrm{degen}}; d, \delta_G, \partial_\sharp\right)$ of $(\Omega_G^{*,*}(\mathrm{H}_\sharp); d, \delta_G, \partial_\sharp)$ is the quotient by the subspace of degenerates where, for each simplicial degree $q \geq 1$, the subspace $\Omega_G^{*,*}(\mathrm{H}_q)^{\mathrm{degen}}$ of degenerates is the sum of the images of the degeneracy operations s_j from $\Omega_G^{*,*}(\mathrm{H}_{q-1})$ to $\Omega_G^{*,*}(\mathrm{H}_q)$, for $0 \leq j \leq q-1$. Here the notation $d, \delta_G, \partial_\sharp$ is abused. Ignoring the equivariant theory for the moment, we recall [8] that the *realization* $(|\Omega(\mathcal{H})|, D)$ of $\Omega(\mathcal{H}) = (\Omega^*(\mathrm{H}_\sharp), d, \partial_\sharp)$ is the total cochain complex of the normalized bicomplex

$$\Omega^*(\mathrm{H}_0) \xleftarrow{\partial_\sharp} \overline{\Omega^*(\mathrm{H}_1)} \xleftarrow{\partial_\sharp} \dots \xleftarrow{\partial_\sharp} \overline{\Omega^*(\mathrm{H}_q)} \xleftarrow{\partial_\sharp} \dots$$

whose vertical differentials are the de Rham operators and whose horizontal ones $\partial_\sharp$ are induced by the alternating sums of the simplicial operations $\partial_p\colon \Omega^*(\mathrm{H}_q) \to \Omega^*(\mathrm{H}_{q-1})$. The graded module $|\Omega(\mathcal{H})|$ underlying the *total complex*

$$|(\Omega^*(\mathcal{H}), d, \partial_\sharp)| = (|\Omega(\mathcal{H})|, D)$$

of this bicomplex is by definition in degree r the direct *sum* (not the product) of the $\overline{\Omega^p(\mathrm{H}_q)}$ for $p - q = r$. Thus

$$|\Omega(\mathcal{H})|^r = \Omega^r(\mathrm{H}_0) \oplus \overline{\Omega^{r+1}(\mathrm{H}_1)} \oplus \dots \oplus \overline{\Omega^{r+q}(\mathrm{H}_q)} \oplus \dots$$

Notice when K has a finite set of free generators this sum is finite in each degree. Moreover this construction has an obvious extension

$$|\Omega_G(\mathcal{H})| = (|\Omega_G^{*,*}(\mathcal{H})|, D) = \left|(\Omega_G^{*,*}(\mathrm{H}_\sharp); d, \delta_G, \partial_\sharp)\right|$$

to the equivariant theory so that

$$|\Omega_G^{*,*}(\mathcal{H})|^r = |\Omega_G^{*,*}|^r(\mathrm{H}_0) \oplus \overline{|\Omega_G^{*,*}|^{r+1}(\mathrm{H}_1)} \oplus \dots \oplus \overline{|\Omega_G^{*,*}|^{r+q}(\mathrm{H}_q)} \oplus \dots$$

The compatibility of (4.3) with all the operators entails that after totalization and normalization we arrive at the pairing

$$(4.4)\quad \left|(\Omega_G^{*,*}(G^\natural); d, \delta_G, \delta^\natural)\right| \otimes |NK| \to \mathrm{N}_\sharp\left([|(\Omega_G^{*,*}(\mathrm{H}_\sharp); d, \delta_G)| \otimes (C_\natural(K_\sharp), \partial_\natural)], \partial_\sharp\right)$$

where $N_\sharp$ refers to normalization in the $\sharp$-direction. The generalized Eilenberg-Zilber theorem [58] yields a natural chain equivalence from the right-hand side of (4.4) onto $(|\Omega_G^{*;*}(\mathcal{H})|, D) \otimes |NK|$. Hence (4.4) combined with this surjection yields the pairing

$$\left|(\Omega_G^{*;*}(G^\natural); d, \delta_G, \delta^\natural)\right| \otimes |NK| \to (|\Omega_G^{*;*}(\mathcal{H})|, D) \otimes |NK| \tag{4.5}$$

When we combine it with the chain map $\mathrm{Id} \otimes \varepsilon$ where ε is the augmentation map from $|NK|$ to the reals induced by the obvious projection from NK to a point, we arrive at the pairing

$$\langle \cdot, \cdot \rangle \colon \left|(\Omega_G^{*;*}(G^\natural); d, \delta_G, \delta^\natural)\right| \otimes |NK| \to (|\Omega_G^{*;*}(\mathcal{H})|, D) \tag{4.6}$$

This pairing can be understood without explicit reference to the generalized Eilenberg-Zilber theorem: it amounts to picking the components of the right-hand side of (4.4) which involve only $C_0(K_\sharp)$ and ignoring the rest, but the generalized Eilenberg-Zilber theorem provides the appropriate formal circumstances. The precise geometric analogue

$$\Omega_G^{*;*}(BG) \otimes C_*(M) \to \Omega_G^{*;*}(\mathrm{Smooth}^o(M, BG))$$

of (4.6) for a smooth manifold M arises from the evaluation pairing from $\mathrm{Smooth}^o(M, BG) \times M$ to BG combined with integration against chains on M and subsequent composition with the chain map induced by the augmentation map from $C_*(M)$ to the reals $\mathbf{R}$.

The pairing (4.6) produces equivariant forms on the realization $\left|(\overline{\Omega}_G^{*,*}(\mathrm{H}_\sharp); d, \delta_G, \partial_\sharp)\right|$ and hence, as we shall see later, on the cosimplicial manifold $\mathcal{H} = \mathrm{Hom}(K, G)$, in the following way: Let u and r be positive integers, let Ω be an equivariant form, that is, an element of

$$\left|(\Omega_G^{*;*}(G^\natural); d, \delta_G, \delta^\natural)\right| = (|\Omega_G^{*;*}(G^\natural)|, D)$$

of total degree $u + r$, and let c be a chain of $|NK|$ of total degree $r \geq 1$, cf. (3.5). For k fixed, let $\Omega^k \in \oplus_{i+j=u+r-k} \Omega_G^{i,j}(G^k)$ be the indicated component. Then

$$\langle \Omega, c \rangle = \langle \Omega^1, c_{1,r-1} \rangle + \cdots + \langle \Omega^r, c_{r,0} \rangle \in |\overline{\Omega}_G^{*;*}(\mathrm{H}_\sharp)|.$$

Lemma 4.7. *Suppose Ω and c are closed. Then $\langle \Omega, c \rangle$ is a closed form in*

$$\left(\left|\overline{\Omega}_G^{*;*}(\mathrm{H}_\sharp)\right|, D\right) = \left|\left(\overline{\Omega}_G^{*;*}(\mathrm{H}_\sharp), d, \delta_G, \partial_\sharp\right)\right|$$

of total degree u.

Proof. This follows at once from the identity

$$D\langle \Omega, c \rangle = \langle d_G \Omega, c \rangle + (-1)^{|\Omega|} \langle \Omega, \partial c \rangle.$$

5. Realization and integration

A standard construction endows $(|\Omega^*(\mathcal{H})|, D)$ with a structure of differential graded algebra; we shall explain this below in the equivariant setting. Before doing so for illustration, we point out that, for K the Kan group on the 2-sphere with its standard decomposition with two cells, for a Lie group G, $|\mathrm{Hom}(K,G)|$ amounts to the space $\Omega G = \mathrm{Map}^o(S^1, G)$ of based loops on G and we have on the one hand the bar construction $B\Omega^* G$ on the de Rham complex $\Omega^* G$, with its shuffle multiplication, as a model for the algebra of cochains on the based loop space. On the other hand, as a cosimplicial space, $\mathcal{H} = \mathrm{Hom}(K, G)$ looks like $(o, G, G^2, \dots, G^q, \dots)$, and the realization of the simplicial differential graded algebra $\Omega^*(\mathcal{H})$ (whose algebra structure is yet to be explained) has constituents $\Omega^*(G^q)$. The canonical maps from $(\Omega^*(G))^q$ to $\Omega^*(G^q)$ now induce a morphism of differential graded algebras from $B\Omega^* G$ to $(|\Omega^*(\mathcal{H})|, D)$ which is a homology isomorphism.

We now recall the construction of differential graded algebra structure on $(|\Omega^{*,*}_G(\mathcal{H})|, D)$: For each pairs (i,j) and (i',j') of bidegrees, we have the simplicial vector spaces $\Omega^{i,j}_G(\mathrm{H}_\sharp)$ and $\Omega^{i',j'}_G(\mathrm{H}_\sharp)$, and for each pair (q,q') the *shuffle map* ∇ yields a natural morphism

$$\nabla \colon \Omega^{i,j}_G(\mathrm{H}_q) \otimes \Omega^{i',j'}_G(\mathrm{H}_{q'}) \to \Omega^{i,j}_G(\mathrm{H}_{q+q'}) \otimes \Omega^{i',j'}_G(\mathrm{H}_{q+q'})$$

of vector spaces which, combined with usual multiplication of forms, yields a pairing

$$\Omega^{i,j}_G(\mathrm{H}_q) \otimes \Omega^{i',j'}_G(\mathrm{H}_{q'}) \to \Omega^{i+i',j+j'}_G(\mathrm{H}_{q+q'}).$$

This pairing endows $(|\Omega^{*,*}_G(\mathcal{H})|, D)$ with a structure of differential graded commutative algebra which is natural in the data; by construction, it arises from a differential trigraded algebra structure.

We now relate this algebra with forms on the geometric realization. To this end, pick $q \geq 0$ and consider the evaluation mapping from $\Delta_q \times \mathrm{Smooth}(\Delta_q, \mathrm{H}_q)$ to H_q. For each $r \geq 0$, integration over Δ_q induces a map

$$I_q \colon \Omega^{r+q}(\mathrm{H}_q) \to \Omega^r(\mathrm{Smooth}(\Delta_q, \mathrm{H}_q))$$

with a suitable interpretation of forms on the mapping spaces. Using the theory of *differentiable space* [56], [57] or, what amounts to the same, that of *"diffeological" space* ("espace difféologique") [63], [64], forms on the mapping spaces admit a purely finite dimensional interpretation and do *not* require infinite dimensional techniques. With a suitable interpretation of forms $\Omega^*(|\mathcal{H}|_{\mathrm{smooth}})$ on the geometric realization $|\mathcal{H}|_{\mathrm{smooth}}$, the integration maps assemble to a morphism

$$I \colon |(\Omega^*(\mathcal{H}); d, \partial_\sharp)| = (|\Omega^*(\mathcal{H})|, D) \to (\Omega^*(|\mathcal{H}|_{\mathrm{smooth}}), d) \tag{5.1}$$

of differential graded algebras, cf. Section 5 of [8]. The problem here is that the geometric realization $|\mathcal{H}|_{\mathrm{smooth}}$ will have singularities. This difficulty is overcome by means of the already cited concept of differentiable space, in the following way: Recall that a *plot* for $|\mathcal{H}|_{\mathrm{smooth}}$ is a map F from a smooth finite dimensional manifold W to $|\mathcal{H}|_{\mathrm{smooth}}$ which is smooth in the sense that the adjoint

$$F^\natural_q \colon W \times \Delta_q \to \mathrm{H}_q$$

of each component

$$F_q: W \to \mathrm{Smooth}(\Delta_q, \mathrm{H}_q)$$

of F is smooth [56, 57]; in this theory, a *form* on $|\mathcal{H}|_{\mathrm{smooth}}$ is the assignment of a form on W to each plot which is natural for smooth maps in the domains of the plots. With this interpretation of forms on $(\Omega^*(|\mathcal{H}|_{\mathrm{smooth}}), d)$ the above integration mapping I makes strict sense. In particular, when $\mathcal{M}$ is a subspace of $|\mathcal{H}|_{\mathrm{smooth}}$ which *is* smooth we can combine the integration mapping with restriction, and there results a morphism I of differential graded algebras from $(|\Omega^*(\mathcal{H})|, D)$ to $(\Omega^*(\mathcal{M}), d)$. Furthermore the whole construction is G-invariant whence, with the appropriate notion of G-equivariant plots, we finally obtain a morphism

$$(5.2) \qquad I: |(\Omega_G^{*,*}(\mathcal{H}); d, \delta_G, \partial_\sharp)| = (|\Omega_G^{*,*}(\mathcal{H})|, D) \to |(\Omega_G^{*,*}(|\mathcal{H}|_{\mathrm{smooth}}); d, \delta_G)|$$

of differential bigraded algebras.

Under our circumstances, G-equivariant plots admit a natural interpretation as *G-equivariant families of principal bundles with connection*: a G-equivariant plot $F: W \to |\mathcal{H}|_{\mathrm{smooth}}$, combined with the map Φ (cf. (1.6)), yields a map from W to $\mathrm{Smooth}^o(Y, BG)$ having a "smooth" G-equivariant adjoint

$$\widetilde{F}: W \times Y \to BG$$

satisfying $\widetilde{F}(w, o) = o$. Consequently a G-equivariant plot F for $|\mathcal{H}|_{\mathrm{smooth}}$ defined on W amounts to a smooth G-equivariant family of G-bundles with connection on Y parametrized by W.

A cosimplicial space is said to *converge* [1], [8], [10], when integration yields a cohomology equivalence from the cohomology of the realization of the simplicial de Rham algebra to the cohomology of the realization. The cosimplicial space $\mathcal{H}$ will rarely have this property; however see Section 7 below.

6. Extended moduli spaces for a closed surface and generalizations

Let Σ be a closed topological surface of genus $\ell \geq 0$, endowed with the usual CW-decomposition with a single 0-cell o, with 1-cells $u_1, v_1, \ldots, u_\ell, v_\ell$, and with a single 2-cell c, and let

$$\mathcal{P} = \langle x_1, y_1, \ldots, x_\ell, y_\ell; r \rangle$$

be the corresponding presentation for the fundamental group π of Σ. We maintain the notation as in Section 2 of [60] without repeating it. Our aim is to show how the results of [21], [25] may at once be deduced from our general theory: Write Q for the given bilinear 2-form on g, and let

$$\Omega_Q = Q^{0,3,1} + Q^{2,1,1} + Q^{0,2,2} + Q^{2,0,2} \in |\Omega_G^{*,*}(G^\natural)|$$

be the resulting equivariantly closed form of total degree 4, cf. (2.17). Since $\mathrm{H}_2(\Sigma)$ is infinite cyclic, in view of (3.12), there is a 2-cycle

$$c = c_{2,0} + c_{1,1}, \quad c_{2,0} \in C_2(K_0), \ c_{1,1} \in \overline{C}_1(K_1),$$

of $|NK|$ which under the deformation retraction onto the cellular chains of Σ goes to a 2-cycle representing a generator. It is very easy to manufacture such a 2-cycle:

Let $c_{2,0} \in C_2(K_0) = C_2(F)$ be a 2-chain with $\partial_\natural c_{2,0} = \Pi[x_j, y_j] \in F$; such a $c_{2,0}$ exists since $\Pi[x_j, y_j]$ is zero in $\mathrm{H}_1(F) = F^{\mathrm{Ab}}$; moreover, let $c_{1,1} = r \in K_1$ so that, by construction, $\partial_\sharp c_{1,1} = \Pi[x_j, y_j] \in K_0$. Then c is closed in $|NK|$, and $\langle Q, c\rangle$ is a closed element of $\left|(\Omega_G^{*,*}(\mathrm{H}_\sharp); d, \delta_G, \partial_\sharp)\right|$ of degree 2. Notice when $\ell = 0$ we have $c_{2,0} = 0$.

Embed the Lie algebra g into $\mathrm{Smooth}(\Delta_1, G)$ by the assignment to $X \in g$ of the corresponding path $t \mapsto \exp(tX)$, let $O \subseteq g$ be the subspace where the exponential mapping is regular, and let $\mathcal{M}$ be the subspace of $|\mathcal{H}|_{\mathrm{smooth}}$ consisting of pairs $(w, X) \in G^{2\ell} \times O$ so that $\exp(X) = r(w)$. This is a smooth finite dimensional G-manifold and the inclusion F from $\mathcal{M}$ to $|\mathcal{H}|_{\mathrm{smooth}}$ is a G-equivariant plot. By construction, the equivariantly closed form

$$I\langle Q, c\rangle \in \left|\overline{\Omega}_G^{*,*}(\mathcal{M})\right|$$

of degree 2 has components

$$\omega_c = I\langle Q^{0,3,1}, c_{1,1}\rangle + I\langle Q^{0,2,2}, c_{2,0}\rangle \in \Omega_G^{0,2}(\mathcal{M})$$
$$\mu^\sharp = I\langle Q^{2,1,1}, c_{1,1}\rangle + I\langle Q^{2,0,2}, c_{2,0}\rangle \in \Omega_G^{2,0}(\mathcal{M})$$

so that $\mu^\sharp\colon g \to C^\infty(\mathcal{M})$ is the adjoint of a smooth map μ from $\mathcal{M}$ to g. The sum $\omega_c + \mu^\sharp$ to be equivariantly closed amounts to the closedness of ω_c in the usual sense together with the property that

$$\delta_G \omega_c = d\mu^\sharp$$

which is the momentum mapping property. In particular, the integration mapping from the realization of the equivariant simplicial differential graded de Rham algebra to the equivariant differential graded de Rham algebra on the realization reproduced in Section 5 above now provides a natural explanation for the operation of integration along linear paths in g which in [21], [25] seemed somewhat ad hoc. The term $I\langle Q^{2,0,2}, c_{2,0}\rangle$ is actually irrelevant and may be ignored; it amounts to a constant modification of the momentum mapping.

The whole approach may be extended to arbitrary 2-complexes Y with a single 0-cell and a single 2-cell; write r for the corresponding relator. The fundamental group π of such a 2-complex is a one-relator group. It is well known that Y and π have second homology group a copy of the integers if and only if the exponent sum of each generator in the relator r equals zero. In this case the above construction carries over verbatim and there results an extended moduli space $\mathcal{M}$ together with a closed 2-form ω and momentum mapping. However in order for ω to be non-degenerate we must require the relevant cup pairings on $\mathrm{H}^1(\pi, \cdot)$ to be non-degenerate. This is related with the question whether π is a 2-dimensional Poincaré duality group over the reals. We do not pursue this question here.

7. Cohomology

Let $r \geq 1$, let c be a cellular r-cycle of Y representing an integral homology class, and let c_K be an r-cycle of $|NK|$ which under the deformation retraction onto the cellular chains of Y goes to c, cf. (3.12). For every invariant polynomial

Q on g of degree u, with $r \leq 2u$, by (4.7), $\langle \Omega_Q, c_K \rangle$ is a closed element of $(|\Omega_G^{*,*}(\mathcal{H})|, D)$ of degree $2u - r$; here Ω_Q refers to the corresponding closed elements of $|(\Omega_G^{*,*}(G^\natural); d, \delta_G, \delta^\natural)|$, cf. (2.17). Recall that in Section 1 of [60] we constructed a weak chain equivalence between the space of based maps from Y to BG and the realization of $\mathcal{H}$.

Theorem 7.1. *As a graded commutative algebra, the equivariant cohomology of each connected component of $|\mathcal{H}|$ and hence of the space of based maps from Y to BG is freely generated by the classes of the elements $I\langle \Omega_Q, c_K \rangle$ where Q runs through a set of invariant polynomials on g and c through a set of representatives in degree ≥ 1 of the real homology of Y subject to the restriction $|c| < 2|Q|$, together with the invariant polynomials Q viewed as elements of $\Omega_G^{*,0}(G^0)$.*

Proof. This is proved by induction on dimension, with reference to the fibration (1.8.1). The argument is formally the same as that hinted at on p. 181 of [15], cf. also Note 5.1.2 on p. 206. The induction starts with the observation that $\mathrm{Hom}(KY^1, G)$ amounts to a product of as many copies of G as Y has 1-cells and that, for a circle $K = S^1$, when c represents the generator of its first homology group, the element $\langle \Omega_Q, c_K \rangle$ yields the exterior generator of $\mathrm{H}^{2|Q|-1}(G)$ which transgresses to the class of $[Q]$ in $\mathrm{H}^{2|Q|}(BG)$. We leave the details to the reader. □

Here the realization $|\mathcal{H}|_{\mathsf{smooth}}$ of $\mathcal{H}$ is viewed as a space with differentiable structure in the sense of [56, 57, 63, 64] as explained in Section 5 above.

ILLUSTRATION. Let Y be a closed surface Σ of genus $\ell \geq 0$, let $G = \mathrm{U}(n)$, the unitary group, let $Q_1, \ldots, Q_n$ be the Chern polynomials, and let $\Omega_1, \ldots, \Omega_n$ be the corresponding closed elements of $|(\Omega_G^{*,*}(G^\natural); d, \delta_G, \delta^\natural)|$. Maintaining the notation in the previous Section, for $r = 1, \ldots, n$ and $j = 1, \ldots, \ell$, we get the elements

$$\begin{aligned} f_r &= \langle \Omega_r, c \rangle, \quad |f_r| = 2r - 2, \\ b_r^j &= \langle \Omega_r, u_j \rangle, \quad |b_r^j| = 2r - 1, \\ b_r^{j+\ell} &= \langle \Omega_r, v_j \rangle, \quad |b_r^{j+\ell}| = 2r - 1, \\ a_r &= Q_r, \quad |a_r| = 2r, \quad \text{viewed as an element of } \Omega_G^{2r,0}(G^0). \end{aligned}$$

They freely generate the equivariant cohomology of the space $\mathrm{Map}^o(\Sigma, BG)$ or, what amounts to the same, of the union over all topological types of G-bundles of spaces of based gauge equivalence classes, cf. what is said in Section 1. Notice f_1 picks the topological type of connected component. The generators f_r and b_r^j coincide with those constructed in [62], see also Section 2 of [2] and [28]. Likewise, for genus $\ell = 0$ and arbitrary connected G, when Q denotes the given invariant symmetric bilinear form on g so that $Q^{0,3,1}$ is E. Cartan's fundamental 3-form on G, cf. what was said at the end of Section 2, the resulting 2-form on ΩG restricts to the Kirillov form on each connected component of $\mathrm{Hom}(S^1, G)$, when identified with the adjoint orbits in g generated by some X with $\exp(X) = e$.

When G is simply connected the cosimplicial space $\mathcal{H} = \mathrm{Hom}(K\Sigma, G)$ converges, that is, the integration mapping (5.2) is a cohomology equivalence from the cohomology of the realization of the simplicial de Rham algebra to the cohomology of the realization.

8. 3-complexes and 3-manifolds

Let Y be a 3-complex with a single 3-cell, for example, a closed compact 3-manifold, endowed with a regular CW-decomposition with a single 0-cell o, with 1-cells $u_1, \ldots, u_\ell$, 2-cells $c_1, \ldots, c_\ell$, and a single 3-cell c, and write

$$\mathcal{S} = \langle x_1, \ldots, x_\ell; r_1, \ldots, r_\ell; \sigma \rangle$$

for the corresponding spine for Y; in particular, (i) the data $\mathcal{P} = \langle x_1, \ldots, x_\ell; r_1, \ldots, r_\ell \rangle$ constitute a presentation of the fundamental group π of Y so that the attaching maps of the 2-cells assign a word w_j in the free group F on the generators to each relator r_j, and (ii) the attaching map σ of the single 3-cell assigns an *identity among relations*

$$i = z_1 r_{j_1}^{\varepsilon_1} z_1^{-1} \ldots z_m r_{j_m}^{\varepsilon_m} z_m^{-1}$$

to c representing the element of the second homotopy group $\pi_2(Y^2)$ of the 2-skeleton Y^2 of Y which is killed by the 3-cell c. See Section 3 of [60] for more details and notation. We shall give a purely finite dimensional description of the Chern-Simons function. This will be the assignment to every G-equivariant plot

$$F: W \to |\mathcal{H}|_{\text{smooth}}$$

of a smooth G-invariant map Ψ from W to the circle S^1 which is natural in G-equivariant plots.

Write Q for the given bilinear 2-form on g, and let

$$\Omega_Q = Q^{0,3,1} + Q^{2,1,1} + Q^{0,2,2} + Q^{2,0,2} \in \left|\Omega_G^{*,*}(G^\natural)\right|$$

be the resulting equivariantly closed form of total degree 4, cf. (2.17). Suppose $\mathrm{H}_3(Y)$ infinite cyclic; for example this will be true when Y is an orientable 3-manifold. In view of (3.12), there is a 3-cycle

$$c = c_{3,0} + c_{2,1} + c_{1,2}, \quad c_{3,0} \in C_3(K_0), \ c_{2,1} \in \overline{C}_2(K_1), \ c_{1,2} \in \overline{C}_1(K_2),$$

of $|NK|$ which under the deformation retraction onto the cellular chains of Y goes to a cellular 3-cycle representing a generator of $\mathrm{H}_3(Y)$. An explicit such a 3-cycle is obtained as follows: Let $c_{1,2} = \sigma \in C_1(K_2)$; then

$$\partial_\sharp \sigma = (s_0 z_1) r_{j_1}^{\varepsilon_1} (s_0 z_1)^{-1} \ldots (s_0 z_m) r_{j_m}^{\varepsilon_m} (s_0 z_m)^{-1} \in C_1(K_1)$$

and the class of the latter in $\mathrm{H}_1(K_1)$ is zero. In fact, $\mathrm{H}_1(K_1)$ is the free abelian group on the relators $r_1, \ldots, r_\ell$ and the degeneracies $s_0 x_1, \ldots, s_0 x_\ell$ of the generators in $\mathcal{P}$, and the subgroup generated by the relators amounts to the group $C_2(Y)$ of cellular 2-chains of Y. The assignment to σ of the image of $\partial_\sharp \sigma$ in $C_2(Y)$ under the map from $C_1(K_1)$ to $\mathrm{H}_1(K_1)$ combined with the projection onto $C_2(Y)$, cf. (3.10) above, is the value of the boundary $\partial \sigma \in C_2(Y)$ under the cellular boundary $\partial: C_3(Y) \to C_2(Y)$ and this is zero since σ represents a 3-cycle. However, the image of $\partial_\sharp \sigma$ in $\mathrm{H}_1(K_1)$ lies in the subgroup of $\mathrm{H}_1(K_1)$ generated by the relators $r_1, \ldots, r_\ell$ and this subgroup is mapped isomorphically onto $C_2(Y)$ whence the image of $\partial_\sharp \sigma$

in $\mathrm{H}_1(K_1)$ is zero. Since the sequence (3.10) is exact, we conclude that there is a chain $c_{2,1} \in C_2(K_1)$ with

$$\partial_\flat c_{2,1} = \partial_\sharp c_{1,2} \in C_1(K_1).$$

Next,

$$\partial_\flat \partial_\sharp c_{2,1} = \partial_\sharp \partial_\flat c_{2,1} = \partial_\sharp \partial_\sharp c_{1,2} = 0$$

whence, again in view of the exactness of (3.10), there is a chain $c_{3,0} \in C_3(K_0)$ with

$$\partial_\flat c_{3,0} = -\partial_\sharp c_{2,1} \in C_2(K_0).$$

Then

$$c = c_{3,0} + c_{2,1} + c_{1,2}$$

is the desired 3-cycle in $|NK|$, and $\langle Q, c\rangle$ is a closed element of $\left|(\Omega_G^{*;*}(\mathrm{H}_\sharp); d, \delta_G, \partial_\sharp)\right|$ of degree 1. In some more detail, $\langle Q, c\rangle$ has components

$$\begin{aligned}
&\langle Q^{0,3,1}, c_{1,2}\rangle \in \Omega^{0,3}(\mathrm{H}_2),\\
&\langle Q^{0,2,2}, c_{2,1}\rangle \in \Omega_G^{0,2}(\mathrm{H}_1),\\
&\langle Q^{2,1,1}, c_{1,2}\rangle \in \Omega^{2,1}(\mathrm{H}_2),\\
&\langle Q^{2,0,2}, c_{2,1}\rangle \in \Omega^{2,0}(\mathrm{H}_1).
\end{aligned}$$

Notice that here $\langle Q^{0,3,1}, c_{1,2}\rangle \in \Omega^{0,3}(\mathrm{H}_2)$ is just the form pulled back from Cartan's form $\lambda \in \Omega^3(G)$ via the canonical projection from H_2 onto its primitive part $P_2 = G$. Thus, keeping in mind that $\mathrm{H}_1 = G^{2\ell}$, under the present circumstances, the construction yields the 2-form

$$\alpha = \langle Q^{0,2,2}, c_{2,1}\rangle \in \Omega_G^{0,2}(G^{2\ell})$$

having the property that

$$d\alpha = i^*\lambda \in \Omega_G^{0,3}(G^{2\ell})$$

where i refers to the smooth map from $G^{2\ell}$ to G induced by the identity among relations (3.1) in [60]. Notice also that there is no component involving a form on H_0. This relies on the fact that the (non-equivariant) Shulman construction (2.16) yields only non-zero forms in $\Omega^j(G^k)$ for $j \geq k$. Thus $(\alpha, \lambda) \in \Omega^2(\mathrm{H}_1) \times \Omega^3(G)$ is a pair of forms which yields an equivariant form in

$$|(\Omega_G^{*;*}(\mathcal{H}); d, \delta_G, \partial_\sharp)| = (|\Omega_G^{*;*}(\mathcal{H})|, D)$$

of total degree 1, and, under the integration mapping (5.2), this form passes to an equivariant 1-from in $|(\Omega_G^{*;*}(|\mathcal{H}|_{\mathrm{smooth}}); d, \delta_G)|$. Hence, given a G-equivariant plot

$$F\colon W \to |\mathcal{H}|_{\mathrm{smooth}},$$

its adjoint $F^\natural$ has components

$$F_q^\natural\colon W \times \Delta_q \to \mathrm{H}_q$$

and here only $F_1^\natural$ and $F_2^\natural$ are relevant; they fit into the commutative diagram

$$\begin{array}{ccc} W \times \Delta_1 & \xrightarrow{\mathrm{Id}\times\varepsilon^2} & W \times \Delta_2 \\ {\scriptstyle F_1^\natural}\big\downarrow & & {\scriptstyle \mathrm{pr}F_2^\natural}\big\downarrow \\ \mathrm{H}_1 & \longrightarrow & G \end{array}$$

where pr refers to the canonical projection from H_2 onto its primitive part $P_2 = G$. Now

$$\psi = \int_{\Delta_2} (\mathrm{pr}F_2^\natural)^*\lambda + \int_{\Delta_1} (F_1^\natural)^*\alpha$$

is a closed G-equivariant 1-form on W having integral periods and hence integrates to a smooth map Ψ from W to the circle S^1. Moreover, Ω_Q is equivariantly closed, the term $Q^{2,0,2}$ is irrelevant, and, for every $X \in g$, the value $\delta_G\psi(X) = -\psi(X_W)$ is calculated by

$$dI\langle Q^{2,1,1}, c_{1,2}\rangle$$

where $\langle Q^{2,1,1}, c_{1,2}\rangle \in \Omega_G^{2,1}(\mathrm{H}_2)$ and where d is the de Rham operator; however, for degree reasons, $I\langle Q^{2,1,1}, c_{1,2}\rangle$ is zero whence, for every $X \in g$, the value $\delta_G\psi(X) = -\psi(X_W)$ is zero and hence Ψ is constant on G-orbits, that is to say, G-equivariant. Thus, the choice of the cycle c determines, for every G-equivariant plot F defined on a smooth G-manifold W, a smooth G-equivariant map Ψ from W to the circle S^1 which is natural in G-equivariant plots. This is our description of the Chern-Simons function.

We conclude this Section with an observation: Formally, we can interpret the smooth map i from $G^{2\ell}$ to G induced by the above mentioned identity among relations as arising from the presentation

$$\widetilde{\mathcal{P}} = \langle x_1, \dots, x_\ell, r_1, \dots, r_\ell; s\rangle, \quad s = z_1 r_{j_1}^{\varepsilon_1} z_1^{-1} \dots z_m r_{j_m}^{\varepsilon_m} z_m^{-1}$$

of a one relator group arising from K_1, the free group on $x_1, \dots, x_\ell, r_1, \dots, r_\ell$, by interpreting the identity among relations (3.1) in [60] as a relation among the generators of K_1. The map i from from $G^{2\ell}$ to G then amounts to the word map given by the association

$$(a_1, \dots, a_\ell, b_1, \dots, b_\ell) \longmapsto w_1(a) b_{j_1}^{\varepsilon_1} (w_1(a))^{-1} \dots w_m(a) b_{j_m}^{\varepsilon_m} (w_m(a))^{-1} \in G$$

where $w_k(a) \in G$ is obtained by substituting each occurrence of x_j in z_k by a_j. Plainly, then

$$\begin{aligned} \omega &= \langle Q^{0,3,1}, c_{1,2}\rangle + \langle Q^{0,2,2}, c_{2,1}\rangle \in \Omega_G^{0,3}(\mathrm{H}_2) + \Omega_G^{0,2}(\mathrm{H}_1) \\ \mu^\sharp &= \langle Q^{2,1,1}, c_{1,2}\rangle \in \Omega_G^{2,1}(\mathrm{H}_2) \end{aligned}$$

satisfy

$$\delta_G \omega = \pm d\mu^\sharp$$

and ω is a closed 2-form in the appropriate sense, just as in the case of surface groups studied before.

9. Simply connected 4-manifolds

As in Section 4 of [60], we describe a simply connected 4-manifold Y as the cofibre of a map f from the 3-sphere S^3 to a bunch $\vee_\ell S^2_j$ of ℓ copies of the 2-sphere. We maintain the notation in [ibidem] but do not reproduce it. When Y underlies a smooth 4-manifold, the space of based maps from Y to BG is a model for the space of based gauge equivalence classes of connections on all topological types of bundles on Y and Theorem 7.1 above gives a complete description of its real equivariant cohomology. For intelligibility, we recall that the smooth geometric realization of our cosimplicial manifold $\mathcal{H} = \mathrm{Hom}(KY, G)$ may be described as the space of pairs (ϕ_1, ϕ_3) of smooth maps $\phi_1 \colon \Delta_1 \to \mathrm{H}_1 = G^\ell$ and $\phi_3 \colon \Delta_3 \to G$ subject to the conditions

(1) $\phi_1(0) = \phi_1(1) = e$,
(2) ϕ_3 has constant value e on the first three faces of Δ_3, and
(3) the diagram

$$\begin{array}{ccc} \Delta_2 & \xrightarrow{\epsilon^3} & \Delta_3 \\ {\scriptstyle (\phi_1 \circ \eta^0, \phi_1 \circ \eta^1)}\downarrow & & \downarrow{\scriptstyle \phi_3} \\ G^\ell \times G^\ell & \xrightarrow[r]{} & G \end{array} \tag{9.1}$$

is commutative; see Section 4 of [60] for details. Perhaps moduli spaces of based gauge equivalence classes of ASD-connections can be found within the geometric realization of $\mathcal{H}$ in the following way: Endow the 4-manifold Y with a metric as usual and view Y as a compactification of the interior $\mathbf{R}^4$ of the 4-ball, with the induced metric. When this metric is flat up to a diffeomorphism of $\mathbf{R}^4$, standard constructions yield all finite energy ASD-connections on $\mathbf{R}^4$. Whether or not this happens to be the case, finite energy ASD-connections should correspond to certain maps of the kind ϕ_3. In the flat case, such connections can be extended over the 4-sphere, by Uhlenbeck's removable singularities theorem. In general there are presumably additional constraints corresponding to the requirement that for a choice of ϕ_1 determined by the data the diagram (9.1) be commutative. Perhaps these additional constraints explain the additional term $-db_2^+$ in the formula for the dimension of the moduli space which comes from the index theorem where d refers to the dimension of the structure group and b_2^+ to the rank of the self dual part of $\mathrm{H}^2(Y)$ as usual. Also in special cases the description of ASD-connections on the 4-sphere and related 4-manifolds as holomorphic maps from the 2-sphere or related 2-manifolds to ΩG might be relevant here, cf. [52], [61]. Another question is whether anything reasonable can be said about the component ϕ_1, which is a point of the product of ℓ copies of the loop space ΩG. Is there a way to relate Yang-Mills theory over Y with Yang-Mills theory over the embedded 2-spheres where critical points are known to correspond to geodesics or homomorphisms? The map Φ from the realization of $\mathcal{H}$ to $\mathrm{Map}^o(Y, BG)$ [60] involves choices, in particular a choice of homotopy inverse from Y to $|SY|$ of the canonical map from $|SY|$ to Y, and this choice destroys the symmetry of the situation. The ASD-condition will presumably only come down to a certain fixed map from S^1 to G for each 2-sphere in Y. A finer decomposition, e. g. a triangulation of Y, will provide a canonical map from

Y to $|SY|$ and hence might restore the missing symmetry. We hope to settle these issues eventually elsewhere.

References

References [1] – [51] are given in the first part [60].

52. M. F. Atiyah, *Instantons in two and four dimensions*, Comm. Math. Phys. **93** (1984), 437–451.
53. M. Atiyah and R. Bott, *The moment map and equivariant cohomology*, Topology **23** (1984), 1–28.
54. N. Berline, E. Getzler, and M. Vergne, *Heat kernels and Dirac Operators*, Springer, Berlin · Heidelberg · New York · Tokyo, 1992.
55. R. Bott and H. Samelson, *Applications of the theory of Morse to symmetric spaces*, Amer. J. of Math. **80** (1958), 954–1029.
56. K. T. Chen, *Iterated path integrals*, Bull. Amer. Math. Soc. **83** (1977), 831–879.
57. K. T. Chen, *Degeneracy indices and Chern classes*, Adv. in Math. **45** (1982), 73–91.
58. A. Dold und D. Puppe, *Homologie nicht-additiver Funktoren*, Annales de l'Institut Fourier **11** (1961), 201–313.
59. K. Guruprasad, J. Huebschmann, L. Jeffrey, and A. Weinstein, *Group systems, groupoids, and moduli spaces of parabolic bundles*, preprint, 1994.
60. J. Huebschmann, *Extended moduli spaces and the Kan construction*, MPI preprint, 1995, dg-ga/9505005.
61. A. D. King, *Instantons and holomorphic bundles on the blown-up plane*, D. Phil. Thesis, Oxford University, April 1989.
62. P. E. Newstead, *Characteristic classes of stable bundles over an algebraic curve*, Trans. Amer. Math. Soc. **169** (1972), 337–345.
63. J. M. Souriau, *Groupes différentiels*, in: Lecture Notes in Mathematics, No. 836 (1981), Springer, Berlin-Heidelberg-New York-Tokyo, 81.
64. J. M. Souriau, *Groupes différentiels et physique mathématique*, in: Feuilletages et quantification géométrique, P. Dazord & N. Desolneux-Moulis, eds., Travaux en cours (1984), Herman, Paris.
65. E. Witten, *Two-dimensional gauge theories revisited*, J. Geom. and Physics **9** (1992), 303–368.