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Abstract

Inspired by the work of Silverman on the geometry and the arithmetic of monomial maps and also on the translated maps on Abelian varieties, we generalize his results to the case of the translated monomial maps.

Contents

1 Introduction

Let $\varphi : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ be a dominant rational map. The dynamical degree of φ is defined by

$$\delta_\varphi = \lim_{n\to\infty} (\deg(\varphi^n))^{\frac{1}{n}}.$$

It is conjectured that δ_φ is an algebraic integer. Let $Z(\varphi)$ be the indeterminacy locus of φ. For $x \in \mathbb{P}^N(\overline{\mathbb{Q}})$, $\mathcal{O}_\varphi(x)$ denotes the orbit of x under φ. We set the notation

$$\mathbb{P}^N(\overline{\mathbb{Q}})_\varphi = \{x \in \mathbb{P}^N(\overline{\mathbb{Q}}) |\, \mathcal{O}_\varphi(x) \cap Z(\varphi) = \emptyset\}.$$

Let

$$h : \mathbb{P}^N(\overline{\mathbb{Q}}) \to [0, \infty)$$

denote the usual Weil height.

We denote by $\mathrm{Mat}_N^+(\mathbb{Z})$ the set of $N \times N$ matrices with integer coefficients and non-zero determinant. To $A = (a_{ij}) \in \mathrm{Mat}_N^+(\mathbb{Z})$ and $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{G}_m^N(\overline{\mathbb{Q}})$, we associate a dominant map on $\mathbb{G}_m^N(\overline{\mathbb{Q}})$ denoted by $\varphi_{A,\alpha}$ and called the translated monomial map, that is the map

$$\varphi_{A,\alpha}(x_1, \ldots, x_N) = (\alpha_1 x^{a_1}, \ldots, \alpha_N x^{a_N}) = (\alpha_1 \prod_{i=1}^N x_i^{a_{1i}}, \ldots, \alpha_N \prod_{i=1}^N x_i^{a_{Ni}}).$$

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By induction, we have for any $n \geq 2$ and any $x = (x_1, \ldots, x_N) \in \mathbb{G}_m^N(\overline{\mathbb{Q}})$

$$\varphi_{A,\alpha}^{(n)}(x) = (\alpha_1 \alpha^{\sum_{k=1}^{n-1} a_1^{(k)}} x^{a_1^{(n)}}, \ldots, \alpha_N \alpha^{\sum_{k=1}^{n-1} a_N^{(k)}} x^{a_N^{(n)}}) = \varphi_{\sum_{j=0}^{n-1} A^j}(\alpha) * \varphi_{A^n}(x)^a. \quad (1)$$

with $\varphi_{A,\alpha}^{(n)} = \varphi_{A,\alpha} \circ \cdots \circ \varphi_{A,\alpha}$ is the composition of $\varphi_{A,\alpha}$ n times and $a_i^{(n)}$ is the i-th row of A^n. The associated rational map $\varphi_{A,\alpha} : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ is dominant. We can easily see that

$$\deg(\varphi_{A,\alpha}^n) = \deg(\varphi_A^n), \quad \text{and then} \quad \delta_{\varphi_{A,\alpha}} = \delta_{\varphi_A}. \quad (2)$$

Definition 1.1. *Let $\varphi : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ be a dominant rational map defined over $\overline{\mathbb{Q}}$, and let $x \in \mathbb{P}^N(\overline{\mathbb{Q}})_\varphi$. The arithmetic degree of φ at x is the quantity*

$$\alpha_\varphi(x) = \limsup_{n \to \infty} h(\varphi^n(x))^{\frac{1}{n}}.$$

By [2, Proposition 12], we know that

$$\alpha_\varphi(x) \leq \delta_\varphi, \quad x \in \mathbb{P}^N(\overline{\mathbb{Q}})_\varphi. \quad (3)$$

In the following conjecture, Silverman gives a sufficient condition for equality in 3.

Conjecture 1.2. *[2, Conjecture 1] Let $\varphi : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ be a dominant rational map.*

(a) The set

$$\{\alpha_\varphi(x) |\, x \in \mathbb{P}^N(\mathbb{Q})\}$$

is a finite set of algebraic numbers.

(b) Let $x \in \mathbb{P}^N(\overline{\mathbb{Q}})$ be a point such that $\mathcal{O}_\varphi(x)$ is Zariski dense in $\mathbb{P}^N(\overline{\mathbb{Q}})$. Then $\alpha_\varphi(x) = \delta_\varphi$.

The second conjecture is a necessary step toward the definition of a good notion of canonical height associated to φ.

Conjecture 1.3. *[2, Conjecture 2] Let $\varphi : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ be a dominant rational map. Then the infimum*

$$l_\varphi = \inf\{l \geq 0 |\, \sup_{n \geq 1} \frac{\deg(\varphi^n)}{n^l \delta_\varphi^n} < \infty\},$$

exists and is an integer satisfying $0 \leq l_\varphi \leq N$.

When φ is a monomial map, Favre and Wulcan [1] proved Conjecture 1.3.
Conjecture 1.2 is proved in the case of monomial maps by Silverman in [2, §7].

Under Conjecture 1.3, the canonical height of $x \in \mathbb{P}^N(\overline{\mathbb{Q}})_\varphi$ with respect to φ is given as follows

$$\widehat{h}_\varphi(x) = \limsup_{n \to \infty} \frac{1}{n^{l_\varphi} \delta_\varphi^n} h(\varphi^n(x)).$$

We have (see [2, Proposition 19]),

$$\widehat{h}_\varphi(\varphi(x)) = \delta_\varphi \widehat{h}_\varphi(x).$$

In [2, p. 649], it is suspected that $\widehat{h}_\varphi(x)$ is finite when $\delta_\varphi > 1$. This holds for monomial maps as shown in [2, Proposition 25]. In the following theorem, we generalize this result to translated monomial maps.

[a] For $\alpha = (\alpha_1, \ldots, \alpha_N), \beta = (\beta_1, \ldots, \beta_N) \in \mathbb{G}_m^N(\overline{\mathbb{Q}})$, and $c = (c_1, \ldots, c_N) \in \mathbb{Z}^N$ we denote $\alpha^c = \prod_{i=1}^N \alpha_i^{c_i}$. We denote $\alpha * \beta = (\alpha_1\beta_1, \ldots, \alpha_N\beta_N)$.

Theorem 1.4 (see Theorem 2.8)**.** *Let $A \in \mathrm{Mat}_N^+(\mathbb{Z})$ with $\rho(A) > 1$. Let $F_A \in \mathbb{Z}[X]$ be the characteristic polynomial of A. We write $F_A(X) = F_1(X)F_2(X)$ with F_1 and F_2 are two polynomials in $\mathbb{Z}[X]$ such that $F_1(X) = (X-1)^r$ and $F_2(1) \neq 0$. We have, for any $x \in \mathbb{G}_m^N(\overline{\mathbb{Q}})$, $\widehat{h}_{\varphi_{A,\alpha}}(x)$ is finite and $\alpha_{\varphi_{A,\alpha}}(x)$ is an algebraic integer.*

This theorem confirms (b) of Conjecture 1.2 in the case of translated monomial maps. If $\delta_\varphi = 1$, then it is possible to have $l_\varphi \geq 1$ and $\widehat{h}_\varphi(x) = \infty$ as shown in [2, Example 17]. In Theorem 2.4, we produce more examples of rational maps φ with $\delta_\varphi = 1$ and $l_\varphi \geq 1$ but $\widehat{h}_\varphi(x) = \infty$ for any $x \in \mathbb{G}_m^N(\overline{\mathbb{Q}})$.

A fundamental property of the canonical height for morphisms is that height zero characterizes points with finite orbits. For any dominant rational maps φ with $\delta_\varphi > 1$ or $l_\varphi > 0$, we have

$$x \in \mathrm{PrePer}(\varphi) \Longrightarrow \widehat{h}_\varphi(x) = 0,$$

but the converse is not true in general, as noted by Silverman. This leads him to the following conjecture.

Conjecture 1.5. *Let $\varphi : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ be a dominant rational map with dynamical degree $\delta_\varphi > 1$, let $x \in \mathbb{P}^N(\overline{\mathbb{Q}})_\varphi$ be a point whose orbit $\mathcal{O}_\varphi(x)$ is Zariski dense in $\mathbb{P}^N(\overline{\mathbb{Q}})$. Then $\widehat{h}_\varphi(x) > 0$.*

When φ is a monomial map, then Conjecture 1.5 is true (see [2, Corollary 29]). We generalize [2, Corollary 29] to the case of translated monomial maps (see Corollary 2.10).

2 The arithmetic of translated monomial maps.

The following lemma can be seen as an analogue of [3, Lemma 5].

Lemma 2.1. *Let $\mathbb{G}_m^N(\overline{\mathbb{Q}})$ the torus of dimension N over $\overline{\mathbb{Q}}$, let $\varphi_A : \mathbb{G}_m^N(\overline{\mathbb{Q}}) \to \mathbb{G}_m^N(\overline{\mathbb{Q}})$ be a monomial map with $A \in \mathrm{Mat}_N^+(\mathbb{Z})$. Let $F(X) \in \mathbb{Z}[X]$ be a polynomial such that $\varphi_{F(A)}(x) = (1, \ldots, 1), \forall x \in \mathbb{G}_m^N(\overline{\mathbb{Q}})$. Suppose that F factors as*

$$F(X) = F_1(X)F_2(X) \quad \textit{with}\, F_1, F_2 \in \mathbb{Z}[X] \textit{ and } \gcd(F_1, F_2) = 1,$$

where the gcd *is computed in $\mathbb{Q}[X]$. Let*

$$G_1 = \varphi_{F_1(A)}\mathbb{G}_m^N(\overline{\mathbb{Q}}) \quad \textit{and} \quad G_2 = \varphi_{F_2(A)}\mathbb{G}_m^N(\overline{\mathbb{Q}}),$$

so G_1 and G_2 are subgroups of $\mathbb{G}_m^N(\overline{\mathbb{Q}})$. Then we have:

(a) $\mathbb{G}_m^N(\overline{\mathbb{Q}}) = G_1 \cdot G_2$.

(b) $G_1 \cap G_2$ is finite. More precisely, if we let $\rho = \mathrm{Res}(F_1, F_2)$, then $G_1 \cap G_2 \subset \mathbb{G}_m^N[\rho]$, where $[\rho] := \varphi_{\rho I_N}$ and I_N is the unit matrix.

The following map is an isogeny

$$\lambda : G_1 \times G_2 \to \mathbb{G}_m^N(\overline{\mathbb{Q}}), \quad \lambda(x, y) = x * y.$$

Proof. We have

$$\varphi_A(G_1) \subset G_1 \quad \text{and} \quad \varphi_A(G_2) \subset G_2. \tag{4}$$

These inclusions follow from the following identities $\varphi_A \circ \varphi_B = \varphi_{AB}$ and $\varphi_A \cdot \varphi_B = \varphi_{A+B}$ for A and B two matrices in $\mathrm{Mat}_N^+(\mathbb{Z})$ (see [2, p. 659 (9)]).

$$G_1(X)F_1(X) + G_2(X)F_2(X) = \rho = \mathrm{res}(F_1, F_2).$$

We have $\varphi_{\rho I_N} = \varphi_{F_1 G_1} \cdot \varphi_{F_2 G_2}$. This implies

$$\mathbb{G}_m^N(\overline{\mathbb{Q}}) = \varphi_{\rho I_N}(\mathbb{G}_m^N) \subset G_1 \cdot G_2. \tag{5}$$

Let $x \in G_1 \cap G_2$, then $x = \varphi_{F_i(A)}(x_i)$ for some $x_i \in G_i$ with $i = 1, 2$. We have

$$\begin{aligned} \varphi_{\rho I_N}(x) &= \varphi_{G_1(A)}(\varphi_{F_1(A)}(x)) \cdot \varphi_{G_2(A)}(\varphi_{F_2(A)}(x)) \\ &= \varphi_{G_1(A)}(\varphi_{F_1(A)F_2(A)}(y)) \cdot \varphi_{G_2(A)}(\varphi_{F_2(A)F_1(A)}(x)) \\ &= \varphi_{G_1(A)}(\varphi_0(x))\varphi_{G_2(A)}(\varphi_0(x)) \\ &= 1. \end{aligned}$$

Then $G_1 \cap G_2 \subset \mathbb{G}_m^N[\rho]$. We use (5) to deduce that the map λ is onto, and then to conclude that λ is an isogeny. $\square$

We can find a pair $(\alpha_1, \alpha_2) \in G_1 \times G_2$ satisfying $\lambda(\alpha_1, \alpha_2) = \alpha$, i.e $\alpha_1 * \alpha_2 = \alpha$. Since φ_A commutes with $F_1(\varphi_A)$ and $F_2(\varphi_A)$, we write ϕ_1 and ϕ_2 for the restrictions of φ_A to G_1 and to G_2 respectively, and we define maps φ_1 and φ_2 as follows

$$\varphi_1 : G_1 \to G_1 \quad \varphi_1(x) = \phi_1(x) * \alpha_1, \tag{6}$$

$$\varphi_2 : G_2 \to G_2 \quad \varphi_2(y) = \phi_2(y) * \alpha_2. \tag{7}$$

We have, for any $x \in G_1$ and $y \in G_2$

$$\begin{aligned} \lambda \circ (\varphi_1 \times \varphi_2)(x, y) &= \lambda\big(\phi_1(x) * \alpha_1, \phi_2(y) * \alpha_2\big) \\ &= \phi_1(x) * \alpha_1 * \phi_2(y) * \alpha_2 \\ &= \varphi_A(x * y) * \alpha \\ &= \varphi_{A,\alpha} \circ \lambda(x, y). \end{aligned}$$

This shows that the following diagram is commutative

$$\begin{array}{ccc} G_1 \times G_2 & \xrightarrow{\lambda} & \mathbb{G}_m^N \\ \Big\downarrow{\scriptstyle \varphi_1 \times \varphi_2} & & \Big\downarrow{\scriptstyle \varphi_{A,\alpha}} \\ G_1 \times G_2 & \xrightarrow{\lambda} & \mathbb{G}_m^N \end{array}$$

Then,

$$\varphi_{A,\alpha}^n \circ \lambda(\alpha_1, \alpha_2) = \lambda \circ (\varphi_1^n \times \varphi_2^n)(x, y) \quad \forall n \in \mathbb{N}. \tag{8}$$

Lemma 2.2. *Let $A \in \mathrm{Mat}_N^+(\mathbb{Z})$ and $\alpha \in \mathbb{G}_m^N(\overline{\mathbb{Q}})$. We consider the map $\varphi_{A,\alpha}$. We have for any $x \in \mathbb{G}_m^N(\overline{\mathbb{Q}})$*

$$\widehat{h}_{\varphi_{A,\alpha}}(1) \le \widehat{h}_{\varphi_{A,\alpha}}(x) + \widehat{h}_{\varphi_A}(x^{-1}) \quad \textit{and} \quad \widehat{h}_{\varphi_{A,\alpha}}(x) \le \widehat{h}_{\varphi_{A,\alpha}}(1) + \widehat{h}_{\varphi_A}(x). \tag{9}$$

In particular, $\widehat{h}_{\varphi_{A,\alpha}}(x)$ is finite if and only if $\widehat{h}_{\varphi_{A,\alpha}}(1)$ is finite.

$$\alpha_{\varphi_{A,\alpha}}(1) \le \alpha_{\varphi_{A,\alpha}}(x) + \alpha_{\varphi_A}(x^{-1}) \quad \textit{and} \quad \alpha_{\varphi_{A,\alpha}}(x) \le \alpha_{\varphi_{A,\alpha}}(1) + \alpha_{\varphi_A}(x). \tag{10}$$

Proof. Recall that $\varphi^{(n)}_{A,\alpha}(x) = \varphi_{\sum_{j=0}^{n-1} A^j}(\alpha) * \varphi_{A^n}(x)$, so $\varphi^{(n)}_{A,\alpha}(1) = \varphi_{\sum_{j=0}^{n-1} A^j}(\alpha)$. By the definition of Weil height, it is easy to get the following

$$h(\varphi^n_{A,\alpha}(1)) \leq h(\varphi^n_{A,\alpha}(x)) + h(\varphi_{A^n}(x^{-1})). \tag{11}$$

and

$$h(\varphi^n_{A,\alpha}(x)) \leq h(\varphi^n_{A,\alpha}(1)) + h(\varphi_{A^n}(x)). \tag{12}$$

So the inequalities of the lemma follow easily. We know that $\widehat{h}_{\varphi_A}$ is finite by [2, Proposition 25]. Then, $\widehat{h}_{\varphi_{A,\alpha}}(x)$ is finite if and only if $\widehat{h}_{\varphi_{A,\alpha}}(1)$ is finite. $\square$

Claim 2.3. *Fix $r \geq 1$, we have*

$$\sum_{k=0}^{n-1} X^k \equiv \binom{n}{r}(X-1)^{r-1} + P_{n,r}(X)\ [mod\,(X-1)^r]. \tag{13}$$

with $P_{n,r}(X) = \sum_{k=0}^{r-2} d_{r,j}(n)X^k$ is a polynomial in $\mathbb{Z}[X]$ with $d_{r,j}(n) = O(n^{r-1})$ for $n \gg 1$.

Proof. We have

$$\begin{aligned}\sum_{k=0}^{n-1} X^k &\equiv \frac{X^n-1}{X-1}\\ &\equiv \sum_{k=0}^{r-1}\binom{n}{k+1}(X-1)^k\ [\mathrm{mod}\,(X-1)^r]\\ &\equiv \sum_{k=0}^{r-1}\sum_{j=0}^{k}\binom{n}{k+1}\binom{k}{j}(-1)^{k-j}X^j\ [\mathrm{mod}\,(X-1)^r]\\ &\equiv \sum_{j=0}^{r-1}\binom{n}{r}\binom{r-1}{j}(-1)^{r-1-j}X^j + \sum_{k=0}^{r-2}\sum_{j=0}^{k}\binom{n}{k+1}\binom{k}{j}(-1)^{k-j}X^j\ [\mathrm{mod}\,(X-1)^r].\end{aligned}$$

$\square$

In [2, Example 17], an example of a rational map on $\mathbb{P}^3$ is given, satisfying $\delta_\varphi = 1$ and $l_\varphi > 0$ but $\widehat{h}_\varphi$ takes an infinite value at a point in $\mathbb{P}^3(\overline{\mathbb{Q}})_\varphi$. The following result gives examples of rational maps φ with $\delta_\varphi = 1$ and $l_\varphi > 0$ but the canonical height $\widehat{h}_\varphi$ takes infinite values.

Theorem 2.4. *Let $A \in \mathrm{Mat}_N^+(\mathbb{Z})$ and F_A its characteristic polynomial. We suppose that $F_A(X) = (X-I)^r$ with $r \in \mathbb{N}_{\geq 2}$. Let $\alpha \in \mathbb{G}_m^N(\overline{\mathbb{Q}})$. We have for $x \in \mathbb{G}_m^N(\overline{\mathbb{Q}})$, the canonical height $\widehat{h}_{\varphi_{A,\alpha}}(x)$ is finite if and only if $\varphi_{(A-I)^{r-1}}(\alpha) \in \mathbb{G}_m^N(\overline{\mathbb{Q}})_{\mathrm{tors}}$ (equivalently, $\log|\alpha| \in \ker_{\mathbb{C}}(A-I)^{r-1}$).*

Proof. Using Claim 2.3, we have for any $v \in \mathcal{M}_K$, $\sum_{k=0}^{n-1} A^k \log\|\alpha\|_v{}^b = \binom{n}{r}(A-I)^{r-1}\log\|\alpha\|_v + P_{n,r}(A)\log\|\alpha\|_v$. If there exists $v_0 \in \mathcal{M}_K$ such that $(A-I)^{r-1}\log\|\alpha\|_{v_0}$ has a positive coordinate, then we can find a positive constant c such that

$$h(\varphi^n_{A,\alpha}(1)) = \sum_{v\in\mathcal{M}_K}\max_{1\leq i\leq N}(0, (\sum_{k=0}^{n-1} A^k\log\|\alpha\|_v)_i) \geq cn^r \quad \forall n \gg 1.$$

[b] By definition, $\log\|\alpha\|_v$ is the transpose of $(\log\|\alpha_1\|_v, \ldots, \log\|\alpha_N\|_v)$.

But $l_{\varphi_{A,\alpha}} = l_{\varphi_A} = l_A$ which is less than $r-1$. Then, by the definition of the canonical height, we get

$$\widehat{h}_{\varphi_{A,\alpha}}(1) = \infty. \tag{14}$$

Since $\widehat{h}_{\varphi_A}$ is finite ([2, Proposition 25]), and by 9 we conclude that

$$\widehat{h}_{\varphi_{A,\alpha}}(x) = \infty \quad \forall\, x \in \mathbb{G}_m^N(\overline{\mathbb{Q}}). \tag{15}$$

If $(A-I)^{r-1}\log\|\alpha\|_v \leq 0$ for any $v \in \mathcal{M}_K$. This implies that $\varphi_{(A-I)^{r-1}}(\alpha)$ is a torsion point in $\mathbb{G}_m^N(\overline{\mathbb{Q}})$. We deduce that the limit

$$\limsup_{n\to\infty} \frac{h(\varphi_{A,\alpha}^n(1))}{n^{r-1}}, \tag{16}$$

is finite. Then, $\widehat{h}_{\varphi_{A,\alpha}}(x)$ is finite for any $x \in \mathbb{G}_m^N(\overline{\mathbb{Q}})$.

□

Remark 2.5. *A similar formula can be obtained for an $A \in \mathrm{Mat}_N^+(\mathbb{Z})$ satisfying $(A^s - I)^r = 0$ with $s \in \mathbb{N}_{\geq 2}$.*

Proposition 2.6. *Let $A \in \mathrm{Mat}_N^+(\mathbb{Z})$ and F_A its characteristic polynomial. We suppose that $F_A(1) \neq 0$. We have*

$$\widehat{h}_{\varphi_{A,\alpha}}(x) = \frac{1}{\det(A-I)}\widehat{h}_{\varphi_A}(\beta * x^{\det(A-I)}) \quad \forall x \in \mathbb{G}_m^N(\overline{\mathbb{Q}}), \tag{17}$$

where $\beta = \varphi_{t_{\mathrm{Com}(A-I)}}(\alpha)$. In particular, the canonical height $\widehat{h}_{\varphi_{A,\alpha}}(x)$ is finite. We have,

$$\alpha_{\varphi_{A,\alpha}}(x) = \alpha_{\varphi_A}(\beta * x^{\det(A-I)}).$$

and $\left\{\alpha_{\varphi_{A,\alpha}}(x) | x \in \mathbb{P}^N(\overline{\mathbb{Q}})_{\varphi_{A,\alpha}}\right\}$ is a finite set of algebraic integers.

Proof. By assumption, we can find $\beta \in \mathbb{G}_m^N(\overline{\mathbb{Q}})$ such that $\varphi_{A-I}(\beta) = \alpha^{\det(A-I)}$. In fact we can take $\beta = \varphi_{t_{\mathrm{Com}(A-I)}}(\alpha)$. Then $\left(\varphi_{A,\alpha}^{(n)}(x)\right)^{\det(A-I)} = \varphi_{A^n-I}(\beta) * \varphi_{A^n}(x^{\det(A-I)}) = \varphi_{A^n}(\beta * x^{\det(A-I)}) * \varphi_{-I}(\beta)$. From this, we get two inequalities

$$\det(A-I)h(\varphi_{A,\alpha}^{(n)}(x)) \leq h(\varphi_A^{(n)}(\beta * x^{\det(A-I)})) + h(\beta^{-1}), \tag{18}$$

and

$$h(\varphi_A^{(n)}(\beta * x^{\det(A-I)})) \leq \det(A-I)h(\varphi_{A,\alpha}^{(n)}(x)) + h(\beta). \tag{19}$$

We conclude that

$$\widehat{h}_{\varphi_{A,\alpha}}(x) = \frac{1}{\det(A-I)}\widehat{h}_{\varphi_A}(\beta * x^{\det(A-I)}). \tag{20}$$

Using 18 and 19, we deduce

$$\alpha_{\varphi_{A,\alpha}}(x) = \limsup_{n\to\infty} h(\varphi_{A,\alpha}^{(n)}(x))^{\frac{1}{n}} = \limsup_{n\to\infty} h(\varphi_A^{(n)}(\beta * x^{\det(A-I)}))^{\frac{1}{n}} = \alpha_{\varphi_A}(\beta * x^{\det(A-I)}). \tag{21}$$

By [2, Corollary 32], we conclude the proof of the proposition. □

Corollary 2.7. *Let $\alpha \in \mathbb{G}_m^N(\overline{\mathbb{Q}})$. Let $A \in \mathrm{Mat}_N^+(\mathbb{Z})$ whose characteristic polynomial is irreducible over $\mathbb{Q}$. Let $x \in \mathbb{G}_m^N(\overline{\mathbb{Q}})$. Then*

$$\widehat{h}_{\varphi_{A,\alpha}}(x) = 0 \Longleftrightarrow x \in \mathrm{PrePer}(\varphi_{A,\alpha}).$$

Proof. Let F_A be the characteristic polynomial of A. By assumption, F_A is irreducible over $\mathbb{Q}$. In particular, $F_A(1) \neq 0$. The proof of the corollary follows from Proposition 2.6 and [2, Corollary 31]. □

The following theorem gives examples of rational maps satisfying [2, Question 18. p.658]

Theorem 2.8. *Let $A \in \mathrm{Mat}_N^+(\mathbb{Z})$ with $\rho(A) > 1$. Let $F_A \in \mathbb{Z}[X]$ be the characteristic polynomial of A. We write $F_A(X) = F_1(X)F_2(X)$ with F_1 and F_2 are two polynomials in $\mathbb{Z}[X]$ such that $F_1(X) = (X-1)^r$ and $F_2(1) \neq 0$. We have, for any $x \in \mathbb{G}_m^N(\overline{\mathbb{Q}})$, $\widehat{h}_{\varphi_{A,\alpha}}(x)$ is finite and $\alpha_{\varphi_{A,\alpha}}(x)$ is an algebraic integer.*

Proof. If $r = 0$, this is Proposition 2.6. We assume that $r \geq 1$. Recall the definitions of φ_1 and φ_2 (see 6 and 7).

We can show that $l_{\varphi_1}, l_{\varphi_2} \leq l_{\varphi_{A,\alpha}}$. Since $\varphi_{F_1(A)}(\phi_2) = 1$ and $\varphi_{F_2(A)}(\phi_1) = 1$, we have G_1 and G_2 are tori, ϕ_1 and ϕ_2 are monomial maps on G_1 and G_2 respectively. If we denote by A_1 (resp. A_2) the associated matrix of ϕ_1 (resp. ϕ_2) then $F_2(A_1) = 0$ and $F_1(A_2) = 0$. By Lemma 2.1, we have

$$h(\varphi_{A,\alpha}^n(x * y)) = h(\varphi_1^n(x)) + h(\varphi_2^n(y)) + O(1), \quad \forall n \in \mathbb{N}. \tag{22}$$

Which gives

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n^{l_A}\rho(A)^n} h(\varphi_{A,\alpha}^n(x * y)) &\leq \limsup_{n\to\infty} \frac{1}{n^{l_A}\rho(A)^n} h(\varphi_1^n(x)) + \limsup_{n\to\infty} \frac{1}{n^{l_A}\rho(A)^n} h(\varphi_2^n(y)) \\ &= \lim_{n\to\infty} \frac{n^{l_{\varphi_1}}}{n^{l_A}} \limsup_{n\to\infty} \frac{h(\varphi_1^n(x))}{\rho(A)^n n^{l_{\varphi_1}}} + \lim_{n\to\infty} \frac{n^{l_{\varphi_2}}}{n^{l_A}} \limsup_{n\to\infty} \frac{h(\varphi_2^n(y))}{\rho(A)^n n^{l_{\varphi_2}}}\end{aligned}$$

By Theorem 2.4 and 12, we have

$$h(\varphi_2^n(y)) = O(n^{l_{\varphi_2}+1}) \quad \text{for } n \gg 1, \tag{23}$$

Recall that $\rho(A) > 1$, then the second term of previous inequality is zero. For the first term, this limit is finite by Proposition 2.6. A simple argument shows that, in fact, we have

$$\widehat{h}_{\varphi_{A,\alpha}}(x * y) = \lim_{n\to\infty} \frac{n^{l_{\varphi_1}}\delta_{\varphi_1}^n}{n^{l_A}\rho(A)^n} \widehat{h}_{\varphi_1}(x). \tag{24}$$

In particular,

$$\lim_{n\to\infty} \frac{n^{l_{\varphi_1}}\delta_{\varphi_1}^n}{n^{l_A}\rho(A)^n} = 1. \tag{25}$$

Then

$$\widehat{h}_{\varphi_{A,\alpha}}(x * y) = \widehat{h}_{\varphi_1}(x) < \infty.$$

We have

$$h(\varphi_{A,\alpha}^n(x * y))^{\frac{1}{n}} = h(\varphi_1^n(x))^{\frac{1}{n}} \Big(1 + \frac{h(\varphi_2^n(y))}{h(\varphi_1^n(x))} + o(1)\Big)^{\frac{1}{n}}. \tag{26}$$

By 23 and 25, we obtain

$$\limsup_{n\to\infty} h(\varphi^n_{A,\alpha}(x*y))^{\frac{1}{n}} = \limsup_{n\to\infty} h(\varphi_1^n(x))^{\frac{1}{n}}. \tag{27}$$

That is,

$$\alpha_{\varphi_{A,\alpha}}(x*y) = \alpha_{\varphi_1}(x). \tag{28}$$

Since $\alpha_{\varphi_1}(x)$ is an algebraic integer (see Proposition 2.6), we conclude that $\alpha_{\varphi_{A,\alpha}}(z)$ is an algebraic integer for any $z \in \mathbb{G}_m^N(\overline{\mathbb{Q}})$. $\square$

Proposition 2.9. *Let $\alpha \in \mathbb{G}_m^N(\overline{\mathbb{Q}})$. Let $A \in \mathrm{Mat}_N^+(\mathbb{Z})$ and $\varphi_{A,\alpha}$ the associated monomial map. We assume that $\delta_{\varphi_{A,\alpha}} > 1$. There exist $\overline{r}(A)$ a positive integer and an algebraic subgroup $G \subset \mathbb{G}_m^N(\overline{\mathbb{Q}})$, with dimension $\dim G \geq N - \overline{r}(A)$ such that*

$$\left\{x \in \mathbb{G}_m^N(\overline{\mathbb{Q}}) \,|\, \widehat{h}_{\varphi_{A,\alpha}}(x) = 0\right\} \subset G(\overline{\mathbb{Q}})^{\mathrm{div}} * (\varphi_{{}^t\mathrm{Com}(A-I)}(\alpha))^{-\frac{1}{\det(A-I)}}. \tag{29}$$

Proof. The proof of the proposition follows easily from [2, Theorem 27] combined with Proposition 2.6. $\square$

This proposition has the following corollary,

Corollary 2.10. *Let $\alpha \in \mathbb{G}_m^N(\overline{\mathbb{Q}})$. Let $A \in \mathrm{Mat}_N^+(\mathbb{Z})$ and $\varphi_{A,\alpha}$ the associated monomial map. We assume that $\delta_{\varphi_{A,\alpha}} > 1$. If $\widehat{h}_{\varphi_{A,\alpha}}(x) = 0$ then the orbit $\mathcal{O}_{\varphi_{A,\alpha}}(x)$ is not Zariski dense in $\mathbb{G}_m^N(\overline{\mathbb{Q}})$.*

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