Secondary invariants and the singularity of the Ruelle zeta-function in the central critical point

Andreas Juhl

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Germany

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by

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Abstract. The Ruelle zeta-function of the geodesic flow on the sphere bundle S(X) of an even-dimensional compact locally symmetric space X of rank 1 is a meromorphic function in the complex plane that satisfies a functional equation relating its values in s and -s. The multiplicity of its singularity in the central critical point s = 0 only depends on the hyperbolic structure of the flow and can be calculated by integrating a secondary characteristic class canonically associated to the flow-invariant foliations of S(X) for which a representing differential form is given.

Let Y = G/K be a rank one symmetric space of the non-compact type, i.e., Y is a real, complex or quaternionic hyperbolic space or the (16-dimensional) hyperbolic Cayley-plane. Let Γ be a uniform lattice in the (connected simple) isometry group G of Y without torsion. Γ acts properly discontinuous on Y = G/K (K a maximal compact subgroup of G) and $X = \Gamma\backslash G/K$ is a compact locally symmetric space. We consider X as a Riemannian manifold with respect to an arbitrary (constant) multiple g of the metric g_o induced by the Killing form on the Lie algebra of G. Then the Riemannian manifold (X,g) is a space of negative curvature.

The negativity of the curvature of the metric g on X implies the existence of an infinite countable set of prime closed geodesics in X with a discrete set of prime periods accumulating at infinity.

Let Φ_t be the geodesic flow on the unit sphere bundle S(X) of the space (X,g).

The prime period of a periodic orbit of Φ_t on S(X) coincides with the length of the closed geodesic in X obtained by projecting the periodic orbit into X. Now we use these periods to define the zeta-function

$$(1) \qquad Z_R(s) = \prod_c (1-\exp(-sl_c))^{-1}$$

for $s \in \mathbb{C}$ such that Re(s)>h (h being the topological entropy of the geodesic flow Φ_t on S(X)). The product in (1) runs over all closed *oriented* geodesics c in X and l_c denotes the length of c as a curve in X. Note that for each (unoriented) closed geodesic c in X there are two lifts of c as periodic orbits of Φ_t in correspondence with the two possibilities to orient c.

The function Z_R is well-known as the Ruelle zeta-function of the geodesic flow ([F1]). The Euler product (1) defines a holomorphic function in the half-plane Re(s)>h.

Now by symbolic dynamics the zeta-function Z_R can be written for large Re(s) as an alternating product of zeta-functions associated to suspensions of subshifts of finite type. The latter zeta-functions coincide with Fredholm-determinants $\det(1-L_*(s))$ (in sense of Grothendieck) of holomorphic families $L_*(s)$ of nuclear transfer operators on certain spaces of differential forms. This representation implies that Z_R has a meromorphic continuation to the complex plane (see [F1], [R]). While these arguments establish the existence of a meromorphic continuation it seems to be rather difficult to prove results on the positions and the multiplicities of the singularities of Z_R by the same method.

On the other hand, the zeta-function Z_R can be written as a product of generalized Selberg zeta-functions. Generalized Selberg zeta-functions are also defined by Euler products similar to (1) but with more complicated local Euler factors containing monodromy contributions of the loops c in certain vector bundles on S(X) (see [F1], [G], [W]). The generalized Selberg zeta-functions in turn can be investigated by using trace formula techniques which are at present the only known methods to uncover, for instance, the deeper relations between periodic orbits and the geometry and topology of the underlying space. However, in contrast to the dynamical point of view, along the usual trace formula arguments the relation between the hyperbolic structure and the analytical properties of the zeta-function remains mysterious.

The main result of the present note describes a direct relation between the multiplicity of the singularity of Z_R in the central critical point s = 0 and the hyperbolic structure of the flow.

Let us first consider the well-known special case of the Ruelle zeta-function of the geodesic flow of a compact Riemannian surface $X = \Gamma\backslash H^2$ of negative Euler characteristic $\chi(X)$. Consider X as a Riemannian manifold with the metric inherited from the hyperbolic metric

(2) $$y^{-2}(dx^2+dy^2)$$

of constant curvature -1 on the upper half plane H^2. Then the product (1) defines a holomorphic function in the half-plane Re(s)>1. Moreover, $Z_R(s)$ satisfies the functional equation

(3) $$Z_R(s)Z_R(-s) = ((1-\exp(2\pi is))(1-\exp(-2\pi is))^{2-2g}.$$

In particular, for the multiplicity m_o of its singularity in s = 0 we have

(4) $$m_o = 2-2g = \chi(X).$$

The functional equation (3) is a consequence of the functional equation

$$Z_S(1-s) = Z_S(s) \exp(2(2-2g)\int_o^{s-1/2}(\pi t)\tan(\pi t)dt),$$

for the classical Selberg zeta-function

(5) $$Z_S(s) = \prod_c \prod_{N\geq 0} (1-\exp(-(s+N)l_c)), \quad Re(s)>1, \ s \in \mathbb{C},$$

(see [S]) and the obvious relation $Z_R(s) = Z_S(s+1)/Z_S(s)$.

The Ruelle zeta-function Z_R always satisfies a functional equation similar to (3) relating $Z_R(s)$ to $Z_R(-s)$. However, the general theory of the functional equation for Z_R will not be discusssed here. Instead we shall only discuss certain formulas for the multiplicity m_o of the singularity of Z_R in $s = 0$ generalizing formula (4).

To formulate the main result we shall use group-theoretical descriptions of the geodesic flow and its hyperbolic structure (Anosov property).

Let g_o and k_o be the respective Lie algebras of G and K and let $g_o = k_o \oplus p_o$ be the Cartan decomposition of g_o being orthogonal with respect to the Killing form. Identify p_o with the tangent space $T_{eK}(Y)$. Let $a_o \subset p_o$ be a one-dimensional abelian subspace and let $M \subset K$ be the centralizer of a_o in K.

Now consider the action

$$\Phi_A: A \times \Gamma\backslash G/M \longrightarrow \Gamma\backslash G/M, \quad A = \exp(a_o)$$

defined by

$$(6) \qquad (a,\Gamma gM) \longmapsto \Phi_a(\Gamma gM) = \Gamma g a^{-1} M, \quad a \in A.$$

The definition of the action Φ_A is independent of the choice of a metric on X and we shall denote this action in the following as the *abstract* geodesic flow.

Once and for all we fix an orientation of the flow Φ_A which amounts to fixing an (open) Weyl-chamber a_o^+ in a_o, called the positive chamber, and we shall restrict attention only to the action of $A^+ = \exp(a_o^+) \subset A$.

Now in terms of the abstract geodesic flow Φ_A on $\Gamma\backslash G/M$ the hyperbolic structure can be described as follows.

Let n_o^+ and n_o^- be the subspaces of g_o on which ad(X) for $X \in a_o^+$ acts by positive and negative eigenvalues $\alpha(X)$ $(\alpha \in \Delta(g_o,a_o))$, respectively. The nilpotent Lie algebras $n_o^{\pm}$ are real MA-modules with respect to the adjoint action.

Let $\mathcal{P}^{\pm}$ be the locally homogeneous vector bundles

$$\Gamma\backslash G\times_M(n_o^{\pm}) \longrightarrow \Gamma\backslash G/M \tag{7}$$

regarded as subbundles of the tangent bundle $T(\Gamma\backslash G/M)$. The real vector bundles $\mathcal{P}^{\pm}$ are integrable and the foliation of $\Gamma\backslash G/M$ obtained by integrating $\mathcal{P}^{+}$, resp. $\mathcal{P}^{-}$, is the Φ_A-invariant unstable, resp. stable, foliation of $\Gamma\backslash G/M$. More precisely, the tangent bundle $T(\Gamma\backslash G/M)$ admits a $d(\Phi_A)$-equivariant decomposition

$$T(\Gamma\backslash G/M) = \mathcal{P}^{+} \oplus T^{o}(\Gamma\backslash G/M) \oplus \mathcal{P}^{-} \tag{8}$$

into the the direct sum of the stable subbundle $\mathcal{P}^{-}$, the central subbundle $T^{o}(\Gamma\backslash G/M)$ and the unstable subbundle $\mathcal{P}^{+}$. Tangent vectors in $\mathcal{P}^{-}$, resp. $\mathcal{P}^{+}$, are contracted, resp. expanded, exponentially by the differential $d(\Phi_a), a \in A^{+}$.

Note that the stable and the unstable leaves of $\Gamma\backslash G/M$ are smoothly embedded smooth submanifolds.

Now we shall associate to both foliations of $\Gamma\backslash G/M$ canonical differential forms

$$\Omega_R(\mathcal{P}^{\pm}) \in C^{\infty}(\Lambda^{2d-2}T^{*}(\Gamma\backslash G/M)),\ d = \dim(X).$$

We begin with the construction of a left G-invariant and $\mathrm{End}(\hat{\mathcal{P}}^{\pm})$-valued 2-forms

$$\omega_R^{\pm} \in C^{\infty}(\Lambda^{2}T^{*}(G/MA) \otimes \mathrm{End}(\hat{\mathcal{P}}^{\pm})),$$

where $\hat{\mathcal{P}}^{\pm}$ denotes the G-homogeneous vector bundle

$$G\times_{MA}(n_o^{\pm}) \longrightarrow G/MA.$$

By G-invariance it suffices to define $\omega_R^{\pm}$ in $\underline{e}$ = eMA.

We choose a (real) basis $\{Z_j\}$ of the space $n_o^{\pm}$. Let $\{Z^j\}$ be the dual basis of $(n_o^{\pm})^*$. The linear forms $Z^j \in (n_o^{\pm})^*$ will also be regarded as linear forms on g_o annihilating $n_o^{\mp}$, m_o (= Lie algebra of M) and a_o. Set

(9) $$(\omega_R^{\pm})_{\underline{e}} = ((\omega_R^{\pm})_{\underline{e}}(\cdot,\cdot)_j^k)_{j,k}, \quad j,k = 1,\ldots,\dim(n_o^{\pm}) = d-1,$$

where

(10) $$(\omega_R^{\pm})_{\underline{e}}(\hat{X}_{\underline{e}},\hat{Y}_{\underline{e}})_j^k = \langle Z^k, -[[X,Y],Z_j]\rangle$$

for $X,Y \in n_o^+ \oplus n_o^-$, and $\hat{X}_{\underline{e}} = d_e(\pi)(X)$ (π: G $\longrightarrow$ G/MA being the canonical projection). Regard the matrix $(\omega_R^{\pm})_{\underline{e}}$ as an End($n_o^{\pm}$)-valued alternating 2-form on $T_{\underline{e}}$(G/MA).

Now extend $(\omega_R^{\pm})_{\underline{e}}$ to a G-invariant and End($\hat{\mathcal{P}}^{\pm}$)-valued 2-form $\omega_R^{\pm}$ on G/MA.

Next lift $\omega_R^{\pm}$ via G/M $\longrightarrow$ G/MA to an End($\mathcal{P}^{\pm}$)-valued G-invariant differential 2-form on G/M. The latter 2-form drops down to an End($\mathcal{P}^{\pm}$)-valued 2-form on $\Gamma\backslash$G/M also denoted by $\omega_R^{\pm}$.

Now define

(11) $$\Omega_R(\mathcal{P}^{\pm}) = \det((i/2\pi)\omega_R^{\pm}).$$

Then the forms $\Omega_R(\mathcal{P}^{\pm}) \in C^{\infty}(\Lambda^{2d-2}T^*(\Gamma\backslash G/M))$ are *closed basic* forms with respect to the foliation of $\Gamma\backslash$G/M by the orbits of the abstract geodesic flow Φ_A, i.e., $\Omega_R(\mathcal{P}^{\pm})$ is Φ_A-invariant and $i_{\hat{X}}(\Omega_R(\mathcal{P}^{\pm})) = 0$ for all sections $\hat{X}$ of $T^o(\Gamma\backslash G/M)$.

Let the dimension of X be even. Then it follows that

$$\Omega_R(\mathcal{P}^+) = - \Omega_R(\mathcal{P}^-)$$

and the uniquely determined *real eigenvalue* $\neq 0$ of the End($\mathcal{P}^{\pm}$)-valued 2-form $\omega_R^{\pm}$ on $\Gamma\backslash$G/M is a well-defined Φ_A-invariant

2-form

$$\mu_R^{\pm} \in C^{\infty}(\Lambda^2 T^*(\Gamma\backslash G/M))$$

on $\Gamma\backslash G/M$. Since $\mu_R^{\pm}$ is an *exact* 2-form it follows that there exists a (uniquely determined) left G-invariant and right Φ_A-invariant 1-form $\alpha_R^{\pm}$ on G/M such that $\alpha_R^{\pm}$ drops down to a Φ_A-invariant 1-form

$$\alpha_R^{\pm} \in C^{\infty}(T^*(\Gamma\backslash G/M))$$

which satisfies

$$d\alpha_R^{\pm} = (i/2\pi)\, \mu_R^{\pm}. \tag{12}$$

Note that the main reason to formulate these constructions by using the abstract geodesic flow Φ_A instead of the geodesic flow Φ_t is the obvious fact that the multiplicity m_o is independent of the choice of any scaling of the negative curvature metric on X.

Now let $\phi : S(X) \longrightarrow \Gamma\backslash G/M$ be the diffeomorphism obtained by composing the canonical isomorphism of S(X) and $\Gamma\backslash S(Y)$ with the G-equivariant map $S(Y) \ni g(eK,X) \longmapsto gM \in G/M$, $X \in a_o^+$ of S(Y) onto G/M. The stable and unstable foliations of $\Gamma\backslash G/M$ then obviously correspond to the stable and unstable foliations of S(X) associated to the geodesic flow on S(X).

Theorem 1. Let the dimension of X *be even. Then the multiplicity* m_o *of the singularity of* Z_R *in* s = 0 *is given by the formula*

$$m_o = \int_{S(X)} \phi^*(\Omega_R(\mathcal{P}^+)\wedge\alpha_R^+) = \int_{S(X)} \phi^*(\Omega_R(\mathcal{P}^-)\wedge\alpha_R^-). \tag{13}$$

Moreover, by using the functional equation of Z_R (not given here) it can be proved that *all* singularities of Z_R outside the critical strip $Re(s) \in [-h,h]$ have multiplicity $2m_o$.

The differential forms $\phi^*(\Omega_R(\mathcal{P}^+)\wedge\alpha_R^+)$ and $\phi^*(\Omega_R(\mathcal{P}^-)\wedge\alpha_R^-)$ should be regarded as representing a (top-degree) *secondary characteristic* *class* of the normal bundle of the weak-stable and weak-unstable foliation of S(X), respectively.

There is an equivalent definition of $\Omega_R(\mathcal{P}^{\pm})$ which emphasizes the analogy of the forms $\Omega_R(\mathcal{P}^{\pm})$ with the Pfaffian of the curvature of the Levi-Civita connection of a Riemannian manifold.

In fact, consider the involution J on the tangent bundle T(G/MA) defined by $J|\hat{\mathcal{P}}^{\pm} = \pm\, id|\hat{\mathcal{P}}^{\pm}$ and set $B(X,Y) = \Omega(X,JY)$ for a G-invariant symplectic form Ω on G/MA such that the leaves of the stable and unstable foliations (of G/MA) are Lagrangian submanifolds (one can construct Ω by reduction of the canonical symplectic form on T(G/K)\0). Then B is an invariant pseudo-Riemannian metric of signature (d-1,d-1). The curvature 2-form ω_D of the corresponding torsion-free pseudo-Riemannian connection D (also considered in [K]) splits as

$$(14)\qquad \omega_D = \begin{pmatrix} \omega_D^+ & 0 \\ 0 & \omega_D^- \end{pmatrix}, \quad \omega_D^- = -\,\omega_D^+$$

according to the G-invariant decomposition $T(G/MA) = \hat{\mathcal{P}}^+ \oplus \hat{\mathcal{P}}^-$. Then the lift (via G/M $\longrightarrow$ G/MA) of the determinant of $(i/2\pi)\,\omega_D^{\pm}$ coincides with the form $\Omega_R(\mathcal{P}^{\pm})$.

The differential forms $\Omega_R(\mathcal{P}^{\pm})\wedge\alpha_R^{\pm}$ are the top-degree components of A-equivariantly closed forms (of mixed degree) on $\Gamma\backslash G/M$. Therefore *theorem 1* can be regarded as a *regularized* analog of a localization formula in equivariant cohomology (see [BGV]). In particular, it is natural to regard the multiplicity formula (13) as an analog of the Poincaré-Hopf formula for the sum of the indices of a (non-degenerate) vector field.

In the case of a compact Riemannian surface $X = \Gamma\backslash H^2$ we have $S(X)\ (\simeq \Gamma\backslash G/M) \simeq \Gamma\backslash PSL(2,\mathbb{R})$ and equation (4) follows from *theorem 1* by an elementary calculation. In fact, in terms of the basis $\{Y_+, Y_-, Y_o\}$

$$Y_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ Y_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ Y_o = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

of $sl(2,\mathbb{R})$ and the dual basis $\{Y^+, Y^-, Y^o\}$ of $sl(2,\mathbb{R})^*$ we have on $G/M = PSL(2,\mathbb{R})$

$$\Omega_R(\mathscr{P}^+)_e = -(i/\pi)(Y^+ \wedge Y^-).$$

The A-invariant 1-form α_R^+ pairs with the tangent vector Y_o in $e \in PSL(2,\mathbb{R})$ to give the value i/π and by an easy calculation (using Cayley transformation and partial integration) we obtain by Gauss-Bonnet

$$\int_{S(X)} \phi^*(\Omega_R(\mathscr{P}^+) \wedge \alpha_R^+) = -(2\pi)^{-1} \int_X y^{-2}(dx \wedge dy) = 2-2g = \chi(X).$$

From the point of view of secondary characteristic classes the differential form

$$(4\pi^2)\phi^*(\Omega_R(\mathscr{P}^+) \wedge \alpha_R^+)$$

represents the Godbillon-Vey class of the weak-stable foliation of the geodesic flow on the sphere bundle $S(X)$ of the surface X. Thus *theorem 1* can be interpreted as the assertion that the Godbillon-Vey invariant of the weak-stable foliation can be calculated from the closed orbits of the geodesic flow. In foliation theory the relation between the Godbillon-Vey invariant of the weak-stable foliation of $S(X)$ and the Euler characteristic of the surface X is a very well-known result due to Roussarie (see [HK]).

By the method of symbolic dynamics it follows that m_o coincides with the alternating sum of the (finite) dimensions of the generalized eigenspaces of the transfer operators $L_*(0)$ for the eigenvalue 1. Recall that the operators $L_*(0)$ do *not* depend on return times (see [F1])! Therefore *theorem 1* also can be regarded as a formula for the integer associated to the flow by forming this *analytical index*. Although the definition of the latter index strongly depends on the choice of a Markov-family of local sections it is, in fact, independent of the ambiguities involved in the construction of the local sections. Moreover, *theorem 1* implies that the analytical index coincides with the integer defined by integrating a secondary characteristic cohomology class that only depends on the hyperbolic structure of the flow.

Theorem 1 is but a special case of more general formulas relating the multiplicities of the singularities of generalized Selberg zeta-functions at special points to integrals of canonically associated secondary characteristic classes.

Next we combine *theorem 1* with proportionality theory. Let Y^d be the compact dual symmetric space of Y. Y^d is a rank one space and all geodesics are closed and have the same length. Let Y^d_{geo} be the space of all (oriented) geodesics in Y^d (see [B]).

Theorem 2. Let X *be as in theorem 1. Then*

$$m_o = (\chi(X)/\chi(Y^d))\ \chi(Y^d_{geo}), \tag{15}$$

where χ *always denotes Euler characteristic.*

In fact, since $(g_o/m_o)_{\mathbb{C}} \simeq (g^d_o/m_o)_{\mathbb{C}}$ the G-invariant differential forms $\Omega_R(\mathcal{F}^{\pm})\wedge\alpha^{\pm}_R$ on G/M canonically correspond to G^d-invariant volume forms on $G^d/M \simeq S(Y^d)$. By the Gauss-Bonnet-Chern formula the integral of the latter forms coincide with $i^d\ \chi(Y^d_{geo})$, $d = \dim(X)$. Now by *theorem 1* we obtain

$$m_o = i^d\ \chi(Y^d_{geo})\ \mathrm{vol}(\Gamma\backslash G/M)/\mathrm{vol}(G^d/M) =$$

$$= (-1)^{d/2}\ \chi(Y^d_{geo})\ \mathrm{vol}(\Gamma\backslash G/K)/\mathrm{vol}(G^d/K),$$

where the volumes are defined with respect to compatible measures.

Then elliptic proportionality theory implies formula (15).

The calculation of the Euler characteristics of the compact homogeneous spaces in (15) yields

Corollary. Let X *be as in theorem 2. Then*

$$m_o = (\dim(X)/2)\,\chi(X). \tag{16}$$

Note that, in contrast to the even-dimensional case, for an odd-dimensional *real* hyperbolic space we have the following formula.

Theorem 3. Let $X = \Gamma\backslash H^{2n+1}$ *be a compact real hyperbolic space of dimension* 2n+1. *Then the multiplicity of the singularity of* Z_R *in* s = 0 *is given by*

$$2((-1)^{n+1}b_{n+1}(X)+\ldots+(-1)^{2n+1}(n+1)b_{2n+1}(X)) \tag{17}$$

where $b_p(X)$ *is the* p*-th Betti-number of* X.

On the proofs.

Our proof of *theorem 1* is, unfortunately, much more complicated than the result itself suggests. Thus we only can give here rough hints how the assertion can be proved. More details can be found in [J].

The proof rests on a cohomological trace formula which can be regarded as a common (non-commutative) generalization of the Poisson summation formula and the Lefschetz fixed-point formula. It implies that the zeta-function Z_R is closely related to the alternating product of infinite-dimensional (regularized) characteristic determinants of global Frobenius-operators (canonically determined by the action of the geodesic flow) on the cohomology groups of some differential complexes associated to the invariant foliations of S(X). More precisely, it yields a cohomological formula for the multiplicity m_o in terms of the

Lie-algebra-cohomology of $n^{\pm}$ with values in the Harish-Chandra modules of the irreducible representations of G in $L^2(\Gamma\backslash G)$.

Now if dim(X) is even then G has a compact Cartan subgroup H. The cohomological formula for m_o turns out to be connected with an analogous cohomological formula obtained by replacing $n^{\pm}$-cohomology by Lie-algebra-cohomology with respect to the nilradicals of the (complex) Borel algebras containing the complexified Lie algebra of H. This can be proved by using Osborne's character-formula (see [HS]) and suitable patching conditions for characters on neighbouring Cartan subgroups. But the latter number coincides with the analytical index of the (elliptic) deRham complex on the space $\Gamma\backslash G/H$. By working backwards with the corresponding index-form (given by Gauss-Bonnet) one finally ends up with the formula (13).

If the dimension of X is odd then one can explicate the cohomological formula for m_o directly by using results from [C].

Also it should be possible, of course, to give more traditional proofs by applying suitable explicit Selberg trace formulas. In fact, a proof of *theorem 3* along these lines can be found in [F2]. However, by using the same method in the even-dimensional case, the relation of m_o to the hyperbolic structure in terms of the integral of the canonical curvature-form given in (13) would remain mysterious.

References

[BGV] N.Berline, E.Getzler and M.Vergne, Heat kernel and Dirac operators (1992), Springer, Berlin.

[B] A.Besse, Manifolds all of whose geodesics are closed, Erg. d. Math. 93 (1978), Springer, Berlin.

[C] D.Collingwood, Representations of rank one Lie groups, Research Notes in Math . 137 (1985), Pitman, Boston.

[F1] D.Fried, The zeta functions of Ruelle and Selberg, I, Ann. scient. Ec. Norm. Sup., 19 (1986), 491-517.

[F2] D.Fried, Analytic torsion and closed geodesics on hyperbolic manifolds, Invent. math. 84 (1986), 523-540.

[G] R.Gangolli, Zeta functions of Selberg's type for compact space forms of symmetric spaces of rank one, Illionois J. Math. 21 (1977), 1-41.

[GV] C.Godbillon and J.Vey, Un invariant de feuilletages de codimension 1, C.R.Acad.Sci., Paris 273 (1971), 92-95.

[HK] S.Hurder and A.Katok, Differentiability, rigidity and Godbillon-Vey classes for Anosov flows, Publ. IHES (1990), 5-61.

[HS] H.Hecht and W.Schmid, Characters, asymptotics and n-homology of Harish-Chandra modules, Acta math. 151 (1983), 49-151.

[J] A.Juhl, Zeta-Funktionen, Indextheorie und hyperbolische Dynamik, Preprint (1992).

[K] M.Kanai, Geodesic flows of negatively curved manifolds with smooth stable and unstable foliations, Ergodic Theory and Dynamical Systems 8 (1988), 215-239.

[R] D.Ruelle, Zeta functions for expanding maps and Anosov flows, Invent. math. 34 (1976), 231-242.

[S] A.Selberg, Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series, J. Indian Math. Soc. 20 (1956), 47-87.

[W] M.Wakayama, Zeta functions of Selberg's type associated with homogeneous vector bundles, Hiroshima Math. J. 15,2 (1985), 235-295.

Andreas Juhl

Max-Planck-Institut für Mathematik

Gottfried-Claren-Str.26

W-5300 Bonn 3

Germany