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REDUCTION OF MODULI SCHEMES OF ABELIAN VARIETIES WITH DEFINITE QUATERNION MULTIPLICATIONS: THE MINIMAL CASE

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Abstract. In this paper we make an initial study on type D moduli spaces in positive characteristic $p \neq 2$, where we allow the prime p ramified in the defining datum. We classify explicitly the isogeny classes of p-divisible groups with additional structures in question. We also study the reduction of the type D moduli spaces of minimal rank.

1. Introduction

1.1. PEL moduli spaces parametrize abelian varieties with additional structures of polarizations, endomorphisms and level structures. They are divided into types A, C and D according to the Dynkin type of their defining algebraic groups. Previous studies of these moduli spaces and their integral models are mainly focusing on the spaces of types A and C in the case of good reduction. There is comparatively less known about type D moduli spaces in the literature. Certain important results on all smooth PEL-type moduli spaces, which of course include the case of type D, have been obtained by Wedhorn [32, 33] and Moonen [15, 16, 17], where they concern the density of the μ-ordinary locus and the Ekedahl-Oort (EO) strata. In this paper we study the type D moduli spaces in mainly positive characteristic and certain basic classification problems for abelian varieties and associated p-divisible groups with additional structures in question. A main point here is that we allow the prime p ramified in the definite quaternion algebra concerned. In a very special case (of the minimal rank), we also exhibit a method for studying the case with arbitrary polarization degree, different from some previous studies limited to polarizations of prime-to-p degree.

Through out this paper, let p denote an odd prime number. Let F be a totally real algebraic number field and O_F the ring of integers. Let B be a totally definite quaternion algebra over F and let $*$ be the canonical involution on B, which is the unique positive involution on B. Let O_B be an O_F-order in B which is stable under the involution $*$ and maximal at p, that is, the completion $O_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$ at p is a maximal order in the algebra $B_p := B \otimes_{\mathbb{Q}} \mathbb{Q}_p$. A polarized abelian O_B-variety is a tuple (A, λ, ι), where (A, λ) is a polarized abelian variety and $\iota : O_B \to \mathrm{End}(A)$ is a ring monomorphism such that $\lambda \circ \iota(b^*) = \iota(b)^t \circ \lambda$, i.e. the map ι is compatible with the involution $*$ and the Rosati involution induced by the polarization λ. Clearly,

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this notion can be defined over any base scheme and one can study families of such objects.

If (A, λ, ι) is a complex polarized abelian O_B-variety, then one has $\dim A = 2m[F : \mathbb{Q}]$ for some positive integer m. A type D moduli space is the moduli space parametrizing $2m[F:\mathbb{Q}]$-dimensional polarized abelian O_B-varieties, for some integer $m \geq 1$, with auxiliary structures and certain conditions. These conditions are imposed in order to make more precise study of the moduli space. The minimal case we refer to in the title is the case where $m = 1$. Although the minimal type D moduli spaces are special cases of type D moduli spaces, some speculation indicates that the geometry of this family behaves quite different from that of the non-minimal cases. Therefore, it would be good to have an individual and detailed study for this particular family.

The main contents of this paper settle the following two basic problems:

- **(a)** Classify explicitly the isogeny classes of quasi-polarized p-divisible groups with additional structures in question over an algebraically closed field of characteristic p. Here we are not limited to the minimal case.
- **(b)** Study the reduction of the minimal type D moduli spaces.

As the reader can see from known results of classical moduli spaces like Siegel or Hilbert moduli spaces, the results obtained so far in the type D moduli spaces are comparably weaker. However, several points of the present paper are already subtle and technical. Below we illustrate main results.

1.2. Part (a). Let k be an algebraically closed field of characteristic p. For a polarized abelian O_B-variety $\underline{A} = (A, \lambda_A, \iota_A)$ over k, the associated p-divisible group $(H, \lambda_H, \iota_H) := (A, \lambda_A, \iota_A)[p^\infty]$ with additional structures is a quasi-polarized p-divisible $O_B \otimes \mathbb{Z}_p$-module (see Section 5.1). We like to determine the slope sequences and isogeny classes of these quasi-polarized p-divisible $O_B \otimes \mathbb{Z}_p$-modules. As the first standard step, we can decompose these p-divisible groups with additional structures and study the problems for each component independently. Write

$$F \otimes_{\mathbb{Q}} \mathbb{Q}_p = \prod_{v|p} F_v, \quad B \otimes_{\mathbb{Q}} \mathbb{Q}_p = \prod_{v|p} B_v, \quad O_B \otimes_{\mathbb{Z}} \mathbb{Z}_p = \prod_{v|p} \mathcal{O}_{B_v} \tag{1.1}$$

and we get a decomposition $(H, \lambda_H, \iota_H) = \prod_{v|p}(H_v, \lambda_{H_v}, \iota_{H_v})$. The slope sequence $\underline{\nu}(\underline{A})$ of $\underline{A}$ is defined to be a collection of slope sequences $\nu(H_v)$ indexed by the set of places $v|p$ of F. So we reduce the problems for quasi-polarized p-divisible $\mathcal{O}_{B_v}$-modules. We shall write $\mathbf{B}$, $\mathbf{F}$ and $\mathcal{O}_{\mathbf{B}}$ for B_v, F_v and $\mathcal{O}_{B_v}$, respectively for brevity. In this part we do the following:

1. Study the structure of skew-Hermitian $\mathcal{O}_{\mathbf{B}} \otimes W$-modules and quasi-polarized Dieudonné $\mathcal{O}_{\mathbf{B}}$-modules (see Section 5.1), where W denotes the ring of Witt vectors over k. See Sections 4 and 5.
2. Determine all possible slope sequences of quasi-polarized Dieudonné $\mathcal{O}_{\mathbf{B}}$-modules of rank $4dm$, where $d = [\mathbf{F} : \mathbb{Q}_p]$. Moreover, we show that these slope sequences can be also realized by *separably* quasi-polarized Dieudonné $\mathcal{O}_{\mathbf{B}}$-modules. See Theorems 6.4 and 7.3 for the precise statement; also see Corollaries 7.7 and 7.8 for the list of all possible slope sequences in the cases $m = 1, 2$.
3. Classify the isogeny classes of quasi-polarized Dieudonné $\mathcal{O}_{\mathbf{B}}$-modules of rank $4dm$. See Section 9.

The method of finding possible slope sequences in (2) is using a criterion for embeddings of a simple algebra into another one over a local field (see [39] and Section 6.3). This gives a description for possible slope sequences. Then we construct a separably quasi-polarized Dieudonné $\mathcal{O}_{\mathbf{B}}$-module realizing each possible slope sequence. The construction is divided into the supersingular part and non-supersingular part. For the supersingular part we write down a separably quasi-polarized superspecial Dieudonné $\mathcal{O}_{\mathbf{B}}$-module that also satisfies the determinant condition. For the definition of the determinant condition; see Section 5.2. The construction of such a Dieudonné module is given in Section 8. For the non-supersingular part we use the "double construction"; see Lemma 7.1. The construction easily produces a separable $\mathcal{O}_{\mathbf{B}}$-linear polarization. However, a Dieudonné $\mathcal{O}_{\mathbf{B}}$-module obtained in this way rarely satisfies the determinant condition; also see Remark 8.1. In fact, given a possible slope sequence ν as in (2) or in Theorem 7.3, it is not always possible to construct a Dieudonné $\mathcal{O}_{\mathbf{B}}$-module M with slope sequence ν which *both* admits a separable $\mathcal{O}_{\mathbf{B}}$-linear quasi-polarization and satisfies the determinant condition. We will discuss this in more details in the minimal case later.

To classify the isogeny classes of the p-divisible groups with additional structures, it suffices to classify those with a fixed slope sequence ν. Rapoport and Richartz [28] have given a cohomological description of this finite set $I(\nu)$ (for the defining $\mathbb{Q}_p$-groups which is non-connected, or not quasi-split, see Kottwitz [13]). We carry out a more elementary approach through invariants as what is done for quadratic forms and Hermitian forms. One can first reduce to the case where ν is supersingular; see Lemma 9.1. Then we construct a bijection between the set of $I(\nu)$ (ν is supersingular) and the set of isomorphism classes of skew-Hermitian $\mathbf{B}'$-modules for some quaternion $\mathbf{F}$-algebra $\mathbf{B}'$ which is easily determined; see Theorem 9.2. When $\mathbf{B}'$ is the matrix algebra, one reduces to classify quadratic forms over $\mathbf{F}$, and we apply the classical theory of quadratic forms over local fields (cf. O'Meara [23]). When $\mathbf{B}'$ is the quaternion division algebra, we use the work of Tsukamoto [30].

1.3. Part (b). In this part we restrict ourselves to the minimal case. Part (b) consists of Section 3 and Sections 10–14 of this paper.

Let $\mathcal{M}$ be the coarse moduli scheme over $\operatorname{Spec}\mathbb{Z}_{(p)}$ of $2[F:\mathbb{Q}]$-dimensional polarized abelian O_B-varieties (A,λ,ι). Let $\mathcal{M}^{(p)}\subset\mathcal{M}$ be the open and closed subscheme consisting of objects (A,λ,ι) with prime-to-p polarization degree. Both moduli spaces $\mathcal{M}$ and $\mathcal{M}^{(p)}$ are schemes locally of finite type. Each of them is an union of infinitely many open and closed subschemes of finite type:

$$\mathcal{M}=\coprod_{D\geq 1}\mathcal{M}_D,\quad \mathcal{M}^{(p)}=\coprod_{D\geq 1, p\nmid D}\mathcal{M}_D, \tag{1.2}$$

where $\mathcal{M}_D$ is the subscheme parametrizing objects (A,λ,ι) in $\mathcal{M}$ with $\deg\lambda=D^2$.

Let $\mathcal{M}_K\subset\mathcal{M}$ be the closed subscheme parametrizing objects in $\mathcal{M}$ that satisfy a determinant condition; see Section 2. Let $\mathcal{M}_K^{(p)}=\mathcal{M}^{(p)}\cap\mathcal{M}_K$ be the scheme-theoretic intersection.

First of all, all geometric fibers of the moduli scheme $\mathcal{M}_K^{(p)}$ are non-empty; see Lemma 2.3. To show this, one uses the analytic construction and Grothendieck's semi-stable reduction [7]. We also show that any $2[F:\mathbb{Q}]$-dimensional abelian O_B-variety over any field is potentially of CM type; see Proposition 3.3 and also

Section 3 for some discussions. This indicates that objects in a minimal type D moduli space have rich arithmetic properties. This is of course expected as these moduli spaces are expected to be zero-dimensional (though this is not true; see Theorem 1.3).

As one of the main results in this part, we prove the following result.

Theorem 1.1.

(1) *Suppose that p is unramified in B, that is, the algebra $B \otimes \mathbb{Q}_p$ is a product of matrix algebras over unramified field extensions of $\mathbb{Q}_p$. Then the moduli scheme $\mathcal{M}_K \to \operatorname{Spec} \mathbb{Z}_{(p)}$ is flat and every connected component is projective of relative dimension zero.*

(2) *The moduli scheme $\mathcal{M}_K^{(p)} \to \operatorname{Spec} \mathbb{Z}_{(p)}$ is flat and every connected component is projective of relative dimension zero.*

Theorem 1.1 confirms some cases of the Rapoport-Zink conjecture on integral models of Shimura varieties; see [29]. This implies that all geometrically fibers of the above moduli spaces are zero-dimensional. Theorem 1.1 (1) is proved in [34]. The proof of Theorem 1.1 (2), given in Section 10, uses the theory of local models (see Rapoport-Zink [29]). More precisely, let $\mathbf{M}_\Lambda$ be the local model over $\operatorname{Spec} \mathbb{Z}_p$ associated to a unimodular skew-Hermitian free $O_B \otimes \mathbb{Z}_p$-module Λ of rank one. Following Rapoport and Zink we have the following local model diagram

$$\mathcal{M}_K^{(p)} \otimes \mathbb{Z}_p \xleftarrow{\varphi^{\mathrm{mod}}} \widetilde{\mathcal{M}} \xrightarrow{\varphi^{\mathrm{loc}}} \mathbf{M}_\Lambda. \tag{1.3}$$

See Section 10 for more precise descriptions. The morphism φ^{mod} is a $\mathcal{G}$-torsor, where $\mathcal{G}$ is the automorphism group scheme of Λ over $\operatorname{Spec} \mathbb{Z}_p$, and the morphism φ^{loc} is $\mathcal{G}$-equivariant. Locally in etale topology singularities of $\mathcal{M}_K^{(p)}$ are governed by the local model $\mathbf{M}_\Lambda$, so one reduces to prove the flatness of $\mathbf{M}_\Lambda$, which is done in Section 11. For this, we show that any point in the special fiber $\mathbf{M}_\Lambda(k)$ can be lifted to characteristic zero and that all its geometric fibers are zero-dimensional.

Returning to the local model diagram (1.3), it is a basic question whether the morphism φ^{loc} is surjective. That is, whether the following induced map

$$\theta_k : \mathcal{M}_K^{(p)}(k) \to \mathcal{G}(k)\backslash \mathbf{M}_\Lambda(k) \tag{1.4}$$

surjective. This map factors through the map

$$\alpha : \mathrm{Dieu}^{O_B \otimes \mathbb{Z}_p}(k) \to \mathcal{G}(k)\backslash \mathbf{M}_\Lambda(k), \tag{1.5}$$

where $\mathrm{Dieu}^{O_B \otimes \mathbb{Z}_p}(k)$ denotes the set of isomorphism classes of separably quasi-polarized Dieudonné $O_B \otimes \mathbb{Z}_p$-modules of rank $4d$ satisfying the determinant condition. We show that α is surjective; see Section 12 and Proposition 12.1. That is, the surjectivity of φ^{loc} is confirmed at the level for p-divisible groups with the additional structures. We plan to further work out the surjectivity of this morphism.

For the possible slope sequences of objects in the minimal case, we have the following refined results (cf. Theorem 7.3).

Theorem 1.2 (Theorem 12.3)**.** *Let M be a separably quasi-polarized Dieudonné $\mathcal{O}_{\mathbf{B}}$-module of rank $4d$ satisfying the determinant condition, where $d = [\mathbf{F} : \mathbb{Q}_p]$.*

(1) *If $\mathbf{B}$ is the matrix algebra, then*

$$\nu(M) = \left\{ \left(\frac{a}{d}\right)^{2d}, \left(\frac{d-a}{d}\right)^{2d} \right\},$$

where a can be any integer with $0 \le a < d/2$, or

$$\nu(M) = \left\{ \left(\frac{1}{2}\right)^{4d} \right\}.$$

(2) *If $\mathbf{B}$ is the division algebra, then*

$$\nu(M) = \left\{ \left(\frac{a}{2d}\right)^{2d}, \left(\frac{2d-a}{2d}\right)^{2d} \right\},$$

where a can be any odd integer with $2[e/2]f \le a < d$, or

$$\nu(M) = \left\{ \left(\frac{1}{2}\right)^{4d} \right\}.$$

Here e and f are the ramification index and the inertia degree of $\mathbf{F}$ over $\mathbb{Q}_p$, respectively.

In the remaining of this part (Sections 13 and 14) we limit ourselves to the case $F = \mathbb{Q}$. The goal is to determine the dimensions of the special fibers of above moduli schemes. Write $\mathcal{M}_{\overline{\mathbb{F}}_p}$ for the base change $\mathcal{M} \otimes \overline{\mathbb{F}}_p$; similarly we do this for $\mathcal{M}^{(p)}_{\overline{\mathbb{F}}_p}$, $\mathcal{M}_{K,\overline{\mathbb{F}}_p}$ etc. Theorem 1.1 (2) implies that $\dim \mathcal{M}^{(p)}_K \otimes \overline{\mathbb{F}}_p = 0$. For the remaining cases we prove the following result.

Theorem 1.3 (Theorem 14.1 and Proposition 14.7). *Let notations be as above. Assume that $F = \mathbb{Q}$.*

(1) *If p is unramified in B, then* $\dim \mathcal{M}_{\overline{\mathbb{F}}_p} = 0$.
(2) *If p is ramified in B, then* $\dim \mathcal{M}_{\overline{\mathbb{F}}_p} = 1$.
(3) *We have* $\dim \mathcal{M}^{(p)}_{\overline{\mathbb{F}}_p} = 0$.
(4) *If p is ramified, then* $\dim \mathcal{M}_{K,\overline{\mathbb{F}}_p} = 1$.

We explain the ideas of the proof. For (1) any object $\underline{A} = (A, \lambda, \iota) \in \mathcal{M}(k)$ is either ordinary or superspecial. Then we use the canonical lifting for ordinary abelian varieties and the fact that the dimension of the generic fiber has dimension zero. For the other case we use the fact that the superspecial locus has dimension zero. For (2) and (4), we construct a Moret-Bailly family of supersingular polarized abelian O_B-surfaces. See the construction in Section 13. This produces a non-constant one-dimensional family in the moduli space $\mathcal{M}_{\overline{\mathbb{F}}_p}$. A close exam shows that this $\mathbf{P}^1$-family actually lands in the locus $\mathcal{M}_{K,\overline{\mathbb{F}}_p} \subset \mathcal{M}_{\overline{\mathbb{F}}_p}$; see Lemma 14.6. This gives a lower bound for the dimensions

$$1 \le \dim \mathcal{M}_{K,\overline{\mathbb{F}}_p} \le \dim \mathcal{M}_{\overline{\mathbb{F}}_p}.$$

For the other bound, we consider the finite morphism $f : \mathcal{M}_{\overline{\mathbb{F}}_p} \to \mathcal{A}_{2,\overline{\mathbb{F}}_p}$ to the moduli space $\mathcal{A}_{2,\overline{\mathbb{F}}_p}$ of polarized abelian surfaces, through forgetting the endomorphism structure. As p is ramified, every object in $\mathcal{M}(k)$ is supersingular, cf. Corollary 7.7 and hence the whole space $\mathcal{M}_{\overline{\mathbb{F}}_p}$ is supersingular. The image of $\mathcal{M}_{\overline{\mathbb{F}}_p}$ in $\mathcal{A}_{2,\overline{\mathbb{F}}_p}$ then lands in the supersingular locus $\mathcal{S}_2$ of $\mathcal{A}_{2,\overline{\mathbb{F}}_p}$. Then we use a result of Norman-Oort [22] (also cf. Katsura and Oort [9] for principally polarized case) that $\dim \mathcal{S}_2 = 1$ to get the other bound $\dim \mathcal{M}_{\overline{\mathbb{F}}_p} \le 1$. This shows (2) and (4). This above result on

$\dim \mathcal{S}_2 = \dim \mathcal{A}^{(0)}_{2,\overline{\mathbb{F}}_p}$ is a special case of a theorem of Norman and Oort which states that the p-rank zero locus $\mathcal{A}^{(0)}_{g,\overline{\mathbb{F}}_p}$ of the Siegel moduli space $\mathcal{A}_{g,\overline{\mathbb{F}}_p}$ has co-dimension g. (3) We only need to treat the case when p is ramified case. In this case we show that any separably quasi-polarized $O_B \otimes \mathbb{Z}_p$-Dieudonné module (of rank $4d$) is superspecial, and again use the dimension zero of the superspecial locus.

We end this part with a few remarks about Theorem 1.3.

(1) Theorem 1.3 (1) yields another proof of the result $\dim \mathcal{M}^{(p)}_{K,\overline{\mathbb{F}}_p} = 0$, which follows from Theorem 1.1.
(2) When p is ramified in B, the one-dimensional moduli spaces $\mathcal{M}_{\overline{\mathbb{F}}_p}$ and $\mathcal{M}_{K,\overline{\mathbb{F}}_p}$ both contain components of dimension zero and one. This follows from the fact that the both moduli spaces $\mathcal{M}^{(p)}_{\overline{\mathbb{F}}_p}$ and $\mathcal{M}^{(p)}_{K,\overline{\mathbb{F}}_p}$ are zero-dimensional and non-empty.
(3) Suppose p is ramified in B. By Theorem 1.3 (2) and (4), we conclude that the moduli schemes $\mathcal{M}$ and $\mathcal{M}_K$ are not flat over $\operatorname{Spec} \mathbb{Z}_{(p)}$. Moreover, there is a polarized abelian O_B-surface satisfying the determinant condition which can not be lifted to characteristic zero.
(4) When p is unramified in B, we have $\mathcal{M}^{(p)} = \mathcal{M}^{(p)}_K$ and hence the moduli scheme $\mathcal{M}^{(p)}$ is flat over $\operatorname{Spec} \mathbb{Z}_{(p)}$. When p is ramified in B, we construct a superspecial prime-to-p degree polarized abelian O_B-surface which does not satisfy the determinant condition; see Lemma 14.3. In particular, this point can not be lifted to characteristic zero. This shows that the inclusion $\mathcal{M}^{(p)}_K(k) \subset \mathcal{M}^{(p)}(k)$ is strict. This phenomenon is different from the reduction modulo p of Hilbert moduli schemes or Hilbert-Siegel moduli schemes. In the Hilbert-Siegel case, any separably polarized abelian varieties with RM by O_F of a totally real algebraic number field F satisfies the determinant condition automatically; see Yu [35], Görtz [5] and Vollaard [31].
(5) To generalize Theorem 1.3 to the case where B is a quaternion algebra over any totally real field F, one can construct Moret-Bailly families to get a lower bound for the dimensions. However, we do not know how to produce the other good bound.

2. Moduli spaces

2.1. Moduli spaces. Let p be an odd prime number. Let F be a totally real field of degree $d = [F : \mathbb{Q}]$ and O_F the ring of integers. Let B be a totally definite quaternion algebra over F and let $*$ be the canonical involution on B, which is the unique positive involution on B. Let O_B be an order in B which is stable under the involution $*$ and maximal at p, that is, $O_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is a maximal order in the algebra $B_p := B \otimes_{\mathbb{Q}} \mathbb{Q}_p$. A polarized abelian O_B-scheme over a base scheme S is a tuple (A, λ, ι), where

- (A, λ) is a polarized abelian scheme over S, and
- $\iota : O_B \to \operatorname{End}_S(A)$ is a ring monomorphism that satisfies the compatibility condition $\lambda \circ \iota(b^*) = \iota(b)^t \circ \lambda$.

The pair (A, ι), where A and ι are as above, is called an abelian O_B-scheme. A polarization λ on an abelian O_B-scheme (A, ι) satisfying the above compatibility condition is said to be *O_B-linear*. Similar objects can be also defined when the algebra B is replaced by an arbitrary semi-simple $\mathbb{Q}$-algebra together with a positive involution.

Let $m \ge 1$ be a positive integer, and let $\mathcal{M}$ be the coarse moduli scheme over $\operatorname{Spec}\mathbb{Z}_{(p)}$ of $2dm$-dimensional polarized abelian O_B-varieties (A,λ,ι). Let $\mathcal{M}^{(p)} \subset \mathcal{M}$ be the subscheme parametrizing the objects in $\mathcal{M}$ which have a prime-to-p degree polarization. The moduli spaces $\mathcal{M}$ and $\mathcal{M}^{(p)}$ are schemes both locally of finite type. Each of them is an union of infinitely many open and closed subschemes which are of finite type:

$$\mathcal{M} = \coprod_{D\ge 1} \mathcal{M}_D, \quad \mathcal{M}^{(p)} = \coprod_{D\ge 1, p\nmid D} \mathcal{M}_D, \tag{2.1}$$

where D runs through all positive integers and $\mathcal{M}_D \subset \mathcal{M}$ is the subscheme parametrizing the objects (A,λ,ι) with polarization degree $\deg\lambda = D^2$.

2.2. Study of $\mathcal{M}_{\mathbb{C}}$. Let (V,ψ) be a $\mathbb{Q}$-valued non-degenerate skew-Hermitian B-module of B-rank m. That is, $\psi : V\times V \to \mathbb{Q}$ is a non-degenerate alternating pairing such that $\psi(ax,y) = \psi(x,a^*y)$ for all $a\in B$ and $x,y\in V$. Let $G_1 \subset G$ be the algebraic groups over $\mathbb{Q}$ defined as follows: For any commutative $\mathbb{Q}$-algebra R, one has

$$\begin{aligned} G(R) &= \{ g \in \operatorname{Aut}_{B\otimes_{\mathbb{Q}} R}(V\otimes_{\mathbb{Q}} R) \mid c(g) = g'g \in R^\times \}, \\ G_1(R) &= \{ g \in \operatorname{Aut}_{B\otimes_{\mathbb{Q}} R}(V\otimes_{\mathbb{Q}} R) \mid g'g = 1 \}, \end{aligned} \tag{2.2}$$

where $g\mapsto g'$ is the adjoint involution with respect to ψ. Denote by $c: G\to \mathbb{G}_{\mathrm{m}}$ the multiplier homomorphism. We have an exact sequence of algebraic groups

$$1 \longrightarrow G_1 \longrightarrow G \xrightarrow{\ c\ } \mathbb{G}_{\mathrm{m}} \longrightarrow 1. \tag{2.3}$$

One can easily show that $G_1\otimes\overline{\mathbb{Q}}$ is isomorphic to the product of d-copies of the orthogonal group O_{2m} over $\overline{\mathbb{Q}}$ (see Lemma 3.1). Therefore, both G_1 and G have 2^d connected components (cf. [12, Section 7]).

There is a unique B-valued skew-Hermitian pairing $\psi_B : V\times V\to B$, i.e.

$$\psi_B(a_1x, a_2y) = a_1\psi_B(x,y)a_2^*, \quad \forall\, a_1,a_2\in B, \quad x,y\in V,$$

such that $\psi(x,y) = \operatorname{Trd}\psi_B(x,y)$, where Trd is the reduced trace from B to $\mathbb{Q}$. Note that the property $\psi_B(ax,y) = \psi_B(x,a^*y)$ for all $a\in B$ and $x,y\in V$ does not hold anymore. We can choose an orthogonal basis $\{e_i\}$ for ψ_B and put $b_i := \psi_B(e_i,e_i)$. Then $b_i^* = -b_i$ for $i=1,\dots,m$ and

$$\psi\left(\sum_{i=1}^m x_ie_i, \sum_{i=1}^m y_ie_i\right) = \sum_{i=1}^m \operatorname{Trd}(x_ib_iy_i^*).$$

For any anti-symmetric element $b\in B^\times$, i.e. $b^*=-b$, we define a ($\mathbb{Q}$-valued) rank-one skew-Hermitian B-module (B,ψ_b), where $\psi_b(x,y) := \operatorname{Trd}(xby^*)$. Then we have a decomposition of skew-Hermitian B-modules

$$(V,\psi) = \bigoplus_{i=1}^m (B,\psi_{b_i}). \tag{2.4}$$

Lemma 2.1. *There is a $B\otimes\mathbb{R}$-linear complex structure J_0 on $V_{\mathbb{R}} = V\otimes_{\mathbb{Q}}\mathbb{R}$ such that $\psi(J_0x,J_0y) = \psi(x,y)$ for $x,y\in V_{\mathbb{R}}$ and the symmetric bilinear form $(x,y) := \psi(x,J_0y)$ is negative definite.*

PROOF. By (2.4) we may assume that $m = 1$ and $(V, \psi) = (B, \psi_b)$. Let J_0 be the right multiplication of the element $b/\sqrt{\mathrm{Nr}_{B/F}(b)}$ in $B \otimes \mathbb{R}$, where $\mathrm{Nr}_{B/F}$ is the reduced norm from B to F. Then one obtains $\psi_b(x, J_0 y) = -\operatorname{Trd}(xy^*)$, which is negative definite. ∎

We call a complex structure J_0 as in Lemma 2.1 an *admissible* complex structure on $(V_\mathbb{R}, \psi)$. The group $G_1(\mathbb{R})$ of real points acts transitively on the set of all admissible complex structures on $(V_\mathbb{R}, \psi)$ by conjugation (see [12, Lemma 4.3]). It is well known that the Hermitian symmetric space

$$X_1 := G_1(\mathbb{R})/K_\infty$$

has dimension $dm(m-1)/2$, where K_∞ is the stabilizer of a fixed admissible complex structure J_0. Fix an O_B-lattice Λ such that $\psi(\Lambda, \Lambda) \subset \mathbb{Z}$ and let $\Gamma_\Lambda \subset G_1(\mathbb{Q})$ be the arithmetic subgroup which stabilizes the lattice Λ. The natural map $g \mapsto (V_\mathbb{R}/\Lambda, \operatorname{Int}(g)J_0, \psi)$ induces an open and closed immersion of analytic spaces

$$\Phi_{(\Lambda,\psi)} : \Gamma_\Lambda \backslash X_1 \hookrightarrow \mathcal{M}(\mathbb{C}).$$

Let $\mathcal{M}_{(\Lambda,\psi)}$ denote the open and closed subscheme of $\mathcal{M}_\mathbb{C}$ over $\mathbb{C}$ whose underlying space is the image of $\Phi_{(\Lambda,\psi)}$. Then we have a decomposition

$$\mathcal{M}_\mathbb{C} = \coprod_{(\Lambda,\psi)} \mathcal{M}_{(\Lambda,\psi)}.$$

of $\mathcal{M}_\mathbb{C}$ into open and closed subschemes, where (Λ, ψ) runs through the isomorphism classes of all $\mathbb{Z}$-valued non-degenerate skew-Hermitian O_B-lattices of rank m.

Lemma 2.2.

(1) *There is an anti-symmetric element $b \in B^\times$ such that* (a) $\psi_b(O_B, O_B) \subset \mathbb{Z}$ *and* (b) $O_B \otimes \mathbb{Z}_p$ *is a self-dual lattice with respect to* ψ_b.

(2) *The moduli space $\mathcal{M}^{(p)}_\mathbb{C}$ is non-empty*

PROOF. (1) We have the decomposition $O_B \otimes \mathbb{Z}_p = \oplus_{v|p} O_{B_v}$ with respect to $O_F \otimes \mathbb{Z}_p = \prod_{v|p} O_{F_v}$. We first show that for each place v of F over p, one can choose an anti-symmetric element $b_v \in B_v^\times$ so that ψ_{b_v} is a $\mathbb{Z}_p$-valued perfect paring on O_{B_v}. When v is unramified in B, we are reduced to finding a $\mathbb{Z}_p$-valued perfect symmetric pairing on $O_{F_v}^2$ which clearly exists, and let b_v be the element corresponding to this perfect pairing. When v is ramified in B, one may choose a prime element Π_v of B_v so that Π_v^2 is a uniformizer of F_v, and let $b_v := \delta_v \Pi_v^{-1}$, where δ_v is a generator of the inverse difference $\mathcal{D}^{-1}_{F_v/\mathbb{Q}_p}$.

Using the weak approximation, there is an anti-symmetric element $b \in B^\times$ close to b_v for each place $v|p$. Replacing b by a prime-to-p multiple bM of b, one gets a pairing ψ_b that satisfies the properties (a) and (b).

(2) Choose a pairing $\psi = \psi_b$ on B as in (1). Then the triple $(V_\mathbb{R}/O_B, J_0, \psi)$ defines a $2d$-dimensional polarized complex abelian O_B-variety $(A_0, \lambda_0, \iota_0)$. Put $\underline{A} = (A_0^m, \lambda_0^m, \iota)$, where $\iota : O_B \to \operatorname{End}(A_0^m)$ is the diagonal embedding. This gives an object in $\mathcal{M}^{(p)}(\mathbb{C})$ and hence that $\mathcal{M}^{(p)}_\mathbb{C}$ is non-empty. ∎

2.3. Moduli spaces with the determinant condition. Let (V,ψ) be any skew-Hermitian B-module of B-rank m. Let J_0 be an admissible complex structure on $V_{\mathbb{R}}$. Let $V_{\mathbb{C}} = V^{-1,0}\oplus V^{0,-1}$ be the eigenspace decomposition where J_0 acts respectively by i and $-i$ on $V^{-1,0}$ and $V^{0,-1}$. Let $\Sigma := \mathrm{Hom}(F,\mathbb{R}) = \mathrm{Hom}(F,\mathbb{C})$ be the set of real embeddings of F. We have

$$B_{\mathbb{C}} := B\otimes_{\mathbb{Q}}\mathbb{C} = \prod_{\sigma\in\Sigma} B_\sigma, \quad B_\sigma := B\otimes_{F,\sigma}\mathbb{C}\simeq \mathrm{Mat}_2(\mathbb{C}).$$

The action of $B_{\mathbb{C}}$ on $V^{-1,0}$ gives the decomposition

$$V^{-1,0} = \sum_{\sigma\in\Sigma} V_\sigma,$$

where each subspace V_σ is a B_σ-module of $\mathbb{C}$-dimension $2m$ and it is isomorphic to the direct sum of m-copies of the simple B_σ-module $\mathbb{C}^2$. If $a\in B$, then the characteristic polynomial of a on V_σ is equal to $\sigma(\mathrm{char}_F(a)^m)$, where $\mathrm{char}_F(a)\in F[T]$ is the reduced characteristic polynomial of a. Therefore, the characteristic polynomial of $a\in B$ on $V^{-1,0}$ is given by

$$\mathrm{char}\,(a|V^{-1,0}) = \prod_{\sigma\in\Sigma}\sigma(\mathrm{char}_F(a))^m = \mathrm{char}\,(a)^m\in\mathbb{Q}[T],$$

where $\mathrm{char}\,(a) = N_{F/\mathbb{Q}}\,\mathrm{char}_F(a)\in\mathbb{Q}[T]$ is the reduced characteristic polynomial of a from B to $\mathbb{Q}$.

Let $\underline{A}_S = (A,\lambda,\iota)\in\mathcal{M}(S)$ be a polarized abelian O_B-scheme over S, where S is a connected locally Noetherian $\mathbb{Z}_{(p)}$-scheme. Since the Lie algebra $\mathrm{Lie}(A)$ is a locally free $\mathcal{O}_S$-module, the characteristic polynomial $\mathrm{char}\,(\iota(a)|\,\mathrm{Lie}(A))$, for any element $a\in O_B$, is defined and it is a polynomial in $\mathcal{O}_S[T]$ of degree $2dm$. The *determinant condition* for $\underline{A}_S$ is the quality of the following two polynomials

$$\text{(K)}\quad \mathrm{char}\,(\iota(a)|\,\mathrm{Lie}(A)) = \mathrm{char}\,(a)^m\in\mathcal{O}_S[T],\quad \forall\, a\in O_B. \tag{2.5}$$

This is a closed condition and it only depends on the rank of the skew-Hermitian B-module (V,ψ).

Let $\mathcal{M}_K\subset\mathcal{M}$ (resp. $\mathcal{M}_K^{(p)}\subset\mathcal{M}^{(p)}$) denote the closed subscheme parametrizing the objects $\mathcal{A}$ in $\mathcal{M}$ (resp. $\mathcal{M}^{(p)}$) that satisfy the determinant condition (K).

Let $L\subset B$ be a maximal subfield such that any place v of F lying over p is unramified in L and that the order $L\cap O_B$ is maximal at p. We can construct such a maximal subfield L by

(a) constructing a maximal commutative semi-simple subalgebra $\mathbf{L}\subset B\otimes\mathbb{Q}_p$ such that $\mathbf{L}$ is the unramified quadratic extension of $F\otimes\mathbb{Q}_p$ and the maximal order $\mathcal{O}_{\mathbf{L}}$ of $\mathbf{L}$ is contained in $O_B\otimes\mathbb{Z}_p$, and
(b) applying the weak approximation.

Clearly the condition (K) implies the trace condition

$$\text{(T)}\quad \mathrm{Tr}(\iota(a)|\,\mathrm{Lie}(A)) = m\,\mathrm{Tr}_{L/\mathbb{Q}}(a)\in\mathcal{O}_S,\quad \forall\, a\in O_L. \tag{2.6}$$

Similarly, let $\mathcal{M}_T\subset\mathcal{M}$ (resp. $\mathcal{M}_T^{(p)}\subset\mathcal{M}^{(p)}$) denote the closed subscheme parametrizing the objects $\mathcal{A}$ in $\mathcal{M}$ (resp. $\mathcal{M}^{(p)}$) that satisfy the trace condition (T).

Lemma 2.3. *The special fiber $\mathcal{M}_K^{(p)}\otimes\overline{\mathbb{F}}_p$ is non-empty.*

PROOF. We may assume that $m=1$ by the same reduction step we show in the non-emptiness of $\mathcal{M}^{(p)}_{\mathbb{C}}$. In this case, $\mathcal{M}(\mathbb{C})=\mathcal{M}(\overline{\mathbb{Q}})$ as each subscheme $\mathcal{M}_D\otimes\mathbb{Q}$ is zero-dimensional and of finite type. By Grothendieck's semi-stable reduction theorem [7], any object $(A,\lambda,\iota)\in\mathcal{M}(\overline{\mathbb{Q}_p})$ has good reduction as the $\mathbb{Z}$-rank of O_B is larger than $\dim A$. Since $\mathcal{M}^{(p)}(\overline{\mathbb{Q}_p})$ is non-empty (Lemma 2.2), the reduction modulo p gives some point x in $\mathcal{M}^{(p)}(\overline{\mathbb{F}}_p)$. Since x is the specialization of an object in characteristic zero, it lands in the $\mathcal{M}^{(p)}_K$. Thus, the special fiber $\mathcal{M}^{(p)}_K\otimes\overline{\mathbb{F}}_p$ is also non-empty. ∎

2.4. Study of $\mathcal{M}_{(\Lambda,\psi)}$. Let $\mathbb{H}$ denote the real Hamilton quaternion algebra. One has

$$\mathbb{H}=\mathbb{C}+\mathbb{C}\,\mathbf{j},\quad \mathbf{j}\,a=\bar a\mathbf{j}\,,\quad a\in\mathbb{C}. \tag{2.7}$$

It is a standard fact that any non-degenerate skew-Hermitian $\mathbb{H}$-module of rank m is isomorphic to $(\mathbb{H}^m,\psi_0)$, where $\psi_0(e_i,e_j)=\mathbf{j}\,\delta_{i,j}$. Put $\mathbf{J}_m:=\mathrm{diag}(\mathbf{j},\dots,\mathbf{j})\in\mathrm{Mat}_m(\mathbb{H})$. We extend the canonical involution $*$ on $\mathrm{Mat}_m(\mathbb{H})$ by $(a_{ij})^*=(a'_{ij})$, where $a_{ij}\in\mathbb{H}$ and $a'_{ij}:=a^*_{ji}$. Let O^*_{2m} denote the algebraic $\mathbb{R}$-group of isometries of $(\mathbb{H}^m,\psi_0)$; one has

$$\mathrm{O}^*_{2m}(\mathbb{R})=\{A\in\mathrm{GL}_m(\mathbb{H})\mid A\mathbf{J}_mA^*=\mathbf{J}_m\,\}. \tag{2.8}$$

The group $G_1\otimes\mathbb{R}$ is isomorphic to $\prod_{\sigma\in\Sigma}\mathrm{O}^*_{2m}$.

Lemma 2.4. *One has $G_1(\mathbb{R})=G_1^0(\mathbb{R})$, $G(\mathbb{R})=G^0(\mathbb{R})$ and $c(G(\mathbb{R}))=\mathbb{R}^\times$.*

PROOF. Let $\mathrm{SO}^*_{2m}=\{A\in\mathrm{O}^*_{2m}|\,\mathrm{Nrd}(A)=1\}$. The group SO^*_{2m} is a form of SO_{2m} and hence it is the connected component of O^*_{2m}. We show that if $g\in\mathrm{O}^*_{2m}(\mathbb{R})$ then $\mathrm{Nrd}(g)=1$. It suffices to show that for any element $g\in\mathrm{GL}_m(\mathbb{H})$ one has $\mathrm{Nrd}(g)>0$. Since the set $\mathrm{GL}_m(\mathbb{H})^{\mathrm{ss}}\subset\mathrm{GL}_m(\mathbb{H})$ of semi-simple elements is open and dense in the classical topology, it suffices to show $\mathrm{Nrd}(g)>0$ for $g\in\mathrm{GL}_m(\mathbb{H})^{\mathrm{ss}}$. Since such g is contained in a maximal commutative semi-simple subalgebra of $\mathrm{Mat}_m(\mathbb{H})$, which is isomorphic to $\mathbb{C}^m$, one has $\mathrm{Nrd}(g)>0$. Therefore,

$$G_1(\mathbb{R})=\prod_\sigma\mathrm{O}^*_{2\mathrm{m}}(\mathbb{R})=\prod_\sigma\mathrm{SO}^*_{2\mathrm{m}}(\mathbb{R})=\mathrm{G}_1^0(\mathbb{R}). \tag{2.9}$$

It follows from $G_1(\mathbb{R})=G_1^0(\mathbb{R})$ that $G(\mathbb{R})=G^0(\mathbb{R})$. For the last statement we just need to find an element g such that $c(g)<0$. Consider diagonal elements $x=\mathrm{diag}(y,\dots,y)$ and we are reduced to show this in the case where $m=1$. In this case one has $\mathbf{iji}^*=-\mathbf{j}$ and hence $c(\mathbf{i})=-1$. This proves the lemma. ∎

Lemma 2.5.

(1) *The Lie group $G_1(\mathbb{R})$ is connected.*
(2) *The Lie group $G(\mathbb{R})$ has two connected components with the neutral component*

$$G(\mathbb{R})^+=\{g\in G(\mathbb{R})\mid c(g)>0\,\}.$$

PROOF. (1) It suffices to show that the Lie group $\mathrm{O}^*_{2m}(\mathbb{R})=\mathrm{SO}^*_{2m}(\mathbb{R})$ is connected. We embed $\mathbb{H}\hookrightarrow\mathrm{Mat}_2(\mathbb{C})$ as

$$\mathbb{H}\ni a+b\mathbf{j}\mapsto\begin{pmatrix}a & b\\ -\bar b & \bar a\end{pmatrix}\in\mathrm{Mat}_2(\mathbb{C}),$$

and have $\mathrm{Mat}_m(\mathbb{H}) \subset \mathrm{Mat}_{2m}(\mathbb{C})$. Let J and J_m the image of $\mathbf{j}$ and $\mathbf{J}_m$ in $\mathrm{Mat}_2(\mathbb{C})$ and $\mathrm{Mat}_{2m}(\mathbb{C})$, respectively. Clearly, $J_m^t = -J_m$ and $J_m^{-1} = -J_m$. The complex conjugation on $\mathrm{Mat}_2(\mathbb{C})$ coming from the $\mathbb{R}$-structure of $\mathbb{H}$ is given by $J\overline{A}J^{-1}$, where $A \mapsto \overline{A}$ is the usual complex conjugation. Thus, we can recover $\mathrm{Mat}_m(\mathbb{H})$ from $\mathrm{Mat}_{2m}(\mathbb{C})$ by

$$\mathrm{Mat}_m(\mathbb{H}) = \{A \in \mathrm{Mat}_{2m}(\mathbb{C}) \mid J_m\overline{A}J_m^{-1} = A\}. \tag{2.10}$$

We have $A^* = \overline{A^t}$ for $A \in \mathrm{Mat}_m(\mathbb{H})$. By (2.8) and (2.10), one gets

$$\mathrm{O}^*_{2m}(\mathbb{R}) = \{A \in \mathrm{Mat}_{2m}(\mathbb{C}) \mid AJ_m\overline{A^t} = J_m,\ J_m\overline{A}J_m^{-1} = A,\ \det(A) = 1\}. \tag{2.11}$$

The first condition in (2.11) gives

$$J_m\overline{A^t}J_m^{-1} = A^{-1}. \tag{2.12}$$

Taking the transpose, the second condition in (2.11) becomes the condition

$$A^t = J_m^{-t}\overline{A^t}J_m^t = J_m^{-1}\overline{A^t}J_m = J_m\overline{A^t}J_m^{-1} \tag{2.13}$$

With (2.12), the condition (2.13) becomes $A^tA = I_{2m}$. Therefore,

$$\mathrm{O}^*_{2m}(\mathbb{R}) = \{A \in \mathrm{Mat}_{2m}(\mathbb{C}) \mid AJ_m\overline{A^t} = J_m,\ A^tA = I_{2m},\ \det(A) = 1\}. \tag{2.14}$$

This is the group $\mathrm{SO}^*(2m)$ defined in [8, Chapter X, Section 2]. By [8, Chapter X, Lemma 2.4], the Lie group $\mathrm{O}^*_{2m}(\mathbb{R})$ is connected.

(2) This follows from (1) and Lemma 2.4. ∎

Remark 2.6. Using the complex coordinates as in the proof of Proposition 2.5, one sees that there is a section for $c : G \to \mathbb{G}_{\mathrm{m}}$ over $\mathbb{C}$. Thus, $G_{\mathbb{C}} \simeq G_1 \rtimes \mathbb{G}_{\mathrm{m}}$ over $\mathbb{C}$.

We show that there is no section over $\mathbb{R}$ when m is odd. Suppose that there is a section over $\mathbb{R}$. Then one has $G \simeq G_1 \rtimes \mathbb{G}_{\mathrm{m}}$ over $\mathbb{R}$. The restriction of the reduced norm map Nrd to $\mathbb{G}_{\mathrm{m}}$ gives the character $\mathrm{Nrd}(t) = t^m$. Write $y = (x, t) \in G(\mathbb{R}) = G_1(\mathbb{R}) \rtimes \mathbb{G}_{\mathrm{m}}(\mathbb{R})$ and we have $\mathrm{Nrd}(y) = \mathrm{Nrd}(x)t^m = t^m$. As $\mathrm{Nrd}(y) > 0$, we have $t^m > 0$ for all $t \in \mathbb{R}^\times$. This is possible only when m is even.

Proposition 2.7. *Let (Λ, ψ) be a non-degenerate skew-Hermitian O_B-lattice. The subscheme $\mathcal{M}_{(\Lambda,\psi)}$ of $\mathcal{M}_{\mathbb{C}}$ is irreducible. In particular $\mathcal{M}_{(\Lambda,\psi)}$ is defined over $\overline{\mathbb{Q}}$.*

Proof. This follows immediately from Lemma 2.5 (1). ∎

Corollary 2.8. *If $m = 1$, then the group $G_1(\mathbb{R})$ is isomorphic to $(\mathbb{C}_1^\times)^d$ and the quotient space $X_1 = G_1(\mathbb{R})/K_\infty$ consists of one point, where $\mathbb{C}_1^\times = \{z \in \mathbb{C}^\times | z\bar{z} = 1\}$.*

The following result is an immediate consequence of Proposition 2.7.

Proposition 2.9. *Suppose that $m = 1$. The map that sends each object $(A, \lambda, \iota) \in \mathcal{M}(\overline{\mathbb{Q}}) = \mathcal{M}(\mathbb{C})$ to its first homology group $(H_1(A(\mathbb{C}), \mathbb{Z}), \psi_\lambda)$ with the Riemann form induces a bijection between the space $\mathcal{M}(\overline{\mathbb{Q}})$ and the discrete set of isomorphism classes of $\mathbb{Z}$-valued rank one skew-Hermitian O_B-modules $(V_{\mathbb{Z}}, \psi)$.*

Moreover, the subspace $\mathcal{M}^{(p)}(\overline{\mathbb{Q}})$ corresponds to the subset of classes with the property that $V_{\mathbb{Z}} \otimes \mathbb{Z}_p$ is self-dual with respect to the pairing ψ.

2.5. Connection with the adelic description. Let (V,ψ), G_1, G, J_0 be as before. Let $h_0 : \mathbb{C} \to \mathrm{End}_{B\otimes\mathbb{R}}(V_{\mathbb{R}})$ be the $\mathbb{R}$-algebra homomorphism defined by $h_0(i) = J_0$ and denote again by $h_0 : \mathbb{C}^\times \to G_{\mathbb{R}}$ the homomorphism of $\mathbb{R}$-groups. Let X be the $G(\mathbb{R})$-conjugacy class of h_0. Fix an O_B-lattice Λ_0 in V and let $U \subset G(\mathbb{A}_f)$ be the open and compact subgroup that stabilizes the lattice $\Lambda_0 \otimes \hat{\mathbb{Z}}$. Here $\hat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$ and $\mathbb{A}_f = \hat{\mathbb{Z}} \otimes \mathbb{Q}$ is the finite adele ring of $\mathbb{Q}$. One forms a Shimura variety

$$\mathrm{Sh}_U(G,X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/U.$$

Lemma 2.10. *The Hermitian symmetric space X is $G(\mathbb{R})/\mathbb{R}^\times K_\infty$ and it has two connected components.*

Proof. Since $c(G(\mathbb{R})) = \mathbb{R}^\times$ (Lemma 2.4), the closed immersion $G \to \mathrm{GSp}(V,\psi)$ induces a surjective map $\pi_0(X) \to \pi_0(\mathbb{H}_g^\pm)$, where $\mathbb{H}_g^\pm$ is the Siegel double space. On the other hand $G(\mathbb{R})$ has two connected components (Lemma 2.5). Therefore X has two connected components and $X = G(\mathbb{R})/Z(G(\mathbb{R}))K_\infty = G(\mathbb{R})/\mathbb{R}^\times K_\infty$. ∎

By Lemma 2.5, the group $G(\mathbb{Q})$ is dense in $G(\mathbb{R})$. Thus,

$$\begin{aligned}\mathrm{Sh}_U(G.X) &= G(\mathbb{Q})^+\backslash X_1 \times G(\mathbb{A}_f)/U \\ &= \coprod_{i=1}^h \Gamma_i\backslash X_1, \quad \Gamma_i = G(\mathbb{Q})^+ \cap c_i U c_i^{-1},\end{aligned} \tag{2.15}$$

where $G(\mathbb{Q})^+ = G(\mathbb{Q}) \cap G(\mathbb{R})^+$ and $c_1, \dots, c_h$ are coset representatives for the finite set $G(\mathbb{Q})^+\backslash G(\mathbb{A}_f)/U$.

We now describe (2.15) in terms of lattices. We say two O_B-lattices Λ and Λ' in V are *similar* (resp. *strictly similar*), denote $\Lambda \sim \Lambda'$ (resp. $\Lambda \sim_s \Lambda'$), if there is an element $g \in G(\mathbb{Q})$ (resp. $g \in G(\mathbb{Q})^+$) such that $\Lambda' = g\Lambda$. We say Λ and Λ' are in the same *idealcomplex* if $\Lambda_v \sim \Lambda'_v$ for all finite places v of F, where $\Lambda_v := \Lambda \otimes O_{F_v}$. Let

$$\mathfrak{I} := \{\Lambda \subset V \mid \Lambda_v \sim \Lambda_{0,v}\ \forall\, v\}$$

be the idealcomplex containing the O_B-lattice Λ_0. The map $c \mapsto c\Lambda_0$, where $c \in G(\mathbb{A}_f)$, induces a bijection between the double coset space $G(\mathbb{Q})^+\backslash G(\mathbb{A}_f)/U$ and the set of strict similitude classes in $\mathfrak{I}$. In particular, the complex Shimura variety $\mathrm{Sh}_U(G,X)$ has $h(\mathfrak{I})$ connected components, where $h(\mathfrak{I})$ is the strict class number of $\mathfrak{I}$.

Put $\Lambda_i = c_i\Lambda_0$ and $\{\Lambda_1, \dots, \Lambda_h\}$ represents the strict similitude classes of $\mathfrak{I}$. After rescaling we may assume that ψ takes $\mathbb{Z}$-values on Λ_i for all i. It is easy to verify $\Gamma_i = \mathrm{Aut}(\Lambda_i,\psi) = \Gamma_{\Lambda_i}$. Thus, we get

$$\mathrm{Sh}_U(G,X) \simeq \coprod_{i=1}^h \mathcal{M}_{(\Lambda_i,\psi)}. \tag{2.16}$$

3. Arithmetic properties

In this section we study polarized abelian O_B-varieties from the arithmetic point of view.

3.1. Let $\underline{A}=(A,\lambda,\iota)$ be a $2dm$-dimensional polarized abelian O_B-variety over K, where K a field of finite type over its prime field. Let K_s be a separable closure of K, and let $\mathcal{G}_K$ denote the Galois group of K_s over K. Let T_ℓ denote the Tate module of A_K, and put $V_\ell := T_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, where ℓ is a rational prime with $\ell \neq \operatorname{char} K$. Finally we let

$$\rho_\ell : \mathcal{G}_K \to \operatorname{Aut}(T_\ell)$$

be the attached ℓ-adic representation of the Galois group $\mathcal{G}_K$.

Let $\langle\, ,\rangle_A : T_\ell(A) \times T_\ell(A^t) \to \mathbb{Z}_\ell(1)$ be the canonical pairing, where A^t denotes the dual abelian variety of A. We may identify $T_\ell(A^t)$ with the linear dual $T_\ell(A)^* := \operatorname{Hom}(T_\ell(A), \mathbb{Z}_\ell(1))$. The polarization λ induces an alternating non-degenerate (i.e. with non-zero discriminant) pairing $\langle\, ,\rangle : T_\ell(A) \times T_\ell(A) \to \mathbb{Z}_\ell(1)$ by $\langle x,y\rangle := \langle x, \lambda y\rangle_A$. Let $g \mapsto g'$ be the adjoint of the pairing on $g \in \operatorname{End}(V_\ell)$; one has the relation $g' = \lambda^{-1} g^t \lambda$, where $g^t \in \operatorname{End}(V_\ell^*)$ is the pull-back map. The pairing also respects the $\mathcal{G}_K$-action:

$$\langle \rho_\ell(\sigma)x, \rho_\ell(\sigma)y\rangle = \rho_\ell(\sigma)(\langle x,y\rangle) = \chi_\ell(\sigma)\langle x,y\rangle \tag{3.1}$$

for all $x,y \in V_\ell$ and $\sigma \in \mathcal{G}_K$, where $\chi_\ell : \mathcal{G}_K \to \mathbb{Q}_\ell^\times$ is the ℓ-adic cyclotomic character. This shows that

$$\rho_\ell(\sigma)'\rho_\ell(\sigma) = \chi_\ell(\sigma). \tag{3.2}$$

Put $B_\ell := B \otimes \mathbb{Q}_\ell$, and let G_ℓ be the group of B_ℓ-linear similitudes on V_ℓ. Due to the relation (3.2) the ℓ-adic representation ρ_ℓ factors through this subgroup

$$\rho_\ell : \mathcal{G}_K \to G_\ell(\mathbb{Q}_\ell).$$

We have the following basic properties:

Lemma 3.1. *Let notations be as above.*

1. *T_ℓ is a free $O_F \otimes \mathbb{Z}_\ell$-module of rank $4m$.*
2. *V_ℓ is a free $O_B \otimes \mathbb{Q}_\ell$-module of rank m.*
3. *If O_B is maximal at ℓ, i.e. $O_B \otimes \mathbb{Z}_\ell$ is a maximal order, then T_ℓ is a $O_B \otimes \mathbb{Z}_\ell$-module of rank m.*
4. *If $m=1$, then the connected component G_ℓ^0 of G_ℓ is a torus and $G_\ell(\mathbb{Q}_\ell)/G^0(\mathbb{Q}_\ell)$ is a finite elementary 2-group.*
5. *The center of $G_\ell(\mathbb{Q}_\ell)$ is given by*

$$Z(G_\ell(\mathbb{Q}_\ell)) = \{x \in (F \otimes \mathbb{Q}_\ell)^\times \mid x^2 \in \mathbb{Q}_\ell^\times\}.$$

Proof. The statement (1) follows from the fact that $\operatorname{Tr}(a; V_\ell; \mathbb{Q}_\ell) = 4m \operatorname{Tr}_{F/\mathbb{Q}}(a)$ for all $a \in O_F$ and that $O_F \otimes \mathbb{Z}_\ell$ is a maximal order. The statements (2) and (3) are obvious.

(4) Let $V_\ell = B_\ell$ as a left B_ℓ-module. Let $(\, ,) : B_\ell \times B_\ell \to B_\ell$ be the lifting of $\langle\, ,\rangle$. One has $\langle x,y\rangle = \operatorname{Trd}_{B_\ell/\mathbb{Q}_\ell}(x\alpha y^*)$. where $\alpha := (1,1)$ with $\alpha^* = -\alpha$. Any element in $\operatorname{End}_{B_\ell}(V_\ell)$ is a right translation ρ_g by an element $g \in B_\ell$. The condition $\langle xg, yg\rangle = c(g)\langle x,y\rangle$ gives the relation $g\alpha g^* = c(g)\alpha$. Replacing g by g^{-1}, the group G_ℓ is identified with the subgroup of $B_\ell^\times$ defined by the relation $g\alpha g^* = c(g)\alpha$ for some $c(g) \in \mathbb{G}_{\mathrm{m}}$.

For each embedding $\sigma \in \Sigma := \operatorname{Hom}(F_\ell, \overline{\mathbb{Q}}_\ell)$, put $B_\sigma = B_\ell \otimes_{F_\ell,\sigma} \overline{\mathbb{Q}}_\ell \simeq M_2(\overline{\mathbb{Q}}_\ell)$. Let $j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $g \in B_\sigma$, one has $jg^*j^{-1} = g^t$. Write $\alpha = \beta j$, then $\beta^t = \beta$

and the relation defining G_ℓ becomes $g\beta g^t = c(g)\beta$ for some $c(g)\in \mathbb{G}_{\mathrm{m}}$. Therefore,

$$G_{\overline{\mathbb{Q}}_\ell} \simeq \{(g_\sigma)\in \mathrm{GL}_2^\Sigma \ ; \ g_\sigma g_\sigma^t = c \ \text{ for some } c\in \overline{\mathbb{Q}}_\ell^\times, \forall\, \sigma\in\Sigma\}, \quad \text{and}$$

$$G^0_{\overline{\mathbb{Q}}_\ell} \simeq \left\{ \begin{pmatrix} a_\sigma & b_\sigma \\ -b_\sigma & a_\sigma \end{pmatrix} \in \mathrm{GL}_2^\Sigma \ ; \ a_\sigma^2 + b_\sigma^2 = c \ \text{ for some } c\in \overline{\mathbb{Q}}_\ell^\times, \forall\, \sigma\in\Sigma \right\}.$$

This shows that G^0_ℓ is a torus. The second statement follows from $G(\mathbb{Q}_\ell)/G^0(\mathbb{Q}_\ell)\subset G/G^0(\overline{\mathbb{Q}}_\ell)\simeq (\mathbb{Z}/2\mathbb{Z})^d$.

(5) This follows directly from the computation in (4). ∎

Remark 3.2. (1) The group $G_\ell(\mathbb{Q}_\ell)$ need not to be Zariski dense in G_ℓ. For example, if $F_\ell = F\otimes \mathbb{Q}_\ell$ remains a field and B splits at ℓ, then $[G_\ell(\mathbb{Q}_\ell) : G^0_\ell(\mathbb{Q}_\ell)] = 1$ or 2. However, we always have $[G_\ell : G^0_\ell] = 2^d$.

Recall that an abelian variety A over any field k is said to have *sufficient many complex multiplications* or be *of CM-type* over k if there is a semi-simple commutative $\mathbb{Q}$-subalgebra $L\subset \mathrm{End}^0_k(A) = \mathrm{End}_k(A)\otimes_{\mathbb{Z}}\mathbb{Q}$ such that $[L:\mathbb{Q}] = 2\dim A$. It is said to have *potentially sufficient many complex multiplications* or be *potentially of CM-type* if there is a finite field extension k_1 over k so that the base change A_{k_1} is of CM type over k_1.

Proposition 3.3. *Let $\underline{A} = (A,\lambda,\iota)$ be a $2d$-dimensional polarized abelian O_B-variety over K, where K a field of finite type over its prime field. Then A is potentially of CM-type.*

PROOF. Let $\mathbb{Q}_\ell[\mathcal{G}_\ell]$ be the subalgebra of $\mathrm{End}(V_\ell)$ generated by the image $\mathcal{G}_\ell := \rho_\ell(\mathcal{G}_K)$. Replacing K by a finite extension of K, we may assume that $\mathcal{G}_\ell\subset G^0_\ell(\mathbb{Q}_\ell)$ is abelian. By the semi-simplicity of Tate modules due to Faltings and Zarhin (see [3] and [40]), $\mathbb{Q}_\ell[\mathcal{G}_\ell]$ is a commutative and semi-simple subalgebra. Let L be a maximal semi-simple commutative subalgebra in $\mathrm{End}^0(A)$, then so is $L\otimes\mathbb{Q}_\ell\subset \mathrm{End}^0(A)\otimes_{\mathbb{Q}}\mathbb{Q}_\ell$. By the theorem of Faltings and Zarhin on Tate's conjecture (see [3] and [40]), we have $\mathrm{End}^0(A)\otimes\mathbb{Q}_\ell = \mathrm{End}_{\mathbb{Q}_\ell[\mathcal{G}_\ell]}(V_\ell)$. As $\mathbb{Q}_\ell[\mathcal{G}_\ell]$ is commutative and semi-simple, any maximal semi-simple commutative subalgebra in $\mathrm{End}_{\mathbb{Q}_\ell[\mathcal{G}_\ell]}(V_\ell)$ has degree $2\dim A$ over $\mathbb{Q}_\ell$. This shows $[L:\mathbb{Q}] = 2\dim A$ and finishes the proof of the proposition. ∎

Corollary 3.4. *Let $\underline{A}$ and K be as in Proposition 3.3.*

(1) *If $\mathrm{char}\, K = 0$, then $\underline{A}$ is, up to a finite extension of K, defined over a number field. That is, there are a finite field extension K_1 of K and an abelian variety $\underline{A}_0$ with additional structure over a number field K_0 contained in K_1 such that there is an isomorphism $\underline{A}\otimes_K K_1 \simeq \underline{A}_0\otimes_{K_0} K_1$.*

(2) *If $\mathrm{char}\, K = p > 0$, then $\underline{A}$ is, up to a finite extension of K and up to isogeny, defined over a finite field. That is, there are a finite field extension K_1 of K and an abelian variety $\underline{A}_0$ with additional structure over a finite field K_0 contained in K_1 such that there is an isogeny $\underline{A}\otimes_K K_1 \sim \underline{A}_0\otimes_{K_0} K_1$.*

PROOF. The statement (1) follows from Proposition 3.3 and a basic fact in the theory of complex multiplication. The statement (2) follows from Proposition 3.3 and a result of Grothendieck on isogeny classes of CM abelian varieties in positive characteristic (see [24] and [36]). ∎

4. Skew-Hermitian $O_B \otimes W$-modules

4.1. In this section we investigate the basic properties of related local modules with additional structures. We use the following notations.

Let k be an algebraically closed field of characteristic $p > 0$. Let $W = W(k)$ be the ring of Witt vectors over k, and $B(k) := \mathrm{Frac}\, W$ the fraction field of W. Let σ be the Frobenius map on W and on $B(k)$. Let $\mathbf{F}$ be a finite field extension of $\mathbb{Q}_p$ and $\mathcal{O}$ the ring of integers. Let e and f be the ramification index and inertial degree of $\mathbf{F}/\mathbb{Q}_p$, respectively, and π a uniformizer of $\mathcal{O}$. Let $\mathbf{F}^{\mathrm{nr}}$ denote the maximal unramified subfield extension of $\mathbb{Q}_p$ in $\mathbf{F}$, and put $\mathcal{O}^{\mathrm{nr}} := \mathcal{O}_{\mathbf{F}^{\mathrm{nr}}}$ the ring of integers.

Let $\mathbf{B}$ be a quaternion algebra over $\mathbf{F}$ and $\mathcal{O}_{\mathbf{B}}$ be a maximal order. As before, we denote by $*$ the canonical involution. If $\mathbf{B}$ is the matrix algebra, then we fix an isomorphism $\mathbf{B} = \mathrm{Mat}_2(\mathbf{F})$ with $\mathcal{O}_{\mathbf{B}} = \mathrm{Mat}_2(\mathcal{O})$.

Choose an unramified maximal subfield $\mathbf{L} \subset \mathbf{B}$ so that the integral ring $\mathcal{O}_{\mathbf{L}}$ is contained in $\mathcal{O}_{\mathbf{B}}$. If $\mathbf{B}$ is a division algebra, then $\mathcal{O}_{\mathbf{L}}$ is contained in $\mathcal{O}_{\mathbf{B}}$ always. In this case we choose a presentation

$$\mathcal{O}_{\mathbf{B}} = \mathcal{O}_{\mathbf{L}}[\Pi] = \{a + b\Pi; a, b \in \mathcal{O}_{\mathbf{L}}\} \tag{4.1}$$

with the relations

$$\Pi^2 = -\pi \quad \text{and} \quad \Pi a = \bar{a}\Pi, \quad \forall\, a \in \mathcal{O}_{\mathbf{L}}, \tag{4.2}$$

where $a \mapsto \bar{a}$ is the non-trivial automorphism of $\mathbf{L}/\mathbf{F}$.

Indeed we first choose a presentation of $\mathcal{O}_{\mathbf{B}}$ as (4.1) with relations $\Pi a = \bar{a}\Pi$ and $\Pi^2 = -\pi u$ for some element $u \in \mathcal{O}^{\times}$. Then replacing Π by $\alpha\Pi$ for some element $\alpha \in \mathcal{O}_{\mathbf{L}}^{\times}$, one gets the relation $\Pi^2 = -\pi$. Similarly, one could also choose a presentation but with the relation $\Pi^2 = \pi$ instead. Nevertheless we simply fix the presentation of $\mathbf{B}$ as (4.1) and (4.2).

We may regard $\mathcal{O}_{\mathbf{B}}$ as an $\mathcal{O}$-subalgebra of $\mathrm{Mat}_2(\mathcal{O}_{\mathbf{L}})$ by sending

$$\Pi \mapsto \begin{pmatrix} 0 & -1 \\ \pi & 0 \end{pmatrix}, \quad \text{and} \quad a \mapsto \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}, \quad \forall\, a \in \mathcal{O}_{\mathbf{L}}.$$

Thus,

$$\mathcal{O}_{\mathbf{B}} = \left\{ a + b\Pi = \begin{pmatrix} a & -b \\ \pi\bar{b} & \bar{a} \end{pmatrix} \,\middle|\, a, b \in \mathcal{O}_{\mathbf{L}} \right\} \subset \mathrm{Mat}_2(\mathcal{O}_{\mathbf{L}}).$$

We also have the following properties

$$\Pi^* = -\Pi, \quad \text{and} \quad (a + b\Pi)^* = \bar{a} - b\Pi.$$

4.2. Let $\Sigma_0 := \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{O}^{\mathrm{nr}}, W)$ be the set of embeddings of $\mathcal{O}^{\mathrm{nr}}$ into W. Write $\Sigma_0 = \{\sigma_i\}_{i \in \mathbb{Z}/f\mathbb{Z}}$ in the way that $\sigma\sigma_i = \sigma_{i+1}$ for all $i \in \mathbb{Z}/f\mathbb{Z}$. For any W-module M together with a W-linear action of $\mathcal{O}^{\mathrm{nr}}$, write

$$M^i := \{x \in M \mid ax = \sigma_i(a)x, \ \forall\, a \in \mathcal{O}^{\mathrm{nr}}\} \tag{4.3}$$

for the σ_i-component, and we have the decomposition

$$M = \bigoplus_{i \in \mathbb{Z}/f\mathbb{Z}} M^i. \tag{4.4}$$

If V is a finite-dimensional k-vector space with a k-linear action of $\mathbb{F}_{p^f}$, we write

$$V = k^{m_0} \oplus \cdots \oplus k^{m_{f-1}} \tag{4.5}$$

for the decomposition $V = \oplus V^i$ as in (4.4) with $m_i = \dim_k V^i$ for all $i \in \mathbb{Z}/f\mathbb{Z}$.

Let $P(T) \in \mathcal{O}^{\rm nr}[T]$ be the minimal polynomial of π; one has $\mathcal{O} = \mathcal{O}^{\rm nr}[\pi] = \mathcal{O}^{\rm nr}[T]/P(T)$. For any $i \in \mathbb{Z}/f\mathbb{Z}$, set $W^i := W[T]/(\sigma_i(P(T))$ and denote again by π the image of T in W^i. Each W^i is a complete discrete valuation ring and one has the decomposition

$$\mathcal{O} \otimes_{\mathbb{Z}_p} W = \prod_{i\in \mathbb{Z}/f\mathbb{Z}} W^i. \tag{4.6}$$

The action of the Frobenius map σ on $\mathcal{O} \otimes_{\mathbb{Z}_p} W$ through the right factor gives a map $\sigma : W^i \to W^{i+1}$ which sends a to $\sigma(a)$ for $a \in W$ and $\sigma(\pi) = \pi$. If M is an $\mathcal{O} \otimes_{\mathbb{Z}_p} W$-module, then we have the decomposition (4.4) with each component M^i a W^i-module. Note that the structure of M as an $\mathcal{O} \otimes_{\mathbb{Z}_p} W$-module is determined by the structure of each M^i as a W^i-module for all $i \in \mathbb{Z}/f\mathbb{Z}$.

Let $\mathbf{L}^{\rm nr}$ be the maximal unramified extension of $\mathbb{Q}_p$ in $\mathbf{L}$, and let $\mathcal{O}_{\mathbf{L}^{\rm nr}}$ be the ring of integers. Let $\Sigma := \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{O}_{\mathbf{L}^{\rm nr}}, W)$ be the set of embeddings of $\mathcal{O}_{\mathbf{L}^{\rm nr}}$ into W. Write $\Sigma = \{\tau_j\}_{j\in\mathbb{Z}/2f\mathbb{Z}}$ in the way that $\sigma\tau_j = \tau_{j+1}$ and $\tau_j|_{\mathcal{O}^{\rm nr}} = \sigma_i$ where $i = j$ mod f for all j. The Galois group $\mathrm{Gal}(\mathbf{L}/\mathbf{F})$ acts on the set Σ by composing with the conjugate: $\bar\tau_i(x) := \tau_i(\bar x)$. One has $\bar\tau_i = \sigma^f \circ \tau_i = \tau_{i+f}$. For any W-module M together with a W-linear action of $\mathcal{O}_{\mathbf{L}}$, write

$$M^j := \{x \in M \mid ax = \tau_j(a)x,\ \forall\, a \in \mathcal{O}_{\mathbf{L}^{\rm nr}}\} \tag{4.7}$$

for the τ_j-component, and we have the decomposition

$$M = \bigoplus_{j\in\mathbb{Z}/2f\mathbb{Z}} M^j. \tag{4.8}$$

Similarly, each M^j is a W^i-module where $i = j \mod f$ and the structure of M as an $\mathcal{O}_{\mathbf{L}} \otimes_{\mathbb{Z}_p} W$ is determined by the structure of each M^j as a W^i-module for all $j \in \mathbb{Z}/2f\mathbb{Z}$.

4.3. Finite $\mathcal{O}_{\mathbf{B}} \otimes_{\mathbb{Z}_p} W$-modules. Suppose M is a finite W-module together with a W-linear action by $\mathcal{O}_{\mathbf{B}}$.

If $\mathbf{B}$ is the matrix algebra, then one has the decomposition

$$M = e_{11}M \oplus e_{22}M =: M_1 \oplus M_2 \tag{4.9}$$

where e_{11} and e_{22} are standard idempotents of $\mathrm{Mat}_2(\mathcal{O})$, and M_1 and M_2 are finite W-modules with a W-linear action by $\mathcal{O}$ with $\mathrm{rank}_W M_1 = \mathrm{rank}_W M_2$. The Morita equivalence states that the module M is uniquely determined by the $\mathcal{O} \otimes_{\mathbb{Z}_p} W$-module M_1. Furthermore, the structure of M_1 as an $\mathcal{O} \otimes_{\mathbb{Z}_p} W$-module is given by its decomposition $M_1 = \oplus_{i\in\mathbb{Z}/f\mathbb{Z}} M_1^i$ as W^i-submodules and the W^i-module structure of each component M_1^i. This describes finite $\mathcal{O}_{\mathbf{B}} \otimes W$-modules when $\mathbf{B}$ is the matrix algebra. In particular, if M is free as a W-module, then M is uniquely determined by the numbers $\mathrm{rank}_{W^i} M^i$, which equals $2\,\mathrm{rank}_{W^i} M_1^i$, up to isomorphism (and these ranks can be arbitrary even non-negative integers). The module M is a free $\mathcal{O}_{\mathbf{B}}\otimes W$-module if and only if the ranks $\mathrm{rank}_{W^i} M^i$ are constant.

Suppose $\mathbf{B}$ is the division algebra. Write $\mathcal{O}_{\mathbf{B}} = \mathcal{O}_{\mathbf{L}}[\Pi]$ as in Subsection 4.1. The action by $\mathcal{O}_{\mathbf{L}^{\rm nr}}$ gives the decomposition

$$M = \bigoplus_{j\in\mathbb{Z}/2f\mathbb{Z}} M^j \tag{4.10}$$

with each component M^j a finite W^i-module. Moreover, one has

$$\Pi : M^j \to M^{j+f}, \quad \Pi^2 = -\pi$$

for all $j \in \mathbb{Z}/2f\mathbb{Z}$. To see this, if $x \in M^j$, then for $a \in \mathcal{O}_{\mathbf{L}^{\mathrm{nr}}}$,

$$a \cdot \Pi x = \Pi\, \bar{a} \cdot x = \Pi\, (\tau_j(\bar{a}))x = \Pi\, \tau_{j+f}(a)x = \tau_{j+f}(a)\Pi x.$$

As a consequence, we obtain

$$\operatorname{rank}_{W^i} M^j = \operatorname{rank}_{W^i} M^{j+f}, \quad \forall\, j \in \mathbb{Z}/2f\mathbb{Z}. \tag{4.11}$$

This is the only constraint for M to be an $\mathcal{O}_{\mathbf{B}} \otimes W$-module. Put

$$a_j := \dim_k M^j/\Pi M^{j-f} \tag{4.12}$$

If M is free as a W-module, then M is uniquely determined by the numbers $\{a_j\}_j$ up to isomorphism. The number a_j can be arbitrary between 0 and $\operatorname{rank}_{W^i} M^j$ subject to the condition $a_j + a_{j+f} = \operatorname{rank}_{W^i} M^j$, Furthermore, M is a free $\mathcal{O}_{\mathbf{B}} \otimes W$-module if and only if the numbers a_j are constant.

4.4. Skew-Hermitian $\mathcal{O}_{\mathbf{B}} \otimes_{\mathbb{Z}_p} W$-module. Let M be a finite non-degenerate skew-Hermitian $\mathcal{O}_{\mathbf{B}}$-module over W, that is, it is a finite and free W-module with a W-linear action of $\mathcal{O}_{\mathbf{B}}$ and together with an alternating non-degenerate bilinear pairing

$$\psi : M \times M \to W$$

satisfying the condition

$$\psi(bx, y) = \psi(x, b^* y), \quad \forall\, x, y \in M, b \in \mathcal{O}_{\mathbf{B}}. \tag{4.13}$$

(non-degeneracy here means that the induced map $M \to M^t := \operatorname{Hom}_W(M, W)$ is injective). If the pairing ψ is perfect, we call M *self-dual*.

Suppose $\mathbf{B}$ is the matrix algebra. Let $C := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ be the Weyl element. Put $\varphi(x, y) := \psi(x, Cy)$. Then the pairing $\varphi : M \times M \to W$ is symmetric and the decomposition $M = M_1 \oplus M_2$ in (4.9) respects the pairing φ; indeed, this follows from the property $C^* = -C$. One also has $C^{-1}a^*C = a^t$ for any $a \in \mathbf{B} = \operatorname{Mat}_2(\mathbf{F})$. The Morita equivalence then reduces to describe the symmetric $\mathcal{O} \otimes W$-module M_1. The condition (4.13) then becomes $\varphi(ax, y) = \varphi(x, ay)$ for $x, y \in M_1$ and $a \in \mathcal{O}$ and this implies that $\varphi(M_1^i, M_1^{i'}) = 0$ for $i \neq i'$ in $\mathbb{Z}/f\mathbb{Z}$, where M_1^i's are the components in the decomposition (4.4). Consider the restriction of φ to each component M_1^i. Then there is a unique W^i-bilinear pairing

$$\varphi_i : M_1^i \times M_1^i \to \mathcal{D}^{-1}_{W^i/W}$$

such that $\varphi = \operatorname{Tr}_{W_i/W} \varphi_i$ on each M_1^i, where $\mathcal{D}^{-1}_{W^i/W}$ is the inverse difference of W^i over W. Then it suffices to describe symmetric W^i-modules M_1^i and this description is well known; see O'Meara [23]. As the ground field k is algebraically closed, if M_1^i is self-dual (with respect to the values in $\mathcal{D}^{-1}_{W^i/W}$), then the isomorphism class of M_1^i is determined by its rank $\operatorname{rank}_{W^i} M_1^i$. Note that M is self-dual with respect to the pairing ψ if and only if each submodule M_1^i is self-dual with respect to the pairing φ_i (with values in $\mathcal{D}^{-1}_{W^i/W}$). Following from this, we conclude the following result.

Lemma 4.1. *Assume* $\mathbf{B}$ *is the matrix algebra. Any two self-dual skew-Hermitian* $\mathcal{O}_{\mathbf{B}} \otimes W$*-modules* M *and* N *are isomorphic if and only if* $\mathrm{rank}_{W^i} M^i = \mathrm{rank}_{W^i} N^i$ *for all* $i \in \mathbb{Z}/f\mathbb{Z}$*. Moreover, for any given non-negative even integers* n_i *for* $i \in \mathbb{Z}/f\mathbb{Z}$*, there is a unique up to isomorphism self-dual skew-Hermitian* $\mathcal{O}_{\mathbf{B}} \otimes W$*-modules* M *such that* $\dim_{W^i} M^i = n_i$ *for all* $i \in \mathbb{Z}/f\mathbb{Z}$*.*

Suppose $\mathbf{B}$ is the division algebra. Let $M = \oplus_j M^j$ be the decomposition by the action of $\mathcal{O}_{\mathbf{L}^{\mathrm{nr}}}$ as (4.8). It is easy to see using (4.13) that $\psi(M^{j_1}, M^{j_2}) = 0$ if $j_1 - j_2 \neq f$ in $\mathbb{Z}/2f\mathbb{Z}$ and hence ψ is determined by its restriction

$$\psi : M^j \times M^{j+f} \to W$$

for $0 \leq j < f$. Note that $\mathrm{rank}_{W^i} M^j = \mathrm{rank}_{W^i} M^{j+f}$ (4.11). Let $\mathcal{D}^{-1}_{W^i/W} = W^i \delta_i$, where δ_i is a generator. Then there is a unique W^i-bilinear pairing

$$\psi_i : M^j \times M^{j+f} \to W^i$$

where $i := j \mod f$ such that $\psi = \mathrm{Tr}_{W^i/W}(\delta_i \psi_i)$. Clearly, the module M is self-dual with respect to the pairing ψ if and only if each ψ_i is a perfect pairing.

Now we only consider the case where M is *self-dual.* For any $x, y \in M^j$, one easily sees

$$\psi_i(x, \Pi y) = \psi_i(\Pi^* x, y) = \psi_i(y, \Pi x),$$

so the pairing

$$\varphi_i : M^j \times M^j \to W^i, \quad \varphi_i(x, y) := \psi_i(x, \Pi y),$$

is symmetric. Put $\overline{M^j} := M^j / \pi M^j$. Then ψ_i induces the perfect pairing which we still denote by $\psi_i : \overline{M^j} \times \overline{M^{f+j}} \to k$. Recall $a_j := \dim_k M^j / \Pi M^{j+f}$. From the isomorphisms

$$\Pi : M^{j+f} / \Pi M^j \simeq \Pi M^{j+f} / \pi M^j$$

we get

$$a_j + a_{j+f} = \dim_k \overline{M^j} = \dim_k \overline{M^{j+f}}.$$

Lemma 4.2. *For* $j \in \mathbb{Z}/2f\mathbb{Z}$ *and let notations be as above. Then there are* W^i*-bases*

$$\{x^j_1, \dots, x^j_{a_j + a_{j+f}}\}, \quad \{x^{j+f}_1, \dots, x^{j+f}_{a_j + a_{j+f}}\}$$

for M^j *and* M^{j+f}*, respectively, where the positive integers* a_j *and* a_{j+f} *are as above, such that*

$$\begin{cases} \Pi(x^j_k) = x^{j+f}_k, & \forall\, 1 \leq k \leq a_j, \\ \Pi(x^{j+f}_{a_j+k}) = x^j_{a_j+k}, & \forall\, 1 \leq k \leq a_{j+f}, \end{cases} \tag{4.14}$$

and for $1 \leq k, l \leq a_j + a_{j+f}$*, one has*

$$\psi_i(x^j_k, x^{j+f}_l) = \delta_{k,l}. \tag{4.15}$$

Proof. Consider the induced symmetric pairing $\varphi_i : \overline{M^j} \times \overline{M^j} \to k$. Since $\Pi\overline{M^j}$ and $\Pi\overline{M^{j+f}}$ are mutual orthogonal complemented with respect to the pairing ψ_i, one obtains a *non-degenerate* symmetric pairing

$$\varphi_i : M^j / \Pi M^{j+f} \times M^j / \Pi M^{j+f} \to k. \tag{4.16}$$

We prove the statement by induction on the rank of M^j (or of M^{j+f}). Suppose $a_j > 0$, using (4.16) we may choose an element $x_1^j \in M^j$ such that $\varphi_i(x_1^j, x_1^j) = 1$. This is because $W^{i\times} = (W^{i\times})^2$. Put $x_1^{j+f} := \Pi x_1^j \in M^{j+f}$. Then we have

$$M^j \oplus M^{j+f} = N \oplus N^\perp,$$

where N is the W^i-submodule generated by x_1^j and x_1^{j+f} and $N^\perp$ is the orthogonal complement of N. Clear N is stable under the $\mathcal{O}_\mathbf{B}$-action and hence so $N^\perp$. By induction we can choose bases $\{x_1^j, \dots, x_{a_j+a_{j+f}}^j\}$ and$\{x_1^{j+f}, \dots, x_{a_j+a_{j+f}}^{j+f}\}$ for M^j and M^{j+f}, respectively, satisfying (4.14) and (4.15). If $a_j = 0$ then $a_{j+f} > 0$ and we do the same for M^{j+f}. This completes the proof of the lemma. ∎

We obtain the following classification result.

Corollary 4.3. *Assume that* $\mathbf{B}$ *is the division algebra. Any two self-dual skew-Hermitian* $\mathcal{O}_\mathbf{B} \otimes W$*-modules* M *and* N *are isomorphic if and only if*

$$\dim_k M^j/\Pi M^{j+f} = \dim_k N^j/\Pi N^{j+f}, \quad \textit{for all } j \in \mathbb{Z}/2f\mathbb{Z}.$$

Moreover, for any given non-negative integers a_j *for* $j \in \mathbb{Z}/2f\mathbb{Z}$*, there is a unique up to isomorphism self-dual skew-Hermitian* $\mathcal{O}_\mathbf{B} \otimes W$*-modules* M *such that* $a_j = \dim_k M^j/\Pi M^{j+f}$*for all* $j \in \mathbb{Z}/2f\mathbb{Z}$.

5. Quasi-polarized Dieudonné $\mathcal{O}_\mathbf{B}$-modules

5.1. All Dieudonné modules in this paper are assumed to be finite and free as W-modules. For basic theory of Dieudonné modules, we refer to Manin [14] and Zink [41]. When working with Dieudonné modules, we use the standard notations F and V for the Frobenius and Verschiebung operators, respectively. This should not bring much danger of confusion with the totally real starting with.

By a Dieudonné $\mathcal{O}_\mathbf{B}$-module we mean a Dieudonné module M together with a ring monomorphism $\mathcal{O}_\mathbf{B} \to \mathrm{End}_{\mathrm{DM}}(M)$ of $\mathbb{Z}_p$-algebras. Recall that a quasi-polarization on a Dieudonné module M is an alternating non-degenerate W-bilinear form

$$\langle\ ,\ \rangle : M \times M \to W,$$

such that $\langle Fx, y\rangle = \langle x, Vy\rangle^\sigma$ for all $x, y \in M$; a quasi-polarization is called $\mathcal{O}_\mathbf{B}$-*linear* if it satisfies the condition

$$\langle bx, y\rangle = \langle x, b^* y\rangle, \quad \forall\, x, y \in M,\ b \in \mathcal{O}_\mathbf{B}. \tag{5.1}$$

A quasi-polarized Dieudonné $\mathcal{O}_\mathbf{B}$-module is a Dieudonné $\mathcal{O}_\mathbf{B}$-module M together with an $\mathcal{O}_\mathbf{B}$-linear quasi-polarization.

We also recall that a quasi-polarized p-divisible $\mathcal{O}_\mathbf{B}$-module is a triple (H, λ, ι) where G is a p-divisible group, $\iota : \mathcal{O}_\mathbf{B} \to \mathrm{End}(H)$ is a ring monomorphism of $\mathbb{Z}_p$-algebras and $\lambda : H \to H^t$ is a quasi-polarization (i.e. an isogeny λ satisfying $\lambda^t = -\lambda$) such that $\lambda \circ \iota(b^*) = \iota(b)^t \circ \lambda$ for all $b \in \mathcal{O}_\mathbf{B}$, where H^t is the Cartier dual of H. For a quasi-polarized p-divisible $\mathcal{O}_\mathbf{B}$-module (H, λ, ι), the associated (covariant) Dieudonné module $M = M(H)$ with the additional structures is a quasi-polarized Dieudonné $\mathcal{O}_\mathbf{B}$-module.

Clearly these notions can be defined for orders in general semi-simple $\mathbb{Q}_p$-algebras with involutions (i.e. for general PEL data); cf. [36, Section 2.1].

Let $(M,\langle\,,\rangle)$ be a (not necessarily separably) quasi-polarized Dieudonné $\mathcal{O}_{\mathbf{B}}$-module.

Suppose $\mathbf{B}$ is the matrix algebra. Similarly to Subsection 4.4 we define the symmetric pairing $(x,y):=\langle x, Cy\rangle$ and have the decomposition $M=e_{11}M\oplus e_{22}M=: M_1\oplus M_2$, which respects the pairing $(\,,\,)$, where C is the Weyl element. Hence the Morita equivalence reduces to considering the *anti-quasi-polarized* Dieudonné $\mathcal{O}$-module M_1. It admits the properties

$$(ax,y)=(x,ay) \text{ and } (Fx,y)=(x,Vy)^\sigma \quad \forall\, a\in\mathcal{O},\ x,y\in M_1. \tag{5.2}$$

Let $M_1=\oplus_{i\in\mathbb{Z}/f\mathbb{Z}}M_1^i$ be the decomposition by the action of $\mathcal{O}^{\rm nr}$. Then one has

$$F: M^i\to M^{i+1},\quad V: M^{i+1}\to M^i$$

and $(M_1^{i_1},M_1^{i_2})=0$ if $i_1\neq i_2$ in $\mathbb{Z}/f\mathbb{Z}$. This shows that the ranks $\mathrm{rank}_{W^i} M_1^i$ (or equivalently $\mathrm{rank}_{W^i} M^i$) for $i\in\mathbb{Z}/f\mathbb{Z}$ are constant. As a result, the module M_1 (or M) is free as an $\mathcal{O}\otimes_{\mathbb{Z}_p} W$-module (This only uses the property that M is a Dieudonné $\mathcal{O}$-module).

Lemma 5.1. *Assume that* $\mathbf{B}$ *is the matrix algebra. Let* M *and* N *are two* separably *quasi-polarized Dieudonné* $\mathcal{O}_{\mathbf{B}}$*-modules, then* M *and* N *are isomorphic as skew-Hermitian* $\mathcal{O}_{\mathbf{B}}\otimes W$*-modules.*

Proof. This follows from the fact that M_1 (or M) is free as an $\mathcal{O}\otimes_{\mathbb{Z}_p} W$-module and Lemma 4.1. ∎

Suppose $\mathbf{B}$ is the division algebra. Let $M=\oplus_{j\in\mathbb{Z}/2f\mathbb{Z}}M^j$ be the decomposition as (4.10). Then we have

$$F: M^j\to M^{j+1},\quad V: M^{j+1}\to M^j,$$

and $\langle M^{j_1},M^{j_2}\rangle=0$ if $j_1-j_2\neq f$ in $\mathbb{Z}/2f\mathbb{Z}$. This shows that the ranks $\mathrm{rank}_{W^i} M^j$ for $j\in\mathbb{Z}/2f\mathbb{Z}$ are constant. As a result, M is free as an $\mathcal{O}_{\mathbf{L}}\otimes_{\mathbb{Z}_p} W$-module (This only uses the property that M is a Dieudonné $\mathcal{O}_{\mathbf{L}}$-module).

5.2. From now on until Section 9 we assume that $\mathrm{rank}_W M=m[\mathbf{B}:\mathbb{Q}_p]$ for some positive integer m. We say a Dieudonné $\mathcal{O}_{\mathbf{B}}$-module M satisfies the *determinant condition* if one has the equality of the characteristic polynomials, cf. (2.4)

$$\text{(K)}\quad \mathrm{char}\,(\iota(a)|M/VM)=\mathrm{char}\,(a)^m\in k[T],\quad \forall\, a\in\mathcal{O}_{\mathbf{B}}, \tag{5.3}$$

where $\mathrm{char}\,(a)\in\mathbb{Z}_p[T]$ is the reduced characteristic polynomial of a from $\mathbf{B}$ to $\mathbb{Q}_p$, which is of degree $[\mathbf{B}:\mathbb{Q}_p]/2$. If we let $d=[\mathbf{F}:\mathbb{Q}_p]$, then the above polynomials are of degree $2dm$.

Lemma 5.2. *Let* M *be a Dieudonné* $\mathcal{O}_{\mathbf{B}}$*-module.*

1. *If* $\mathbf{B}$ *is the matrix algebra, then* M *satisfies the determinant condition (K) if and only if for all* $i\in\mathbb{Z}/f\mathbb{Z}$*, one has* $\dim_k(M_1/VM_1)^i=em$.
2. *If* $\mathbf{B}$ *is the division algebra, then* M *satisfies the determinant condition (K) if and only if for all* $j\in\mathbb{Z}/2f\mathbb{Z}$*, one has* $\dim_k(M/VM)^j=em$.

Proof. (1) Using the Morita equivalence, M satisfies the condition (K) if and only if M_1 satisfies (K) for all $a\in\mathcal{O}$. Choose an algebraically closure $B(k)^{\rm alg}$ of

$B(k) := W(k)[1/p]$ and put $\Sigma_{\mathbf{F}} := \mathrm{Hom}(\mathbf{F}, B(k)^{\mathrm{alg}})$. For $a \in \mathcal{O}$, the left hand side of the equation (5.3) is

$$\prod_{i\in\mathbb{Z}/f\mathbb{Z}} (T-\widetilde{\sigma}_i(a))^{\dim_k(M_1/VM_1)^i},$$

where $\widetilde{\sigma}_i \in \Sigma_{\mathbf{F}}$ is any lift of $\sigma_i \in \Sigma_0$. The right hand side of the equation (5.3) is equal to

$$\prod_{\sigma\in\Sigma_{\mathbf{F}}} (T-\sigma'(a))^m = \prod_{i\in\mathbb{Z}/f\mathbb{Z}} (T-\widetilde{\sigma}_i(a))^{em} .$$

Therefore, the condition (K) is satisfied if and only if $\dim_k(M_1/VM_1)^i = em$ for all $i \in \mathbb{Z}/f\mathbb{Z}$.

(2) As $\mathbf{B}$ is generated by the element Π over $\mathbf{L}$, it suffices the check the equality (5.3) for $a = \Pi$ and all $a \in \mathcal{O}_{\mathbf{L}}$. Since $\iota(\Pi)$ on M/VM is nilpotent, its characteristic polynomial is T^{2dm}, where $d = [\mathbf{F} : \mathbb{Q}_p]$. The (reduced) characteristic polynomial of Π is the product of $(T^2 + \pi)$ which is also equal to T^{2dm} in $k[T]$. The same proof of (1) then shows that M satisfies the determinant condition if and only if $\dim_k(M/VM)^j = em$ for all $j \in \mathbb{Z}/2f\mathbb{Z}$. ∎

Lemma 5.3. *Suppose M is a separably quasi-polarized Dieudonné $\mathcal{O}_{\mathbf{B}}$-module.*

1. *If $\mathbf{B}$ is the matrix algebra, then for any $i \in \mathbb{Z}/f\mathbb{Z}$, one has $\dim_k(M_1/VM_1)^i = em$.*
2. *If $\mathbf{B}$ is the division algebra, then for any $i \in \mathbb{Z}/f\mathbb{Z}$, one has $\dim_k(M/VM)^i = 2em$ (recall that $(M/VM)^i$ denotes the σ_i-component of M/VM).*

Proof. (1) Consider M_1 as a separably anti-quasi-polarized Dieudonné $\mathcal{O}$-module. Using the similar proof of [35, Lemma 2.6 (2)], we show that the numbers $\dim_k(M_1/VM_1)^i$ are constant, therefore, $\dim_k(M_1/VM_1)^i = em$ for all $i \in \mathbb{Z}/f\mathbb{Z}$.

(2) Consider M as a separably quasi-polarized Dieudonné $\mathcal{O}$-module (the Hilbert-Siegel analogue). Again using the similar proof of [35, Lemma 2.6 (2)], we conclude that the numbers $\dim_k(M/VM)^i$ are constant, therefore, $\dim_k(M/VM)^i = 2em$ for all $i \in \mathbb{Z}/f\mathbb{Z}$. ∎

Remark 5.4. According to Lemma 5.3, if $\mathbf{B}$ is the matrix algebra, then any separably quasi-polarized Dieudonné $\mathcal{O}_{\mathbf{B}}$-module satisfies the determinant condition. However, in the case where $\mathbf{B}$ is the division algebra, there are a few possibilities for $\dim_k(M/VM)^j$, so that the determinant condition would impose a further condition for separably quasi-polarized Dieudonné $\mathcal{O}_{\mathbf{B}}$-modules.

5.3. We discuss the relationship between the numbers $a_j := \dim_k(M/\Pi M)^j$ and the numbers $\dim_k(M/VM)^j$ when $\mathbf{B}$ is the division algebra. Put

$$c_j := \dim_k(M/VM)^j \quad \text{for } j \in \mathbb{Z}/2f\mathbb{Z}. \tag{5.4}$$

Write $V_j : M^{j+1} \to M^j$ for the restriction of V on M^{j+1}, and $\Pi_j : M^j \to M^{j+f}$ for that of Π on M^j. We have the commutative diagram

$$\begin{tikzcd} M^j \arrow[d, "\Pi_j"'] & M^{j+1} \arrow[l, "V_j"'] \arrow[d, "\Pi_{j+1}"] \\ M^{j+f} & M^{j+f+1}. \arrow[l, "V_{j+f}"] \end{tikzcd} \tag{5.5}$$

Let ord be the normalized valuation on W^i, that is, one has $\mathrm{ord}(\pi)=1$. Let $\mathrm{ord}\det\Pi_j$ denote the valuation of $\det(A_j)$, where A_j is the representative matrix of the map Π_j with respect to a set of W^i-bases for M^j and M^{j+f} respectively; this is well-defined. Similarly we define $\mathrm{ord}\det V_j$ for suitable bases of M^{j+1} and M^j. It is easy to see that

$$\mathrm{ord}\det V_j = c_j,\ \text{ and }\ \mathrm{ord}\det \Pi_j = a_{j+f}, \quad \forall\, j\in \mathbb{Z}/2f\mathbb{Z}. \tag{5.6}$$

As $\Pi_{j+1}=V_{j+f}^{-1}\circ\Pi_j\circ V_j$, one has the relation

$$a_{j+f+1}=a_{j+f}+c_j-c_{j+f},$$

or equivalently

$$a_{j+1}=a_j+c_{j+f}-c_j, \quad \forall\, j\in \mathbb{Z}/2f\mathbb{Z}. \tag{5.7}$$

Since $\mathrm{rank}_{W^i} M^j=2m$, it follows from $\Pi_j\circ\Pi_{j+f}=-\pi$ that

$$a_j+a_{j+f}=2m. \tag{5.8}$$

Since $a_j's$ are integers between 0 and $2m$, it follows from (5.7) that

$$\Big|\sum_{i=j}^{j+r}(c_{i+f}-c_i)\Big|\le 2m, \quad \forall\, j\in \mathbb{Z}/2f\mathbb{Z},\ 0\le r\le f-1. \tag{5.9}$$

On the other hand, the collection $\{c_j\}$ satisfies the condition

$$\sum_{j\in\mathbb{Z}/2f\mathbb{Z}} c_j = 2dm. \tag{5.10}$$

If M admits an $\mathcal{O}_B$-linear quasi-polarization which is *separable*, or called separably quasi-polarizable, then one has the additional property

$$c_j+c_{j+f}=2em, \quad \forall\, j\in \mathbb{Z}/2f\mathbb{Z}. \tag{5.11}$$

Lemma 5.5. *Notations being as above, the sets of numbers $\{a_j\}$ and $\{c_j\}$ satisfy the conditions (5.7)–(5.10). Moreover, if M is separably quasi-polarizable, then one has in addition the condition (5.11).*

Proposition 5.6. *Let M be a Dieudonné $\mathcal{O}_{\mathbf{B}}$-module of rank $4dm$, where $d=[\mathbf{F}:\mathbb{Q}_p]$.*

1. *Suppose $\mathbf{B}$ is the matrix algebra.*
   - (a) *The module M is free as an $\mathcal{O}_{\mathbf{B}}\otimes W$-module.*
   - (b) *The module M is separably quasi-polarizable if and only if it satisfies the determinant condition.*
2. *Suppose $\mathbf{B}$ is the division algebra. Then M is free as an $\mathcal{O}_{\mathbf{B}}\otimes W$-module if and only if it satisfies the determinant condition.*

Proof. (1) Part (a) is discussed in the paragraph before Lemma 5.1. (b) The only if part follows from Lemmas 5.2 and 5.3. For the if part, we refer to the discussion in [35, Lemma 2.6].

(2) If M satisfies the determinant condition. then each c_j is equal to em by Lemma 5.2. By (5.7) and (5.8), each a_j is equal to m. Therefore, M is free as an $\mathcal{O}_{\mathbf{B}}\otimes W$-module. Conversely, if M is free as an $\mathcal{O}_{\mathbf{B}}\otimes W$-module, then $c_j=c_{j+f}$ for all j by (5.7). By (5.11), each c_j is equal to em. It follows from Lemma 5.2 that the Dieudonné $\mathcal{O}_{\mathbf{B}}$-module M satisfies the determinant condition. ∎

Corollary 5.7. *Let Λ be a $\mathbb{Z}_p$-valued unimodular skew-Hermitian $\mathcal{O}_{\mathbf{B}}$-module of $\mathbb{Z}_p$-rank $4dm$. For any separably quasi-polarized Dieudonné $\mathcal{O}_{\mathbf{B}}$-module M of rank $4dm$ that satisfies the determinant condition, one has an isomorphism $M \simeq \Lambda \otimes_{\mathbb{Z}_p} W$ as skew-Hermitian $\mathcal{O}_{\mathbf{B}} \otimes W$-modules.*

PROOF. By Proposition 5.6, M is free as an $\mathcal{O}_{\mathbf{B}} \otimes W$-module. When $\mathbf{B}$ is the matrix algebra, the assertion follows from Lemma 4.1. When $\mathbf{B}$ is the division algebra, by Proposition 5.6 M is a free $\mathcal{O}_B \otimes W$-module. The assertion then follows from Lemma 4.2. ∎

6. Isogeny classes of p-divisible $\mathcal{O}_{\mathbf{B}}$-modules

6.1. Keep the notations of the previous section. Our goal is to classify the isogeny classes of quasi-polarized p-divisible $\mathcal{O}_{\mathbf{B}}$-modules $\underline{H} = (H, \lambda, \iota)$ over an algebraically closed field k of characteristic p. Let $(M, \langle\, , \rangle, \iota)$ be the associated Dieudonné module with the additional structures. Assume that $\mathrm{rank}_W M = m[\mathbf{B} : \mathbb{Q}_p]$ for some integer $m \geq 1$. Note that this is the general type D case (in the local situation). Let $d := [\mathbf{F} : \mathbb{Q}_p]$. So the p-divisible group H has height $4dm$.

The slope sequence (or Newton polygon) of a p-divisible group H is denoted by $\nu(H)$. Write

$$\{\, \beta_i^{m_i} \,\}_{1 \leq i \leq t}$$

for the slope sequence with each slope β_i of multiplicity m_i.

Two quasi-polarized p-divisible $\mathcal{O}_{\mathbf{B}}$-modules $\underline{H} = (H, \lambda, \iota)$ and $\underline{H'} = (H', \lambda', \iota')$ are said to be *isogenous* if there is an $\mathcal{O}_{\mathbf{B}}$-linear quasi-isogeny $\varphi : H \to H'$ such that $\varphi^* \lambda' = \lambda$. This is equivalently saying that the associated F-isocrystals $\underline{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \underline{M}' \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ are isomorphic compatible with the additional structures. Similarly, one defines the isogenies for p-divisible $\mathcal{O}_{\mathbf{B}}$-modules (H, ι). Clearly, the slope sequence of a (quasi-polarized) p-divisible $\mathcal{O}_{\mathbf{B}}$-module is determined by its isogeny class. The relationship between the Newton polygon and isogeny class of p-divisible groups with additional structures (in the general setting of F-isocrystals with G-structure) is known due to the works of Kottwitz [11, 13] and Rapoport-Richartz [28]. Here we describe the image of the Newton map ν for p-divisible $\mathcal{O}_{\mathbf{B}}$-modules.

6.2. We describe the isogeny classes of p-divisible $\mathcal{O}_{\mathbf{B}}$-modules.

Lemma 6.1. *Let (H, ι) and (H', ι') be two p-divisible $\mathcal{O}_{\mathbf{B}}$-modules of same height. Then (H, ι) is isogenous to (H', ι') if and only if $\nu(H) = \nu(H')$.*

PROOF. The direction $\Longrightarrow$ is obvious and we show the other direction. Since $\nu(H) = \nu(H')$, we may choose an isogeny $\varphi : H \to H'$. Then we have an isomorphism $j : \mathrm{End}^0(H') \simeq \mathrm{End}^0(H)$ of $\mathbb{Q}_p$-algebras, sending $a \mapsto \varphi^{-1} a \varphi$, where $\mathrm{End}^0(H) := \mathrm{End}(H) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Put $\iota_1 := j \circ \iota' : \mathbf{B} \to \mathrm{End}^0(H)$. Since the center of $\mathrm{End}^0(H)$ is a product of copies of $\mathbb{Q}_p$, by the Noether-Skolem theorem, there is an element $\alpha \in \mathrm{End}^0(H)^\times$ such that $\iota_1 = \mathrm{Int}(\alpha) \circ \iota$. For $b \in B$, one has

$$\varphi^{-1}\, \iota'(b)\, \varphi = \iota_1(b) = \alpha\, \iota(b)\, \alpha^{-1}.$$

Therefore, $\varphi \circ \alpha : H \to H'$ is an $\mathcal{O}_{\mathbf{B}}$-linear quasi-isogeny. ∎

6.3. The isoclinic case. Let H be an isoclinic p-divisible $\mathcal{O}_{\mathbf{B}}$-module of height $h>0$ of the slope β, and let M be the associated Dieudonné $\mathcal{O}_{\mathbf{B}}$-module. Let $E := \mathrm{End}^0(H)$ be the endomorphism algebra. Write $E = \mathrm{Mat}_n(\Delta)$, where Δ is a central division $\mathbb{Q}_p$-algebra with $\mathrm{inv}(\Delta)=\beta$.

Suppose $\mathbf{B}$ is the matrix algebra. Then $H = H_1 \oplus H_2$, H_1 has height $h_1 := h/2$ and $\deg(\mathrm{End}^0(H_1)) = h_1$. It is well known that there is an monomorphism $\mathbf{F} \to \mathrm{End}^0(H_1)$ if and only if $d|h_1$, or equivalently $2d|h$. As $\deg(\mathrm{End}^0(H_1)) = h_1$, one has $\beta = a/h_1$ for some integer $0 \le a \le h_1$. Therefore,

$$\nu(H) = \left\{ \left(\frac{a}{h_1}\right)^h \right\}$$

for some integer $0 \le a \le h_1$. Conversely, suppose $2d|h$ and we are given a slope sequence $\nu = \{(2a/h)^h\}$ for some integer $0 \le a \le h/2$. Then there is a p-divisible group H of height h and with an monomorphism $\mathbf{B} \to \mathrm{End}^0(H)$ such that $\nu(H) = \nu$. Replacing H by another p-divisible group in its isogeny class, the map $\mathbf{B} \to \mathrm{End}^0(H)$ can be extended to a map $\iota : \mathcal{O}_{\mathbf{B}} \to \mathrm{End}(H)$.

Suppose that $\mathbf{B}$ is the division algebra. Write $\beta = a/h$ for some integer $0 \le a \le h$. Suppose

$$\Delta \otimes_{\mathbb{Q}_p} \mathbf{B}^{\mathrm{op}} = \Delta_{\mathbf{F}} \otimes_{\mathbf{F}} \mathbf{B}^{\mathrm{op}} = \mathrm{Mat}_c(\Delta'),$$

where $\Delta_{\mathbf{F}} := \Delta \otimes_{\mathbb{Q}_p} \mathbf{F}$, $\mathbf{B}^{\mathrm{op}}$ is the opposite algebra of $\mathbf{B}$, Δ' is a central division $\mathbf{F}$-algebra.

By an embedding theorem for general simple algebras [39, Theorem 2.7], there is an embedding of $\mathbf{B}$ into $E = \mathrm{Mat}_n(\Delta)$ if and only if the following condition holds

$$[\mathbf{B} : \mathbb{Q}_p] \mid nc. \tag{6.1}$$

Write $\delta := \deg(\Delta)$ and $\delta' := \deg(\Delta')$. As the field $\mathbf{L}$ can be embedded into E, one has $2d|h$. So we can put $h = 2dh'$ for some integer h'.

Lemma 6.2. *The condition (6.1) is equivalent to the condition*

$$a \equiv h' \pmod 2 \tag{6.2}$$

Proof. We have $c\delta' = 2\delta$ and $h = n\delta$. Then

$$4d|nc \iff 4d\delta'|nc\delta' \iff 2d\delta'|n\delta = h$$

and this is equivalent to the condition

$$\delta'|h'. \tag{6.3}$$

As δ' is the denominator of $\mathrm{inv}(\Delta')$, the condition (6.3) holds if and only if $h' \cdot \mathrm{inv}(\Delta') \in \mathbb{Z}$. We compute

$$\mathrm{inv}(\Delta') = d \cdot \frac{a}{h} - \frac{1}{2} = \frac{a-h'}{2h'} \quad \text{and hence} \quad h' \cdot \mathrm{inv}(\Delta') = \frac{a-h'}{2}.$$

Therefore, the condition (6.3) holds if and only if the condition (6.2) holds. This proves the lemma. ∎

By Lemma 6.2 one has

$$\nu(H) = \left\{ \left(\frac{a}{h}\right)^h \right\}$$

for some integer $0 \le a \le h$ with $a \equiv h' \pmod 2$. Conversely, suppose $h = 2dh'$ for some $h' \in \mathbb{Z}_{\ge 1}$ and we are given a slope sequence $\nu = \{(a/h)^h\}$ for some integer

$0 \le a \le h$ with $a \equiv h' \pmod 2$. Then there is a p-divisible $\mathcal{O}_{\mathbf{B}}$-module H of height h such that $\nu(H) = \nu$.

We conclude the discussion in the following proposition.

Proposition 6.3. *Let $h = 2dh'$ with $h' \in \mathbb{Z}_{\ge 1}$, and let (H, ι) be an isoclinic p-divisible $\mathcal{O}_{\mathbf{B}}$-module of height h over k.*

(1) *If $\mathbf{B}$ is the matrix algebra, then*

$$\nu(H) = \left\{ \left(\frac{a}{dh'} \right)^h \right\},$$

where a can be integer with $0 \le a \le dh'$.

(2) *If $\mathbf{B}$ is the division algebra, then*

$$\nu(H) = \left\{ \left(\frac{a}{h} \right)^h \right\},$$

where a can be any integer with $0 \le a \le h$ and $a \equiv h' \pmod 2$.

6.4. The general case. It is not hard to state the general case for possible slope sequences of p-divisible $\mathcal{O}_{\mathbf{B}}$-modules based on the isoclinic case. One simply considers the decomposition $H \sim H_1 \times H_2 \times \cdots \times H_t$ into the isoclinic components in the isogeny class.

Theorem 6.4. *Let $h = 2dh'$ with $h' \in \mathbb{Z}_{\ge 1}$, and let (H, ι) be a p-divisible $\mathcal{O}_{\mathbf{B}}$-module of height h over k.*

(1) *If $\mathbf{B}$ is the matrix algebra, then*

$$\nu(H) = \left\{ \left(\frac{a_i}{dh'_i} \right)^{2dh'_i} \right\}_{1 \le i \le t}, \tag{6.4}$$

where $h'_1 + \cdots + h'_t = h'$ is any partition of the integer h' and a_i can be any integer with $0 \le a_i \le dh'_i$. Moreover, after combining the indices i with same slope a_i/dh'_i and rearranging the indices, we may assume that

$$\frac{a_i}{dh'_i} < \frac{a_{i+1}}{dh'_{i+1}}, \quad i = 1, \dots, t-1.$$

(2) *If $\mathbf{B}$ is the division algebra, then*

$$\nu(H) = \left\{ \left(\frac{a_i}{2dh'_i} \right)^{2dh'_i} \right\}_{1 \le i \le t}, \tag{6.5}$$

where $h'_1 + \cdots + h'_t = h'$ is any partition of the integer h' and a_i can be any integer with $0 \le a_i \le 2dh'_i$ and $a_i \equiv h'_i \pmod 2$. Similarly, we may rearrange the indices so that

$$\frac{a_i}{2dh'_i} < \frac{a_{i+1}}{2dh'_{i+1}}, \quad i = 1, \dots, t-1.$$

7. Slope sequences of quasi-polarized Dieudonné $\mathcal{O}_{\mathbf{B}}$-modules

Let $h = 4dm$ with $m \in \mathbb{Z}_{\geq 1}$, and let (H, λ, ι) be a quasi-polarized p-divisible $\mathcal{O}_{\mathbf{B}}$-module of height h over k. Then the slope sequence $\nu(H)$ of H is given as Theorem 6.4 together with the symmetric condition: for all $0 \leq i, j \leq t$ with $i + j = t + 1$, one has $h'_i = h'_j$ and

$$\begin{aligned} &\frac{a_i}{dh'_i} + \frac{a_j}{dh'_j} = 1 \quad \text{in the matrix algebra case,} \\ &\frac{a_i}{2dh'_i} + \frac{a_j}{2dh'_j} = 1 \quad \text{in the division algebra case,} \end{aligned} \tag{7.1}$$

where $h'_1 + \cdots + h'_t = 2m$ is a partition of $2m$.

Note that the integer t is even if and only if H has no supersingular component. For any symmetric slope sequence ν which has the form in Theorem 6.4, we write

$$\nu = \nu_n \cup \nu_s, ,$$

where ν_n consists of all slopes in ν which are not $1/2$ and ν_s consists of all slopes $1/2$ in ν.

Lemma 7.1. *Let h be a positive integer and β a positive rational number so that there exists an isoclinic p-divisible $\mathcal{O}_{\mathbf{B}}$-module of height h and with slope β. Then there exists a separably quasi-polarized p-divisible $\mathcal{O}_{\mathbf{B}}$-module (H, λ, ι) of height $2h$ such that $\nu(H) = \{\beta^h, (1-\beta)^h\}$.*

Proof. Choose an isoclinic p-divisible $\mathcal{O}_{\mathbf{B}}$-module (H_1, ι_1) of height h and with the slope β. Put $H_2 := H_1^t$, which is an isoclinic p-divisible $\mathcal{O}_{\mathbf{B}}$-module of height h and with the slope $1 - \beta$. The monomorphism $\iota_2 : \mathcal{O}_{\mathbf{B}} \to \mathrm{End}(H_2)$ is given by $\iota_2(a) := \iota_1(a^*)^t$ for $a \in \mathcal{O}_{\mathbf{B}}$. Put $H := H_1 \times H_2$, and then $H^t = H_1^t \times H_2^t$. Let $\lambda = (\lambda_1, \lambda_2) : H \to H^t$ be an $\mathcal{O}_{\mathbf{B}}$-linear isogeny, where $\lambda_1 : H_1 \to H_2^t$ and $\lambda_2 : H_2 \to H_1^t$ are $\mathcal{O}_{\mathbf{B}}$-linear isogenies. Then $\lambda^t = (\lambda_2^t, \lambda_1^t)$, so $\lambda^t = -\lambda$ if and only if $\lambda_2 = -\lambda_1^t$. Choose λ_1 an $\mathcal{O}_{\mathbf{B}}$-linear isomorphism and put $\lambda = (\lambda_1, -\lambda_1^t)$. Then λ is a separably $\mathcal{O}_{\mathbf{B}}$-linear quasi-polarization. ∎

Note that the construction in Lemma 7.1 works for any finite-dimensional simple $\mathbb{Q}_p$-algebra $\mathbf{B}$ with involution. This method of construction appears quite often in dealing with symmetric slope sequences with two slopes and we call this *the double construction.*

For the supersingular case, we have the following result.

Theorem 7.2. *For any positive integer m, there exists a superspecial separably quasi-polarized Dieudonné $\mathcal{O}_{\mathbf{B}}$-module M of rank $4dm$ that satisfies the determinant condition (K).*

The proof of this theorem is placed in the next section. We conclude the main result in this section.

Theorem 7.3. *Let $h = 4dm$ with any $m \in \mathbb{Z}_{\geq 1}$. Let ν be a slope sequence of the form in Theorem 6.4 that satisfies the symmetric condition (7.1), where $h'_1 + \cdots + h'_t = 2m$ is any partition of $2m$. Then there exists a separably quasi-polarized Dieudonné $\mathcal{O}_{\mathbf{B}}$-module M of rank h and with $\nu(M) = \nu$.*

PROOF. Write $\nu = \nu_n + \nu_s$ into non-supersingular part and supersingular part, say of length $4dm_n$ and $4dm_s$, respectively. As each isoclinic component $\{\beta_i^{m_i}\}$ of ν_n can be realized by a p-divisible $\mathcal{O}_{\mathbf{B}}$-module (Theorem 6.4), by Lemma 7.1 there is a separably quasi-polarized p-divisible $\mathcal{O}_{\mathbf{B}}$-module $(H_n, \lambda_n, \iota_n)$ of height $4dm_n$ such that $\nu(H_n) = \nu_n$. On the other hand, by Theorem 7.2, there is a superspecial separably quasi-polarized p-divisible $\mathcal{O}_{\mathbf{B}}$-module $(H_s, \lambda_s, \iota_s)$ of height $4dm_s$ that satisfies the determinant condition. The product $(H_n, \lambda_n, \iota_n) \times (H_s, \lambda_s, \iota_s)$ satisfies the desired properties. ∎

Remark 7.4.

(1) In Lemma 7.1 one may choose H so that H is a *minimal* p-divisible group in the sense of Oort, that is, the endomorphism ring $\mathrm{End}(H)$ of H is a maximal $\mathbb{Z}_p$-order in the semi-simple $\mathbb{Q}_p$-algebra $\mathrm{End}^0(H)$. This follows from the construction of the minimal isogeny; see Section 4 (particularly Proposition 4.8) in [38]. Therefore, the p-divisible $\mathcal{O}_{\mathbf{B}}$-module in Theorem 7.3 can be chosen to be minimal.

(2) We shall see that when $\mathbf{B}$ is the division algebra, the determinant condition (K) will rule out some possibilities of the slope sequences that are realized by separably quasi-polarized Dieudonné $\mathcal{O}_{\mathbf{B}}$-module in Theorem 7.3. That is, not all symmetric slope sequences in Theorem 7.3 occurring as those of separably quasi-polarized Dieudonné $\mathcal{O}_{\mathbf{B}}$-modules that satisfy *the determinant condition.* We refer to Section 12 for more details in the case of rank $4d$.

Corollary 7.5. *There is an ordinary separably quasi-polarized p-divisible $\mathcal{O}_{\mathbf{B}}$-module of height $4dm$ if and only if one of the following holds:*

(1) $\mathbf{B}$ *is the matrix algebra;*
(2) $\mathbf{B}$ *is the division algebra and m is even.*

PROOF. Any ordinary p-divisible $\mathcal{O}_{\mathbf{B}}$-module always admits an $\mathcal{O}_{\mathbf{B}}$-linear separable quasi-polarization.

If $\mathbf{B}$ is the matrix algebra, then the ordinary slope sequence appears in Theorem 6.4 (or in Proposition 6.3). Therefore, by Theorem 7.3 there is an ordinary separably quasi-polarized p-divisible $\mathcal{O}_{\mathbf{B}}$-module of height $4dm$.

Suppose $\mathbf{B}$ is the division algebra. Then the slope sequence $\{(a_1/2dh_1')^{2dh_1'}, (a_2/2dh_2'))^{2dh_2'}\}$ of the form (6.5) can be the ordinary slope sequence if and only if $h_1' = h_2' = m$ and $m \equiv 2dm \pmod 2$ (taking $a_2 = 2dh_2'$ and $h_2' = m$). That is, m is even. ∎

Remark 7.6. For a smooth PEL-type moduli space $\mathcal{M}$, the ordinary locus of $\mathcal{M} \otimes k(v)$ is non-empty if and and if $E_v = \mathbb{Q}_p$, where E is the reflex field. Corollary 7.5 shows that the latter condition is not sufficient for the non-emptiness of the ordinary locus in ramified cases.

We give a few examples of possible slope sequences of quasi-polarized Dieudonné $\mathcal{O}_{\mathbf{B}}$-modules.

Corollary 7.7 ($m = 1$)**.** *Let (H, λ, ι) be a quasi-polarized p-divisible $\mathcal{O}_{\mathbf{B}}$-module of height $4d$.*

(1) *If $\mathbf{B}$ is the matrix algebra, then*

$$\nu(H) = \left\{ \left(\frac{a}{d}\right)^{2d}, \left(\frac{d-a}{d}\right)^{2d} \right\},$$

where a is any integer with $0 \le a < d/2$, or

$$\nu(H) = \left\{ \left(\frac{1}{2}\right)^{4d} \right\}.$$

(2) *If* $\mathbf{B}$ *is the division algebra, then*

$$\nu(H) = \left\{ \left(\frac{a}{2d}\right)^{2d}, \left(\frac{2d-a}{2d}\right)^{2d} \right\},$$

where a is any integer with $0 \le a < d$ with $a \equiv 1 \pmod 2$, or

$$\nu(H) = \left\{ \left(\frac{1}{2}\right)^{4d} \right\}.$$

Corollary 7.8 ($m = 2$)**.** *Let (H, λ, ι) be a quasi-polarized p-divisible $\mathcal{O}_{\mathbf{B}}$-module of height $8d$.*

(1) *If* $\mathbf{B}$ *is the matrix algebra, then we have the following possibilities of $\nu(H)$:*
(a) *one slope case:*

$$\nu(H) = \left\{ \left(\frac{1}{2}\right)^{8d} \right\}.$$

(b) *two slopes case:*

$$\nu(H) = \left\{ \left(\frac{a}{2d}\right)^{4d}, \left(\frac{2d-a}{2d}\right)^{4d} \right\}, \quad 0 \le a < d,\ a \in \mathbb{Z}.$$

(c) *three slopes case:*

$$\nu(H) = \left\{ \left(\frac{a}{d}\right)^{2d}, \left(\frac{1}{2}\right)^{4d} \left(\frac{d-a}{d}\right)^{2d} \right\}, \quad 0 \le a < \frac{d}{2},\ a \in \mathbb{Z}.$$

(d) *four slopes case:*

$$\nu(H) = \left\{ \left(\frac{a}{d}\right)^{2d}, \left(\frac{b}{d}\right)^{2d}, \left(\frac{d-b}{d}\right)^{2d}, \left(\frac{d-a}{d}\right)^{2d} \right\},$$

where a and b are any integers with $0 \le a < b < d/2$.

(2) *If* $\mathbf{B}$ *is the division algebra, then we have the following possibilities of $\nu(H)$:*
(a) *one slope case:*

$$\nu(H) = \left\{ \left(\frac{1}{2}\right)^{8d} \right\}.$$

(b) *two slopes case:*

$$\nu(H) = \left\{ \left(\frac{a}{4d}\right)^{4d}, \left(\frac{4d-a}{4d}\right)^{4d} \right\}, \quad 0 \le a < 2d,\ a \in \mathbb{Z},\ a \equiv 0 \pmod 2.$$

(c) *three slopes case:*

$$\nu(H) = \left\{ \left(\frac{a}{2d}\right)^{2d}, \left(\frac{1}{2}\right)^{4d} \left(\frac{2d-a}{2d}\right)^{2d} \right\},$$

where a is any integer with $0 \le a < d$ and $a \equiv 1 \pmod 2$.

(d) *four slopes case:*

$$\nu(H)=\left\{\left(\frac{a}{2d}\right)^{2d},\left(\frac{b}{2d}\right)^{2d},\left(\frac{2d-b}{2d}\right)^{2d},\left(\frac{2d-a}{2d}\right)^{2d}\right\},$$

where a and b are any integers with $0\le a<b<d$ and $a,b\equiv 1 \pmod 2$.

8. Proof of Theorem 7.2

In this section we construct a superspecial separably quasi-polarized Dieudonné $\mathcal{O}_{\mathbf{B}}$-module of rank $4dm$ that satisfies the determinant condition for any $m\in\mathbb{Z}_{\ge 1}$. It suffices to construct such a Dieudonné module M of rank $4d$ and we take $M^{\oplus m}$.

8.1. The matrix algebra case. Suppose $\mathbf{B}$ is the matrix algebra. Using the Morita equivalence, one reduces to construct Dieudonné $\mathcal{O}$-modules in question. We shall construct a superspecial separably anti-quasi-polarized Dieudonné $\mathcal{O}$-module $(M,(\,,))$ of rank $2d$. Then such a M is a free $\mathcal{O}\otimes W=\prod_{i\in\mathbb{Z}/f\mathbb{Z}}W^i$-module of rank 2 and $(M^{i_1},M^{i_2})=0$ if $i_1\neq i_2$ in $\mathbb{Z}/f\mathbb{Z}$. We shall also impose the condition that M/VM is a free $\mathcal{O}\otimes_{\mathbb{Z}_p}k=\prod_i k[\pi]/(\pi^e)$-module. Let

$$M=\oplus_{i\in\mathbb{Z}/f\mathbb{Z}}M^i$$

where each M^i is a free rank two W^i-module generated by elements X_i and Y_i. We define the symmetric pairing $(\,,):M^i\times M^i\to W$, for each $i\in\mathbb{Z}/f\mathbb{Z}$, with

$$(X_i,\pi^{e-1}Y_i)=1\quad\text{and } (X_i,\pi^bY_i)=0,\quad \forall\, 0\le b\le e-2. \tag{8.1}$$

Note that the symmetric pairing $(\,,)$ with the property (5.2) is uniquely determined by its values (X_i,π^bY_i) for $0\le b\le e-1$. As $(\pi^aX_i,\pi^bY_i)\equiv 0 \pmod p$ for all $a+b>e-1$, the pairing $(\,,)$ is a perfect one. We define the Frobenius map by

$$FX_i=Y_{i+1},\quad FY_i=pX_{i+1},\quad \forall\, i\in\mathbb{Z}/f\mathbb{Z}. \tag{8.2}$$

This defines a superspecial Dieudonné $\mathcal{O}$-module M. If $f=2r$ is even, one has

$$F^{2r}X_0=p^rX_0,\quad F^{2r}Y_0=p^rY_0.$$

If $f=2r+1$ is odd, one has

$$F^{2r+1}X_0=p^rY_0,\quad F^{2r+1}Y_0=p^{r+1}X_0.$$

One easily checks the compatibility

$$p\,(\pi^aX_i,\pi^bY_i)^\sigma=(F\pi^aX_i,F\pi^bY_i)=p\,(\pi^aY_{i+1},\pi^bX_{i+1})=p\,(\pi^aX_{i+1},\pi^bY_{i+1}).$$

for all $0\le a,b\le e-1$, and in particular for $a=0$ and $0\le b\le e-1$, cf. (8.1). This gives a superspecial separably quasi-polarized Dieudonné $\mathcal{O}$-module of rank $2d$ such that M/VM is a free $\mathcal{O}\otimes k$-module. We have completed the construction for the case where $\mathbf{B}$ is the matrix algebra.

8.2. The division algebra case. Suppose $\mathbf{B}$ is the division algebra. We want to construct a superspecial separably quasi-polarized Dieudonné $\mathcal{O}_{\mathbf{B}}$-module of rank $4d$ that satisfies the determinant condition. Let

$$M = \bigoplus_{j \in \mathbb{Z}/2f\mathbb{Z}} M^j,$$

where each M^j is a free rank two W^i-module generated by elements X_j and Y_j, where $i = j \mod f$. We need to construct

(i) a Frobenius map F on M such that $F^2 M \subset pM$ and that $\dim_k (M/VM)^j = e$ for all $j \in \mathbb{Z}/2f\mathbb{Z}$,

(ii) a map Π on M which shifts the degree by f (i.e. $\Pi : M^j \to M^{j+f}$) such that $\Pi^2 = -\pi$, and

(iii) a perfect W^i-bilinear pairing

$$\langle\, ,\rangle_{\mathbf{F}} : M^j \times M^{j+f} \to W^i, \quad \forall\, j \in \mathbb{Z}/2f\mathbb{Z},$$

such that

(a) $\langle X, Y\rangle_{\mathbf{F}} = -\langle Y, X\rangle_{\mathbf{F}}$,

(b) $\langle FX, FY\rangle_{\mathbf{F}} = p\,\langle X, Y\rangle_{\mathbf{F}}^{\sigma}$, and

(c) $\langle \Pi X, \Pi Y\rangle_{\mathbf{F}} = \pi \langle X, Y\rangle_{\mathbf{F}}$

for all $X \in M^j$, $Y \in M^{j+f}$ and $j \in \mathbb{Z}/2f\mathbb{Z}$.

Choose a generator δ of $\mathcal{D}^{-1}_{\mathcal{O}/\mathbb{Z}_p}$ and one has $\mathcal{D}^{-1}_{\mathcal{O}/\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} W = \mathcal{O} \otimes_{\mathbb{Z}_p} W \cdot \delta$. The pairing $\langle\, ,\rangle_{\mathbf{F}}$ above defines a self-dual alternating $\mathcal{O}$-bilinear pairing

$$\langle\, ,\rangle_{\mathbf{F}} : M \times M \to \mathcal{O} \otimes_{\mathbb{Z}_p} W.$$

Put

$$\langle x, y\rangle := \mathrm{Tr}_{\mathbf{F}/\mathbb{Q}_p}(\delta \langle x, y\rangle_{\mathbf{F}}) : M \times M \to W. \tag{8.3}$$

Then one would obtain a desired Dieudonné $\mathcal{O}_{\mathbf{B}}$-module.

Write $F_j : M^j \to M^{j+1}$ for the restriction of the Frobenius map F on M^j, and $\Pi_j : M^j \to M^{j+f}$ for that of Π on M^j. As the maps F and Π commute, one has

$$\Pi_{j+1} F_j = F_{j+f} \Pi_j : M^j \to M^{j+f+1}. \tag{8.4}$$

Therefore, if the Frobenius map F has been chosen, then the map Π on M is uniquely determined by the map Π_0; of course the maps F and Π_0 should be chosen so that each map Π_j defined by the recursive formula (8.4) sends the lattice M^j into M^{j+f}. Observe that if $\Pi_f \circ \Pi_0 = -\pi$, then $\Pi_{j+f} \circ \Pi_j = -\pi$ for all j, as one easily checks

$$\Pi_{j+f} \circ \Pi_j = F^j\, \Pi_f\, F^{-j}\, F^j \Pi_0\, F^{-j} = -\pi.$$

Suppose the maps $F^f : M^0 \to M^f$ and $F^f : M^f \to M^0$ have the representative matrices in the form

$$A_0 = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}, \quad A_f = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \tag{8.5}$$

with respect to the bases $\{X_0, Y_0\}$ and $\{X_f, Y_f\}$, where a and b are non-zero elements in W^0. Set

$$\Pi_0 = \begin{pmatrix} 1 & 0 \\ 0 & -\pi \end{pmatrix}. \tag{8.6}$$

Then one computes

$$\Pi_f = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -\pi \end{pmatrix} \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}^{-1} = \begin{pmatrix} -\pi & 0 \\ 0 & 1 \end{pmatrix},$$

and gets $\Pi_f \circ \Pi_0 = -\pi$.

Suppose the maps $F^f : M^0 \to M^f$ and $F^f : M^f \to M^0$ have the representative matrices in the form

$$A_0 = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad A_f = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \tag{8.7}$$

where a is a non-zero element in W^0. Set

$$\Pi_0 = \begin{pmatrix} 0 & -\pi \\ 1 & 0 \end{pmatrix}. \tag{8.8}$$

Then one computes $\Pi_f = \begin{pmatrix} 0 & -\pi \\ 1 & 0 \end{pmatrix}$ and gets $\Pi_f \circ \Pi_0 = -\pi$.

Let $c := [e/2]$. Suppose the Frobenius map F on M has the following representative matrix (with respect to the bases $\{X_j, Y_j\}$ of M^j):

$$F_j = \begin{pmatrix} 0 & a_j \\ b_j & 0 \end{pmatrix}, \quad \forall\, j \in \mathbb{Z}/2f\mathbb{Z} \tag{8.9}$$

for some elements a_j, b_j with $\mathrm{ord}(b_j) = c$ and $\mathrm{ord}(a_j) = e - c$. Then F satisfies the property (i).

Consider the case where f is even. Put

$$F_j = \begin{pmatrix} 0 & -p\pi^{-c} \\ \pi^c & 0 \end{pmatrix}, \quad \forall\, j \in \mathbb{Z}/2f\mathbb{Z}. \tag{8.10}$$

One computes that the matrices A_0 and A_f have the form as in (8.7). Set

$$\Pi_0 = \begin{pmatrix} 0 & -\pi \\ 1 & 0 \end{pmatrix}$$

and we have

$$\Pi_j = \begin{cases} \begin{pmatrix} 0 & -\pi \\ 1 & 0 \end{pmatrix}, & \text{if } j \text{ is even,} \\ \begin{pmatrix} 0 & -p\pi^{-2c} \\ \pi^{2c+1}p^{-1} & 0 \end{pmatrix}, & \text{if } j \text{ is odd.} \end{cases} \tag{8.11}$$

Clearly, the properties (i) and (ii) for M are satisfied.

For each $j \in \mathbb{Z}/2f\mathbb{Z}$, define a W^i-bilinear pairing

$$\langle\, ,\rangle_{\mathbf{F}} : M^j \times M^{j+f} \to W^i$$

by

$$\langle X_j, X_{j+f}\rangle_{\mathbf{F}} = \langle Y_j, Y_{j+f}\rangle_{\mathbf{F}} = 0, \quad \textit{and} \quad \langle X_j, Y_{j+f}\rangle_{\mathbf{F}} = 1.$$

Then one has $\langle Y_j, X_{j+f}\rangle_{\mathbf{F}} = -1$ for all $j \in \mathbb{Z}/2f\mathbb{Z}$. We check the condition (b):

$$p = \langle FX_j, FY_{j+f}\rangle_{\mathbf{F}} = \langle \pi^c Y_{j+1}, -p\pi^{-c}X_{j+1+f}\rangle_{\mathbf{F}} = p.$$

We check the condition (c) by (8.11)

$$\langle \Pi X_j, \Pi Y_{j+f}\rangle_{\mathbf{F}} = \begin{cases} \langle Y_{j+f}, -\pi X_j\rangle_{\mathbf{F}} = \pi, & \text{if } j \text{ is even,} \\ \langle \pi^{2c+1}p^{-1}Y_{j+f}, -p\pi^{-2c}X_j\rangle_{\mathbf{F}} = \pi, & \text{if } j \text{ is odd.} \end{cases} \tag{8.12}$$

This finishes the case where f is even.

Now consider the case where $f=2r+1$ is odd.

Suppose first that $e=2c$ is even. Let F be such that F_j is the matrix (8.10). One computes that the matrices A_0 and A_f have the form as in (8.5). Set Π_0 to be the matrix (8.6). We have

$$\Pi_j = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & -\pi \end{pmatrix}, & \text{if } j \text{ is even,} \\ \begin{pmatrix} -\pi & 0 \\ 0 & 1 \end{pmatrix}, & \text{if } j \text{ is odd.} \end{cases} \tag{8.13}$$

Clearly, the properties (i) and (ii) for M hold.

For each $j\in \mathbb{Z}/2f\mathbb{Z}$, define a W^i-bilinear pairing

$$\langle\, ,\rangle_{\mathbf{F}} : M^j \times M^{j+f} \to W^i$$

by

$$\langle X_j, Y_{j+f}\rangle_{\mathbf{F}} = \langle Y_j, X_{j+f}\rangle_{\mathbf{F}} = 0$$

and

$$\begin{cases} \langle X_j, X_{j+f}\rangle_{\mathbf{F}} = (-1)^j, \\ \langle Y_j, Y_{j+f}\rangle_{\mathbf{F}} = (-1)^{j+1} p\pi^{-2c}. \end{cases}$$

We check the compatibility:

$$\begin{aligned} \langle X_{j+f}, X_j\rangle_{\mathbf{F}} &= -\langle X_j, X_{j+f}\rangle_{\mathbf{F}} = (-1)^{j+f}, \\ \langle Y_{j+f}, Y_j\rangle_{\mathbf{F}} &= (-1)\langle Y_j, Y_{j+f}\rangle_{\mathbf{F}} = (-1)^{j+1+f} p\pi^{-2c}. \end{aligned}$$

We check the condition (b):

$$\begin{aligned} p(-1)^j &= \langle FX_j, FX_{j+f}\rangle_{\mathbf{F}} = \langle \pi^c Y_{j+1}, \pi^c Y_{j+1+f}\rangle_{\mathbf{F}} = (-1)^j p. \\ p(-1)^{j+1} p\pi^{-2c} &= \langle FY_j, FY_{j+f}\rangle_{\mathbf{F}} \\ &= \langle -p\pi^{-c} X_{j+1}, -p\pi^{-c} X_{j+1+f}\rangle_{\mathbf{F}} = (-1)^{j+1} p^2 \pi^{-2c}. \end{aligned}$$

It is easy to check the condition (c) by (8.13). This finishes the case where $e=2c$ is even.

Suppose $e=2c+1$ is odd. Define the maps F and Π as follows:

$$F_j = \begin{pmatrix} 0 & -\pi^{c+1} \\ \pi^c & 0 \end{pmatrix}, \quad \Pi_j = \begin{pmatrix} 0 & -\pi \\ 1 & 0 \end{pmatrix} \quad \forall\, j \in \mathbb{Z}/2f\mathbb{Z}.$$

It is easy to see that the maps F and Π commute and that the properties (i) and (ii) for M hold.

We choose elements u_j and v_j in ${W^i}^\times$, where $j\in\mathbb{Z}/2f\mathbb{Z}$, such that

$$u_{j+f} = -v_j, \quad pu_j^\sigma = -\pi^{2c+1} v_{j+1} \quad \text{and} \quad pv_j^\sigma = -\pi^{2c+1} u_{j+1}. \tag{8.14}$$

for all $j\in \mathbb{Z}/2f/Z$. This can be done by an analogue of Hensel's lemma and we leave the details to the reader.

For each $j\in \mathbb{Z}/2f\mathbb{Z}$, define a W^i-bilinear pairing

$$\langle\, ,\rangle_{\mathbf{F}} : M^j \times M^{j+f} \to W^i$$

by

$$\langle X_j, X_{j+f}\rangle_{\mathbf{F}} = \langle Y_j, Y_{j+f}\rangle_{\mathbf{F}} = 0$$

and

$$\langle X_j, Y_{j+f}\rangle_{\mathbf{F}} = u_j, \quad \langle Y_j, X_{j+f}\rangle_{\mathbf{F}} = v_j.$$

We check the condition (b):

$$pu_j^\sigma = \langle FX_j, FY_{j+f}\rangle_{\mathbf{F}} = \langle \pi^c Y_{j+1}, -\pi^{c+1}X_{j+1+f}\rangle_{\mathbf{F}} = -\pi^{2c+1}v_{j+1},$$

$$pv_j^\sigma = \langle FY_j, FX_{j+f}\rangle_{\mathbf{F}} = \langle -\pi^{c+1}X_{j+1}, \pi^{c+}Y_{j+1+f}\rangle_{\mathbf{F}} = -\pi^{2c+1}u_{j+1}.$$

It is easy to check the condition (c). This finishes the case where $e = 2c+1$ is odd.

This way we construct a separable $\mathcal{O}_{\mathbf{B}}$-linear quasi-polarization on M. This completes the construction of a superspecial separably quasi-polarized Dieudonné $\mathcal{O}_{\mathbf{B}}$-module of rank $4d$ in the case of the division algebra.

The proof of Theorem 7.2 is complete.

Remark 8.1. (1) When $\mathbf{B}$ is a division algebra and ef is odd, the double construction as in Lemma 7.1 provides an alternative way to produce a separably quasi-polarized superspecial Dieudonné $\mathcal{O}_{\mathbf{B}}$-module M. However, we checked that such a Dieudonné module M rarely satisfies the determinant condition.

(2) We refer to [37] for a classification of superspecial quasi-polarized Dieudonné $O_F \otimes \mathbb{Z}_p$-modules of HB type.

9. Isogeny classes of quasi-polarized p-divisible $\mathcal{O}_{\mathbf{B}}$-modules

Consider rational quasi-polarized Dieudonné $\mathbf{B}$-modules N with $B(k)$-rank $= 4dm$, or quasi-polarized $\mathbf{B}$-linear F-isocrystals. When all slopes of N are between 0 and 1, there is a Dieudonné $\mathcal{O}_B$-lattice M in N, so $N = M \otimes_{W(k)} B(k)$ for some quasi-polarized Dieudonné $\mathcal{O}_{\mathbf{B}}$-module. We like to classify the isomorphism classes of these rational Dieudonné $\mathbf{B}$-modules with a fixed slope sequence ν. This gives the classification of isogeny classes of quasi-polarized p-divisible $\mathcal{O}_{\mathbf{B}}$-modules of height $4dm$.

Let ν be a symmetric slope sequence as in Theorem 7.3. Let $I(\nu)$ denote the set of isogeny classes of quasi-polarized p-divisible $\mathcal{O}_{\mathbf{B}}$-modules (H, λ, ι) of height $4dm$ such that $\nu(H) = \nu$. Rapoport and Richartz [28] have obtained a description for $I(\nu)$ in terms of a Galois cohomology set $H^1(\mathbb{Q}_p, J)$ for a certain reductive group J over $\mathbb{Q}_p$ when the structure group is connected. The description in terms of Galois cohomology set helps us to understand the classification problem. For the present case one still needs to work a bit more as the structure group is not connected, though a similar description is expected.

We shall work along with Dieudonné modules and translate the classification problem into the theory of (skew-)Hermitian forms over local fields; see Theorem 9.2.

Write $\nu = \nu_n + \nu_s$ into the non-supersingular and supersingular parts.

Lemma 9.1. *We have $I(\nu) = I(\nu_n) \times I(\nu_s)$ and $I(\nu_n)$ consists of one isogeny classes.*

Proof. Suppose N_1 and N_2 are rational quasi-polarized Dieudonné $\mathbf{B}$-modules with $\nu(N_1) = \nu(N_2) = \nu$. Write

$$N_1 = N_1^{ns} \oplus N_1^{ss}, \quad N_2 = N_2^{ns} \oplus N_2^{ss}$$

into the non-supersingular component and supersingular component, respectively. Clearly, $N_1 \simeq N_2$ if and only if $N_1^{ns} \simeq N_2^{ns}$ and $N_1^{ss} \simeq N_2^{ss}$. This shows the first part.

For the second part, we decompose the rational Dieudonné modules

$$N_1^{ns} = \oplus_{\beta<1/2}(N_{1,\beta} \oplus N_{1,\beta}^t), \quad N_2^{ns} = \oplus_{\beta<1/2}(N_{2,\beta} \oplus N_{2,\beta}^t)$$

into isotypic components. By Lemma 6.1, we have an isomorphism $N_{1,\beta} \simeq N_{2,\beta}$ as rational Dieudonné $\mathbf{B}$-modules for each $\beta < 1/2$. It follows that $N_1^{ns} \simeq N_2^{ns}$ as quasi-polarized Dieudonné $\mathbf{B}$-modules. ∎

By Lemma 9.1, one reduces to classify the isomorphism classes of supersingular rational quasi-polarized Dieudonné $\mathbf{B}$-modules of $B(k)$-rank $4dm_s$, where $4dm_s$ is the length of the supersingular part ν_s.

Let N be a supersingular rational quasi-polarized Dieudonné $\mathbf{B}$-module of $B(k)$-rank $4dm_s$. Put

$$\widetilde{N} := \{x \in N | F^2x = px\}.$$

This is a $B(\mathbb{F}_{p^2})$-vector space of dimension $4dm_s$ such that

- $W(k) \otimes_{B(\mathbb{F}_{p^2})} \widetilde{N} = N$,
- $F = V$ on $\widetilde{N}$, and
- the action of $\mathbf{B}$ leaves $\widetilde{N}$ invariant.

Let $\mathbf{D}$ be the quaternion division algebra over $\mathbb{Q}_p$. We can write $\mathbf{D} = B(\mathbb{F}_{p^2})[F]$ with relations $F^2 = p$ and $Fa = \sigma(a)F$ for all $a \in B(\mathbb{F}_{p^2})$. Then $\widetilde{N}$ naturally becomes a left $\mathbf{D}$-module of $\mathbb{Q}_p$-rank $8dm_s$. Define the involution $*_{\mathbf{D}}$ on $\mathbf{D}$ by $(a + bF)^{*_{\mathbf{D}}} := \sigma(a) + bF$. This is an orthogonal involution as the fixed subspace is 3-dimensional. As the actions of $\mathbf{B}$ and $\mathbf{D}$ commute, $\widetilde{N}$ becomes a left $\mathbf{B} \otimes_{\mathbb{Q}_p} \mathbf{D}$-module. Write

$$\mathbf{B} \otimes_{\mathbb{Q}_p} \mathbf{D} = \mathbf{B} \otimes_{\mathbf{F}} (\mathbf{F} \otimes_{\mathbb{Q}_p} \mathbf{D}) \simeq \mathrm{Mat}_2(\mathbf{B}'),$$

where $\mathbf{B}'$ is a quaternion algebra over $\mathbf{F}$. One can easily determine whether $\mathbf{B}'$ splits or not by the following

$$\mathrm{inv}(\mathbf{B}') = 1/2[\mathbf{F} : \mathbb{Q}_p] - \mathrm{inv}(\mathbf{B}).$$

The alternating pairing

$$\langle\, ,\rangle : \widetilde{N} \times \widetilde{N} \to B(\mathbb{F}_{p^2})$$

has values in $B(\mathbb{F}_{p^2})$ satisfying $\langle Fx, y\rangle = \langle x, Vy\rangle^\sigma$. Define

$$\psi(x, y) := \mathrm{Tr}_{B(\mathbb{F}_{p^2})/\mathbb{Q}_p}\langle x, Fy\rangle.$$

One has the following properties: For all $x, y \in \widetilde{N}$, $a \in \mathbf{D}$, and $b \in \mathbf{B}$ one has

(i) $\psi(y, x) = -\psi(x, y)$,
(ii) $\psi(ax, y) = \psi(x, a^{*_{\mathbf{D}}}y)$, and
(iii) $\psi(bx, y) = \psi(x, b^*y)$.

That is, $\widetilde{N}$ is a $\mathbb{Q}_p$-valued skew-Hermitian $\mathbf{B} \otimes_{\mathbb{Q}_p} \mathbf{D}$-module with respect to the product involution $* \otimes *_{\mathbf{D}}$. We check (i)–(iii). For (i), one has

$$\psi(y, x) = \mathrm{Tr}\langle y, Fx\rangle = \mathrm{Tr}\langle Fy, x\rangle^\sigma = -\mathrm{Tr}\langle x, Fy\rangle = -\psi(x, y).$$

For (ii), one has for $a \in B(\mathbb{F}_{p^2})$

$$\psi(ax, y) = \mathrm{Tr}\langle ax, Fy\rangle = \mathrm{Tr}\langle x, Fa^\sigma y\rangle = \psi(x, a^\sigma y),$$

$$\psi(Fx,y) = \mathrm{Tr}\langle Fx, Fy\rangle = \mathrm{Tr}\, p\langle x,y\rangle^\sigma = \mathrm{Tr}\langle x, F^2 y\rangle = \psi(x, Fy).$$

For (iii), one has

$$\psi(bx,y) = \mathrm{Tr}\langle bx, Fy\rangle = \mathrm{Tr}\langle x, Fb^* y\rangle = \psi(x, b^* y).$$

Note that if we replace $\widetilde{N}$ by $\widetilde{N}' := \{x \in N | F^2x + px = 0\}$, then $F = -V$ on $\widetilde{N}'$ and the pairing $\psi'(x,y) := \mathrm{Tr}\langle x, Fy\rangle$ becomes Hermitian instead of skew-Hermitian. Moreover, the adjoint involution $*'_{\mathbf{D}}$ on $\mathbf{D}$ is the canonical involution.

Since the canonical involution $*$ is symplectic and $*_{\mathbf{D}}$ is orthogonal, the product involution $*\otimes *_{\mathbf{D}}$ is symplectic. Therefore, we can choose an $\mathbf{F}$-algebra isomorphism $\mathbf{B}\otimes_{\mathbb{Q}_p}\mathbf{D} \simeq \mathrm{Mat}_2(\mathbf{B}')$ so that the induced involution is the map $(b_{ij}) \mapsto (b_{ji}^{*'})$, where $*'$ is the canonical involution on $\mathbf{B}'$.

Let e_{11} and e_{22} be the standard idempotent of $\mathrm{Mat}_2(\mathbf{B}')$ and let $\widetilde{N} = \widetilde{N}_1 \oplus \widetilde{N}_2$ be the corresponding decomposition. We have proven

Theorem 9.2. *The association $(N, \langle\, ,\rangle) \mapsto (\widetilde{N}_1, \psi)$ gives rise to a bijection between the set $I(\nu_s)$ and the set of isomorphism classes of $\mathbb{Q}_p$-valued skew-Hermitian free $\mathbf{B}'$-modules of $\mathbf{B}'$-rank m_s, where $\mathbf{B}'$ is the quaternion algebra (unique up to isomorphism) over $\mathbf{F}$ with $\mathrm{inv}(\mathbf{B}') = 1/2[\mathbf{F} : \mathbb{Q}_p] - \mathrm{inv}(\mathbf{B})$.*

Corollary 9.3.

(1) *If $\mathbf{B}'$ is the matrix algebra, then there is a natural bijection between the set $I(\nu_s)$ and the set of isomorphism of non-degenerate symmetric space over $\mathbf{F}$ of dimension $2m_s$.*

(2) *If $\mathbf{B}'$ is the quaternion division algebra, then there is a natural bijection between the set $I(\nu_s)$ and the set of isomorphism classes of non-degenerate skew-Hermitian $\mathbf{B}'$-modules of $\mathbf{B}'$-rank m_s.*

PROOF. For the matrix algebra case, we do the Morita equivalence again as before. The corollary follows from Theorem 9.2. ∎

In the following we use the theory of quadratic forms and the skew-Hermitian quaternionic forms over local fields; see O'Meara [23, Chapter IV] and Tsukamoto [30].

Consider non-degenerate symmetric spaces V of dimension n_0 over a non-Archimedean local field k_0 of characteristic different from 2. Recall the discriminant $\delta V \in k_0^\times/k_0^{\times 2}$ of V is defined by

$$\delta V := (-1)^{[n_0/2]} \det V.$$

Note that we have $\delta V = [1]$ when V is the hyperbolic plane. Let $S\,V \in \{\pm 1\}$ denote the Hasse symbol of V (see [23, p. 167]). Denote by $Q(n_0)$ the set of isomorphism classes of non-degenerate symmetric spaces V of dimension n_0 over k_0.

Theorem 9.4. *Notations as above.*

(1) *For any $n_0 \geq 1$, the map $(\delta, S) : Q(n_0) \to k_0^\times/k_0^{\times 2} \times \{\pm 1\}$ is injective. This map is also surjective for any $n_0 \geq 3$.*

(2) *For $n_0 = 1$, the map $\delta : Q(1) \simeq k_0^\times/k_0^{\times 2}$ is a bijection.*

(3) *For $n_0 = 2$, the image of the map (δ, S) is*

$$\{([a], \pm 1); [a] \neq [1]\} \cup \left\{([1], \left(\frac{-1,-1}{k_0}\right))\right\}.$$

PROOF. See Theorems 63:20, 63:22 and 63:23 of [23, p. 170-171].

Corollary 9.5. *If k_0 is non-dyadic, then one has*

$$|Q(1)| = 4, \quad |Q(2)| = 7, \quad \textit{and} \quad |Q(n_0)| = 8, \quad \forall\, n_0 \geq 3.$$

Let B_0 be the quaternion division algebra over k_0 together with the canonical involution $*$. Denote by $SQ(n_0)$ the set of isomorphism classes of skew-Hermitian B_0-modules (V, ψ) of rank n_0 for $n_0 \geq 1$. The discriminant $\delta V \in k_0^\times / k_0^{\times 2}$ is defined by

$$\delta V := (-1)^{[n_0/2]} \operatorname{Nr}\left(\psi(e_i, e_j)\right),$$

where $\{e_i\}$ is a basis for V over B_0 and $\operatorname{Nr} : \operatorname{Mat}_{n_0}(B_0) \to k_0$ is the reduced norm.

Theorem 9.6.

(1) *For $n_0 \geq 2$, the map $\delta : SQ(n_0) \to k_0^\times / k_0^{\times 2}$ is a bijection.*
(2) *For $n_0 = 1$, the map $\delta : SQ(n_0) \to k_0^\times / k_0^{\times 2}$ is injective and its image is equal to $\{[a]; [a] \neq [1]\}$.*

Proof. This is Theorem 3 in [30].

Corollary 9.7. *If k_0 is non-dyadic, then one has*

$$|SQ(1)| = 3, \quad \textit{and} \quad |SQ(n_0)| = 4, \quad \forall\, n_0 \geq 2.$$

Theorem 9.8. *Let notations be as above.*

(1) *If $\mathbf{B}'$ is the matrix algebra, then we have*

$$|I(\nu_s)| = \begin{cases} 7 & \textit{if } m_s = 1, \\ 8 & \textit{if } m_s \geq 2. \end{cases} \tag{9.1}$$

(2) *If $\mathbf{B}'$ is the quaternion division algebra, then we have*

$$|I(\nu_s)| = \begin{cases} 3 & \textit{if } m_s = 1, \\ 4 & \textit{if } m_s \geq 2. \end{cases} \tag{9.2}$$

Proof. These follow from Corollaries 9.3, 9.5 and 9.7. ∎

Combining Lemma 9.1, Theorem 9.2, Corollary 9.3 and Theorem 9.8, we obtain an explicit classification of isogeny classes of quasi-polarized p-divisible $\mathcal{O}_{\mathbf{B}}$-modules.

10. Integral model $\mathcal{M}_K^{(p)}$

For the remaining of this paper we restrict ourselves to the minimal case $m = 1$. We shall use the notations in Section 2 and in Subsection 4.1.

10.1. Local models. Let Λ be a free $O_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$-module of rank one together with a perfect $\mathbb{Z}_p$-valued skew-Hermitian pairing

$$\psi : \Lambda \times \Lambda \to \mathbb{Z}_p.$$

For such a lattice Λ, we define, following Rapoport and Zink [29], a projective $\mathbb{Z}_p$-scheme $\mathbf{M}_\Lambda$, called the *local model associated to* Λ (and ψ), which represents the following functor. For any $\mathbb{Z}_p$-scheme S, $\mathbf{M}_\Lambda(S)$ is the set of locally free $\mathcal{O}_S$-submodules $\mathcal{F} \subset \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S$ of rank $[B : \mathbb{Q}]/2 = 2d$ such that

(i) $\mathcal{F}$ is isotropic with respect to the pairing ψ;
(ii) locally for Zariski topology on S, $\mathcal{F}$ is a direct summand of $\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S$;
(iii) $\mathcal{F}$ is invariant under the O_B-action;

(iv) $\mathcal{F}$ satisfies the determinant condition (cf. Subsection 2.3):

$$(K) \quad \mathrm{char}\,(a|\Lambda\otimes\mathcal{O}_S/\mathcal{F}) = \mathrm{char}\,(a)\in\mathcal{O}_S[T], \quad \forall\, a\in O_B.$$

Recall that for an abelian scheme A over a base scheme S, we have the Hodge filtration

$$0\to\omega_{A/S}\to H^1_{\mathrm{DR}}(A/S)\to\mathrm{Lie}(A^t/S)\to 0.$$

Taking the dual one obtains the short exact sequence

$$0\to\omega_{A^t/S}\to H_1^{\mathrm{DR}}(A/S)\to\mathrm{Lie}(A/S)\to 0.$$

If M is the covariant Dieudonné module of an abelian variety A over a perfect field k_0, then there is a canonical isomorphism $M/pM\simeq H_1^{\mathrm{DR}}(A/k_0)$ with the Hodge filtration VM/pM corresponding to ω_{A^t}. This justifies the definition of the determinant condition for objects in the local model $\mathbf{M}_\Lambda$ in (iv).

By an automorphism of the lattice $\Lambda\otimes\mathcal{O}_S$, where S is a $\mathbb{Z}_p$-scheme, we mean an $O_B\otimes\mathbb{Z}_p$-linear automorphism of the $\mathcal{O}_S$-module $\Lambda\otimes\mathcal{O}_S$ that preserves the pairing ψ. We denote by $\mathrm{Aut}_{O_B\otimes\mathcal{O}_S}(\Lambda\otimes\mathcal{O}_S,\psi)$ the group of automorphisms of $\Lambda\otimes\mathcal{O}_S$.

Let $\mathcal{G}=\mathrm{Aut}_{O_B\otimes\mathbb{Z}_p}(\Lambda,\psi)$ be the group scheme over $\mathbb{Z}_p$ that represents the group functor

$$S\mapsto\mathrm{Aut}_{O_B\otimes\mathcal{O}_S}(\Lambda\otimes\mathcal{O}_S,\psi).$$

We know that $\mathcal{G}$ is an affine smooth group scheme over $\mathbb{Z}_p$ whose generic fiber $\mathcal{G}_{\mathbb{Q}_p}$ is a $\mathbb{Q}_p$-form of $(\mathrm{Res}_{F/\mathbb{Q}}\,O_{2,F})\otimes_{\mathbb{Q}}\mathbb{Q}_p$; see Section 3. The group scheme $\mathcal{G}$ acts naturally on $\mathbf{M}_\Lambda$ on the left.

10.2. Local model diagrams. Let S be a $\mathbb{Z}_p$-scheme and $\underline{A}=(A,\lambda,\iota)$ be an object in $\mathcal{M}_K^{(p)}(S)$. A *trivialization* γ of the de Rham homology $H_1^{\mathrm{DR}}(A/S)$ by $\Lambda\otimes_{\mathbb{Z}_p}\mathcal{O}_S$ is an $O_B\otimes\mathbb{Z}_p$-linear isomorphism $\gamma: H_1^{\mathrm{DR}}(A/S)\to\Lambda\otimes_{\mathbb{Z}_p}\mathcal{O}_S$ of $\mathcal{O}_S$-modules such that $\psi(\gamma(x),\gamma(y))=\langle x,y\rangle_\lambda$ for $x,y\in H_1^{\mathrm{DR}}(A/S)$, where

$$\langle\,,\rangle_\lambda: H_1^{\mathrm{DR}}(A/S)\times H_1^{\mathrm{DR}}(A/S)\to\mathcal{O}_S$$

is the perfect alternating pairing induced by λ.

Let $\widetilde{\mathcal{M}}=\widetilde{\mathcal{M}_K^{(p)}}$ denote the moduli space over $\mathbb{Z}_p$ that parametrizes equivalence classes of objects $(\underline{A},\gamma)_S$, where

- $\underline{A}=(A,\lambda,\iota)$ is an object over a $\mathbb{Z}_p$-scheme S in $\mathcal{M}_K^{(p)}\otimes\mathbb{Z}_p$, and
- γ is a trivialization of $H_1^{\mathrm{DR}}(A/S)$ by $\Lambda\otimes\mathcal{O}_S$.

The moduli scheme $\widetilde{\mathcal{M}}$ has two natural projections φ^{mod} and φ^{loc}. The morphism

$$\varphi^{\mathrm{mod}}:\widetilde{\mathcal{M}}\to\mathcal{M}_K^{(p)}\otimes\mathbb{Z}_p$$

forgets the trivialization. The morphism

$$\varphi^{\mathrm{loc}}:\widetilde{\mathcal{M}}\to\mathbf{M}_\Lambda$$

sends any object $(\underline{A},\gamma)$ to $\gamma(\omega_{A^t/S})$, where $\omega_{A^t/S}\subset H_1^{\mathrm{DR}}(A/S)$ is the $\mathcal{O}_S$-submodule in the Hodge filtration. Thus, we have the so called *local model diagram*:

$$\mathcal{M}_K^{(p)}\otimes\mathbb{Z}_p \xleftarrow{\varphi^{\mathrm{mod}}} \widetilde{\mathcal{M}} \xrightarrow{\varphi^{\mathrm{loc}}} \mathbf{M}_\Lambda. \tag{10.1}$$

The local model diagram above was introduced by Rapoport and Zink [29] in a more general setting. The moduli scheme $\widetilde{\mathcal{M}}$ also admits a left action by the group scheme $\mathcal{G}$. Recall that k denotes an algebraically closed field of characteristic

$p > 0$ and $W = W(k)$ the ring of Witt vectors over k. Using Corollary 5.7, for any k-valued point $\underline{A}$ in $\mathcal{M}_K^{(p)}$, there is an $O_B \otimes \mathbb{Z}_p$-linear isomorphism of W-modules

$$M(A) \simeq \Lambda \otimes W$$

which is compatible with the alternating pairings. This shows that the morphism φ^{mod} is surjective. It follows that φ^{mod} is a left $\mathcal{G}$-torsor, and hence this morphism is affine and smooth.

By the Grothendieck-Messing deformation theory of abelian schemes (see [6] and [20]), for any k-valued point x of $\mathcal{M}_K^{(p)}$, there is a k-valued point y in $\mathbf{M}_\Lambda$ such that there is a (non-canonical) isomorphism

$$\alpha : \mathcal{M}_K^{(p)}|_x^\wedge \simeq \mathbf{M}_\Lambda|_y^\wedge \tag{10.2}$$

of formal local moduli spaces. This shows particularly that if the local model $\mathbf{M}_\Lambda$ is flat over $\operatorname{Spec} \mathbb{Z}_p$, then the integral model $\mathcal{M}_K^{(p)}$ is flat over $\operatorname{Spec} \mathbb{Z}_p$.

The morphism φ^{loc} is smooth, $\mathcal{G}$-equivariant, and of relative dimension same as φ^{mod}. However, at this moment we do not know whether the morphism φ^{loc} is surjective yet. If this is so, then the integral model $\mathcal{M}_K^{(p)}$ is flat over $\operatorname{Spec} \mathbb{Z}_p$ if and only if the local model $\mathbf{M}_\Lambda$ is flat over $\operatorname{Spec} \mathbb{Z}_p$.

We shall show that the local model $\mathbf{M}_\Lambda$ is finite and flat in the next section (Theorem 11.8). Then we get the main result of this section.

Theorem 10.1. *The moduli scheme $\mathcal{M}_K^{(p)} \to \operatorname{Spec} \mathbb{Z}_{(p)}$ is flat and every connected component is projective and of relative dimension zero.*

11. Computation of local models

11.1. A reduction step. Let Λ and $\mathbf{M}_\Lambda$ be those as in the previous section. Let

$$\Lambda = \oplus_{v|p} \Lambda_v$$

be the decomposition of Λ obtained from the decomposition $O_F \otimes \mathbb{Z}_p = \prod_{v|p} \mathcal{O}_v$, where $\mathcal{O}_v$ is the ring of integers in the local field F_v of F at v. Then we have

$$\mathbf{M}_\Lambda = \prod_{v|p} \mathbf{M}_{\Lambda_v},$$

where the product Π means the fiber product of the schemes $\mathbf{M}_{\Lambda_v}$'s over $\operatorname{Spec} \mathbb{Z}_p$ and $\mathbf{M}_{\Lambda_v}$ is the local model defined by the lattice Λ_v in the same way as $\mathbf{M}_\Lambda$; see Subsection 10.1.

Write $O_B \otimes_{\mathbb{Z}} \mathbb{Z}_p = \prod_{v|p} \mathcal{O}_{B_v}$ for the decomposition with respect to $O_F \otimes \mathbb{Z}_p = \prod_{v|p} \mathcal{O}_v$. Then $\mathcal{O}_{B_v}$ is a maximal order in B_v. Similarly we have the automorphism group scheme $\mathcal{G}_v = \operatorname{Aut}_{\mathcal{O}_{B_v}}(\Lambda_v, \psi_v)$ associated to the local lattice (Λ_v, ψ_v), and have the fiber product decomposition

$$\mathcal{G} = \prod_{v|p} \mathcal{G}_v.$$

Now we fix a place v of F over p. Let $\mathcal{O}_v^{\text{nr}} \subset \mathcal{O}_v$ be the maximal etale extension of $\mathbb{Z}_p$ in $\mathcal{O}_v$ and put

$$I_v := \operatorname{Hom}_{\mathbb{Z}_p}(\mathcal{O}_v^{\text{nr}}, W).$$

Let $e = e_v$ be the ramification index and $f = f_v$ be the inertia degree. Let π be a uniformizer of $\mathcal{O}_v$ and let $P(T)$ be the minimal polynomial of π over $\mathcal{O}_v^{\text{nr}}$. For any

$\sigma \in I_v$ put $W_\sigma := W[T]/(\sigma(P(T)))$ and denote by π again the image of T in W_σ. One has $W_\sigma = W[\pi]$ and the element π satisfies the equation $\sigma(P(T)) = 0$. We have the decomposition

$$\Lambda_v \otimes_{\mathbb{Z}_p} W = \oplus_{\sigma\in I_v}\Lambda_\sigma, \quad \Lambda_\sigma := \Lambda_v \otimes_{\mathcal{O}_v^{\rm nr},\sigma} W.$$

Write

$$\psi_\sigma : \Lambda_\sigma \times \Lambda_\sigma \to W$$

for the induced alternating pairing.

Similarly we define the local model $\mathbf{M}_{\Lambda_\sigma}$ over $\operatorname{Spec} W$ attached to each skew-Hermitian lattice $(\Lambda_\sigma, \psi_\sigma)$. If $\mathcal{F}_v \subset \Lambda_v \otimes \mathcal{O}_S$ is an object in $\mathbf{M}_{\Lambda_v}$ and let $\mathcal{F}_v = \oplus_{\sigma\in I_v}\mathcal{F}_\sigma$ be the natural decomposition, then every factor $\mathcal{F}_\sigma$ is a locally free $\mathcal{O}_S$-module of rank $2e$; this follows from the determinant condition (K). Therefore we have a natural isomorphism

$$f : \mathbf{M}_{\Lambda_v} \otimes W \simeq \prod_{\sigma\in I_v} \mathbf{M}_{\Lambda_\sigma}, \quad \mathcal{F}_v \mapsto (\mathcal{F}_\sigma)_{\sigma\in I_v},$$

where the product Π means the fiber product of the schemes $\mathbf{M}_{\Lambda_\sigma}$'s over $\operatorname{Spec} W$.

We shall compute the special fiber $\mathbf{M}_{\Lambda_\sigma} \otimes k$ of $\mathbf{M}_{\Lambda_\sigma}$. Put $\mathcal{O}_{B_\sigma} := \mathcal{O}_{B_v} \otimes_{\mathcal{O}_v^{\rm nr},\sigma} W$ for $\sigma \in I_v$.

11.2. Unramified case. Suppose v is unramified in B. Then $\mathcal{O}_{B_\sigma} = \operatorname{Mat}_2(W_\sigma)$. By the Morita equivalence reduction as before, we have $\Lambda_\sigma = \Lambda_{\sigma,1} \oplus \Lambda_{\sigma,2}$ and a unimodular Hermitian pairing

$$\varphi_\sigma : \Lambda_{\sigma,1} \times \Lambda_{\sigma,1} \to W.$$

Recall that $\varphi_\sigma(x, y)$ is the restriction of the symmetric pairing $\psi(x, Cy)$ on the first factor $\Lambda_{\sigma,1}$, where C is the Weyl element. The local model $\mathbf{M}_{\varphi_\sigma}$ associated to the symmetric lattice $(\Lambda_{\sigma,1}, \psi_\sigma)$ is defined to parametrize the $W_\sigma \otimes \mathcal{O}_S$-submodules $\mathcal{F}$ of $\Lambda_{\sigma,1} \otimes \mathcal{O}_S$ with the following properties:

(i) $\mathcal{F}$ is a locally free $\mathcal{O}_S$-module of rank e and locally for Zariski topology on S is a direct summand of $\Lambda_{\sigma,1} \otimes \mathcal{O}_S$;

(ii) $\mathcal{F}$ is isotropic with respect to the pairing φ_σ.

Any $\mathcal{F}_\sigma$ is an object in $\mathbf{M}_{\Lambda_\sigma}$ has the decomposition $\mathcal{F}_\sigma = \mathcal{F}_{\sigma,1} \oplus \mathcal{F}_{\sigma,2}$.

Lemma 11.1. *The map which sends any object $\mathcal{F}_\sigma$ in $\mathbf{M}_{\Lambda_\sigma}$ to its first factor $\mathcal{F}_{\sigma,1}$ induces an isomorphism of schemes*

$$\mathbf{M}_{\Lambda_\sigma} \simeq \mathbf{M}_{\varphi_\sigma}.$$

Proof. It suffices to check that $\mathcal{F}_\sigma$ is isotropic with respect to the pairing ψ_σ if and only if $\mathcal{F}_{\sigma,1}$ is isotropic with respect to the pairing $\varphi_\sigma(x, y) = \psi_\sigma(x, Cy)$. Using $e_{11}^* = e_{22}$, we get $\psi_\sigma(\mathcal{F}_\sigma, \mathcal{F}_\sigma) = 0$ if and only if $\psi_\sigma(\mathcal{F}_{\sigma,1}, \mathcal{F}_{\sigma,2}) = 0$. On the other hand, the isomorphism $C : \Lambda_\sigma \xrightarrow{\sim} \Lambda_\sigma$ induces the isomorphism $C : \Lambda_{\sigma,1} \xrightarrow{\sim} \Lambda_{\sigma,2}$. Therefore, $\varphi_\sigma(\mathcal{F}_{\sigma,1}, \mathcal{F}_{\sigma,1}) = 0$ if and only if $\psi_\sigma(\mathcal{F}_{\sigma,1}, \mathcal{F}_{\sigma,2}) = 0$. This shows the lemma. ∎

Put $\overline{\Lambda}_\sigma := \Lambda_\sigma/p\Lambda_\sigma$ and $\overline{\Lambda}_{\sigma,1} := \Lambda_{\sigma,1}/p\Lambda_{\sigma,1}$. Let $\mathcal{D}^{-1}_{W_\sigma/W}$ be the inverse difference of the extension W_σ/W and choose a generator δ_σ of this fractional ideal. Then there is a unique W_σ-valued W_σ-bilinear symmetric pairing

$$\varphi'_\sigma : \Lambda_{\sigma,1} \times \Lambda_{\sigma,1} \to W_\sigma$$

such that $\varphi_\sigma(x,y) = \mathrm{Tr}\,[\delta_\sigma \cdot \varphi'_\sigma(x,y)]$. One can show that a $k[\pi/(\pi^e)$-submodule $\mathcal{F}_{\sigma,1} \subset \overline{\Lambda}_{\sigma,1}$ is isotropic with respect to the pairing φ_σ if and only if so it is for the pairing φ'_σ.

Since $\Lambda_{\sigma,1}$ is a self-dual lattice and k is algebraically closed, we can choose a W_σ-basis x_1, x_2 for $\Lambda_{\sigma,1}$ such that

$$\varphi'_\sigma(x_1,x_1) = \varphi'_\sigma(x_2,x_2) = 0 \quad \text{and} \quad \varphi'_\sigma(x_1,x_2) = \varphi'_\sigma(x_2,x_1) = 1.$$

Denote by $\bar{x}_i$, for $i = 1,2$, the image of x_i in $\overline{\Lambda}_{\sigma,1}$. Let $\mathcal{F} \subset \overline{\Lambda}_{\sigma,1}$ be an object in $\mathbf{M}_{\varphi_\sigma}(k)$. As $\overline{\Lambda}_{\sigma,1}$ is a free $k[\pi]/(\pi^e)$-module of rank two, one has

$$\overline{\Lambda}_{\sigma,1}/\mathcal{F} \simeq k[\pi]/(\pi^{e_1}) \oplus k[\pi]/(\pi^{e_2})$$

for some integers e_1, e_2 with $0 \le e_1 \le e_2 \le e$ and $e_1 + e_2 = e$. The pair (e_1, e_2) will be called the *Lie type* of the object $\mathcal{F}$. We can write

$$\mathcal{F} = \mathrm{Span}\{\pi^{e_1}\bar{y}_1, \pi^{e_2}\bar{y}_2\},$$

where $\bar{y}_1$ and $\bar{y}_2$ generate $\overline{\Lambda}_{\sigma,1}$ over $k[\pi]/(\pi^e)$. Moreover, we can write either

(a) $\bar{y}_1 = \bar{x}_1 + t\bar{x}_2$ and $\bar{y}_2 = \bar{x}_2$, or
(b) $\bar{y}_1 = t\bar{x}_1 + \bar{x}_2$ and $\bar{y}_2 = \bar{x}_1$,

where $t \in k[\pi]/(\pi^e)$. We can represent t as

$$t = t_0 + t_1\pi + \cdots + t_{e-2e_1-1}\pi^{e-2e_1-1}, \quad t_i \in k$$

because if $\mathrm{ord}_\pi(t) \ge e - 2e_1$ then one can replace $\bar{x}_1 + t\bar{x}_2$ by $\bar{x}_1$ in the case (a) (and the same for the case (b)). Now one easily computes that

$$\varphi'_\sigma(\mathcal{F},\mathcal{F}) = 0 \iff 2t\pi^{2e_1} = 0.$$

This condition gives $t_0\pi^{2e_1} + \cdots + t_{e-2e_1-1}\pi^{e-1} = 0$ and hence

$$t_0 = \cdots = t_{e-2e_1-1} = 0.$$

Therefore, we get two objects.

$$\mathcal{F} = \mathrm{Span}\{\pi^{e_1}\bar{x}_1, \pi^{e_2}\bar{x}_2\}, \quad \text{or} \quad \mathcal{F} = \mathrm{Span}\{\pi^{e_1}\bar{x}_2, \pi^{e_2}\bar{x}_1\}.$$

Notice that these two members are in the same orbit under the action of the group $\mathcal{G}_\sigma(k)$ as the automorphism of $\overline{\Lambda}_{\sigma,1}$ switching $\bar{x}_1$ and $\bar{x}_2$ lies in $\mathcal{G}_\sigma(k)$, where

$$\mathcal{G}_\sigma = \mathrm{Aut}_{W_\sigma}(\Lambda_{\sigma,1}, \varphi_\sigma)$$

is the automorphism group scheme of the symmetric lattice $(\Lambda_{\sigma,1}, \varphi_\sigma)$ over W. We obtain the following the result.

Proposition 11.2. *Assume that v is unramified in B and let $\sigma \in I_v$. Then $\mathbf{M}_{\varphi_\sigma}(k)$ consists of the $k[\pi]/(\pi^e)$-submodules*

$$\mathcal{F}_{e_1} = \mathrm{Span}\{\pi^{e_1}\bar{x}_1, \pi^{e-e_1}\bar{x}_2\}, \quad \text{for } 0 \le e_1 \le e.$$

Moreover, two objects $\mathcal{F}_{e_1}$ and $\mathcal{F}_{e'_1}$ are in the same orbit under the action of $\mathcal{G}_\sigma(k)$ if and only if $e_1 = e'_1$ or $e_1 + e'_1 = e$.

Proposition 11.3. *Assume that v is unramified in B, and let $\sigma \in I_v$.*

(1) *The special fiber $\mathbf{M}_{\varphi_\sigma} \otimes_W k$ is zero-dimensional and two objects $\mathcal{F}$ and $\mathcal{F}'$ in $\mathbf{M}_{\varphi_\sigma}(k)$ are in the same orbit under the $\mathcal{G}_\sigma(k)$ if and only if they have the same Lie type.*
(2) *The structure morphism $\mathbf{M}_{\varphi_\sigma} \to \mathrm{Spec}\, W$ is finite and flat.*

PROOF. (1) This follows immediately from Proposition 11.2.

(2) Since the morphism f is quasi-finite and projective, f is finite. We now show that any object $\mathcal{F}_0$ in $\mathbf{M}_{\varphi_\sigma}(k)$ can be lifted to an object $\mathcal{F}_R$ over an integral domain R with residue field k and fraction field K of characteristic zero. Then the coordinate ring of $\mathbf{M}_{\varphi_\sigma}$ is torsion-free as a W-module and hence is flat over W.

By Proposition 11.2, write $\mathcal{F}_0 = \mathrm{Span}\{\pi^{e_1}\bar{x}_1, \pi^{e_2}\bar{x}_2\}$ for two integers e_1, e_2 with $0 \le e_1, e_2 \le e$ and $e_1 + e_2 = e$. Write $W_\sigma = W[T]/(\sigma P(T))$. Let R be the ring of integers in a finite separable field extension K of $B(k) = \mathrm{Frac}(W)$ such that the polynomial $\sigma P(T)$ decomposes completely over R:

$$\sigma P(T) = (T - \pi_1) \cdots (T - \pi_e) \in R[T].$$

Let π_R be a uniformizer of R. We have $W_\sigma \otimes_W R = R[T]/(\sigma P(T))$. As W_σ is a free W-module, we have an exact sequence:

$$0 \longrightarrow W_\sigma \otimes_W (\pi_R) \longrightarrow W_\sigma \otimes_W R \longrightarrow W_\sigma \otimes_W k \longrightarrow 0.$$

So an element $f(T)$ in $R[T]/(\sigma P(T))$ specializes to zero in $W_\sigma \otimes k = k[T]/(T^e)$ if and only if $f(T) \in \pi_R \cdot R[T]/(\sigma P(T))$. We shall construct a $W_\sigma \otimes_W R$-submodule $\mathcal{F}_R \subset \Lambda_{\sigma,1} \otimes_W R$ such that

(i) $\mathcal{F}_R \otimes_R k = \mathcal{F}_0$;
(ii) $\mathcal{F}_R$ and $(\Lambda_{\sigma,1} \otimes_W R)/\mathcal{F}_R$ are both free of rank e over R;
(iii) $\mathcal{F}_R$ is isotropic with respect to the pairing ψ'_σ.

Now we let $\mathcal{F}_R$ be the submodule generated by the elements $(T-\pi_1)\cdots(T-\pi_{e_1})x_1$ and $(T - \pi_{e_1+1}) \cdots (T - \pi_e)x_2$. Clearly $\pi_i \in \pi_R R$ for all i so one has (i). The statement (ii) follows from (i) by the right exactness of the tensor product. To check (iii), as $\mathcal{F}_R \subset \mathcal{F}_K := \mathcal{F}_R \otimes K$, it suffices to check (iii) for $\mathcal{F}_K$. Now we have

$$W_\sigma \otimes_W K = \prod_{i=1}^{e} K \quad \text{and} \quad \mathcal{F}_K = (\mathcal{F}_{K,i})_{1 \le i \le e}.$$

It is easy to see that each component $\mathcal{F}_{K,i}$ is one-dimensional K-subspace generated by either x_1 or x_2 and hence $\mathcal{F}_K$ satisfies the condition (iii). ∎

Let $\mathcal{F}_v$ be an object in $\mathbf{M}_{\Lambda_v}(k)$ and let $\mathcal{F}_v = \oplus_{\sigma \in I_v} \mathcal{F}_\sigma$ be the natural decomposition. The *reduced Lie type of* $\mathcal{F}_v$ is defined to the system of pairs $(e_{\sigma,1}, e_{\sigma,2})$ indexed by I_v, where $(e_{\sigma,1}, e_{\sigma,2})$ is the Lie type of $\mathcal{F}_{\sigma,1}$. Proposition 11.3 immediately gives the following result.

Theorem 11.4. *Suppose that v is unramified in B.*

1. *The special fiber $\mathbf{M}_{\Lambda_v} \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ is zero-dimensional and two objects $\mathcal{F}_v$ and $\mathcal{F}'_v$ in $\mathbf{M}_{\Lambda_v}(k)$ are in the same orbit under the $\mathcal{G}_v(k)$ if and only if they have the same reduced Lie type.*
2. *The structure morphism $\mathbf{M}_{\Lambda_v} \to \mathrm{Spec}\, \mathbb{Z}_p$ is flat and finite.*

11.3. Ramified case. Now we compute the local model $\mathbf{M}_{\Lambda_v}$ for the case where v is ramified in B. Recall that Λ_v is a free $\mathcal{O}_{B_v}$-module of rank one together with a perfect $\mathbb{Z}_p$-valued skew-Hermitian pairing $\psi_v : \Lambda_v \times \Lambda_v \to \mathbb{Z}_p$. We fix a unramified quadratic field extension $L_v \subset B_v$ as in Subsection 4.1. Notice that the ring $\mathcal{O}_{L_v}$ of integers is contained in the unique maximal order $\mathcal{O}_{B_v}$. We choose a presentation $\mathcal{O}_{B_v} = \mathcal{O}_{L_v}[\Pi]$ as in (4.1) and (4.2)

Let $\mathcal{O}_{L_v^{\rm nr}}$ denote the maximal etale extension over $\mathbb{Z}_p$ in L_v, and put $J_v := \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{O}_{L_v^{\rm nr}}, W)$. Let $\mathrm{pr} : J_v \to I_v$ be the restriction map from $\mathcal{O}_{L_v^{\rm nr}}$ to $\mathcal{O}_v^{\rm nr}$; this is a two-to-one map. We have the decomposition

$$\Lambda_v \otimes_{\mathbb{Z}_p} W = \oplus_{\sigma\in I_v} \Lambda_\sigma, \quad \Lambda_\sigma = \Lambda_\tau \oplus \Lambda_{\tau'}$$

where $\{\tau, \tau'\} = \mathrm{pr}^{-1}(\sigma)$ and Λ_τ (resp. $\Lambda_{\tau'}$) is the τ-component (resp. τ'-component) of Λ_v. Notice that the pairing ψ_σ induces a perfect pairing

$$\psi_\sigma : \Lambda_\tau \times \Lambda_{\tau'} \to W.$$

Let

$$\psi'_\sigma : \Lambda_\tau \times \Lambda_{\tau'} \to W_\sigma$$

be the unique W_σ-valued W_σ-bilinear pairing such that $\psi_\sigma(x, y) = \mathrm{Tr}[\delta_\sigma \cdot \psi'_v(x, y)]$.

The local model $\mathbf{M}_{\Lambda_\sigma}$ over $\mathrm{Spec}\, W$ parametrizes the $W_\sigma \otimes_W \mathcal{O}_S$-submodules

$$\mathcal{F}_v = \mathcal{F}_\tau \oplus \mathcal{F}_{\tau'} \subset (\Lambda_\tau \oplus \Lambda_{\tau'}) \otimes \mathcal{O}_S$$

such that

(i) $\mathcal{F}_\tau$ and $\mathcal{F}_{\tau'}$ are locally free $\mathcal{O}_S$-modules of rank e and they are locally direct summands of $\Lambda_\tau \otimes \mathcal{O}_S$ and $\Lambda_{\tau'} \otimes \mathcal{O}_S$, respectively;
(ii) $\Pi(\mathcal{F}_\tau) \subset \mathcal{F}_{\tau'}$ and $\Pi(\mathcal{F}_{\tau'}) \subset \mathcal{F}_\tau$;
(iii) $\psi_\sigma(\mathcal{F}_\tau, \mathcal{F}_{\tau'}) = 0$.

As $\mathcal{F}_\tau$ and $\mathcal{F}_{\tau'}$ are of rank e, the condition (iii) says that one is the orthogonal complement of the other and hence one submodule determines the other.

We check that $\psi_\sigma(\mathcal{F}_\tau, \mathcal{F}_{\tau'}) = 0$ if and only if $\psi'_\sigma(\mathcal{F}_\tau, \mathcal{F}_{\tau'}) = 0$. As $\mathcal{D}^{-1} = \mathcal{D}^{-1}_{W_\sigma/W}$ is the largest W_σ-submodule in $W_\sigma[1/p]$ such that $\mathrm{tr}(\mathcal{D}^{-1}) \subset W$, therefore, $\mathrm{tr}(\pi^{-1}\mathcal{D}^{-1}) = p^{-1}W$. So $\mathrm{tr}(\pi^{e-1}\mathcal{D}^{-1}) = W$. Consider the structure map $\phi : W \to \mathcal{O}_S$. If $\ker \phi = 0$, then $\psi'_\sigma(\mathcal{F}_\tau, \mathcal{F}_{\tau'}) \neq 0$ implies $\psi_\sigma(\mathcal{F}_\tau, \mathcal{F}_{\tau'}) \neq 0$. Suppose $\ker \phi = p^r W$. If $\psi'_\sigma(\mathcal{F}_\tau, \mathcal{F}_{\tau'}) \neq 0$, then

$$\delta_\sigma \psi'_\sigma(\mathcal{F}_\tau, \mathcal{F}_{\tau'}) \supset p^{r-1}\pi^{e-1}\mathcal{D}^{-1} \otimes_W \mathcal{O}_S$$

Taking the trace one gets

$$\psi_\sigma(\mathcal{F}_\tau, \mathcal{F}_{\tau'}) \supset p^{r-1} W \otimes_W \mathcal{O}_S \neq 0.$$

This verifies the assertion.

By Lemma 4.2 we can choose a W_σ-basis x_1, x_2 for Λ_τ and a W_σ-basis x'_1, x'_2 for $\Lambda_{\tau'}$ such that

$$\psi'_\sigma(x_i, x'_j) = \delta_{i,j}, \quad \text{for} \quad 1 \le i, j \le 2 \tag{11.1}$$

and

$$\Pi(x_1) = x'_1, \quad \Pi(x_2) = -\pi x'_2, \quad \Pi(x'_1) = -\pi x_1, \quad \Pi(x'_2) = x_2. \tag{11.2}$$

Put $\overline{\Lambda}_\tau := \Lambda_\tau/p\Lambda_\tau$ and $\overline{\Lambda}_{\tau'} := \Lambda_{\tau'}/p\Lambda_{\tau'}$ Write $\bar{x}_i$ or $\bar{x}'_i$ for the image of x_i or x'_i in $\overline{\Lambda}_\tau$ or $\overline{\Lambda}_{\tau'}$, respectively. Let $\mathcal{F}_\sigma = \mathcal{F}_\tau \oplus \mathcal{F}_{\tau'}$ be an object in $\mathbf{M}_{\Lambda_\sigma}(k)$. One has

$$\overline{\Lambda}_\tau/\mathcal{F}_\tau \simeq k[\pi]/(\pi^{e_1}) \oplus k[\pi]/(\pi^{e_2})$$

as $k[\pi]/(\pi^e)$-modules for some integers e_1, e_2 with $0 \le e_1 \le e_2 \le e$ and $e_1 + e_2 = e$; the pair (e_1, e_2) is called the *Lie type* of $\mathcal{F}_\tau$. It is easy to see that $\mathcal{F}_{\tau'}$ has the same Lie type as $\mathcal{F}_\tau$. The *reduced Lie type* of $\mathcal{F}_\sigma$ is defined to be the Lie type of $\mathcal{F}_\tau$. We call a reduced Lie type (e_1, e_2) of an object $\mathcal{F}_\sigma$ *minimal* if $e_2 - e_1 \in \{0, 1\}$.

Similar to the unramified case, we can write

$$\mathcal{F}_\tau = \mathrm{Span}\{\pi^{e_1}\bar{y}_1, \pi^{e_2}\bar{y}_2\},$$

where $\bar{y}_1$ and $\bar{y}_2$ are in one of the following cases

(a) $\bar{y}_1 = \bar{x}_1 + t\bar{x}_2$ and $\bar{y}_2 = \bar{x}_2$, or
(b) $\bar{y}_1 = t\bar{x}_1 + \bar{x}_2$ and $\bar{y}_2 = \bar{x}_1$,

where $t \in k[\pi]/(\pi^e)$.

In the case (a), we compute

$$\mathcal{F}_{\tau'} = \mathrm{Span}\{\pi^{e_1}(t\bar{x}'_1 - \bar{x}'_2), \pi^{e_2}\bar{x}'_1\}.$$

As $\mathcal{F}_\tau$ and $\mathcal{F}_{\tau'}$ are orthogonal to each other, the condition (ii) is equivalent to

$$\psi'_\sigma(\mathcal{F}_\tau, \Pi\mathcal{F}_\tau) = \psi'_\sigma(\Pi\mathcal{F}_{\tau'}, \mathcal{F}_{\tau'}) = 0. \tag{11.3}$$

This yields the equation

$$\pi^{2e_1}(1 - t\pi^2) = 0. \tag{11.4}$$

If $e = 2c+1$ is odd, then there is no solution for the equation (11.4). If $e = 2c$ is even, then the only one solution is $(e_1, e_2) = (c, c)$ and $t = 0$. That is,

$$\mathcal{F}_\tau = \pi^c\overline{\Lambda}_\tau \quad \text{and} \quad \mathcal{F}_{\tau'} = \pi^c\overline{\Lambda}_{\tau'}. \tag{11.5}$$

In the case (b), we compute

$$\mathcal{F}_{\tau'} = \mathrm{Span}\{\pi^{e_1}(\bar{x}'_1 - t\bar{x}'_2), \pi^{e_2}\bar{x}'_2\}.$$

The condition (11.3) yields the following equation

$$\pi^{2e_1}(t^2 - \pi) = 0. \tag{11.6}$$

If $e = 2c$ is even, then we have only one solution $(e_1, e_2) = (c, c)$ and $t = 0$ and get the object $\mathcal{F}_\sigma$ as in (11.5). If $e = 2c+1$ is odd, then we have $(e_1, e_2) = (c, c+1)$ and $t = 0$. That is,

$$\mathcal{F}_\tau = \mathrm{Span}\{\pi^c\bar{x}_2, \pi^{c+1}\bar{x}_1\} \quad \text{and} \quad \mathcal{F}_{\tau'} = \mathrm{Span}\{\pi^c\bar{x}'_1, \pi^{c+1}\bar{x}'_2\}. \tag{11.7}$$

Proposition 11.5. *Notations as above and assume that v is ramified in B.*

(1) *If $e = 2c$ is even, then $\mathbf{M}_{\Lambda_\sigma}(k)$ consists of the single $k[\pi]/(\pi^e)$-submodule $\mathcal{F}_\sigma = \mathcal{F}_\tau \oplus \mathcal{F}_{\tau'}$ with*

$$\mathcal{F}_\tau = \pi^c\overline{\Lambda}_\tau \quad \textit{and} \quad \mathcal{F}_{\tau'} = \pi^c\overline{\Lambda}_{\tau'}.$$

(2) *If $e = 2c+1$ is odd, then $\mathbf{M}_{\Lambda_\sigma}(k)$ consists of the single $k[\pi]/(\pi^e)$-submodule $\mathcal{F}_\sigma = \mathcal{F}_\tau \oplus \mathcal{F}_{\tau'}$ with*

$$\mathcal{F}_\tau = \mathrm{Span}\{\pi^c\bar{x}_2, \pi^{c+1}\bar{x}_1\} \quad \textit{and} \quad \mathcal{F}_{\tau'} = \mathrm{Span}\{\pi^c\bar{x}'_1, \pi^{c+1}\bar{x}'_2\},$$

where the bases $\{x_i\}$ and $\{x'_i\}$ are chosen as in (11.1) and (11.2).

In particular, only the minimal reduced Lie type can occur in the space $\mathbf{M}_{\Lambda_\sigma}(k)$.

Proposition 11.6. *Assume that v is ramified in B, and let $\sigma \in I_v$. The structure morphism $f : \mathbf{M}_{\Lambda_\sigma} \to \mathrm{Spec}\, W$ is finite and flat.*

PROOF. As f is projective and quasi-finite (Proposition 11.5), the morphism f is finite. Let $B(k)^{\mathrm{alg}}$ be an algebraic closure of the fraction field $B(k) = \mathrm{Frac}(W)$. Since $\mathbf{M}_{\Lambda_\sigma}(k)$ consists of only one element, the specialization map

$$\mathrm{sp} : \mathbf{M}_{\Lambda_\sigma}(B(k)^{\mathrm{alg}}) \to \mathbf{M}_{\Lambda_\sigma}(k)$$

is surjective. Therefore, any (the unique) object in $\mathbf{M}_{\Lambda_\sigma}(k)$ can be lifted to characteristic zero. This shows that the coordinate ring of $\mathbf{M}_{\Lambda_\sigma}$ is torsion free and hence f is flat. ∎

Theorem 11.7. *Suppose v is ramified in B. The structure morphism*

$$f : \mathbf{M}_{\Lambda_v} \to \mathrm{Spec}\, \mathbb{Z}_p$$

is finite and flat.

PROOF. This follows from Proposition 11.6 immediately. ∎

11.4. Flatness of $\mathbf{M}_\Lambda$.

Theorem 11.8. *Let Λ be a free unimodular skew-Hermitian $O_B \otimes \mathbb{Z}_p$-module of rank one and let $\mathbf{M}_\Lambda$ be the associated local model. The structure morphism $f : \mathbf{M}_\Lambda \to \mathrm{Spec}\, \mathbb{Z}_p$ is finite and flat.*

PROOF. This follows from Theorems 11.4 and 11.7. ∎

12. More constructions of Dieudonné modules

In this section we handle two technical problems raised from the results of previous sections.

12.1. Dieudonné $\mathcal{O}_{\mathbf{B}}$-modules with given Lie type. In Section 10 we studied the moduli scheme $\mathcal{M}_K^{(p)}$ through the local models. A basic problem is whether or not the morphism φ^{loc} in the local diagram is surjective on their geometric points. The local model diagram gives rise to a morphism of Artin stacks

$$\theta : \mathcal{M}_K^{(p)} \otimes \mathbb{Z}_p \to [\mathcal{G} \backslash \mathbf{M}_\Lambda], \tag{12.1}$$

and this amounts to ask the surjectivity of the map of the sets of geometric points

$$\theta_k : \mathcal{M}_K^{(p)}(k) \to \mathcal{G}(k) \backslash \mathbf{M}_\Lambda(k). \tag{12.2}$$

Let $\mathrm{Dieu}^{O_B \otimes \mathbb{Z}_p}(k)$ (resp. $\mathrm{Dieu}^{\mathcal{O}_{B_v}}(k)$) denote the set of isomorphism classes of separably quasi-polarized Dieudonné $O_B \otimes \mathbb{Z}_p$-modules (resp. Dieudonné $\mathcal{O}_{B_v}$-modules) of rank $4d$ (resp. of rank $4d_v$) satisfying the determinant condition. The map θ_k factors through the natural map $\mathcal{M}_K^{(p)}(k) \to \mathrm{Dieu}^{O_B \otimes \mathbb{Z}_p}(k)$ and let

$$\alpha : \mathrm{Dieu}^{O_B \otimes \mathbb{Z}_p}(k) \to \mathcal{G}(k) \backslash \mathbf{M}_\Lambda(k) \tag{12.3}$$

be the induced map.

Let $M = \oplus_{v|p} M_v$ be a Dieudonné $O_F \otimes \mathbb{Z}_p$-module of rank $4d$ such that $\mathrm{rank}_W M_v = 4d_v = 4[F_v : \mathbb{Q}_p]$. The *Lie type* of M, denoted by $\underline{e}(M)$, is defined to be a sequence of 4-tuples of non-negative integers indexed by $I := \coprod_{v|p} I_v$

$$\underline{e}(M) := \{\underline{e}_i ; i \in I\}, \quad \underline{e}_i := (e_{i,1}, e_{i,2}, e_{i,3}, e_{i,4}) \tag{12.4}$$

where $e_{i,1} \le e_{i,2} \le e_{i,3} \le e_{i,4}$ are the integers such that

$$(M/VM)^i \simeq k[\pi]/(\pi^{e_{i,1}}) \oplus k[\pi]/(\pi^{e_{i,2}}) \oplus k[\pi]/(\pi^{e_{i,3}}) \oplus k[\pi]/(\pi^{e_{i,4}}),$$

where $(M/VM)^i$ denotes the i-component of the tangent space M/VM. Put $\underline{e}(M_v) := \{\underline{e}_i; i \in I_v\}$ and one has $\underline{e}(M) = (\underline{e}(M_v))_{v|p}$. When $M_v \in \mathrm{Dieu}^{\mathcal{O}_{B_v}}(k)$, there are unique two integers $0 \le e_{i,1} \le e_{i,2} \le e_v$ with $e_{i,1} + e_{i,2} = e_v$ for all $i \in I_v$ such that

$$\underline{e}(M_v) = \{(e_{i,1}, e_{i,1}, e_{i,2}, e_{i,2}); i \in I_v\}.$$

In this case we define the *reduced Lie type of* M_v and that of M, respectively, by

$$\underline{e}^r(M_v) := \{(e_{i,1}, e_{i,2}); i \in I_v\} \quad \text{and} \quad \underline{e}^r(M) := (\underline{e}^r(M_v))_{v|p}. \tag{12.5}$$

The following result gives a partial answer to the above basic problem.

Proposition 12.1. *The map α in (12.3) is surjective.*

PROOF. It suffices to show the surjectivity of the map

$$\alpha_v : \mathrm{Dieu}^{O_{B_v}}(k) \to \mathcal{G}_v(k)\backslash \mathbf{M}_{\Lambda_v}(k) \tag{12.6}$$

for each place $v|p$. The target orbit space in (12.6) is classified by the reduced Lie types of the objects (Theorem 11.4 and Proposition 11.5). When v is unramified in B, this is is a sequence of pairs $(e_{i,1}, e_{i,2})$ of integers indexed by $i \in I_v$ with $0 \le e_{i,1} \le e_{i,2} \le e_v$ and $e_{i,1} + e_{i,2} = e_v$. When v is ramified in B, this is a sequence of pairs $(c, e_v - c)$ indexed by I_v, where $c := [e_v/2]$.

In the ramified case, the construction in Section 8 produces a separably quasi-polarized Dieudonné $\mathcal{O}_{B_v}$-module M with the determinant condition whose Lie type is the minimal one, that is, $(M/VM)^j \simeq k[\pi]/(\pi^c) \oplus k[\pi]/(\pi^{e_v - c})$ for all $j \in \mathbb{Z}/2f_v\mathbb{Z}$. So one has the surjectivity of α_v.

It remains to treat the unramified case. We need to write down a separably anti-quasi-polarized Dieudonné $\mathcal{O}_v$-module M_1 of rank $2d_v$ such that the Lie type $\underline{e}(M_1)$ of M_1 is equal to the given one $\{(e_{i,1}, e_{i,2}); i \in I_v\}$. Fix an identification $I_v \simeq \mathbb{Z}/f_v\mathbb{Z}$. Let $M_1 = \oplus_{i \in \mathbb{Z}/f\mathbb{Z}} M_1^i$, where each M_1^i is a free rank two W^i-module generated by two elements X_i and Y_i. For each $i \in \mathbb{Z}/f\mathbb{Z}$, define a symmetric pairing $(\ ,\ ) : M_1^i \times M_1^i \to W$ by (8.1). Define the Verschiebung map $V : M_1^{i+1} \to M_1^i$ by

$$VX_{i+1} = \pi^{e_{i,1}} X_i, \quad VY_{i+1} = p\pi^{e_v - e_{i,2}} Y_i. \tag{12.7}$$

It is easy to show that $(VX, VY) = p(X,Y)^{\sigma^{-1}}$ for $X, Y \in M_1$ and that the Lie type $\underline{e}(M_1)$ of M_1 is equal to $\{(e_{i,1}, e_{i,2}); i \in I_v\}$. Therefore, one has the surjectivity of α_v. ∎

Remark 12.2. The Dieudonné module M_1 constructed in the proof of Proposition 12.1 has the slope sequence

$$\nu(M_1) = \left\{ \left(\frac{\sum_i e_{i,1}}{d_v} \right)^{d_v}, \left(\frac{\sum_i e_{i,2}}{d_v} \right)^{d_v} \right\}. \tag{12.8}$$

This exhausts all possible slope sequences that can occur in Corollary 7.7 in the case where v is unramified in B.

12.2. Slope sequences of Dieudonné $\mathcal{O}_{\mathbf{B}}$-modules: a refinement. Our goal is to determine all possible slope sequences that can be realized by Dieudonné $O_B \otimes \mathbb{Z}_p$-modules that admit both a separable quasi-polarization and the determinant condition (still in the minimal case $m=1$). This problem is local and one only needs to consider those of Dieudonné $\mathcal{O}_{B_v}$-modules for each place v over p. To simplify the notations as we did in Sections 4 and 9, we write $\mathbf{B}$, $\mathbf{F}$, etc. for B_v, F_v etc. and drop the subscript v from our notations.

Theorem 7.3 determines exactly all possible slope sequences for separably polarized Dieudonné $\mathcal{O}_{\mathbf{B}}$-modules of rank $4dm$, in particular for our current focus case of rank $4d$ (Corollary 7.7). Recall that d, e, f denote the degree, ramification index and inertia degree of $\mathbf{F}$, respectively. The following result settles the case for those Dieudonné modules in addition satisfying the determinant condition.

Theorem 12.3.

(1) *Suppose that $\mathbf{B}$ is the 2×2 matrix algebra. Let ν be a slope sequence as follows:*

$$\nu = \left\{ \left(\frac{1}{2}\right)^{4d} \right\}, \quad or \quad \nu = \left\{ \left(\frac{a}{d}\right)^{2d}, \left(\frac{d-a}{d}\right)^{2d} \right\} \tag{12.9}$$

for an integer a with $0 \le a < d/2$. Then there exists a separably quasi-polarized Dieudonné $\mathcal{O}_{\mathbf{B}}$-module M of rank $4d$ satisfying the determinant condition such that $\nu(M) = \nu$.

(2) *Suppose that $\mathbf{B}$ is the quaternion division algebra. If M is a separably quasi-polarized Dieudonné $\mathcal{O}_{\mathbf{B}}$-module of rank $4d$ satisfying the determinant condition. Then*

$$\nu(M) = \left\{ \left(\frac{1}{2}\right)^{4d} \right\}, \quad or \quad \nu(M) = \left\{ \left(\frac{a}{2d}\right)^{2d}, \left(\frac{2d-a}{2d}\right)^{2d} \right\}, \tag{12.10}$$

for an odd integer a with $2[e/2]f \le a < d$. Conversely, if ν is a slope sequence as (12.10), then there exists a separably quasi-polarized Dieudonné $\mathcal{O}_{\mathbf{B}}$-module M of rank $4d$ satisfying the determinant condition such that $\nu(M) = \nu$.

Proof. (1) This is proved in Proposition 12.1 and Remark 12.2.

(2) Proposition 11.5 asserts that the reduced Lie type $\underline{e}^r(M)$ of M is the minimal one $\{(c, e-c); i \in \mathbb{Z}/f\mathbb{Z}\}$, where $c := [e/2]$. This yields $F^{2f}(M) \subset \pi^{2fc}M$ and hence that smallest slope $\beta \ge 2cf/2d$. Then the first assertion follows from Corollary 7.7.

Suppose that ν is a slope sequence as (12.10). When $e = 2c$ is even, ν is supersingular and the construction in Section 8 produces such a Dieudonné $\mathcal{O}_{\mathbf{B}}$-module. It remains to treat the case where $e = 2c+1$ is odd. We may also assume that ν is non-supersingular as the supersingular case is done in Section 8. Write $a = 2cf + 2r + 1$, where $0 < 2r+1 < f$. Let

$$M = \bigoplus_{j \in \mathbb{Z}/2f\mathbb{Z}} M^j,$$

where each M^j is a free rank two W^i-module generated by elements X_j and Y_j. As before, we fix a presentation $\mathcal{O}_{\mathbf{B}} = \mathcal{O}_{\mathbf{L}}[\Pi]$ as in (4.1) and (4.2). We describe the

Frobenius map F and the map Π by their representative matrices with respect to the bases $\{X_i, Y_i\}$ (see Subsection 8.2):

$$F_j : M^j \to M^{j+1}, \quad \Pi_j : M^j \to M^{j+f}, \quad \forall\, j \in \mathbb{Z}/2f\mathbb{Z}.$$

Put

$$\Pi_j = \begin{pmatrix} 0 & -\pi \\ 1 & 0 \end{pmatrix}, \quad \forall\, j \in \mathbb{Z}/2f\mathbb{Z}. \tag{12.11}$$

For each $j \in \mathbb{Z}/2f\mathbb{Z}$, define a W^i-bilinear pairing

$$\langle\, ,\rangle_{\mathbf{F}} : M^j \times M^{j+f} \to W^i$$

by

(12.12)

$$\langle X_j, X_{j+f}\rangle_{\mathbf{F}} = \langle Y_j, Y_{j+f}\rangle_{\mathbf{F}} = 0, \quad \langle X_j, Y_{j+f}\rangle_{\mathbf{F}} = 1 \quad \text{and} \quad \langle Y_j, X_{j+f}\rangle_{\mathbf{F}} = -1.$$

It is easy to show that $\langle \Pi X, \Pi Y\rangle_{\mathbf{F}} = \pi\langle X, Y\rangle_{\mathbf{F}}$ for $X \in M^j$ and $Y \in M^{j+f}$. So $\langle\, ,\rangle_F$ gives an unimodular skew-Hermitian form on M over $W \otimes \mathcal{O}$. Put

$$F_j = \begin{cases} \begin{pmatrix} 0 & -p\pi^{-c} \\ \pi^c & 0 \end{pmatrix}, & j = 0, \\ \begin{pmatrix} \pi^c & 0 \\ 0 & p\pi^{-c} \end{pmatrix}, & 1 \le j \le r, \\ \begin{pmatrix} p\pi^{-c} & 0 \\ 0 & \pi^c \end{pmatrix}, & r < j < f. \end{cases} \tag{12.13}$$

Using the commutative relation $\Pi_{j+1}F_j = F_{j+1}\Pi_j$ (8.4) we compute

$$F_j = \begin{cases} \begin{pmatrix} 0 & -\pi^{c+1} \\ p\pi^{-c}\pi^{-1} & 0 \end{pmatrix}, & j = f, \\ \begin{pmatrix} \pi^c & 0 \\ 0 & p\pi^{-c} \end{pmatrix}, & f+1 \le j \le f+r, \\ \begin{pmatrix} p\pi^{-c} & 0 \\ 0 & \pi^c \end{pmatrix}, & f+r < j < 2f. \end{cases} \tag{12.14}$$

As the matrix coefficients of F_j lie in the image of $\mathbb{Z}_p[\pi]$ and $\det F_j = p$, one has $\langle FX, FY\rangle_{\mathbf{F}} = p\langle X, Y\rangle_{\mathbf{F}}^{\sigma}$ for $X \in M^j$ and $Y \in M^{j+f}$.

Taking the trace (see (8.3)) we obtain a separable $\mathcal{O}_{\mathbf{B}}$-linear quasi-polarization $\langle\, ,\rangle : M \times M \to W$. It is easy to see that $\dim_k (M/VM)^j = e$ for all $j \in \mathbb{Z}/2f\mathbb{Z}$ and hence M satisfies the determinant condition.

We compute

$$F^f = p^r \begin{pmatrix} 0 & -(p\pi^{-c})^{f-2r} \\ (\pi^c)^{f-2r} & 0 \end{pmatrix} : M^0 \to M^f, \tag{12.15}$$

$$F^f = p^r \begin{pmatrix} 0 & -(\pi^c)^{f-2r}\pi \\ (p\pi^{-c})^{f-2r}\pi^{-1} & 0 \end{pmatrix} : M^f \to M^0, \tag{12.16}$$

and

$$F^{2f} = p^{2r} \begin{pmatrix} -(\pi^c)^{2(f-2r)}\pi & 0 \\ 0 & -(p\pi^{-c})^{2(f-2r)}\pi^{-1} \end{pmatrix} : M^0 \to M^0. \tag{12.17}$$

The valuation of the first diagonal entry of this matrix is

$$2er + 2c(f - 2r) + 1 = 2cf + 2r + 1 = a.$$

This shows that the slope sequence of the Dieudonné module M is equal to ν. ∎

13. Construction of Moret-Bailly families with O_B-action

In Sections 13 and 14 we shall restrict ourselves even to the case where $F = \mathbb{Q}$ (still in the minimal case $m = 1$). Our goal is to determine the dimension of the special fiber of moduli spaces in question.

In this section we assume that p *is ramified in* B. We shall prove

Theorem 13.1. *There is a non-constant family of supersingular polarized abelian O_B-surfaces over $\mathbf{P}^1_k$.*

13.1. Case $B = B_{p,\infty}$. We begin with a construction of Moret-Bailly families for the case where the algebra B is equal to the quaternion algebra $B_{p,\infty}$ over $\mathbb{Q}$ ramified exactly at $\{p, \infty\}$. Choose a supersingular elliptic curve E over k. There is an isomorphism $B \simeq \mathrm{End}^0(E) := \mathrm{End}(E) \otimes \mathbb{Q}$ of $\mathbb{Q}$-algebras and we fix one. Then the endomorphism ring $\mathrm{End}(E)$ is a maximal order O_B of B. The subgroup scheme $E[F] := \ker F = \alpha_p$ is O_B-stable as the Frobenius morphism is functorial. This induces a ring homomorphism

$$\phi : O_B/(p) = \mathbb{F}_{p^2}[\Pi]/(\Pi^2) \to \mathrm{End}_k(\alpha_p) = k. \tag{13.1}$$

Since k is commutative, this map factors through the maximal commutative quotient $(\mathbb{F}_{p^2}[\Pi]/(\Pi^2))^{\mathrm{ab}} = \mathbb{F}_{p^2}[\Pi]/(\Pi^2, I)$, where I is the two-sided ideal of $\mathbb{F}_{p^2}[\Pi]/(\Pi^2)$ generated by elements of the form $ab - ba$ for all $a, b \in \mathbb{F}_{p^2}[\Pi]/(\Pi^2)$. Since $\Pi a - a\Pi = (a^p - a)\Pi$ and $a^p - a$ is invertible if $a \notin \mathbb{F}_p$, the element Π lies in I. This shows that the action of O_B on $E[F] = \alpha_p$ factors through the quotient $O_B \twoheadrightarrow \mathbb{F}_{p^2}$. Put $\mathrm{Hom}_{\mathbb{F}_p}(\mathbb{F}_{p^2}, k) = \{\sigma_1, \sigma_2)$. We may assume this action is given by the embedding $\sigma_1 : \mathbb{F}_{p^2} \to k$.

Let $A_0 := E \times E$ and let $\iota_0 : O_B \to \mathrm{End}(A_0)$ be the diagonal action. Let M be the Dieudonné module of A_0 and the Lie algebra $\mathrm{Lie}(A_0) = M/VM$ has the decomposition of σ_i-components (Section 2.1):

$$\mathrm{Lie}(A_0) = \mathrm{Lie}(A_0)^1 \oplus \mathrm{Lie}(A_0)^2 = k^2 \oplus 0, \tag{13.2}$$

and satisfies the condition

$$\Pi(\mathrm{Lie}(A_0)) = 0. \tag{13.3}$$

Consider the functor

$$\mathcal{X}(S) := \{\varphi : \alpha_{p,S} \hookrightarrow A_0[F]_S = \alpha_p \times \alpha_p \times S \mid \varphi(\alpha_{p,S}) \text{ is } O_B\text{-stable}\}$$

for any k-scheme S. Clearly this functor is representable by a projective variety (again denote by) $\mathcal{X} \subset \mathbf{P}^1$, as the condition that $\varphi(\alpha_{p,S})$ is O_B-stable is closed. Every map φ corresponds to a rank one locally free $\mathcal{O}_S$-submodule $\mathcal{L}$ in $\mathcal{O}_S^2 = \mathrm{Lie}(A[F]_S)$ which locally for Zariski topology is a direct summand. As the ring O_B acts on $\mathrm{Lie}(E[F]_S)$ through the map $\sigma_1 : \mathbb{F}_{p^2} \to k \to O_S$ and by the scalar multiplication, the condition that $\mathcal{L}$ is O_B-stable is automatically satisfied. This shows that $\mathcal{X} = \mathbf{P}^1$.

Let $\varphi^{\rm univ}$ be the universal family, and put $H := \varphi^{\rm univ}(\alpha_{p,\mathbf{P}^1})$, which is an O_B-stable finite flat subgroup scheme of $A_0 \times S$. Let $\mathbf{A} := A_0 \times S/H$. As H is O_B-stable, this defines a family of supersingular O_B-surfaces over $\mathbf{P}^1$. Ignoring the structure of the O_B-action, this family is constructed by Moret-Bailly [18] and it has been shown to be a non-constant. By [12, Lemma 9.2], one can choose an O_B-linear polarization λ_0 on (A_0, ι_0). Replacing λ_0 by $p\lambda_0$ if necessary one may assume that $\ker \lambda_0 \supset A_0[F]$. Since H is isotropic with respect the Weil pairing defined by λ_0 (any rank-p finite group scheme has this property), the polarization $\lambda_0 \times S$ on $A_0 \times S$ descends to a polarization $\lambda_{\mathbf{A}}$ on $\mathbf{A}$ which is also O_B-linear. Therefore, we have constructed a non-constant family of supersingular polarized abelian O_B-surfaces $(\mathbf{A}, \lambda_{\mathbf{A}}, \iota_{\mathbf{A}})$ over $\mathbf{P}^1$ for $B = B_{p,\infty}$.

13.2. Arbitrary case. Now we retain B an arbitrary definite quaternion algebra over $\mathbb{Q}$. Using the construction above, we only need to construct a superspecial abelian O_B-surface (A_0, ι_0) that satisfies the conditions (13.2) and (13.3).

We first find a superspecial p-divisible $O_B \otimes \mathbb{Z}_p$-module (H_2, ι_2) (of height 4) over k such that the conditions (13.2) and (13.3) for $\mathrm{Lie}(H_2)$ are satisfied. One can directly write down a superspecial Dieudonné $O_B \otimes \mathbb{Z}_p$-module of rank 4 with such conditions (see an example in Subsection 13.3) and let (H_2, ι_2) be the corresponding p-divisible $O_B \otimes \mathbb{Z}_p$-module. Alternatively, let $O_{B_{p,\infty}}$ be the maximal order in Subsection 13.1 and (A_0, λ_0) be the superspecial abelian $O_{B_{p,\infty}}$-surface used there. After identifying $O_B \otimes \mathbb{Z}_p$ with $O_{B_{p\infty}} \otimes \mathbb{Z}_p$, the attached p-divisible $O_B \otimes \mathbb{Z}_p$-module $(H_2, \iota_2) := (A_0, \iota_0)[p^\infty]$ shares the desired property.

Choose a supersingular abelian O_B-surface (A_1, ι_1). It exists by the non-emptiness of moduli spaces and Corollary 7.7, or using an elementary proof (Basically it suffices to show that there is an embedding $B \to \mathrm{Mat}_2(B_{p,\infty})$ of $\mathbb{Q}$-algebras and this is easy.) Let $(H_1, \iota_1) := (A_1, \iota_1)[p^\infty]$ be the associated p-divisible $O_B \otimes \mathbb{Z}_p$-module.

Lemma 13.2. *There is an $O_B \otimes \mathbb{Z}_p$-linear isogeny $\varphi : (H_1, \iota_1) \to (H_2, \iota_2)$.*

PROOF. Since H_1 and H_2 are supersingular, one chooses an isogeny $\varphi : H_1 \to H_2$. Define the map $\iota_2' : O_B \otimes \mathbb{Z}_p \to \mathrm{End}^0(H_1)$ so that the following diagram

$$\begin{array}{ccc} H_1 & \xrightarrow{\varphi} & H_2 \\ \downarrow{\scriptstyle \iota_2'(a)} & & \downarrow{\scriptstyle \iota_2(a)} \\ H_1 & \xrightarrow{\varphi} & H_2 \end{array}$$

commutes for all a in an order of $O_B \otimes \mathbb{Z}_p$. We have two algebra homomorphisms

$$\iota_1, \iota_2' : B_p := B \otimes \mathbb{Q}_p \to \mathrm{End}^0(H_1).$$

Since the center of the algebra $\mathrm{End}^0(H_1)$ is $\mathbb{Q}_p$, by the Noether-Skolem theorem there is an element $g \in \mathrm{End}^0(H_1)^\times$ such that

$$\iota_2' = \mathrm{Int}(g) \circ \iota_1 = g \circ \iota_1 \circ g^{-1}.$$

Replacing g by $p^m g$ for some integer m, we may assume that $g \in \mathrm{End}(H_1)$. That is, we have the following commutative diagram

$$\begin{array}{ccccc} H_1 & \xrightarrow{g} & H_1 & \xrightarrow{\varphi} & H_2 \\ \big\downarrow \iota_1(a) & & \big\downarrow \iota_2'(a) & & \big\downarrow \iota_2(a) \\ H_1 & \xrightarrow{g} & H_1 & \xrightarrow{\varphi} & H_2. \end{array}$$

Replacing φ by $\varphi \circ g$, we get an $O_B \otimes \mathbb{Z}_p$-linear isogeny $\varphi : (H_1, \iota_1) \to (H_2, \iota_2)$. ∎

By Lemma 13.2, we choose an $O_B \otimes \mathbb{Z}_p$-linear isogeny $\varphi : (H_1, \iota_1) \to (H_2, \iota_2)$. Let $K := \ker \varphi$; this is an O_B-stable subgroup scheme of A_1. Let $A_0 := A_1/K$ and let $\iota_0 : O_B \to \mathrm{End}(A_0)$ the induced action. Then one has an isomorphism $(A_0, \iota_0)[p^\infty] \simeq (H_2, \iota_2)$. This finds an abelian O_B-surface satisfying the conditions (13.2) and (13.3). Proceed the construction for (A_0, ι_0) in Subsection 13.1 and we have constructed a non-constant family of polarized abelian O_B-surfaces $(\mathbf{A}, \lambda_{\mathbf{A}}, \iota_{\mathbf{A}})$ over $\mathbf{P}^1$. This finishes the proof of Theorem 13.1 ∎

13.3. An example. We write down a superspecial Dieudonné $O_B \otimes \mathbb{Z}_p$-module M so that the conditions (13.2) and (13.3) for M/VM are satisfied. Write $M = M^1 \oplus M^2$ as a free W-module with a $\mathbb{Z}_{p^2}$-action, where M^1 and M^2 are free module generated by elements $\{e_1^1, e_2^1\}$ and $\{e_1^2, e_2^2\}$, respectively. Define the Verschiebung map V by

$$V(e_1^2) = pe_1^1, \quad V(e_1^2) = pe_1^1, \quad V(e_1^1) = e_1^2, \quad V(e_1^1) = e_1^2.$$

This determines the Frobenius map and defines a Dieudonné module with a $\mathbb{Z}_{p^2}$-action which satisfies the condition (13.2) for M/VM. It follows from $\Pi a = \sigma(a)\Pi$ for $a \in \mathbb{Z}_{p^2}$ that we gets $\Pi : M^1 \to M^2$ and $\Pi : M^2 \to M^1$. Since $\Pi^2 = -p$, the restriction of the map Π to M^1, $\Pi|_{M^1} : M^1 \to M^2$ determines Π. The condition (13.3) implies that $\Pi(M^2) = pM^1$ and hence $\Pi(M^1) = M^2$.

It is easy to see that that $\Pi V = V\Pi$ if and only if $\Pi V(e_j^i) = V\Pi(e_j^i)$. Define the map Π by putting $\Pi(e_j^i) := V(e_j^i)$ for all i, j; so the condition $\Pi V = V\Pi$ is satisfied. Therefore, we get a superspecial Dieudonné $O_B \otimes \mathbb{Z}_p$-module so that the conditions (13.2) and (13.3) are fulfilled.

14. Dimensions of special fibers

We keep the setting in the previous section. We have proven that $\dim \mathcal{M}_K^{(p)} \otimes \overline{\mathbb{F}}_p = 0$; see Theorem 10.1. Our goal is to determine the dimensions of the special fibers $\mathcal{M}_{\overline{\mathbb{F}}_p} := \mathcal{M} \otimes \overline{\mathbb{F}}_p$, $\mathcal{M}_{K,\overline{\mathbb{F}}_p} := \mathcal{M}_K \otimes \overline{\mathbb{F}}_p$ and $\mathcal{M}^{(p)}_{\overline{\mathbb{F}}_p} := \mathcal{M}^{(p)} \otimes \overline{\mathbb{F}}_p$.

Theorem 14.1.

(1) *If p is unramified in B, then* $\dim \mathcal{M}_{\overline{\mathbb{F}}_p} = 0$.
(2) *If p is ramified in B, then* $\dim \mathcal{M}_{\overline{\mathbb{F}}_p} = 1$.
(3) *We have* $\dim \mathcal{M}^{(p)}_{\overline{\mathbb{F}}_p} = 0$.

A further exam shows that when p is ramified in B, one has $\dim \mathcal{M}_{K,\overline{\mathbb{F}}_p} = 1$; see Proposition 14.7. This refines the result Theorem 14.1 (2).

14.1. Unramified case. Suppose that p is unramified in B. Let (A, λ, ι) be a polarized abelian O_B-surface over k. By Corollary 7.7 A is either ordinary or supersingular. If A is ordinary, then one has the canonical lifting $(\mathbf{A}, \lambda_{\mathbf{A}}, \iota_{\mathbf{A}})$ over W of (A, λ, ι). Since the generic fiber $\mathcal{M}_{\overline{\mathbb{Q}_p}}$ has dimension zero, each subscheme $\mathcal{M}_{D,\overline{\mathbb{Q}_p}}$ has finitely many points, if it is not empty. Recall that $\mathcal{M}_D \subset \mathcal{M}$ denotes the subscheme parametrizing the objects (A, λ, ι) in $\mathcal{M}$ with polarization degree $\deg \lambda = D^2$. This implies that the ordinary locus $\mathcal{M}^{\mathrm{ord}}_{D,\overline{\mathbb{F}}_p}$ of $\mathcal{M}_{D,\overline{\mathbb{F}}_p}$ has finitely many points and hence it has dimension zero. Therefore, the ordinary locus $\mathcal{M}^{\mathrm{ord}}_{\overline{\mathbb{F}}_p}$ has dimension zero.

Suppose now that A is supersingular. Then A must be superspecial. To see this, let $H := A[p^\infty]$ be the associated p-divisible group. Since $O_B \otimes \mathbb{Z}_p := O_B \otimes \mathbb{Z}_p \simeq \mathrm{Mat}_2(\mathbb{Z}_p)$, the p-divisible group H is isomorphic to $H_1 \times H_2$, where H_1 and H_2 are supersingular p-divisible group of height 2. Therefore, A is superspecial.

For any positive integers g and D, let $\mathcal{A}_{g,D}$ denote the coarse moduli space over $\overline{\mathbb{F}}_p$ of polarized abelian varieties (A, λ) with polarization degree $\deg \lambda = D^2$. Let $\Lambda_{g,D} \subset \mathcal{A}_{g,D}$ be the superspecial locus. It is known that $\Lambda_{g,D}$ is a finite closed subscheme. Let $f : \mathcal{M}_{D,\overline{\mathbb{F}}_p} \to \mathcal{A}_{2,D}$ be the forgetful morphism: $f(A, \lambda, \iota) = (A, \lambda)$. The morphism f induces a map

$$f : \mathcal{M}^{\mathrm{ss}}_{D,\overline{\mathbb{F}}_p} \to \Lambda_{g,D},$$

where $\mathcal{M}^{\mathrm{ss}}_{D,\overline{\mathbb{F}}_p}$ is the supersingular locus of $\mathcal{M}_{D,\overline{\mathbb{F}}_p}$. As $\dim \Lambda_{g,D} = 0$ and the forgetful map f is finite (see [34]), the supersingular locus $\mathcal{M}^{\mathrm{ss}}_{D,\overline{\mathbb{F}}_p}$ also has dimension zero. We conclude that $\mathcal{M}_{\overline{\mathbb{F}}_p}$ has dimension zero. This shows Theorem 14.1 (1).

14.2. Ramified case. Suppose that the prime p is ramified in B. We know that any polarized abelian O_B-surface over k is supersingular (Corollary 7.7). By Theorem 13.1, there is a non-constant family $\underline{\mathbf{A}} \to \mathbf{P}^1_{\overline{\mathbb{F}}_p}$ of supersingular polarized abelian O_B-surfaces. This gives rise to a non-constant moduli map

$$f' : \mathbf{P}^1_{\overline{\mathbb{F}}_p} \to \mathcal{M}_{\overline{\mathbb{F}}_p}.$$

Therefore, $\dim \mathcal{M}_{\overline{\mathbb{F}}_p} \geq \dim f'(\mathbf{P}^1) = 1$.

On the other hand, the forgetful morphism $f : \mathcal{M}_{D,\overline{\mathbb{F}}_p} \to \mathcal{A}_{2,D}$ factors through the supersingular locus $\mathcal{A}^{\mathrm{ss}}_{2,D} \subset \mathcal{A}_{2,D}$. Since f is finite, one gets

$$\dim \mathcal{M}_{D,\overline{\mathbb{F}}_p} \leq \dim \mathcal{A}^{\mathrm{ss}}_{2,D}. \tag{14.1}$$

For any integer i with $0 \leq i \leq g$, let $\mathcal{A}^{(i)}_{g,D} \subset \mathcal{A}_{g,D}$ denote the reduced locally closed subscheme that consists of objects (A, λ) of p-rank equal to i. Norman and Oort [22] showed that the collection of p-strata forms a stratification and for each i

$$\dim \mathcal{A}^{(i)}_{g,D} = g(g-1)/2 + i.$$

When $g = 2$, one has $\mathcal{A}^{(0)}_{2,D} = \mathcal{A}^{\mathrm{ss}}_{2,D}$ and gets $\dim \mathcal{A}^{\mathrm{ss}}_{2,D} = 1$. This shows the other direction

$$\dim \mathcal{M}_{D,\overline{\mathbb{F}}_p} \leq 1.$$

We conclude that $\dim \mathcal{M}_{\overline{\mathbb{F}}_p} = 1$. This shows Theorem 14.1 (2).

14.3. Dimension of $\mathcal{M}^{(p)}_{\overline{\mathbb{F}}_p}$. We come to show Theorem 14.1 (3) For the case where the prime p is unramified in B, we have shown in Theorem 14.1 (1) that $\mathcal{M}_{\overline{\mathbb{F}}_p}$ is zero-dimensional. Therefore, $\dim \mathcal{M}^{(p)}_{\overline{\mathbb{F}}_p} = 0$ in the unramified case.

We now treat the other case where p is ramified in B.

Proposition 14.2. *Assume that p is ramified in B. Any prime-to-p degree polarized abelian O_B-surface over k is superspecial.*

PROOF. Let M be the associated covariant Dieudonné $O_B \otimes \mathbb{Z}_p$-module with a separable quasi-polarization $\langle\ ,\ \rangle$. Recall that $O_B \otimes \mathbb{Z}_p = \mathbb{Z}_{p^2}[\Pi]$, $\Pi^2 = -p$, $\Pi a = \sigma(a)\Pi$ for $a \in \mathbb{Z}_{p^2}$.

Suppose $M/VM = k^2 \oplus 0$ with respect to the action of $\mathbb{F}_{p^2} \otimes_{\mathbb{F}_p} k = k \times k$. Then $VM^2 = pM^1$ and $VM^1 = M^2$, and hence $FM^1 = M^2$ and $FM^2 = pM^1$. This shows that $FM = VM$ and that M is superspecial

We show that if $\Pi(M/VM) = 0$, then M is superspecial. Indeed, it follows that $\Pi M \subset VM$. From $\dim M/\Pi M = 2$ (as $\Pi^2 = -p$) and $\dim M/VM = 2$ it follows that $\Pi M = VM$. Then $V^2M = V\Pi M = \Pi VM = \Pi^2 M = pM$ and hence M is superspecial. So far we have not used the separability of polarizations.

Suppose that $M/VM = k \oplus k$. Consider the induced perfect pairing $\langle\,,\rangle : \overline{M} \times \overline{M} \to k$, where $\overline{M} = M/pM$. Since Π is nilpotent on M/VM one may assume for example that $\Pi\overline{M^2} = V\overline{M^2}$. Taking the orthogonal complements of $\Pi\overline{M^2}$ and $V\overline{M^2}$, we have $\Pi\overline{M^1} = V\overline{M^1}$. This shows that $\Pi(M/VM) = 0$. By the above argument, M is superspecial. ∎

Since the superspecial locus of the Siegel moduli space is zero-dimensional, the superspecial locus of any PEL-type moduli space is zero-dimensional, too. Proposition 14.2 implies that $\dim \mathcal{M}^{(p)}_{\overline{\mathbb{F}}_p} = 0$. This shows Theorem 14.1 (3) and hence completes the proof of Theorem 14.1. ∎

Lemma 14.3. *Assume that p is ramified in B. There is a prime-to-p degree polarized superspecial abelian O_B-surface that does not satisfy the determinant condition.*

PROOF. Using the construction in Section 13.1, we have a superspecial p-divisible $O_B \otimes \mathbb{Z}_p$-module (H, ι_H) of height 4 such that the conditions (13.2) and (13.3) for $\mathrm{Lie}(H)$ are satisfied. Thus, (H, ι_H) does not satisfy the determinant condition. There is a superspecial abelian O_B-surface $(A_0, \iota_0))$ such that $(A_0, \iota_0)[p^\infty] \simeq (H, \iota_H)$. We fix an identification $(A_0, \iota_0)[p^\infty] = (H, \iota_H)$.

We can choose a separable $O_B \otimes \mathbb{Z}_p$-linear quasi-polarization λ_H. Note that (H, ι_H) is isomorphic to the p-divisible $O_{B_{p,\infty}} \otimes \mathbb{Z}_p$-module $E_0[p^\infty]^2$ (E_0 is a supersingular elliptic curve) through the identification $O_{B_{p,\infty}} \otimes \mathbb{Z}_p = O_B \otimes \mathbb{Z}_p$. We can pick the product principal polarization on E_0^2 which yields such a quasi-polarization λ_H.

Choose an O_B-linear polarization λ on (A_0, ι_0) and let $*$ denote the Rosati involution induced by λ. Then $\lambda a_p = \lambda_H$ for some element $a_p \in \mathrm{End}^0_{O_B \otimes \mathbb{Z}_p}(A_0[p^\infty])$ with $a_p^* = a_p$. Since $\mathrm{End}^0_B(A_0) \otimes \mathbb{Q}_p = \mathrm{End}^0_{B \otimes \mathbb{Q}_p}(A_0[p^\infty])$, by the weak approximation we can choose a totally positive symmetric element $a \in \mathrm{End}^0_B(A_0)$ such that a is sufficiently close to a_p. Then we have $(A, \lambda a, \iota)[p^\infty] \simeq (H, \lambda_H, \iota_H)$. Replacing a by Na for a positive prime-to-p integer N if necessary, we get a prime-to-p degree O_B-linear polarization $\lambda_0 = \lambda a$ on (A_0, ι_0). This proves the lemma. ∎

We know that when p is ramified in B, the whole moduli space $\mathcal{M}_{\overline{\mathbb{F}}_p}$ is supersingular. On the other hand when p is unramified in B, the moduli space may have both supersingular and ordinary points according to Proposition 7.7. The following lemma says that this is the case.

Lemma 14.4. *When p is unramified in B, the moduli space $\mathcal{M}^{(p)}_{K,\overline{\mathbb{F}}_p}$ consists of both ordinary and supersingular points.*

PROOF. As p is unramified in B, the determinant condition is automatically satisfied for objects in $\mathcal{M}(k)$. Let (H, λ_H, ι_H) be a supersingular or ordinary separably quasi-polarized p-divisible $O_B \otimes \mathbb{Z}_p$-module. Since B can be embedded into $\mathrm{End}^0(A)$ for any supersingular abelian surface, we can find a supersingular abelian O_B-surface with $(A_0, \iota_0)[p^\infty] \simeq (H, \iota_H)$. We use the argument in the proof of Lemma 14.3 again to obtain a prime-to-p degree O_B-linear polarization. For the ordinary case, we choose any imaginary quadratic field K such that K splits B and p splits in K. Then there is a $\mathbb{Q}$-algebra embedding of B into $\mathrm{Mat}_2(K)$. Choose an ordinary elliptic curve E such that $\mathrm{End}^0(E) \simeq K$ and take the ordinary abelian surface $A = E^2$. As $\mathrm{End}^0_B(A) \otimes \mathbb{Q}_p = \mathrm{End}^0_{B\otimes\mathbb{Q}_p}(A[p^\infty])$, we can repeat the previous argument and get a prime-to-p degree polarized ordinary abelian O_B-surface. ∎

Remark 14.5.
(1) We used local models to show that $\dim \mathcal{M}^{(p)}_K \otimes \overline{\mathbb{F}}_p = 0$. Proposition 14.2 gives a different proof of this result. Lemma 14.3 shows that the inclusion $\mathcal{M}^{(p)}_K(k) \subset \mathcal{M}^{(p)}(k)$ is strict at least for B is a definite quaternion $\mathbb{Q}$-algebra. This phenomenon is different from the reduction modulo p of Hilbert moduli schemes or Hilbert-Siegel moduli schemes. In the Hilbert-Siegel case, any separably polarized abelian varieties with RM by O_F of a totally real field F satisfies the determinant condition automatically; see Yu [35], Görtz [5] and Vollaard [31].

(2) We know that $\mathcal{M}^{(p)}_{\overline{\mathbb{F}}_p}$ is non-empty (Lemma 2.3). When p is ramified in B, the moduli space $\mathcal{M}_{\overline{\mathbb{F}}_p}$ consists of both one-dimensional components (e.g. Moret-Bailly families) and zero-dimensional components (e.g. points in $\mathcal{M}^{(p)}_{\overline{\mathbb{F}}_p}$).

14.4. Dimension of $\mathcal{M}_{K,\overline{\mathbb{F}}_p}$. As before, we only need to treat the ramified case.

Lemma 14.6. *Assume that p is ramified in B. Let M_0 be a Dieudonné $O_B \otimes \mathbb{Z}_p$-module such that*
$$M_0/VM_0 = k^2 \oplus 0,$$
that is the Lie type of M_0 is $(2,0)$. Let M be any Dieudonné module such that $VM_0 \subset M \subset M_0$ and $\dim_k(M_0/M) = 1$. Then one has
$$M/VM = k \oplus k.$$

PROOF. Choose bases $\{X^1_1, X^1_2\}$ and $\{X^2_1, X^2_2\}$ for M^1_0 and M^2_0, respectively. Since $VM \supset V^2M_0 = pM_0$. We can check this in $\overline{M}_0 := M_0/pM_0$. Write x^i_j for the image of X^i_j in $\overline{M}_0$. One has
$$V\overline{M}_0 = \mathrm{Span}_k\{x^2_1, x^2_2\}, \quad \overline{M} := M/pM_0 = \mathrm{Span}_k\{x^2_1, x^2_2, ax^1_1 + bx^1_2\},$$
for some $(a,b) \neq (0,0) \in k^2$. Then
$$V\overline{M} = \mathrm{Span}_k\{V(ax^1_1 + bx^1_2)\} \subset \overline{M}^2_0, \quad \text{and} \quad \dim V\overline{M} = 1.$$

This gives $M/VM = k \oplus k$. ∎

Proposition 14.7. *Assume that p is ramified in B. We have* $\dim \mathcal{M}_{K,\overline{\mathbb{F}}_p} = 1$.

Proof. In the previous section, we constructed a polarized abelian O_B-surface $(\mathbf{A}, \lambda_{\mathbf{A}}, \iota_{\mathbf{A}})$ over $\mathbf{P}^1$ starting from a superspecial abelian surface $(A_0, \lambda_0, \iota_0)$ with additional structures and get a non-constant moduli map $f' : \mathbf{P}^1 \to \mathcal{M}_{\overline{\mathbb{F}}_p}$. The Dieudonné module M_0 of A_0 has the property $M_0/VM_0 = k^2 \oplus 0$. By Lemma 14.6, every fiber of the family $(\mathbf{A}, \lambda_{\mathbf{A}}, \iota_{\mathbf{A}}) \to \mathbf{P}^1$ has Lie type $(1,1)$. Then the image $f'(\mathbf{P}^1)$ is contained in $\mathcal{M}_{K,\overline{\mathbb{F}}_p}$. This shows that $\dim \mathcal{M}_{K,\overline{\mathbb{F}}_p} = 1$. ∎

Remark 14.8. A consequence of Proposition 14.7 asserts that when p is ramified in B, there is a polarized abelian O_B-surface over k with the determinant condition that can not be lifted to a polarized abelian O_B-surface in characteristic zero.

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