# A new construction of a compactfication of $\mathbb{C}^3$ with second Betti number one

(Dedicated to Professor Hirzebruch on his sixtieth birthday)

by

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Introduction. Let $(X,Y)$ be a smooth projective compactification of $\mathbb{C}^3$, namely, $X$ is a smooth projective 3-fold and $Y$ is a subvariety of $X$ such that $X-Y$ is isomorphic to $\mathbb{C}^3$. Assume that $Y$ is normal. Then $X$ is a Fano 3-fold of index $r$ $(1 \leq r \leq 4)$ with $b_2(X) = 1$ and $Y$ is a hyperplane section of $X$, where $b_2(X)$ is the second Betti number of $X$. In the paper [1], we have the following results:

(i) $r = 4 \Rightarrow (X,Y) \cong (\mathbb{P}^3, \mathbb{P}^2)$

(ii) $r = 3 \Rightarrow (X,Y) \cong (\mathbb{Q}^3, \mathbb{Q}^2_0)$, where $\mathbb{Q}^3$ is a smooth quadric hypersurface in $\mathbb{P}^4$ and $\mathbb{Q}^2_0$ is a quadric cone

(iii) $r = 2 \Rightarrow (X,Y) \cong (V_5, H_5)$, where $V_5$ is a Fano 3-fold of degree 5 in $\mathbb{P}^6$ and $H_5$ is a singular del Pezzo surface with exactly one singularity of $A_4$-type

(iv) $r = 1 \Rightarrow (X,Y)$ is not completely determined (see also [2], [3], [6]).

These 3-folds $\mathbb{P}^3$, $\mathbb{Q}^3$, $V_5$ are really compactifications of $\mathbb{C}^3$. In case of $r = 4$, it is clear that $\mathbb{P}^3 - \{\text{a hyperplane } \mathbb{P}^2\} \cong \mathbb{C}^3$. In case of $r = 3$, projecting $\mathbb{Q}^3$ from the vertex of $\mathbb{Q}^2_0$ to $\mathbb{P}^3$, one can see that $\mathbb{Q}^3 - \mathbb{Q}^2_0 \cong \mathbb{C}^3$. In case of $r = 2$, projecting $V_5$ from a line through the singularity of $A_4$-type of $H_5$ to $\mathbb{Q}^3$, one can see that $V_5 - H_5 \cong \mathbb{C}^3$. (see [1]).

In this paper, studying the double projection of $V_5$ from the singularity of $A_4$-type of $H_5$, we will show that the Fano 3-fold $V_5$ can be obtained from a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$ by a flip (cf. [6], [9]) and a blowing down. This gives a new construction of a compactification of $\mathbb{C}^3$ in case of the index $r = 2$. Finally, we remark that in case of the index $r = 1$, by studying the triple projection from a singularity of the bounadry divisor, one can show that such a compactification $(X,Y)$ does not exist (see [2]).

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## § 1. Preliminaries

Let $(X,Y)$ be a smooth projective compactification of $\mathbb{C}^3$ such that $Y$ is normal. Assume that the index $r = 2$. Then $(X,Y) \cong (V_5,H_5)$ (see Introduction). Then the anti-canonical line bundle can be written as follows:

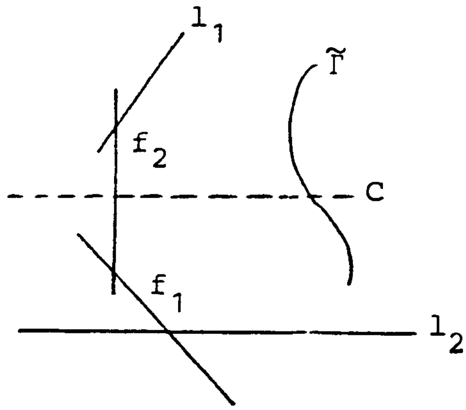
$$-K_Y = \mathcal{O}_Y(\Gamma) ,$$

where $\Gamma$ is an elliptic curve not through the singularity of $Y = H_5$. Thus $\deg Y = (\Gamma^2)_Y = 5$. In particular, the singular locus of $Y$ consists of exactly one point $\{x\}$, which is of $A_4$-type. Let $\alpha : \tilde{Y} \longrightarrow Y$ be the minimal resolution of singularities of $Y$ and put

$$\alpha^{-1}(x) = l_2 \cup f_1 \cup f_2 \cup l_1 ,$$

where $l_i, f_i$ $(1 \le i \le 2)$ are smooth rational curves with the self-intersection number equal to $-2$ and the dual graph of the exceptional divisor $\alpha^{-1}(x)$ is a linear tree (see Figure 1).

On the other hand, $\tilde{Y}$ can be obtained from $\mathbb{P}^2$ by blowing up 4 points (allowed infinitely near points) on a smooth cubic curve $\Gamma_0$ on $\mathbb{P}^2$. Let $\tilde{\Gamma}$ be the proper transform of $\Gamma_0$ in $\tilde{Y}$ (see Figure 1)




(Figure 1)

In Figure 1, there exists an exceptional curve $C$ of the fir kind with $(C \cdot \tilde{\Gamma})_{\tilde{Y}} = 1$ . We put $C_0 = \alpha(C)$ and $\Gamma = \alpha(\tilde{\Gamma})$ . Let $H$ be a general hyperplane section of $X := V_5$ such that $\mathcal{O}_Y(H) = \mathcal{O}_Y(\Gamma)$ . Since

$$\begin{aligned} 1 &= (\tilde{\Gamma} \cdot C)_{\tilde{Y}} \\ &= (\Gamma \cdot C_0)_Y \\ &= (H \cdot C_0)_X \quad . \end{aligned}$$

$C_0$ is a line on $X$ . By [1, Proposition 15], $C_0$ is unique line in $\mathbb{P}^6$ contained in $Y \subset X$ . Since the multiplicity $m(\mathcal{O}_{Y,x})$ of the local ring $\mathcal{O}_{Y,x}$ is equal to two, any line through the point $x$ must be contained in $Y$ . Therefore $C_0$ is unique line in $X$ through the singularity $x$ of $Y = H_5$ .

## § 2. Double projection from a point

We will study the double projection of $X = V_5$ from the singularity $x$ of $A_4$-type of $Y = H_5$. For this, let us consider the linear system

$$|H-2x| = |\mathcal{M}_x^2 \otimes \mathcal{O}_X(H)| ,$$

where $H$ is a hyperplane section of $X$ and $\mathcal{M}_x \subset \mathcal{O}_{X,x}$ is the maximal ideal of the local ring $\mathcal{O}_{X,x}$. Let $\delta_1 : X_1 \longrightarrow X$ be the blowing up at the point $x$ and put $E_1 := \delta_1^{-1}(x) \cong \mathbb{P}^2$. Let $Y_1$ and $C_1$ be the proper transforms of $Y$ and $C_0$ respectively. Then we have

<u>Lemma 1</u>. $\dim |H-2x| = 2$

<u>Proof</u>. Let us consider the exact sequences:

$$0 \longrightarrow \mathcal{O}_{X_1}(\delta_1^*H-E_1) \longrightarrow \mathcal{O}_{X_1}(\delta_1^*H) \longrightarrow \mathcal{O}_{E_1} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_{X_1}(\delta_1^*H-2E_1) \longrightarrow \mathcal{O}_{X_1}(\delta_1^*H-E_1) \longrightarrow \mathcal{O}_{E_1}(1) \longrightarrow 0 .$$

Since $\dim|H-x| = \dim|H| - 1$ , we have

$$H^0(X_1,\mathcal{O}_{X_1}(\delta_1^*H-E_1)) \cong \mathbb{C}^6 , \text{ and}$$

$$H^1(X_1,\mathcal{O}_{X_1}(\delta_1^*H-E_1)) \cong 0 .$$

Let $L := \mathrm{Tr}_{E_1}|\delta_1^*H-E_1| \subseteq |\mathcal{O}_{E_1}(1)|$ be the trace of the linear system $|\delta_1^*H-E_1|$ on $E_1$. Since $|\delta_1^*H-E_1|$ has no fixed component and no base points on $X_1$, so is $L$ on $E_1$. Therefore $L = |\mathcal{O}_{E_1}(1)|$. Thus, we have the surjection

$$H^0(X_1,\mathcal{O}_{X_1}(\delta_1^*H-E_1)) \longrightarrow\!\!\!\!\rightarrow H^0(E_1,\mathcal{O}_{E_1}(1)) \cong \mathbb{C}^3 .$$

This means that

$$H^0(X_1,\mathcal{O}_{X_1}(\delta_1^*H-2E_1)) \cong \mathbb{C}^3 \text{ , and}$$
$$H^1(X_1,\mathcal{O}_{X_1}(\delta_1^*H-2E_1)) = 0 .$$

This completes the proof.

Q.E.D.

By Lemma 1, we have rational maps

$$\Phi := \Phi_{|H-2x|} : X \dashrightarrow \mathbb{P}^2 ,$$
$$\Phi^{(1)} := \Phi_{|\delta_1^*H-2E_1|} : X_1 \dashrightarrow \mathbb{P}^2 .$$

Since $(\delta_1^*H-2E_1)\cdot C_1 = -1 < 0$, $C_1$ is a base curve of the linear system $|\delta_1^*H-2E_1|$.

Next, we will study the singularities of $Y_1$. Let

$\Delta \subset X$ be a small neighbourhood of $x$ in $X$ with a local coordinate system $(Z_1, Z_2, Z_3)$. Since the singularity $x \in Y = H_5$ is of $A_4$-type and $C$ intersects the component $f_2$ of $\alpha^{-1}(x)$ in $\tilde{Y}$ (see Figure 1), we may assume that

(i) $\Delta \cap Y = \{Z_1 \cdot Z_2 = Z_3^5\} \hookrightarrow \Delta$ with $x = (0,0,0)$,

(ii) $C_0 \cap \Delta = \{Z_1 = Z_3^2,\ Z_2 = Z_3^3\} \hookrightarrow \Delta$.

By an easy calculation, we find that $Y_1$ has exactly one singular point $x_1$ of $A_2$-type. Then there exists a birational morphism

$$\mu_1 : \tilde{Y} \longrightarrow Y_1$$

such that

$$\mu_1^{-1}(x_1) = f_1 \cup f_2 \text{, and}$$

$$\tilde{Y} - (f_1 \cup f_2) \overset{\mu_1}{\cong} Y_1 - \{x_1\} \text{ (isomorphic)}.$$

We put $l_i^{(1)} := \mu_1(l_i)$ $(1 \leq i \leq 2)$ and $C_1 = \mu_1(C)$. Then we have

$$E_1 \cdot Y_1 = l_1^{(1)} + l_2^{(1)}, \tag{2.1}$$

in particular, $l_1^{(1)}, l_2^{(1)}$ are two distinct lines on $E_1 \cong \mathbb{P}^2$ and $C_1$ is the proper transform of $C_0$ in $X_1$.

Since $Y_1 \in |\delta_1^* H - 2E_1|$ , by (2.1), we have

$$\mathcal{O}_{Y_1}(Y_1) = \mathcal{O}_{Y_1}(\delta_1^* H - 2E_1)$$
$$= \mathcal{O}_{Y_1}(\Gamma^{(1)} - 2l_1^{(1)} - 2l_2^{(1)}),$$

where $\Gamma^{(1)} = \delta_1^*(Y|_H) = \mu_1(\tilde{\Gamma})$ . We have

$$\mu_1^* \mathcal{O}_{Y_1}(\Gamma^{(2)} - 2l_1^{(1)} - 2l_2^{(1)}) \cong \mathcal{O}_{\tilde{Y}}(\tilde{\Gamma} - 2f_1 - 2f_2 - 2l_1 - 2l_2)$$
$$= \mathcal{O}_{\tilde{Y}}(\tilde{\Gamma} - 2Z) \ , \qquad (2.2)$$

where $Z = f_1 + f_2 + l_1 + l_2$ is the fundamental cycle of the singularity $x$ associated with the resolution $(\tilde{Y}, \alpha)$ . From the exact sequence

$$0 \longrightarrow \mathcal{O}_{X_1} \longrightarrow \mathcal{O}_{X_1}(Y_1) \longrightarrow \mathcal{O}_{Y_1}(Y_1) \longrightarrow 0 \ , \qquad (2.3)$$

we have

$$H^0(Y_1, \mathcal{O}_{Y_1}(Y_1)) \cong H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(\tilde{\Gamma} - 2Z))$$
$$\cong \mathbb{C}^2 \ ,$$

since $H^0(X_1, \mathcal{O}_{X_1}(Y_1)) \cong \mathbb{C}^3$ by Lemma 1. Let $\{\psi_0, \psi_1\}$ be a basis of $H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(\tilde{\Gamma} - 2Z))$ such that

$$\begin{cases} (\psi_0) = 3C + 2f_2 + f_1 + f \\ (\psi_1) = 5C + 4f_2 + 2f_1 + l_1 \ , \end{cases} \qquad (2.4)$$

where $f$ is a smooth rational curve in $\tilde{Y}$ such that $(f^2)_{\tilde{Y}} = 0$ and $(f \cdot l_2)_{\tilde{Y}} = 1$ (in fact, $\tilde{Y}$ can be considered as a ruled surface over a smooth rational curve, which has $f$ as a fiber and $l_2$ as a section). Since

$$(\psi_0) \cap (\psi_1) = C \cup f_1 \cup f_2 ,$$

we have the base locus

$$B_s|\mathcal{O}_{Y_1}(Y_1)| = C_1 \ni x_1 .$$

By (2.2), since $H^1(X_1, \mathcal{O}_{X_1}) = 0$ , we have the base locus

$$B_s|\mathcal{O}_{X_1}(Y_1)| = C_1 \ni x_1 .$$

Since $\operatorname{Pic} X \cong \mathbb{Z}\mathcal{O}_X(H)$ , $|H-2x|$ has no fixed component, hence so is $|\delta_1^* H-2E_1|$ . Thus we have the following

<u>Lemma 2</u>. $|\delta_1^* H-2E_1|$ has no fixed component, but has the base locus

$$B_s|\delta_1^* H-2E_1| = C_1 \ni x_1 .$$

## § 3. Resolution of indeterminancy

We will describe in detail the resolution of indeterminancy of the rational map

$$\Phi^{(1)} : X_1 \dashrightarrow \mathbb{P}^2$$

(see Lemma 2 in § 2).

For this, we need the following

Lemma 3 (Morrison [7]). Let $S$ be a surface with only one singularity $x$ of $A_n$-type in a smooth projective 3-fold. Let $E \subset S \subset X$ be a smooth rational curve in $X$. Let $\mu : \tilde{S} \longrightarrow S$ be the minimal resolution of the singularity of $S$ and put

$$\mu^{-1}(x) = \bigcup_{j=1}^{n+1} C_j$$

where $C_j$'s $(i \leqq j \leqq n+1)$ are smooth rational curve with

$$(C_j^2)_{\tilde{S}} = -2 \quad (1 \leqq j \leqq n+1) ,$$

$$(C_j \cdot C_{j+1})_{\tilde{S}} = 1 \quad (1 \leqq j \leqq n) ,$$

$$(C_i \cdot C_j)_{\tilde{S}} = 0 \quad \text{if} \quad |i-j| \geqq 2 .$$

Let $\tilde{E}$ be the proper transform of $E$ in $\tilde{S}$. Assume that

(i) $N_{\tilde{E}|\tilde{S}} \cong \mathcal{O}_{\tilde{E}}(-1)$ , where $N_{\tilde{E}|\tilde{S}}$ is the normal bundle of $\tilde{E}$ in $\tilde{S}$ , and

(ii) deg $N_{E|X} = -2$ , where $N_{E|X}$ is the normal normal bundle of $E$ in $X$ .

Then we have

(1) $N_{E|X} \cong \mathcal{O}_E \oplus \mathcal{O}_E(-2)$ if $x \in E$ and $(C_j \cdot \tilde{E})_{\tilde{S}} = 1$ for $(j = 1$ or $n+1)$ , or

(2) $N_{E|X} \cong \mathcal{O}_E(-1) \oplus \mathcal{O}_E(-1)$ if $x \notin E$ .

Proof. In the proof of Theorem 3.2 in Morrison [7] we have only to replace the conormal bundle $N^*_{\tilde{E}|\tilde{S}} = \mathcal{O}_{\tilde{E}}(2)$ with $N^*_{\tilde{E}|\tilde{S}} = \mathcal{O}_{\tilde{E}}(1)$ .

Now, we will resolve the indeterminancy.

(Step I). Let $\delta_1 : X_1 \longrightarrow X = V_5$, $E_1$, $Y_1$, $C_1$ and $x_1$ be as in § 2. Let $K_{X_1}$ be the canonical divisor on $X_1$ . Then we have

$$-K_{X_1} = 2\delta_1^*H - 2E_1 .$$

Since

$$(-K_{X_1} \cdot C_1) = 2(\delta_1^*H \cdot C_1) - 2(E_1 \cdot C_1) = 2 - 2 = 0 ,$$

we have

$$\deg N_{C_1|X_1} = -2 .$$

Since $x_1 \in C_1$ and the normal bundle $N_{C|\tilde{Y}} = \mathcal{O}_C(-1)$, by Lemma 3, we have

$$N_{C_1|X_1} \cong \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_1}(-2) .$$

Let $\Delta_1$ be a small neighbourhood of the singularity $x_1$ of $Y_1$ in $X_1$ with a local coordinate system $(Z_1, Z_2, Z_3)$. Since $x_1$ is of $A_2$-type and $(C\cdot Z)_{\tilde{Y}} = (C\cdot f_2)_{\tilde{Y}} = 1$, we may assume that

$$\text{(i)}\quad \Delta_1 \cap Y_1 = \{Z_1\cdot Z_2 = Z_3^3\} \hookrightarrow \Delta_1$$

$$\text{(ii)}\quad \Delta_1 \cap C_1 = \{Z_1 = Z_3,\ Z_2 = Z_3^2\} \hookrightarrow \Delta_1 .$$

(<u>Step II</u>). Let $\delta_2 : X_2 \longrightarrow X_1$ be the blowing up along $C_1$ and put $C_1' = \delta_2^{-1}(C_1)$. By Step I, we have that $C_1' \cong \mathbb{F}_2$. Let $Y_2$, $E_2$ be the proper transform of $Y_1$, $E_1$ in $X_2$ respectively. We find that $Y_2$ has exactly one singularity $x_2$ of $A_1$-type. Then, there exists the birational morphism

$$\mu_2 : \tilde{Y} \longrightarrow Y_2$$

such that

$$\mu_2^{-1}(x_2) = f_2$$

$$\tilde{Y} - f_2 \overset{\mu_2}{\cong} Y_2 - \{x_2\} \quad \text{(isomorphic)} .$$

We put $C_2 := \mu_2(C)$, $f_1^{(2)} := \mu_2(f_1)$, $l_i^{(2)} := \mu_2(l_i)$ $(i = 1,2)$ then $C_1' \cdot Y_2 = f_1^{(2)} + C_2$, in particular, $f_1^{(2)}$ (resp. $C_2$ ) is a fiber (resp. negative section) of the ruled surface $C_1' \cong \mathbb{F}_2$, and $l_i^{(2)}$, $C_2$ are the proper transforms of $l_i^{(1)}$, $C_1$ in $X_2$ respectively. Thus we have $(l_i^{(2)} \cdot l_i^{(2)})_{E_2} = 0$ $(i = 1,2)$, and

$$(f_1^{(2)} \cdot f_1^{(2)})_{E_2} = -1 .$$

Since $K_{X_2} = \delta_2^* K_{X_1} + C_1'$, we have

$$\begin{aligned}(C_2 \cdot K_{X_2}) &= (C_1 \cdot K_{X_1}) + (C_2 \cdot C_1') \\ &= 0 ,\end{aligned}$$

hence

$$\deg N_{C_2|X_2} = -2 .$$

Since $x_2 \in C_2$, applying Lemma 3, we have

$$N_{C_2|X_2} \cong \mathcal{O}_{C_2} \oplus \mathcal{O}_{C_2}(-2) .$$

Let $\Delta_2$ be a small neighbourhood of $x_2$ in $X_2$ with a local coordinate system $(Z_1, Z_2, Z_3)$. Then we may assume that

(i) $\Delta_2 \cap Y_2 = \{Z_1 Z_2 = Z_3^2\} \hookrightarrow \Delta_2$ ,

(ii) $\Delta_2 \cap C_2 = \{Z_1 = Z_2 = Z_3\} \hookrightarrow \Delta_2$ .

(<u>Step III</u>). Let $\delta_3 : X_3 \longrightarrow X_2$ be the blowing up along $C_2$ and put $C_2' = \delta_3^{-1}(C_2)$ . By Step II, we have that $C_2' \cong \mathbb{F}_2$ . Let $Y_3$, $E_3$ be the proper transforms of $Y_2$, $E_2$ in $X_3$ respectively. We find that $Y_3$ is a smooth surface. Then there exists the isomorphism

$$\mu_3 : \tilde{Y} \xrightarrow{\sim} Y_3 .$$

We put $C_3 = \mu_3(C)$, $f_i^{(3)} := \mu_3(f_i)$ $(i = 1,2)$, $l_i^{(3)} = \mu_3(l_i)$ $(i = 1,2)$ . Then we have

$$C_2' \cdot Y_3 = f_2^{(3)} + C_3 ,$$

in particular, $f_2^{(3)}$ (resp. $C_3$ ) is a fiber (resp. negative section) of the ruled surface $C_2' \cong \mathbb{F}_2$, and $f_1^{(3)}, l_i^{(3)}$ $(i = 1,2)$ , $C_3$ are the proper transforms of $f_1^{(2)}$, $l_i^{(2)}$ $(i = 1,2)$, $C_2$ in $X_3$ respectively. Thus we have

$$(l_1^{(3)} \cdot l_1^{(3)})_{E_3} = (f_2^{(3)} \cdot f_2^{(3)})_{E_3} = -1,\ (f_1^{(3)} \cdot f_1^{(3)})_{E_3} = -2 ,$$
$$(l_2^{(3)} \cdot l_2^{(3)})_{E_3} = 0 \quad \text{and}$$

$$(C_3 \cdot l_1^{(3)})_{Y_3} = 0,\ (C_3 \cdot f_2^{(3)})_{Y_3} = 1 .$$

Since $K_{X_3} = \delta_3^* K_{X_2} + C_2'$ , we have

$$\begin{aligned}(C_3 \cdot K_{X_3}) &= (C_2 \cdot K_{X_2}) + (C_3 \cdot C_2') \\ &= 0 ,\end{aligned}$$

hence

$$\deg N_{C_3|X_3} = -2 .$$

Since $Y_3$ is smooth, applying Lemma 3, we have

$$N_{C_3|X_4} \cong \mathcal{O}_{C_3}(-1) \oplus \mathcal{O}_{C_3}(-1) .$$

(Step IV). Let $\delta_4 : X_4 \longrightarrow X_3$ be the blowing up along $C_3$ and put $C_3' = \delta_4^{-1}(C_3)$ . By Step III, we have that $C_3' \cong \mathbb{P}^1 \times \mathbb{P}^1$ . Let $Y_4$, $E_4$ be the proper transforms of $Y_3$, $E_3$ in $X_4$ respectively. Since $Y_3$ is smooth, we have also the isomorphism

$$\mu_4 : \tilde{Y} \xrightarrow{\sim} Y_4 .$$

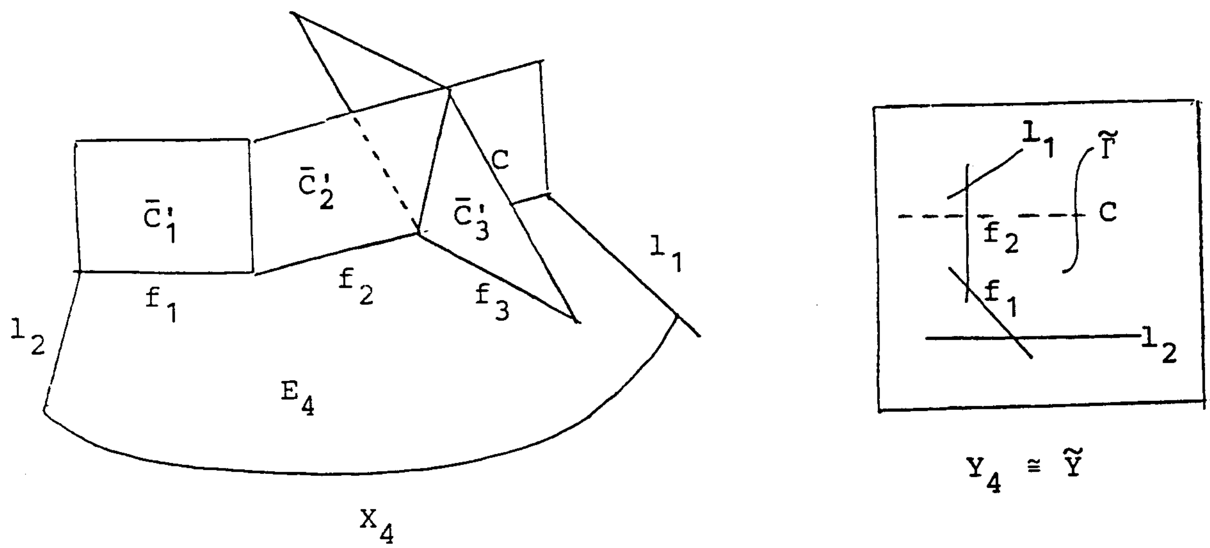
Let $f_3$ be a fiber of the ruled surface $\delta_4|_{C_3} : C_3' \longrightarrow C_3$ . We identify $\tilde{Y}$ with $Y_4$ via the isomorphism $\mu_4$ , and put, for simplicity, $f_i = \mu_4(f_i)$, $l_i = \mu_4(l_i)$ $(i = 1,2)$, $\tilde{\Gamma} = \mu_4(\tilde{\Gamma})$ and $C = \mu_4(C)$ . Then we have

$$C_3' \cdot Y_4 = C ,$$

in particular, $f_i$, $l_i$ $(i = 1,2)$, $C$ are the proper transforms of $f_i^{(3)}$, $l_i^{(3)}$, $C_3$ in $X_4$ respectively, and $(C\cdot C)_{C_3'} = 0$ , $(C\cdot f_3)_{C_3'} = 1$ .

$$\left.\begin{aligned}(l_1 \cdot l_1)_{E_4} &= -1,\ (l_2 \cdot l_2)_{E_4} = 0\\ (f_1 \cdot f_1)_{E_4} &= (f_2 \cdot f_2)_{E_4} = -2\\ (f_3 \cdot f_3)_{E_4} &= -1\ ,\end{aligned}\right\} \qquad (3.1)$$

and the figure below (see also Pagoda (5.8) in Reid [7]), where $\bar{C}_1'$, $\bar{C}_2'$ are the proper transform of $C_1'$, $C_2'$ in $X_4$ .




(Figure 1)

Now, since $Y_{j+1} = \delta_j^* Y_j - C_j'$ $(1 \leq j \leq 3)$ , we have

$$Y_4 = \delta_4^*\delta_3^*\delta_2^*\delta_1^* H - 2\delta_4^*\delta_3^*\delta_2^* E_1 - 3C_3' - 2\bar{C}_2' - \bar{C}_1' \ .$$

Therefore, a general hyperplane section $H$ of $X$ , we have

$$\mathcal{O}_{Y_4}(Y_4) = \mathcal{O}_{Y_4}(\tilde{\Gamma} - 2Z - f_1 - 2f_2 - 3C)\ ,$$

where $Z = l_1 + l_2 + f_1 + f_2$ (see (2.2) in § 2). By (2.4), we have

$$\mathcal{O}_{Y_4}(\Gamma - 2Z - f_1 - 2f_2 - 3C) \cong \mathcal{O}_{Y_4}(f)$$
$$\cong \mathcal{O}_{\tilde{Y}}(f) \ .$$

Since $f$ is a fiber of the ruled surface $\tilde{Y} = Y_4$, $|\mathcal{O}_{Y_4}(f)|$ has no fixed component and no base point. Thus, $|\mathcal{O}_{Y_4}(f)|$ defines a morphism $\varphi := \varphi_{|\mathcal{O}_{Y_4}(f)|} : Y_4 \longrightarrow \mathbb{P}^1$. Then $Y_4$ is a ruled surface over a smooth rational curve $\mathbb{P}^1$ with exactly one singular fiber $2C + 2f_2 + f_1 + l_1$, in particular, $l_2$ is a section. Let us consider the exact sequence:

$$0 \longrightarrow \mathcal{O}_{X_4} \longrightarrow \mathcal{O}_{X_4}(Y_4) \longrightarrow \mathcal{O}_{Y_4}(Y_4) \longrightarrow 0 \ .$$

Since $H^1(X_4, \mathcal{O}_{X_4}) = 0$ and the linear system $|\mathcal{O}_{Y_4}(Y_4)|$ has no fixed component and no base point, so is $|Y_4| = |\mathcal{O}_{X_4}(Y_4)|$. Therefore, it defines a morphism

$$\bar{\Psi} := \bar{\Psi}_{|Y_4|} : X_4 \longrightarrow \mathbb{P}^2$$

of $X_4$ onto $\mathbb{P}^2$ such that

$$\bar{\Psi}^* \mathcal{O}_{\mathbb{P}^2}(1) \cong \mathcal{O}_{X_4}(Y_4) \ .$$

In particular, we have the following diagram:

$$\begin{array}{ccc} X_1 & \xleftarrow{\ \delta\ } & X_4 \\ \downarrow{\scriptstyle \delta_1} & \searrow^{\Phi^{(1)}} & \downarrow{\scriptstyle \bar{\Psi}} \\ X & \xrightarrow[\Phi]{\ -\,-\,-\ } & \mathbb{P}^2 \end{array} \quad ,$$

where $\delta = \delta_2 \circ \delta_3 \circ \delta_4$ .

This is the desired resolution of indeterminancy of the rational map

$$\Phi^{(1)} : X_1 \dashrightarrow \mathbb{P}^2 .$$

## § 4. Structure of $V_5$

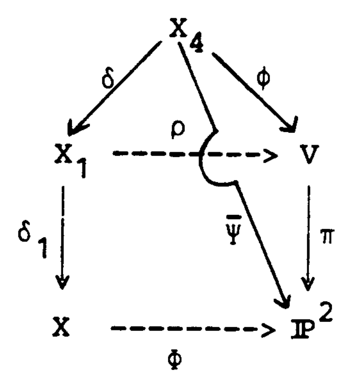
Let $X_4$, $Y_4 = \tilde{Y}$, $C_3' \cong \mathbb{P}^1 \times \mathbb{P}^1$ , and $C$ be as in § 3. Since

$$N_{C_3|X_3} \cong \mathcal{O}_{C_3}(-1) \oplus \mathcal{O}_{C_3}(-1) ,$$

by Corollary 5.6 in Reid [9], there exists the birational morphism $\phi : X_4 \longrightarrow V$ of $X_4$ onto a smooth 3-fold with $b_2(V) = 2$ , and the morphism $\pi : V \longrightarrow \mathbb{P}^2$ of $V$ onto $\mathbb{P}^2$ , and the birational map which is called "flip" $\rho : X_1 \dashrightarrow V$ such that

$$\rho = \phi \circ \delta^{-1} \quad \text{and}$$

$$\bar{\Psi} = \pi \circ \phi$$




In particular, $\bar{f}_3 := \phi(\bar{C}_1' \cup \bar{C}_2' \cup C_3')$ is a smooth rational curve in $V$ , and

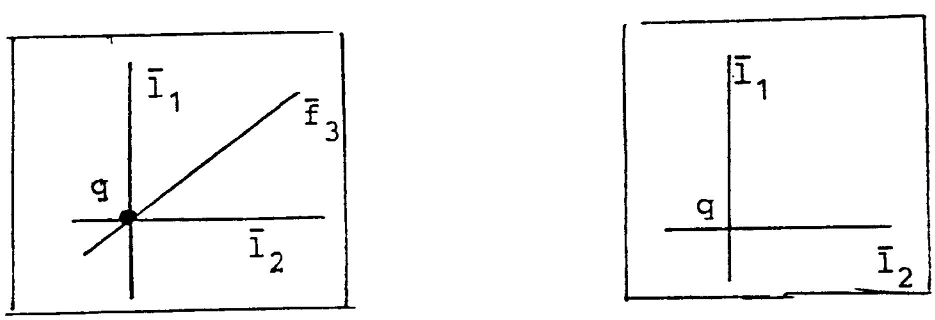
$$X_4 - (\bar{C}_1' \cup \bar{C}_2' \cup \bar{C}_3') \xrightarrow[\sim]{\phi} V - \bar{f}_3 \xleftarrow[\sim]{\rho} X_1 - C_1 . \qquad (4.1)$$

We put $A = \phi(Y_4)$ and $\Sigma = \phi(E_4)$ . Then,

$$-K_V = 2A + 2\Sigma \tag{4.2}$$

$$\mathcal{O}_V(A) = \pi^*\mathcal{O}_{\mathbb{P}^2}(1) \ . \tag{4.3}$$

In fact, since $-K_{X_1} = 2\delta_1^*H - 2E_1 = 2Y_1 + 2E_1$ and $\mathcal{O}_{X_4}(Y_4) \cong \bar{\Psi}^*\mathcal{O}_{\mathbb{P}^2}(1)$, by (4.1), we have (4.2), (4.3). We put $\bar{l}_i = \phi(l_i)$ $(i = 1,2)$ and $L = \pi(\bar{l}_2) \hookrightarrow \mathbb{P}^2$. Then $\bar{l}_i$'s are smooth rational curves in $V$ and $L$ is a line in $\mathbb{P}^2$, in particular, $\pi|_A : A \longrightarrow L$ has a structure of the $\mathbb{P}^1$-bundle $\mathbb{F}_1$ with $\bar{l}_1$ a fiber and $\bar{l}_2$ the negative section. Moreover, $\Sigma$ has only one singularity $q$ of $A_2$-type. The rational curves $\bar{l}_1$, $\bar{l}_2$, $\bar{f}_3$, which are also contained in $\Sigma$, intersect only at the point $q$ (see Figure 2).




Σ A

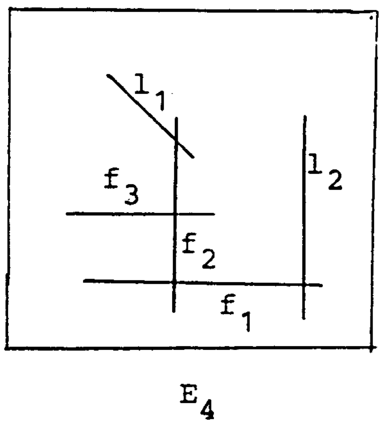
(Figure 2)

By construction, $\sigma := \phi|_{E_4} : E_4 \longrightarrow \Sigma$ is the minimal resolution of the singularity of $\Sigma$ with $\sigma^{-1}(q) = f_1 \cup f_2$, and $\bar{l}_i = \sigma(l_i)$ $(i = 1,2)$, $\bar{f}_3 = \sigma(f_3)$ (see (3.1) and Figure 3).

We put $\lambda := \pi|_{\Sigma} : \Sigma \longrightarrow \mathbb{P}^2$ . Then

$$(\lambda \circ \sigma)(f_1 \cup f_2 \cup l_1) = L \cap \underline{f}_3 = \{p\} \quad \text{(a point)}$$

where $\underline{f}_3 = \pi(\bar{f}_3)$ .




(Figure 3)

For a general fiber $F$ of the morphism $\pi : V \longrightarrow \mathbb{P}^2$ , we have, by (4.2),

$$\begin{aligned} \deg(K_F) &= (K_V \cdot F) = -2(\Sigma \cdot F) \\ &\leqq -2 , \end{aligned}$$

hence $F \cong \mathbb{P}^1$ and $(\Sigma \cdot F) = 1$ , where $K_F$ is the canonical divisor on $F$ . Therefore $\Sigma$ is a meromorphic section of $\pi : V \longrightarrow \mathbb{P}^2$ .

Proposition 1. $\pi : V \longrightarrow \mathbb{P}^2$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$ and $\Sigma$ is a holomorphic section on $\mathbb{P}^2 - \{p\}$ .

Proof. By construction,

$$\mathbb{C}^3 \cong X - Y \overset{\delta_1}{\cong} X_1 - (Y_1 \cup E_1) \overset{\rho}{\cong} V - (A \cup \Sigma) .$$

In particular, $\pi : V - (A \cup \Sigma) \longrightarrow \mathbb{P}^2 - L$ is an affine morphism. Assume that there exists an irreducible divisor $D$ on $V$ such that $\pi(D) = \{\text{one point}\}$ . Then one dimensional scheme $D \cap \Sigma$ is contracted to one point, hence, $\mathrm{Supp}(D \cap \Sigma) = \bar{I}_1$ . Since $\bar{I}_1 \subseteq A = \pi^{-1}(L)$ and $\pi|_A : A \longrightarrow L$ is a $\mathbb{P}^1$-bundle, this is a contradiction. Thus $\pi$ is equi-dimensional, hence, $\pi$ is proper flat morphism. Let $G$ be an arbitrary scheme theoric fiber. Then $(\Sigma \cdot G)_V = 1$ . Since $V - (A \cup \Sigma) \cong \mathbb{C}^3$ contains no compact analytic curve, $G$ must be irreducible. Since $(K_V \cdot G) = -2(\Sigma \cdot G) = -2$, $G$ is a smooth rational curve. Therefore $\pi : V \longrightarrow \mathbb{P}^2$ is a smooth proper morphism. By the upper semicontinuity theorem, we have that $R^1\pi_* \mathcal{O}_V(\Sigma) = 0$ and $\pi_* \mathcal{O}_V(\Sigma)$ is a vector bundle of rank 2 over $\mathbb{P}^2$ . Moreover, for every point $x \in \mathbb{P}^2$ ,

$$\begin{aligned} \pi_* \mathcal{O}_V(\Sigma) \otimes \mathbb{C}(x) &\cong H^0(\pi^{-1}(x), \mathcal{O}_V(\Sigma) \otimes \mathcal{O}_{\pi^{-1}(x)}) \\ &\cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \\ &\cong \mathbb{C}^2 . \end{aligned}$$

Thus the natural homomorphism

$$\pi^* \pi_* \mathcal{O}_V(\Sigma) \longrightarrow\!\!\!\!\!\to \mathcal{O}_V(\Sigma)$$

is surjective and induces the isomorphism $V \cong \mathbb{P}(\pi_* \mathcal{O}_V(\Sigma))$ over $\mathbb{P}^2$ . The rest is clear.

Q.E.D.

Remark. $\pi$ is the contraction of an extremal ray of the smooth projective 3-fold $V$ .

Finally, we will study the vector bundle $\pi_* \mathcal{O}_V(\Sigma)$ of rank 2 over $\mathbb{P}^2$ .

Lemma 4. $\mathcal{O}_\Sigma(\Sigma) = \mathcal{O}_\Sigma(-3\bar{l}_1 + A)$ .

Proof. Since the singularity of $\Sigma$ is rational double point, we have

$$\sigma^* K_\Sigma = K_{E_4}$$

$$= -2f_1 - f_2 - 3l_2 ,$$

hence $K_\Sigma = -3\bar{l}_2$ . On the other hand, since

$$K_\Sigma = K_{V|\Sigma} + \Sigma|_\Sigma$$

$$= -2A|_\Sigma - \Sigma|_\Sigma$$

we have

$$\Sigma|_\Sigma = -2A|_\Sigma - K_\Sigma$$

$$= -2A|_\Sigma + 3\bar{l}_2 \quad .$$

Since $A|_\Sigma = \bar{I}_1 + \bar{I}_2$ , we have

$$\Sigma|_\Sigma = -3\bar{I}_1 + A|_\Sigma ,$$

namely, $\mathcal{O}_\Sigma(\Sigma) = \mathcal{O}_\Sigma(-3\bar{I}_1 + A)$ . Q.E.D.

Let us consider the exact sequence

$$0 \longrightarrow \mathcal{O}_V \longrightarrow \mathcal{O}_V(\Sigma) \longrightarrow \mathcal{O}_\Sigma(\Sigma) \longrightarrow 0 .$$

Taking $\pi_*$ , we have

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow \pi_*\mathcal{O}_V(\Sigma) \longrightarrow \pi_*\mathcal{O}_\Sigma(\Sigma) \longrightarrow 0 . \qquad (4.4)$$

Taking $\pi^*$ in (4.4), we have the diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_V & \longrightarrow & \pi^*\pi_*\mathcal{O}_V(\Sigma) & \longrightarrow & \pi^*\pi_*\mathcal{O}_\Sigma(\Sigma) & \longrightarrow & 0 \\
 & & \wr\| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_V & \longrightarrow & \mathcal{O}_V(\Sigma) & \longrightarrow & \mathcal{O}_\Sigma(\Sigma) & \longrightarrow & 0 ,
\end{array}$$

in particular, we have the surjection

$$\pi^*\pi_*\mathcal{O}_\Sigma(\Sigma) \longrightarrow\!\!\!\!\rightarrow \mathcal{O}_\Sigma(\Sigma) .$$

We put $\lambda := \pi|_\Sigma : \Sigma \longrightarrow \mathbb{P}^2$ . Taking $\lambda^*$ in (4.4), we have the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_\Sigma & \longrightarrow & \lambda^*\pi_*\mathcal{O}_V(\Sigma) & \longrightarrow & \lambda^*\pi_*\mathcal{O}_\Sigma(\Sigma) & \longrightarrow & 0 \\
 & & \downarrow & & \wr\| & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & \lambda^*\pi_*\mathcal{O}_V(\Sigma) & \xrightarrow{\tau} & \mathcal{O}_\Sigma(\Sigma) & \longrightarrow & 0 ,
\end{array}$$

where $L := \ker \tau$ is a line bundle, and the image of the global section 1 of $\mathcal{O}_\Sigma$ via the map $\mathcal{O}_\Sigma \longrightarrow L$ defines an effective Cartier divisor D with $\mathrm{Supp}(D) = \bar{l}_1$ .

<u>Proposition 2</u>. $\lambda^*\pi_*\mathcal{O}_V(\Sigma)$ is an extension of $\mathcal{O}_\Sigma(\Sigma)$ by $\mathcal{O}_\Sigma(3\bar{l}_1)$ .

<u>Proof</u>. We have only to prove that $D = 3\bar{l}_1$ . Since $\lambda^*(\det(\pi_*\mathcal{O}_V(\Sigma))) = \mathcal{O}_\Sigma(\Sigma) \otimes \mathcal{O}_\Sigma(3\bar{l}_1)$, $(\Sigma\cdot\bar{l}_1)_\Sigma + (D\cdot\bar{l}_1)_\Sigma = 0$ . Since $\mathcal{O}_\Sigma(\Sigma) = \mathcal{O}_\Sigma(-3\bar{l}_1 + A)$ by Lemma 4, we must have $D = 3\bar{l}_1$ , and also, by (4.3), we have $\det(\pi_*\mathcal{O}_Y(\Sigma)) = \mathcal{O}_{\mathbb{P}^2}(1)$ .

Q.E.D.

<u>Remark</u>. $\pi_*\mathcal{O}_Y(\Sigma)$ is an extension of $\lambda_*\mathcal{O}_\Sigma(-3\bar{l}_1)\otimes\mathcal{O}_{\mathbb{P}^2}(1)$ by $\mathcal{O}_{\mathbb{P}^2}$ . Therefore $\pi_*\mathcal{O}_Y(\Sigma)$ is a stable vector bundle.

## § 5. A construction of a compactification of $\mathbb{C}^3$

One can easily construct the surfaces $E_3$, $\Sigma$ and the morphisms $\sigma : E_4 \longrightarrow \Sigma$, $\lambda : \Sigma \longrightarrow \mathbb{P}^2$ in § 4 independent of the arguments there. Therefore we may assume the existence of these surfaces and morphisms. We recall some facts on them;

(i) $\Sigma$ has exactly one singular point $q$ of $A_2$-type

(ii) $E_4-(f_1 \cup f_2) \xrightarrow[\sim]{\sigma} \Sigma - \{q\}$ (isomorphic)

(iii) $\Sigma - \bar{l}_1 \xrightarrow[\sim]{\lambda} \mathbb{P}^2 - \{P\}$ (isomorphic)

(iv) $L = \lambda(\bar{l}_2)$, $\underline{f}_3 = \lambda(\bar{f}_3)$ are two lines on $\mathbb{P}^2$ .

<u>Lemma 5</u>. As $\mathbb{Q}$-Cartier divisors, we have

$$\begin{cases} \sigma^*\bar{l}_1 \sim_{\mathbb{Q}} l_1 + \frac{1}{3}f_1 + \frac{2}{3}f_2 \\ \sigma^*\bar{l}_2 \sim_{\mathbb{Q}} l_2 + \frac{2}{3}f_1 + \frac{1}{3}f_2 \\ \sigma^*\bar{f}_3 \sim_{\mathbb{Q}} f_3 + \frac{1}{3}f_1 + \frac{2}{3}f_2 \end{cases} \tag{5.1}$$

and the linear equivalences

$$\begin{cases} l_1 + f_2 + f_3 \sim l_2 \\ \bar{l} \sim \bar{l}_2 + \bar{l}_1 \sim \bar{f}_3 + 2\bar{l}_1 \\ K_{E_4} = \sigma^*K_{\Sigma} \sim \sigma^*(-3\bar{l}) + f_1 + 2f_2 + 3l_1 , \end{cases} \tag{5.2}$$

where $K_{E_4}$ is the canonical divisor on $E_4$ , and $\bar{l} = \lambda^*\mathcal{O}_{\mathbb{P}^2}(1)$ .

Proof. Since $(\sigma^*\bar{l}_1 \cdot f_1) = (\sigma^*\bar{l}_2 \cdot f_i) = (\sigma^*\bar{f}_3 \cdot f_i) = 0$ for $i = 1,2$ , we have (5.1). By a similar calculation we have (5.2).

Now, we will prove the existence of a vector bundle of rank 2 over $\mathbb{P}^2$ which is an extension of $\mathcal{O}_\Sigma(-3\bar{l}_1 + \bar{l})$ by $\mathcal{O}_\Sigma(3\bar{l}_1)$ .

Lemma 6.

1) $\mathrm{Ext}^1_\Sigma(\mathcal{O}_\Sigma(-3\bar{l}_1+\bar{l}), \mathcal{O}_\Sigma(3\bar{l}_1)) \cong \mathrm{Ext}^1_{E_4}(\sigma^*\mathcal{O}_\Sigma(-3\bar{l}_1+\bar{l}), \sigma^*\mathcal{O}_\Sigma(3\bar{l}_1))$ .

2) $\mathrm{Ext}^1_{E_4}(\sigma^*\mathcal{O}_\Sigma(-3\bar{l}_1+\bar{l}), \sigma^*\mathcal{O}_\Sigma(3\bar{l}_1)) \longrightarrow$

$\longrightarrow \mathrm{Ext}^1_{l_1}(\sigma^*\mathcal{O}_\Sigma(-3\bar{l}_1+\bar{l}) \otimes \mathcal{O}_{l_1}, \sigma^*\mathcal{O}_\Sigma(3\bar{l}_1) \otimes \mathcal{O}_{l_1})$ is surjective.

3) $\dim \mathrm{Ext}^1_\Sigma(\mathcal{O}_\Sigma(-3\bar{l}_1+\bar{l}), \mathcal{O}_\Sigma(3\bar{l}_1)) = 3$

$\dim \mathrm{Ext}^1_{l_1}(\sigma^*\mathcal{O}_\Sigma(-3\bar{l}_1+\bar{l}) \otimes \mathcal{O}_{l_1}, \sigma^*\mathcal{O}_\Sigma(3\bar{l}_1) \otimes \mathcal{O}_{l_1}) = 1$ .

Proof. 1) Since $\mathrm{Ext}^1_\Sigma(\mathcal{O}_\Sigma(-3\bar{l}_1+\bar{l}), \mathcal{O}_\Sigma(3l_1)) \cong H^1(\Sigma, \mathcal{O}_\Sigma(6\bar{l}_1-\bar{l}))$ and $\mathrm{Ext}^1_{E_4}(\sigma^*\mathcal{O}_\Sigma(-3\bar{l}_1+\bar{l}), \sigma^*\mathcal{O}_\Sigma(3\bar{l}_1)) \cong H^1(E_4, \sigma^*\mathcal{O}_\Sigma(6\bar{l}_1-\bar{l}))$ , we have only to prove $H^1(\Sigma, \mathcal{O}_\Sigma(6\bar{l}_1-\bar{l})) \xrightarrow{\sim} H^1(E_4, \sigma^*\mathcal{O}_\Sigma(6\bar{l}_i-\bar{l}))$ . Since $R^1\sigma_*\mathcal{O}_{E_4} = 0$ , it is clear. 2) We have only to prove that the morphism $H^1(E_4, \sigma^*\mathcal{O}_\Sigma(6\bar{l}_1-\bar{l})) \longrightarrow H^1(l_1, \sigma^*\mathcal{O}_\Sigma(6\bar{l}_1-\bar{l}) \otimes \mathcal{O}_{l_1})$ is surjective. For this, let us consider the exact sequence:

$$0 \to \sigma^*\mathcal{O}_\Sigma(6\bar{l}_1-\bar{l}) \otimes \mathcal{O}_{E_4}(-l_1) \to \sigma^*\mathcal{O}_\Sigma(6\bar{l}_1-\bar{l}) \to \sigma^*\mathcal{O}_\Sigma(6\bar{l}_1-\bar{l}) \otimes \mathcal{O}_{l_1} \to 0 .$$

By Lemma 5, we have

$$\sigma^*\mathcal{O}_\Sigma(6\bar{l}_1-\bar{l}) \cong \mathcal{O}_{E_4}(6l_1+2f_1+4f_2-\sigma^*l) \cong \mathcal{O}_{E_4}(2K_{E_4}+5\sigma^*\bar{l}) \ ,$$

hence

$$H^2(E_4,\mathcal{O}_{E_4}(2K_{E_4}+5\sigma^*\bar{l}-l_1)) \cong H^0(E_4,\mathcal{O}_{E_4}(-K_{E_4}-5\sigma^*\bar{l}-l_1)) \cong H^0(E_4,\mathcal{O}_{E_4}(-2\sigma^*\bar{l}-f_1-2f_2-2l_1)) \cong 0 \ .$$

Therefore, we have the surjection

$$H^1(E_4,\sigma^*\mathcal{O}_{E_4}(6\bar{l}_1-\bar{l})) \longrightarrow\!\!\!\!\to H^1(l_1,\sigma^*\mathcal{O}_{E_4}(6\bar{l}_1-\bar{l}) \otimes \mathcal{O}_{l_1}) \ .$$

Since

$$(\sigma^*(-3\bar{l}_1+\bar{l}) \cdot l_1)_{E_4} = 1, \ (\sigma^*(3\bar{l}_1) \cdot l_1)_{E_4} = -1 \ ,$$

we have

$$\mathrm{Ext}^1_{l_1}(\sigma^*\mathcal{O}_\Sigma(-3\bar{l}_1+\bar{l}) \otimes \mathcal{O}_{l_1}, \ \sigma^*\mathcal{O}_\Sigma(3\bar{l}_1) \otimes \mathcal{O}_{l_1}) \cong \mathrm{Ext}^1_{\mathbb{P}^1}(\mathcal{O}(1),\mathcal{O}(-1)) \cong H^1(\mathbb{P}^1,\mathcal{O}(-2)) \cong \mathbb{C} \ .$$

Finally, we prove that $H^1(E_4,\mathcal{O}_{E_4}(2K_{E_4} + 5\sigma^*\bar{l})) = \mathbb{C}^3$ . By Lemma 5, $K_{E_4} = -3\sigma^*\bar{l} + f_1 + 2f_2 + 3l_1$ . We put $G := \sigma^*\bar{l}$ . Then $(G^2)_{E_4} = 1$, $(G\cdot f_1)_{E_4} = (G\cdot f_2)_{E_4} = (G\cdot l_1)_{E_4} = 0$ , and $f_1 \cup f_2 \cup l_1$ can be contracted to a smooth point of

We need the following well known

<u>Lemma 7</u>. Let $\nu : S \longrightarrow T$ be the blowing up at the point $P$ on a smooth surface $T$, and put $\nu^{-1}(P) = C$. Them a vector bundle $\mathcal{E}$ on $S$ is the pullback of a vector bundle on $T$ if and only if

$$\mathcal{E}|_C \cong \mathcal{O}_C^{\oplus r} ,$$

where $r = \operatorname{rank} \mathcal{E}$.

Let $E_4 \xrightarrow{\sigma} \Sigma \xrightarrow{\lambda} \mathbb{P}^2$ be as before. We put $\mu := \lambda\circ\sigma$. Then $E_4 - (f_1 \cup f_2 \cup l_1) \overset{\mu}{\cong} \mathbb{P}^2 - \{0\}$, indeed, $E_4$ can be obtained from $\mathbb{P}^2$ by 3 times blowing ups, and $f_1 \cup f_2 \cup l_1$ is the exceptional divisor associated with the blowing ups.

Let $\mathcal{E} = \mathcal{E}_\xi$ be the vector bundle on $E_4$ determined by an element $\xi \in \operatorname{Ext}^1_{E_4}(\sigma^*\mathcal{O}_\Sigma(-3\bar{l}_1+\bar{l}),\ \sigma^*\mathcal{O}_\Sigma(3\bar{l}_1))$, where the image of $\xi$ by the surjection:

$$\operatorname{Ext}^1_{E_4}(\sigma^*\mathcal{O}_\Sigma(-3\bar{l}_1+\bar{l}),\sigma^*\mathcal{O}_\Sigma(3\bar{l}_1)) \twoheadrightarrow \operatorname{Ext}^1_{l_1}(\sigma^*\mathcal{O}_\Sigma(-3\bar{l}_1+\bar{l}),\sigma^*\mathcal{O}_\Sigma(3l_1)) \cong \mathbb{C}$$

is not zero.

Then $\mathcal{E} \otimes \mathcal{O}_{l_1}$ induces the non-split exact sequence

$$0 \longrightarrow \mathcal{O}_{l_1}(-1) \longrightarrow \mathcal{E}\otimes\mathcal{O}_{l_1} \longrightarrow \mathcal{O}_{l_1}(1) \longrightarrow 0 ,$$

hence

a surface. By Lemma 5, we also have

$$2K_{E_4} + 5\sigma^*\bar{l} = -G + 2f_1 + 4f_2 + 6l_1 .$$

Since $f_1 \cup f_2 \cup l_1$ can be contracted to a smooth point, we have

$$H^0(E_4, \mathcal{O}_{E_4}(-G+2f_1+4f_2+6l_1)) = 0 ,$$
$$H^2(E_4, \mathcal{O}_{E_4}(-G+2f_1+4f_2+6l_1)) \cong H^0(E_4, \mathcal{O}_{E_4}(-2G-f_1-2f_2-3l_1)) = 0 .$$

By Riemann-Roch theorem, we have easily

$$\dim H^1(E_4, \mathcal{O}_{E_4}(-G+2f_1+4f_2+6l_1)) = 3 ,$$

hence

$$H^1(E_4, \mathcal{O}_{E_4}(2K_{E_4}+5\sigma^*\bar{l})) \cong \mathbb{C}^3 .$$

Q.E.D.

$$E \otimes \mathcal{O}_{l_1} \cong \mathcal{O}_{l_1} \oplus \mathcal{O}_{l_1} .$$

On the other hand, we have

$$\sigma^*\mathcal{O}_\Sigma(-3\bar{l}_1+\bar{l}) \otimes \mathcal{O}_{f_1} \cong \mathcal{O}_{f_1}$$

$$\sigma^*\mathcal{O}_\Sigma(3l_1) \otimes \mathcal{O}_{f_1} \cong \mathcal{O}_{f_1}$$

$$\sigma^*\mathcal{O}_\Sigma(-3\bar{l}_1+\bar{l}) \otimes \mathcal{O}_{f_2} \cong \mathcal{O}_{f_2}$$

$$\sigma^*\mathcal{O}_\Sigma(3l_1) \otimes \mathcal{O}_{f_2} \cong \mathcal{O}_{f_2} .$$

Thus

$$E \otimes \mathcal{O}_{f_1} \cong \mathcal{O}_{f_1}^{\oplus 2}$$

$$E \otimes \mathcal{O}_{f_2} \cong \mathcal{O}_{f_2}^{\oplus 2} .$$

By Lemma 7, there exist a vector bundle $\tilde{E}$ on $\mathbb{P}^2$ such that $E = \mu^*\tilde{E}$ , and then we have the exact sequence

$$0 \longrightarrow \sigma^*\mathcal{O}_\Sigma(3\bar{l}_1) \longrightarrow \mu^*\tilde{E} \longrightarrow \sigma^*\mathcal{O}_\Sigma(-3\bar{l}_1+\bar{l}) \longrightarrow 0 \qquad (5.1)$$

Taking $\sigma_*$ , we have the exact sequence

$$0 \longrightarrow \mathcal{O}_\Sigma(3\bar{l}_1) \longrightarrow \lambda^*\tilde{E} \longrightarrow \mathcal{O}_\Sigma(-3\bar{l}_1+\bar{l}) \longrightarrow 0 \qquad (5.2)$$

Further, taking $\lambda_*$ , we have

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow \tilde{E} \longrightarrow \lambda_*\mathcal{O}_\Sigma(-3\bar{l}_1)\otimes\mathcal{O}_{\mathbb{P}^2}(1) \longrightarrow 0 , \qquad (5.3)$$

since $R^1\lambda_*\mathcal{O}_\Sigma(3\bar{l}_1) \cong 0$ by Grauert-Riemenschneider vanishing theorem.

We remark that $\lambda : \Sigma \longrightarrow \mathbb{P}^2$ is the blowing up of $\mathbb{P}^2$ by the ideal $J := \lambda_*\mathcal{O}_\Sigma(-3\bar{l}_1)$ . By (5.2), we have the $\mathbb{P}^1$-bundle $V := \mathbb{P}(\tilde{E}) \xrightarrow{\pi} \mathbb{P}^2$ over $\mathbb{P}^2$ and a rational section $E \hookrightarrow V$ .

<u>Lemma 8</u>. $\tilde{E} \otimes \mathcal{O}_L \cong \mathcal{O}_L(1) \oplus \mathcal{O}_L$ .

<u>Proof</u>. Let us consider the exact sequence

$$0 \longrightarrow \mathcal{O}_\Sigma(3\bar{l}_1)\otimes\mathcal{O}_{\bar{l}_2} \longrightarrow \lambda^*E\otimes\mathcal{O}_{\bar{l}_2} \longrightarrow \mathcal{O}_\Sigma(-3\bar{l}_1+\bar{l})\otimes\mathcal{O}_{\bar{l}_2} \longrightarrow 0 .$$

Since $(3l_1\cdot\bar{l}_2)_\Sigma = (\bar{l}\cdot\bar{l}_2)_\Sigma = 1$ , we have

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow \lambda^*E\otimes\mathcal{O}_{\bar{l}_2} \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow 0 \quad \text{(exact)},$$

namely, $\lambda^*E\otimes\mathcal{O}_{\bar{l}_2} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ .

Q.E.D.

By Lemma 8, $\pi^{-1}(L) := A$ is the $\mathbb{P}^1$-bundle $\mathbb{F}_1$ over $L \cong \mathbb{P}^1$ . Since $\lambda^*(L) = \bar{l}_2+\bar{l}_1$ on $\Sigma$ , we have

$$\Sigma \cdot A = \bar{l}_1+\bar{l}_2 .$$

<u>Lemma 9</u>. $N_{\bar{f}_3|V} \cong \mathcal{O}_{\mathbb{P}^1}(-2)\oplus\mathcal{O}_{\mathbb{P}^1}$ , where $N_{\bar{f}_3|V}$ is the normal bundle of $\bar{f}_3 \hookrightarrow \Sigma$ in $V$ .

<u>Proof</u>. Let $K_V$ be the canonical divisor on $V$ . Then

$$K_V = \pi^*(K_{\mathbb{P}^2} + \det \tilde{\mathcal{E}}) - 2\Sigma$$

$$= -2A - 2\Sigma$$

$$\mathcal{O}_\Sigma(\Sigma) = \mathcal{O}_\Sigma(-3\bar{l}_1 + A)$$

$$= \mathcal{O}_\Sigma(-3\bar{l}_1 + \bar{l}) \ .$$

Since $(K_V \cdot \bar{f}_3) = (-4\bar{l} + 6\bar{l}_1 \cdot \bar{f}_3)_\Sigma$

$$= -4 + 4 = 0 \ ,$$

by Lemma 3, we have the claim. Q.E.D.

<u>Lemma 10</u>. $V - (\Sigma \cup A) \cong \mathbb{C}^3$ .

Proof. Since $\pi|_{V-(\Sigma \cup A)} : V - (\Sigma \cup A) \longrightarrow \mathbb{P}^2 - L \cong \mathbb{C}^2$ is an affine $\mathbb{C}$-bundle over $\mathbb{C}^2$ , we have $V - (\Sigma \cup A) \cong \mathbb{C}^3$ .

Q.E.D.

Let $\phi_1 : V_1 \longrightarrow V = \mathbb{P}(\tilde{\mathcal{E}})$ be the blowing up along $\bar{f}_3$ and put $C_1' = \phi_1^{-1}(\bar{f}_3)$ . Then $C_1' \cong \mathbb{F}_2$ by Lemma 9 . Let $\Sigma_1$ be the proper transform of $\Sigma$ in $V_1$ . Then $\Sigma_1$ has the singularity $P_1$ of $A_1$-type, and there exists the birational morphism $\nu_1 : E_4 \longrightarrow \Sigma_1$ such that $\nu_1^{-1}(P_1) = f_2$ and $E_4 - f_2 \overset{\nu_1}{\cong} \Sigma_1 - \{P_1\}$ . We put $f_1^{(1)} = \nu_1(f_1)$ and $f_3^{(1)} := \nu_1(f_3)$ . Then $\Sigma_1 \cdot C_1' = f_1^{(1)} + f_3^{(1)}$ , in particular, $f_1^{(1)}$ is a fiber and $f_3^{(1)}$ is the negative section of $C_1' \cong \mathbb{F}_2$ . Since $P_1 \in f_3^{(1)}$ and $(K_{V_1} \cdot f_3^{(1)}) = (K_V \cdot \bar{f}_3) = 0$ , by Lemma 3, we have

$$N_{f_3^{(1)}|V_1} \cong \mathcal{O}\oplus\mathcal{O}(-2) \ .$$

Let $\phi_2 : V_2 \longrightarrow V_1$ be the blowing up along the curve $f_3^{(1)}$ and put $C_2' = \phi_2^{-1}(f_3^{(1)}) \cong \mathbb{F}_2$ . Let $\Sigma_2$ be the proper transform of $\Sigma_1$ in $V_2$ . Then $\Sigma_2$ is a smooth surface and there exists the isomorphism $\nu_2 : E_4 \xrightarrow{\sim} \Sigma_2$ . We put $f_i^{(2)} := \nu_2(f_i)$ for $i = 1,2,3$ . Then we have $\Sigma_2 \cdot C_2' = f_2^{(2)}+f_3^{(2)}$ , in particular, $f_2^{(2)}$ is a fiber and $f_3^{(2)}$ is the negative section of $C_2' \cong \mathbb{F}_2$ . Since $(K_{V_2} \cdot f_3^{(2)}) = (K_{V_1} \cdot f_3^{(1)}) = 0$ and $\Sigma_2$ is smooth, by Lemma 3, we have

$$N_{f_3^{(2)}|V_2} \cong \mathcal{O}(-1)\oplus\mathcal{O}(-1) \ .$$

Let $\phi_3 : V_3 \longrightarrow V_2$ be the blowing up along the curve $f_3^{(2)}$ and put $C_3' = \phi_3^{-1}(f_3^{(2)}) \cong \mathbb{P}^1 \times \mathbb{P}^1$ . Let $C$ be a fiber of the ruled surface $\phi_{3|C_3'} : C_3' \longrightarrow f_3^{(2)}$ , and $\Sigma_3$ be the proper transform of $\Sigma_2$ in $V_3$ . Then $\Sigma_3$ is a smooth surface and there exists the isomorphism $\nu_3 : E_4 \xrightarrow{\sim} \Sigma_3$ . We put $f_i = \nu_3(f_i)$ $(1 \leq i \leq 3)$, $l_i = \nu_3(l_i)$ $(1 \leq i \leq 2)$ , which are indeed the proper transforms of $f_i^{(2)}$ $(1 \leq i \leq 3)$ , $\nu_2(l_i)$ $(1 \leq i \leq 2)$ in $V_3$ respectively. Then, $\Sigma_3 \cdot C_3' = f_3$ , in particular, $(f_3^2)_{\Sigma_3} = 0$ and $(f_3 \cdot C)_{\Sigma_3} = 1$ (see Step IV and Figure 1 in § 4).

Since $C_3' \cong \mathbb{P}(\mathcal{O}(-1)\oplus\mathcal{O}(-1))$ , by Corollary 5.6 in Reid [7], $C_3'$ can be blown down along the fiber $f_3$ , and, step by step, the blowing down is done, and finally we have the smooth 3-fold $X_1$ with $b_2(X_1) = 2$ and the contraction morphism $\delta : V_3 \longrightarrow X_1$ .

We put $C_1 := \delta(C_3' \cup \bar{C}_2' \cup \bar{C}_1')$, $E_1 := \delta(\Sigma_3)$, and $Y_1 := \delta(A_3)$, where $\bar{C}_j'$ $(j = 1,2)$, $A_3$ are the proper transforms of $C_j'$ $(j = 1,2)$, $A$ in $V_3$ respectively. Then, by construction, one can easily see that $C_1$ is a smooth rational curve in $X_1$ with $C_1 \subset Y_1$, $E_1 \cong \mathbb{P}^2$, and $Y_1$ is a singular del Pezzo surface with an singularity of $A_2$-type. We put $\rho := (\phi_1 \circ \phi_2 \circ \phi_3)^{-1} \circ \delta$. Then $\rho$ is a birational map of $V$ onto $X_1$ such that $\rho : V - \bar{f}_3 \xrightarrow{\sim} X_1 - C$ (isomorphic). Since $K_V = -2A - 2\Sigma$, we have $K_{X_1} = -2Y_1 - 2E_1$. Since $E_1 \cdot Y_1 = l_1^{(1)} + l_2^{(2)}$, by the adjunction formula, $\mathcal{O}_{E_1}(E_1) = \mathcal{O}_{E_1}(-l_j^{(1)})$ for $j = 1,2$, where $l_j^{(1)} := \delta(l_j)$ is a line in $E_1 \cong \mathbb{P}^2$. Thus $E_1$ can be blown down to a point of a smooth projective 3-fold $X$.

Let $\delta_1 : X_1 \longrightarrow X$ be the contraction morphism. Then $Y := \delta_1(Y_1)$ has an singularity of $A_4$-type. Since all the transformations above are done on the divisor $E \hookrightarrow V$, we have $X - Y \xrightarrow{\sim} V - (\Sigma \cup A) \cong \mathbb{C}^3$ (by Lemma 10). Thus, $(X,Y)$ is a smooth projective compactification of $\mathbb{C}^3$ such that $Y$ is a singular del Pezzo surface with an singularity of $A_4$-type. This implies that $X$ is a Fano 3-fold of index $r = 2$ with $\mathrm{Pic}\, X \cong \mathbb{Z} \cdot \mathcal{O}_X(Y)$. Since $Y$ has an singularity of $A_4$-type, we have $\deg N_Y = \deg(-K_Y) = 5$, where $N_Y = [Y]|_Y$ (resp. $K_Y$) is the normal bundle of $Y$ in $X$ (resp. the canonical bundle of $Y$). Thus, $X$ is a Fano 3-fold of degree 5 in $\mathbb{P}^6$ by the anti-canonical embedding. This gives another construction of a compactification of

$\mathbb{C}^3$ in case of the index $r = 2$ (see [1]).

Remark. We put $\bar{A} = \pi^{-1}(\pi(\bar{f}_3)) \hookrightarrow V$ . Then $\bar{A}$ is a $\mathbb{P}^1$-bundle over $\bar{f}_3 \cong \mathbb{P}^1$ . Let $\bar{A}_3$ be the proper transform of $\bar{A}$ in $V_3$ and put $\bar{Y} = \delta(\bar{A}_3)$ . Since $V - (\bar{A} \cup \Sigma) \cong \mathbb{C}^3$ and the transformations above are all on $\Sigma \hookrightarrow V$ , we also have $X - \bar{Y} \overset{\rho}{\cong} V - (\bar{A} \cup \Sigma) \cong \mathbb{C}^3$ . By construction, $\bar{Y}$ is not normal and its normalization is $\mathbb{F}_3$ . This $(X,\bar{Y})$ is the same example of a compactification of $\mathbb{C}^3$ with non-normal boundary as in [1].

Remark. The vector bundle $E_\xi$ is completely determined by $(X,Y)$ . Therefore Theorem 2.4 (b) in [8] is not true.

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