AN ELLIPTIC INCARNATION OF THE BAILEY CHAIN

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Abstract. For the first time a Bailey chain with all entries composed out of the Jacobi theta functions is constructed. This is an elliptic extention of the WP-Bailey chain of Andrews and it generates an infinite sequence of identities for theta hypergeometric series. As a particular example, a new proof of the Frenkel-Turaev's elliptic analogue of the Bailey transformation for a terminating ${}_{10}\Phi_9$ basic hypergeometric series is obtained. An elliptic generalization of the Andrews-Berkovich ${}_{10}\Phi_9 \to {}_{12}\Phi_{11}$ transformation formula is derived by employing an elliptic extention of a Bressoud's Bailey pair.

Contents

1. Introduction

The theory of hypergeometric type series was developing over centuries in two instances—the plain generalized hypergeometric functions ${}_sF_r$ (unilateral series) and ${}_sH_r$ (bilateral series) and their basic analogues ${}_s\Phi_r$ and ${}_s\Psi_r$, known as q-series [AAR, GR, Sl]. The very-well-poised ${}_9F_8$ and ${}_{10}\Phi_9$ series are the most general representatives of these two groups of functions sharing some common properties with the classical Gauss ${}_2F_1$ function [W1, R, W2, RS, GM, SZ1]. About five years ago, it was discovered by Frenkel and Turaev [FT] that there exists an appropriate elliptic generalization of the very-well-poised basic hypergeometric series. These series originated from elliptic solutions of the Yang-Baxter equation [Bax, ABF, D-O1, D-O2]. In particular, Boltzmann weights of the lattice models of statistical mechanics constructed by Date, Jimbo, Kuniba, Miwa, and Okado in [D-O1, D-O2] are expressed in terms of elliptic analogues of the mentioned above ${}_{10}\Phi_9$ functions for particular choice of parameters [FT]. Another theory, where the elliptic hypergeometric series appeared from scratch, is the theory of discrete nonlinear integrable equations associated with biorthogonal rational functions developed by Zhedanov and the present author [SZ1, SZ2]. ¿From the special functions theory point of view, the most striking discoveries of [FT] consisted in elliptic extentions of the Jackson's ${}_8\Phi_7$-sum and of the Bailey's transformation for terminating ${}_{10}\Phi_9$-series [GR].

The plain and basic hypergeometric series are known to be related to some contour integrals. For instance, the Askey-Wilson polynomials expressed through ${}_4\Phi_3$-series are orthogonal with respect to the measure described by the Askey-Wilson integral [AW]. A similar relation holds between the Rahman's family of continuous ${}_{10}\Phi_9$ biorthogonal rational functions [R] and the Nassrallah-Rahman integral, containing the Askey-Wilson integral as a particular subcase [NR, R]. In [S1, S2] the author has constructed an elliptic beta integral, which reduces to the Nassrallah-Rahman integral in some limit (see the Appendix for details). It appears that the elliptic beta integral defines a biorthogonality measure for particular continuous meromorphic functions built out of ${}_{12}E_{11}$ elliptic hypergeometric series [S3]. For special choice of parameters, these functions reduce to the Rahman's ${}_{10}\Phi_9$ rational functions. Elliptic analogues of the Wilson's ${}_9F_8$ and ${}_{10}\Phi_9$ biorthogonal rational functions with the discrete biorthogonality measure [W1, W2] have been constructed earlier in [SZ1, SZ2].

Multivariable extensions of the elliptic hypergeometric series have been considered for the first time by Warnaar [Wa2]. Further developments in this direction were tied to multidimensional generalizations of the elliptic beta integral, also known as elliptic Selberg integrals, constructed by van Diejen and the author in [DS1, DS2]. Multivariable analogues of the Frenkel-Turaev identities were discussed in [Wa2, DS1, Ro, DS3, S4].

Recently, a general theory of theta hypergeometric series of elliptic type has been formulated in [S4]. It has shed new light upon the origins of the notions of balancing and well-poisedness of hypergeometric series [AAR, GR, Sl]. A self-consistent definition of the very-well-poisedness condition for theta hypergeometric series has been recognized as well.

Let us turn now to the main subject of the present paper—the Bailey chain. This is the most powerful tool for derivation of identities in the theory of hypergeometric type series. Its first component, the notion of Bailey pairs and the corresponding Bailey lemma, was discovered in [B1, B2] as a hidden general background mechanism in derivations of hypergeometric type series identites. The second ingredient, giving to this machinery a full power, was recognized by Andrews [A1], who proposed an iterative process of building Bailey pairs leading to infinite chains of complicated identities starting from quite simple initial relations. So far, all known examples of Bailey pairs [AAB, A1, A2, AB2, BMS, BS, Br, BIS, ML1, ML2, P, Sl, Si, Wa1], etc, were expressed through either rational or exponential functions of the parameters adapted, respectively, to various ${}_{r+1}F_r$ or ${}_{r+1}\Phi_r$ series identities. In particular, the celebrated Rogers-Ramanujan identities together with a vast list of their extensions can be proved in this way. A curious relation between Bailey pairs and the notion of supersymmetry deserves the attention of physicists [BMS, AB1].

The proofs of elliptic hypergeometric series identities in [FT] used some graphical calculus which does not belong to the standard set of analytical tools of the special functions theory. Therefore, it was necessary to find more regular proofs admitting generalizations towards more complicated cases. After the discovery of the elliptic beta integral [S1, S2], it was shown in [DS1] that a special limit in the residue calculus for this integral leads to the Frenkel-Turaev sum, giving thus its independent derivation.

In the present paper, we propose an elliptic extention of the most general q-series Bailey chain built by Andrews in [A2]. The latter construction bares the name 'WP-Bailey chain', where 'WP' stands for the term 'well-poised' reflecting the key property of involved plain or basic hypergeometric series. Some origins of the corresponding Bailey pairs can be traced back to Bailey [B1] and Bressoud [Br], but it is only full WP-Bailey chain formulation that gives a complete treatment of q-series identities related to the Bailey's ${}_{10}\Phi_9$-transformation [A2, AB2]. It appears also, that only this WP-Bailey chain admits an elliptic generalization. The key reason for this consists in the fact that it is not possible to simplify elliptic Bailey pairs by taking parameters to zero or infinity—for the existence of such limits requires to remove the ellipticity of the involved functions. As a result, the elliptic Bailey chain produces much smaller number of identities. Among them we find Frenkel-Turaev's elliptic generalization of Bailey's identity for terminating very-well-poised ${}_{10}\Phi_9$ series and an elliptic analogue of the Andrews-Berkovich [AB2] relation between particular terminating ${}_{10}\Phi_9$ and ${}_{12}\Phi_{11}$ series.

The paper is organized as follows. In Sect. 2, we introduce necessary theta hypergeometric series terminology. Sect. 3 is devoted to the general Bailey lemma and its chain type realization at the q-series level. In Sect. 4, we construct elliptic Bailey pairs targeted at the elliptic analogue of the Bailey's ${}_{10}\Phi_9$-series transformation and, as a byproduct, prove a new identity for theta functions. A new complicated identity for theta hypergeometric series is proved in Sect. 5. In the Appendix we describe briefly the elliptic beta integral and indicate how the Frenkel-Turaev sum can be deduced from its residue calculus.

We end this section by a notational layout and a description of the Jacobi theta function properties [WW].

Take two complex variables p and q such that $|p|, |q| < 1$. Modular parameters σ, $\mathrm{Im}(\sigma) > 0$, and τ, $\mathrm{Im}(\tau) > 0$, are introduced as follows

$$p = e^{2\pi i\tau}, \qquad q = e^{2\pi i\sigma}. \tag{1.1}$$

The standard p-shifted factorials have the form [GR]

$$(a;p)_\infty = \prod_{n=0}^{\infty}(1-ap^n), \qquad (a;p)_s = \frac{(a;p)_\infty}{(ap^s;p)_\infty},$$

so that $(a;p)_0 = 1$ and

$$(a;p)_n = \prod_{i=0}^{n-1}(1-ap^i), \qquad (a;p)_{-n} = \frac{1}{(ap^{-n};p)_n}$$

for a positive integer n. In the analysis of hypergeometric type functions, it is useful to take a convention when functions with many arguments mean actually products of one function with different arguments, e.g.

$$(a_1,\dots,a_k;p)_\infty \equiv (a_1;p)_\infty \cdots (a_k;p)_\infty,$$
$$\theta(t_1,\dots,t_k;p) \equiv \prod_{m=1}^{k}\theta(t_m;p).$$

The key Jacobi-type theta function used in this paper has the form

$$\theta(z;p) = (z, pz^{-1};p)_\infty. \tag{1.2}$$

Simple computations yield

$$\theta(pz;p) = \theta(z^{-1};p) = -z^{-1}\theta(z;p). \tag{1.3}$$

Zeros of $\theta(z;p)$ lie on a geometric progression, $\theta(z;p)=0$ for $z=p^M$, $M\in\mathbb{Z}$, and for $p=0$ this function becomes linear, $\theta(z;0)=1-z$. The elliptic shifted factorials (the objects of such type appeared already in the Jacobi's works) are defined as

$$\theta(t;p;q)_n \equiv \prod_{m=0}^{n-1}\theta(tq^m;p), \qquad \theta(t;p;q)_{-n}=\frac{1}{\theta(tq^{-n};p;q)_n},$$

together with the convention

$$\theta(t_0,\dots,t_k;p;q)_{\pm n}\equiv\prod_{m=0}^{k}\theta(t_m;p;q)_{\pm n},$$

where n is a positive integer.

The standard Jacobi θ_1-function [WW] is related to $\theta(z;p)$ as follows

$$\begin{aligned}\theta_1(u;\sigma,\tau) &= -i\sum_{n=-\infty}^{\infty}(-1)^n p^{(2n+1)^2/8}q^{(n+1/2)u}\\ &= p^{1/8}iq^{-u/2}\,(p;p)_\infty\,\theta(q^u;p), \quad u\in\mathbb{C}.\end{aligned}$$

Sometimes, it is convenient to replace the θ_1-symbol in favor of the elliptic numbers notation $[u]$ used in [D-O1, D-O2]:

$$[u]\equiv\theta_1(u) \qquad \text{or} \qquad [u;\sigma,\tau]\equiv\theta_1(u;\sigma,\tau),$$
$$[u_0,\dots,u_k]\equiv\prod_{m=0}^{k}[u_m].$$

The second modular parameter σ was introduced into the definition of θ_1-function for simplification of expressions for elliptic shifted factorials $[u]_{\pm n}$, $n\in\mathbb{N}$, in the additive form:

$$[u]_0=1, \quad [u]_n=[u,u+1,\dots,u+n-1], \quad [u]_{-n}=\frac{1}{[u-n]_n}.$$

Dependence on σ and τ in $[u;\sigma,\tau]$ will be usually dropped in the following.

Complex function $[u]$ is entire, odd $[-u]=-[u]$, and doubly quasiperiodic

$$\begin{aligned}&[u+\sigma^{-1}]=-[u],\\ &[u+\tau\sigma^{-1}]=-e^{-\pi i\tau-2\pi i\sigma u}[u].\end{aligned} \tag{1.4}$$

The modular $PSL(2,\mathbb{Z})$-group,

$$\tau\to\frac{a\tau+b}{c\tau+d}, \qquad \sigma\to\frac{\sigma}{c\tau+d}, \tag{1.5}$$

where $a,b,c,d\in\mathbb{Z}$ and $ad-bc=1$, is generated by two transformations: $\tau\to\tau+1$, $\sigma\to\sigma$, and $\tau\to-\tau^{-1}$, $\sigma\to\sigma\tau^{-1}$. In these two cases the transformation laws of $[u;\sigma,\tau]$ are quite simple

$$[u;\sigma,\tau+1]=e^{\pi i/4}[u;\sigma,\tau], \tag{1.6}$$
$$[u;\sigma/\tau,-1/\tau]=i(-i\tau)^{1/2}e^{\pi i\sigma^2u^2/\tau}[u;\sigma,\tau], \tag{1.7}$$

and the sign of $(-i\tau)^{1/2}$ is fixed from the positivity of its real part condition.

2. Very-well-poised theta hypergeometric series

Throughout this paper, we use the terminology on theta hypergeometric series introduced in [S4]. Its essentials are described in this section together with some new refinements. Although the corresponding approach allows one to construct Jacobi theta functions generalization of unilateral and bilateral basic hypergeometric series ${}_s\Phi_r$, ${}_s\Psi_r$, together with their multivariable partners, we outline here only the one variable unilateral series case. The following formal power series is called *theta hypergeometric series of elliptic type*

$$
\begin{aligned}
&{}_sE_r\left(\begin{matrix} t_0, \ldots, t_{s-1} \\ w_1, \ldots, w_r \end{matrix}; q, p; z\right) \\
&\qquad = \sum_{n=0}^{\infty} \frac{\theta(t_0, t_1, \ldots, t_{s-1}; p; q)_n}{\theta(q, w_1, \ldots, w_r; p; q)_n}\, z^n. \qquad (2.1)
\end{aligned}
$$

All parameters t_i, w_i are assumed to take arbitrary complex values except the singularity points (zeros of denominators). The variable z is independent; if necessary, one may consider it as a function of t_i, w_i, p, q as well.

If we write these series as $\sum_{n\in\mathbb{N}} c_n$, then $h(n) = c_{n+1}/c_n$ is equal to the ratio of s over $r+1$ Jacobi θ_1-functions multiplied by some exponential of n. Thus $h(x)$ is a meromorphic doubly quasiperiodic function of $x \in \mathbb{C}$, and this property may be used for an initial definition of theta hypergeometric series of elliptic type. Such an approach to the ${}_sE_r$-series is close in spirit to general definitions of the plain and basic hypergeometric series through the ratios of series coefficients [AAR, GR, GGR].

Remark 1. The quasiperiodicity of the theta functions shows that the theta hypergeometric series do not admit confluence limits. More precisely, the limits $t_m, w_m \to 0$ or $t_m, w_m \to \infty$ are not well defined and thus do not lead to ${}_sE_r$-series with smaller values of indices r and s.

Consider the limit $\mathrm{Im}(\tau) \to +\infty$ or $p \to 0$. In a straightforward manner one gets

$$
\begin{aligned}
\lim_{p\to 0} {}_sE_r &= {}_s\Phi_r\left(\begin{matrix} t_0, t_1, \ldots, t_{s-1} \\ w_1, \ldots, w_r \end{matrix}; q; z\right) \\
&= \sum_{n=0}^{\infty} \frac{(t_0, t_1, \ldots, t_{s-1}; q)_n}{(q, w_1, \ldots, w_r; q)_n}\, z^n. \qquad (2.2)
\end{aligned}
$$

This is the definition of basic hypergeometric series ${}_s\Phi_r$ used in [Sl]. The definition of ${}_s\Phi_r$ given in [GR] contains under the summation sign the $(r+1-s)$-th power of $(-1)^n q^{n(n-1)/2}$. These two definitions are related to each other by the inversion $q \to q^{-1}$ and a renormalization of the argument z.

Remark 2. The simplest theta hypergeometric series

$$
e(z; q, p) \equiv {}_0E_0(z; q, p) = \sum_{n=0}^{\infty} \frac{z^n}{\theta(q; p; q)_n} \qquad (2.3)
$$

can be considered as a theta analogue of the exponential function or, loosely speaking, an elliptic exponential function. It is not known whether $e(z; q, p)$ satisfies any simple addition formula. However, it provides a formal power series solution to the following q-integral equation

$$\frac{1}{(p;p)_\infty}\sum_{k=-\infty}^{\infty}(-1)^k p^{k(k-1)/2} f(q^k z) = z f(z), \tag{2.4}$$

which is reduced for $p \to 0$ to the equation $f(qz) = (1-z)f(z)$. For $|q| < 1$ the latter yields the standard q-exponential function [GR],

$$e(z;q) = \sum_{n=0}^{\infty}\frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty},$$

as a unique analytical at the $z = 0$ solution normalized by the $f(0) = 1$ condition. The function $e(z;q)$ may be interpreted also as a q-gamma function, since after writing $z = q^u$ it reproduces the most essential part of the Jackson's q-gamma function (see the Appendix).

As usual with the hypergeometric type series, the choice $s = r + 1$ leads to functions with remarkable properties. Let us call the series ${}_{r+1}E_r$ *balanced* if its parameters satisfy the constraint

$$\prod_{m=0}^{r} t_m = q\prod_{k=1}^{r} w_k. \tag{2.5}$$

The following definition justifies the use of the word 'elliptic' in the context of hypergeometric type functions.

Definition. The (unilateral) *elliptic hypergeometric series* is the series $\sum_{n\in\mathbb{N}} c_n$ for which the function $h(n) = c_{n+1}/c_n$ is an elliptic function of the argument n considered as a continuous complex variable.

Denote as σ^{-1} and $\tau\sigma^{-1}$, $\mathrm{Im}(\tau) > 0$, two periods of the elliptic (i.e. meromorphic double periodic) function $h(s)$:

$$h(s+\sigma^{-1}) = h(s), \qquad h(s+\tau\sigma^{-1}) = h(s).$$

The number of poles (or zeros) in the fundamental parallelogram of periods is called the order of $h(s)$, let it be equal to $r+1$. Then $h(s)$ can be factorized as follows [WW]:

$$h(s) = z\prod_{m=0}^{r}\frac{[s+\alpha_m]}{[s+\beta_m]}, \tag{2.6}$$

where the sums of zeros' $\alpha_0, \ldots, \alpha_r$ and poles' $\beta_0, \ldots, \beta_r$ coordinates coincide:

$$\sum_{m=0}^{r}\alpha_m = \sum_{m=0}^{r}\beta_m. \tag{2.7}$$

Let us set $s = n - \beta_0 + 1$, and denote

$$u_m = \alpha_m - \beta_0 + 1, \qquad t_m = q^{u_m},$$
$$v_m = \beta_m - \beta_0 + 1, \qquad w_m = q^{v_m},$$

where $v_0 = 1$ (or $w_0 = q$). Then the condition (2.7) allows one to rewrite (2.6) as follows

$$h(n) = z\prod_{m=0}^{r}\frac{\theta(t_m q^n;p)}{\theta(w_m q^n;p)} = z\prod_{m=0}^{r}\frac{[n+u_m]}{[n+v_m]}, \tag{2.8}$$

which is nothing else than the ratio c_{n+1}/c_n of two successive terms of the balanced ${}_{r+1}E_r$ theta hypergeometric series. Thus the unilateral elliptic hypergeometric series coincide with the balanced theta hypergeometric series ${}_{r+1}E_r$ after placing one of the poles of $h(n)$ into the point $n = -1$.

Theta hypergeometric series are called modular hypergeometric series if they are invariant with respect to the modular group transformations (1.5). It is not difficult to see that ${}_sE_r$-series may be modular only for $s = r + 1$. The modularity of ${}_{r+1}E_r$ follows from the modularity of the function $h(n)$ and vice versa (provided z is an independent variable). From (2.8), the transformation laws (1.6), (1.7), and the demand that the variable z is not touched by the $PSL(2,\mathbb{Z})$ group, one finds that $h(n)$ is modular invariant, provided the parameters u_n, v_n satisfy the balancing condition (2.5), i.e. $\sum_{m=0}^{r} u_m = 1 + \sum_{m=1}^{r} v_m$, and

$$u_0^2 + \ldots + u_r^2 = 1 + v_1^2 + \ldots + v_r^2. \tag{2.9}$$

In the cases when the argument z has some non-trivial modular transformation properties (i.e., if it contains some τ-dependence), the situation is a bit more complicated. The balancing condition should be valid again, but the equality (2.9) is modified according to the tranformation law of z. Thus, the modularity of theta hypergeometric series assumes their ellipticity but the opposite, in general, is not true. Note that the terminating modular hypergeometric series represent particular examples of meromorphic Jacobi forms in the sense of Eichler and Zagier [EZ, Z].

Remark 3. Let us take $r = 3$ and fix parameters in the numerator of $h(n)$ (2.8) as $u_0 = u_1 = 0, u_2 = 1 - XY, u_3 = X + Y$, and in the denominotor $v_1 = -XY, v_2 = X, v_3 = Y$. Evidently, both the balancing and (2.9) conditions are satisfied. Choose $z = (2\pi i n)^{-2}$ and take the limit $n \to 0$. As a result, we get the following $PSL(2,\mathbb{Z})$-invariant function

$$\lim_{n\to 0} h(n) = \frac{\sigma^2 \theta_1'(0)^2 \theta_1(XY-1)\theta_1(X+Y)}{\theta_1(XY)\theta_1(X)\theta_1(Y)\theta_1(1)},$$

which was shown in [Z] to be a particular generating function of the Eichler-Shimura-Manin period polynomials of modular forms (our σ is equal to T in the notations of [Z]). Such a relation shows what kind of information about modular forms may be encoded into the modular hypergeometric series and their identities. Let us mention also an interesting application of the Jacobi forms to the description of elliptic genera of compact complex manifolds (see, e.g. [G]).

Both the q-series ${}_{r+1}\Phi_r$ and their theta analogues ${}_{r+1}E_r$ are called *well-poised* if their parameters satisfy the following constraints

$$u_0 + 1 = u_1 + v_1 = \ldots = u_r + v_r \tag{2.10}$$

in the additive form, or

$$qt_0 = t_1 w_1 = \ldots = t_r w_r \tag{2.11}$$

in the multiplicative form.

Elliptic hypergeometric series ${}_{r+1}E_r$ are called *totally elliptic* if the corresponding series coefficients c_n are elliptic functions of all independent parameters among $u_0, \ldots, u_r$ and $v_1, \ldots, v_r$. Evidently, total ellipticity may take place only under additional constraints imposed upon the parameters u_i, v_i. It can be shown [S4] that the maximal number of free parameters is attained for the well-poised elliptic hypergeometric series ${}_{r+1}E_r$ whose c_n-coefficients are elliptic in $u_0, \ldots, u_{r-5}$ with the same lattice of periods as in n. It is not difficult to see that the totally elliptic (i.e., well-poised balanced theta) hypergeometric series are automatically modular invariant. Such a characterization result brings a new point of view upon the origins of the general concept of well-poisedness itself.

According to the current classification of ${}_{r+1}\Phi_r$-series [GR], they are called *very-well-poised* if, in addition to (2.11), their parameters satisfy the constraints $t_r = qt_0^{1/2}$, $t_{r-1} = -qt_0^{1/2}$. This definiton must be changed at the level of theta hypergeometric series.

Definition. The theta hypergeometric series ${}_{r+1}E_r$ is called *very-well-poised* if, in addition to the constraints (2.10) or (2.11), the following relations hold true

$$\begin{aligned} u_{r-3} &= \frac{1}{2}u_0 + 1, \quad u_{r-2} = \frac{1}{2}u_0 + 1 - \frac{1}{2\sigma}, \\ u_{r-1} &= \frac{1}{2}u_0 + 1 - \frac{\tau}{2\sigma}, \quad u_r = \frac{1}{2}u_0 + 1 + \frac{1+\tau}{2\sigma}, \end{aligned} \tag{2.12}$$

or, in the multiplicative form,

$$\begin{aligned} t_{r-3} &= t_0^{1/2}q, \quad t_{r-2} = -t_0^{1/2}q, \\ t_{r-1} &= t_0^{1/2}qp^{-1/2}, \quad t_r = -t_0^{1/2}qp^{1/2}. \end{aligned} \tag{2.13}$$

Let us substitute constraints (2.13) into ${}_{r+1}E_r$. Applying identities (1.3) and

$$\theta(z, -z, zp^{1/2}, -zp^{1/2}; p) = \theta(z^2; p)$$

one gets

$$\prod_{m=r-3}^{r} \frac{\theta(t_m; p; q)_n}{\theta(qt_0/t_m; p; q)_n} = \frac{\theta(t_0q^{2n}; p)}{\theta(t_0; p)}\,(-q)^n.$$

As a result,

$$\begin{aligned} {}_{r+1}E_r &\left(\begin{matrix} t_0, t_1, \dots, t_{r-4}, qt_0^{1/2}, -qt_0^{1/2}, qp^{-1/2}t_0^{1/2}, -qp^{1/2}t_0^{1/2} \\ qt_0/t_1, \dots, qt_0/t_{r-4}, t_0^{1/2}, -t_0^{1/2}, p^{1/2}t_0^{1/2}, -p^{-1/2}t_0^{1/2} \end{matrix}; q, p; z \right) \\ &= \sum_{n=0}^{\infty} \frac{\theta(t_0q^{2n}; p)}{\theta(t_0; p)} \prod_{m=0}^{r-4} \frac{\theta(t_m; p; q)_n}{\theta(qt_0/t_m; p; q)_n}\,(-qz)^n. \end{aligned} \tag{2.14}$$

It is convenient to introduce the following shorthand notation for the very-well-poised series similar to the one used in [GR]. After replacing z by $-z$ in (2.14) we write

$$\begin{aligned} &{}_{r+1}V_r(t_0; t_1, \dots, t_{r-4}; q, p; z) \\ &\qquad \equiv \sum_{n=0}^{\infty} \frac{\theta(t_0q^{2n}; p)}{\theta(t_0; p)} \prod_{m=0}^{r-4} \frac{\theta(t_m; p; q)_n}{\theta(qt_0t_m^{-1}; p; q)_n}\,(qz)^n. \end{aligned} \tag{2.15}$$

In terms of the elliptic numbers one has

$$\begin{aligned} {}_{r+1}V_r(u_0; u_1, \dots, u_{r-4}; \sigma, \tau; z) &\equiv \sum_{n=0}^{\infty} \frac{[u_0+2n]}{[u_0]} \\ \times \prod_{m=0}^{r-4} &\frac{[u_m]_n}{[u_0+1-u_m]_n}\, z^n q^{n(\sum_{m=1}^{r-4} u_m - (r-7)/2 - u_0(r-5)/2)}. \end{aligned} \tag{2.16}$$

For $p=0$, functions (2.15) reduce to the very-well-poised basic hypergeometric series:

$$
{}_{r-1}W_{r-2}(t_0;t_1,\ldots,t_{r-4};q;qz) \\
= \sum_{n=0}^{\infty} \frac{1-t_0q^{2n}}{1-t_0} \prod_{m=0}^{r-4} \frac{(t_m;q)_n}{(qt_0t_m^{-1};q)_n} (qz)^n,
$$

and the use of the symbol W is standard in this context [GR], note only the replacement of the argument z by qz.

According to the definition given in [GR] the series ${}_{r+1}\Phi_r$ in (2.2) is called balanced provided that

$$
q\prod_{m=0}^{r} t_m = \prod_{k=1}^{r} w_k. \tag{2.17}
$$

(It is usually assumed that simultaneously with (2.17) one has $z=q$, but we prefer not to associate such a constraint with the balancing condition.) For $p\to 0$ the series ${}_{r+1}E_r$ is reduced to ${}_{r+1}\Phi_r$, but the corresponding balancing conditions (2.5) and (2.17) do not match. Surprisingly, this contradiction does not take place for the very-well-poised series.

Indeed, let us impose the balancing constraint (2.5) upon the parameters of ${}_{r+1}V_r$-series (2.16). This yields

$$
\sum_{m=1}^{r-4} u_m = \frac{r-7}{2} + \frac{r-5}{2}u_0 \quad \text{or} \quad \prod_{m=1}^{r-4} t_m = q^{(r-7)/2} t_0^{(r-5)/2},
$$

and the extra factor to the right of z^n in (2.16) disappears. Remarkably, this is precisely the balancing condition for the very-well-poised ${}_{r-1}\Phi_{r-2}$ series [GR]. This fact shows that the very-well-poisedness and balancing conditions are tied to each other more strongly than it was thought before. Moreover, the general balancing condition for hypergeometric type series itself appears to be connected with the ellipticity condition for the ratio of successive terms of the series. Actually, the results of the Sect. 5 show that it is desirable to change definition of balancing for the q-series and to apply this term to all q-series appearing in the $p\to 0$ limit of the elliptic hypergeometric series.

The theta hypergeometric series ${}_sE_r$ are defined as formal infinite series. Because of the quasiperiodicity of theta functions their convergence is questionable. We shall not discuss this problem here since only terminating series will be used in the following (we note, however, that for some choices of parameters the radius of convergence is greater than zero). Truncation of the ${}_sE_r$-series occurs if one of its parameters u_m satisfies the constraint

$$
u_m = -N - K\sigma^{-1} - M\tau\sigma^{-1}, \qquad N\in\mathbb{N}, \quad K,M\in\mathbb{Z} \tag{2.18}
$$

(in the multiplicative form it looks as $t_m = q^{-N}p^{-M}$). The terminating well-poised elliptic hypergeometric series are totally elliptic and, so, do not depend on K, M.

The terminating very-well-poised balanced ${}_{10}\Phi_9$ (or ${}_{10}W_9$) series of the argument $z=q$ is known to satisfy the Bailey transformation identity [GR]. Frenkel and Turaev have found its elliptic generalization [FT], which looks in the notations of the present paper as follows:

$$
\begin{aligned}
{}_{12}V_{11}(t_0;t_1,\dots,t_7;q,p;1) &= \frac{\theta(qt_0,qs_0/t_4,qs_0/t_5,qt_0/t_4t_5;p;q)_N}{\theta(qs_0,qt_0/t_4,qt_0/t_5,qs_0/t_4t_5;p;q)_N} \\
&\quad \times {}_{12}V_{11}(s_0;s_1,\dots,s_7;q,p;1), \qquad (2.19)
\end{aligned}
$$

where $\prod_{m=1}^{7} t_m = t_0^3 q^2$, $t_6 = q^{-N}$, $N \in \mathbb{N}$,

$$
s_0 = \frac{qt_0^2}{t_1t_2t_3}, \quad s_1 = \frac{s_0t_1}{t_0}, \quad s_2 = \frac{s_0t_2}{t_0}, \quad s_3 = \frac{s_0t_3}{t_0},
$$

and s_4, s_5, s_6, s_7 form an arbitrary permutation of t_4, t_5, t_6, t_7. The additional constraint $t_2t_3 = qt_0$ converts the l.h.s. ${}_{12}V_{11}$-series into a terminating ${}_{10}V_9$ series, whereas on the r.h.s. only the first term of the corresponding ${}_{12}V_{11}$-series survives. As a result, one gets the Frenkel-Turaev sum, or an elliptic generalization of the Jackson's sum for terminating very-well-poised balanced ${}_8\Phi_7$-series:

$$
\begin{aligned}
&{}_{10}V_9(t_0;t_1,t_4,t_5,t_6,t_7;q,p;1) \\
&\qquad = \frac{\theta(qt_0,qt_0/t_1t_4,qt_0/t_1t_5,qt_0/t_4t_5;p;q)_N}{\theta(qt_0/t_1t_4t_5,qt_0/t_1,qt_0/t_4,qt_0/t_5;p;q)_N}, \qquad (2.20)
\end{aligned}
$$

where $t_1t_4t_5t_6t_7 = qt_0^2$ and $t_6 = q^{-N}$, $N \in \mathbb{N}$.

3. The Bailey chain in q-series instances

The Bailey approach to derivation of hypergeometric series identities is based upon the very simple statement [B2].

Lemma 1. *Suppose that two sequences β_n, γ_n, $n \in \mathbb{N}$, are defined as*

$$
\beta_n = \sum_{i=0}^{n} u_{n-i} v_{n+i} \alpha_i, \qquad (3.1)
$$

$$
\gamma_n = \sum_{i=n}^{\infty} u_{i-n} v_{i+n} \delta_i, \qquad (3.2)
$$

where $\alpha_i, \beta_i, u_i, v_i$ are some other sequences. Then

$$
\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n, \qquad (3.3)
$$

where it is assumed that all infinite series converge absolutely.

Proof. Consider the l.h.s. sum of (3.3)

$$
\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} u_i v_{i+2n} \delta_{i+n} \alpha_n.
$$

Let us pass from the summation over rows and columns of the infinite square lattice formed by indices $n, i \in \mathbb{N}$ to the summation over its diagonals $k = i + n \in \mathbb{N}$ and $j = 0, \dots, k$. The absolute convergence of infinite sums allows one to change the order of summation and to get

$$
\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{k=0}^{\infty} \sum_{j=0}^{k} u_{k-j} v_{k+j} \alpha_j \delta_k = \sum_{k=0}^{\infty} \beta_k \delta_k, \qquad (3.4)
$$

i.e. the desired equality. $\square$

The sequences α_n, β_n related by (3.1) are said to form a Bailey pair relative to u_n, v_n and the sequences γ_n, δ_n are called a conjugate Bailey pair. Evidently, there is a too large freedom in the choice of all these quantities, so that, from the first glance, it is hard to expect profound consequences from the identity (3.3). However, as it is illustrated in [Sl], even for a very general form of $u_n, v_n, \alpha_n, \delta_n$ associated with plain hypergeometric series, some art in working with *fairly simple summation formulae* allows one to get from (3.3) non-trivial hypergeometric series transformations or *more complicated sums*. Such a process of complication of formulae can be made *iterative*. Let us outline this procedure at the level of q-series.

The following particular choice

$$u_n = \frac{1}{(q;q)_n}, \qquad v_n = \frac{1}{(aq;q)_n}, \tag{3.5}$$

is widely discussed in the literature in connection to the Rogers-Ramanujan type identities [B1, B2, Sl, A1, Br], etc. As shown in [B2], the sequences

$$\delta_i = \frac{(\rho_1, \rho_2; q)_i}{(aq/\rho_1, aq/\rho_2; q)_n} \left(\frac{aq}{\rho_1\rho_2}\right)^i \frac{(aq/\rho_1\rho_2; q)_{n-i}}{(q;q)_{n-i}},$$
$$\gamma_i = \frac{(\rho_1, \rho_2; q)_i}{(aq/\rho_1, aq/\rho_2; q)_i} \left(\frac{aq}{\rho_1\rho_2}\right)^i \frac{1}{(q;q)_{n-i}(aq;q)_{n+i}},$$

where $i = 0, \dots, n$, form a conjugate Bailey pair relative to (3.5) for any ρ_1, ρ_2. If one substitutes these sequences into identity (3.3), then it can be rewritten as follows:

$$\beta_n' = \sum_{i=0}^{\infty} \frac{1}{(q;q)_{n-i}(aq;q)_{n+i}}\, \alpha_i', \tag{3.6}$$

where

$$\alpha_i' = \frac{(\rho_1, \rho_2; q)_i}{(aq/\rho_1, aq/\rho_2; q)_i} \left(\frac{aq}{\rho_1\rho_2}\right)^i \alpha_i, \tag{3.7}$$

$$\beta_n' = \sum_{k=0}^{\infty} \frac{(\rho_1, \rho_2; q)_k (aq/\rho_1\rho_2; q)_{n-k}}{(aq/\rho_1, aq/\rho_2; q)_n (q;q)_{n-k}} \left(\frac{aq}{\rho_1\rho_2}\right)^k \beta_k. \tag{3.8}$$

But relation (3.6) coincides with the definition of Bailey pairs relative to the choice (3.5), i.e. the sequences α_n', β_n' form a Bailey pair similar to α_n, β_n. Since the latter sequences have not been specified yet, equalities (3.7) and (3.8) define a tool for generating an infinite sequence of Bailey pairs relative to (3.5). In this manner one gets an infinite chain of identities of the form (3.6) from some starting sequences α_n, β_n. (We skip explicit examples of such initial Bailey pairs, since they are easily accessible in the literature, and save space for more complicated situations.) Note that the basic relation (3.6) involves only parameters q, a, whereas definitions (3.7), (3.8) contain *new* free parameters ρ_1, ρ_2. Therefore, in the definition of Bailey pair α_n'', β_n'', obtained by replacement of sequences α_n and β_n on the r.h.s of (3.7), (3.8) by α_n' and β_n', respectively, one can replace ρ_1, ρ_2 by arbitrary new parameters. This gives an additional freedom in the construction. The described algorithm reflects the essence of the general Bailey *chain* machinery discovered by Andrews [A1], which boosts greatly the power of Bailey's one-step approach [B2] (some particular cases of this iterative process were recognized also by Paule [P]).

Moreover, sometimes there exist more than one way of building new Bailey pairs α'_n, β'_n relative to some fixed (u_n, v_n)-sequences from given α_n, β_n, i.e. the rules (3.7), (3.8) may have sisters. In this case, the infinite chain of identities becomes a multidimensional lattice [AAB] or a tree [A2].

Clearly, the main difficulty consists in finding appropriate kernel sequences u_n, v_n for which some non-trivial maps (3.6) take place. The following generalization of the Bailey chain (3.5), (3.6), (3.7), (3.8) has been proposed recently by Andrews [A2]. The sequences α_n, β_n, depending at least on three variables q, a, k, are said to form a well-poised or WP-Bailey pair if

$$\beta_n(a,k) = \sum_{i=0}^{n} \frac{(k/a;q)_{n-i}(k;q)_{n+i}}{(q;q)_{n-i}(aq;q)_{n+i}} \alpha_i(a,k). \tag{3.9}$$

The main theorem proved in [A2] states that for a given WP-Bailey pair α_n, β_n the sequences

$$\alpha'_n(a,k) = \frac{(\rho_1, \rho_2; q)_n}{(aq/\rho_1, aq/\rho_2; q)_n} \left(\frac{aq}{\rho_1\rho_2}\right)^n \alpha_n\left(a, \frac{k\rho_1\rho_2}{aq}\right) \tag{3.10}$$

and

$$\begin{aligned}\beta'_n(a,k) = \frac{(k\rho_1/a, k\rho_2/a; q)_n}{(aq/\rho_1, aq/\rho_2; q)_n} \sum_{j=0}^{n} \frac{(1 - k\rho_1\rho_2 q^{2j-1}/a)(\rho_1, \rho_2; q)_j}{(1 - k\rho_1\rho_2/aq)(k\rho_2/a, k\rho_1/a; q)_j} \\ \times \frac{(aq/\rho_1\rho_2; q)_{n-j}(k;q)_{n+j}}{(q;q)_{n-j}(k\rho_1\rho_2/a;q)_{n+j}} \left(\frac{aq}{\rho_1\rho_2}\right)^j \beta_j\left(a, \frac{k\rho_1\rho_2}{aq}\right)\end{aligned} \tag{3.11}$$

form a new WP-Bailey pair as well. For $k = 0$ the WP-Bailey chain formulae are reduced to relations (3.6), (3.7), and (3.8). Consequences of the addition of a new parameter k to (3.5) have been analyzed to some extent by Bailey [B1] and Bressoud [Br], but it is the discovery of the intricate rules (3.10) and (3.11) that constitutes the birth of a new Bailey chain. A number of the corresponding new q-series identities have been found by Andrews and Berkovich in [AB2]. As to the aim of the present paper, it appears that it is only the Andrews' WP-Bailey chain that admits an elliptic generalization, which we are going to describe in the next section.

4. Elliptic Bailey pairs

In the previous section we outlined general origins of Bailey chains but we did not describe any identity following from them. Let us do it directly at the elliptic level. We start from the derivation of a new summation formula associated with the ${}_8V_7$-series.

Theorem 2. *The first m terms of the very-well-poised series ${}_8V_7(a; a/k, kq^n, q^{-n}; q, p; 1)$ are summable to a closed form expression, namely,*

$$\begin{aligned}\sum_{i=0}^{m} \frac{\theta(aq^{2i};p)}{\theta(a;p)} \frac{\theta(a, a/k, kq^n, q^{-n}; p; q)_i}{\theta(q, kq, aq^{1-n}/k, aq^{n+1}; p; q)_i} q^i \\ = \frac{\theta(k;p)\theta(kq^{m+1}, kq^{-m}/a, aq; p; q)_n \theta(q^{1-n}; p; q)_m}{\theta(kq^n;p)\theta(k, k/a, aq^{m+1}; p; q)_n \theta(q;p;q)_m} q^{mn}.\end{aligned} \tag{4.1}$$

Proof. Let us denote the l.h.s. of the relation (4.1) by L_m and the r.h.s. by R_m. For $m=0$ one has obviously $L_0=R_0=1$. Suppose that (4.1) is valid for some fixed value of $m=M$, $0\leq M<n-1$. Let us prove it for the case $m=M+1<n$. For this we consider the ratio L_{m+1}/R_{m+1}:

$$\begin{aligned}\frac{L_{m+1}}{R_{m+1}} = \frac{R_m}{R_{m+1}} &+ \frac{\theta(aq^{2m+2};p)\theta(a,a/k,kq^n,q^{-n};p;q)_{m+1}}{\theta(a;p)\theta(aq^{n+1},aq^{1-n}/k,kq,q^{1-n};p;q)_{m+1}} \\ &\times \frac{\theta(kq^n;p)\theta(k,k/a,aq^{m+2};p;q)_n}{\theta(k;p)\theta(kq^{m+2},kq^{-m-1}/a,aq;p;q)_n}\, q^{(1-n)(m+1)}. \end{aligned} \tag{4.2}$$

The first term may be simplified as follows

$$\frac{R_m}{R_{m+1}} = q^{-n}\frac{\theta(aq^{m+n+1},kq^{m+1},kq^{n-m-1}/a,q^{m+1};p)}{\theta(aq^{m+1},kq^{m+n+1},kq^{-m-1}/a,q^{M+1-n};p)}.$$

After substitution of this expression into (4.2) and various simplifications, we get

$$\begin{aligned}\frac{L_{m+1}}{R_{m+1}} &= \frac{\theta(aq^{m+n+1},kq^{m+1},kq^{n-m-1}/a,q^{m+1};p)-\theta(aq^{2m+2},q^n,k/a,kq^n;p)}{(k/a)\theta(aq^{m+1},kq^{m+n+1},aq^{m+1}/k,q^{n-m-1};p)} \\ &= 1.\end{aligned}$$

The last equality is true because of the well known identity for the product of four theta functions [WW],

$$\begin{aligned}\theta(xw,x/w,yz,y/z;p) &- \theta(xz,x/z,yw,y/w;p) \\ &= yw^{-1}\theta(xy,x/y,wz,w/z;p),\end{aligned} \tag{4.3}$$

where the following identification of variables is needed

$$\begin{aligned} x &= q^{m+1}\sqrt{akq^n}, \qquad \omega = \sqrt{aq^n/k}, \\ y &= \sqrt{kq^n/a}, \quad z = q^{-m-1}\sqrt{kq^n/a}. \end{aligned} \tag{4.4}$$

In order to prove the equality $L_{m+1}=R_{m+1}$ for $m=n-1$ it is necessary to repeat the same procedure after removing from R_{m+1} the factor $\theta(q^{m+1-n};p)$, which is equal to zero for $m=n-1$. Since in this case $\omega=1/z$ in (4.4), the r.h.s. of identity (4.3) vanishes and one finds that $L_n=R_n=0$.

Finally, it is easy to see that for $m>n$ all the terms with $i=n+1,\dots,m$ in the series for L_m vanish. Therefore, in this case $L_m=L_n=R_n=R_m=0$. The theorem is thus proved for arbitrary $m\geq 0$. Note that for $m=n$, the sum (4.1) is equal to the Kronecker delta function, $L_n=\delta_{n0}$. □

For $p=0$ the formula (4.1) reduces to the sum proved by Andrews in [A2].

Let us give a different formulation of this theorem in the case $m=n$. Denote

$$\alpha_n(a,k) = \frac{\theta(aq^{2n};p)\theta(a,a/k;p;q)_n}{\theta(a;p)\theta(q,kq;p;q)_n}\left(\frac{k}{a}\right)^n. \tag{4.5}$$

and

$$\beta_n(a,k) = \delta_{n0}. \tag{4.6}$$

Then the $m=n$ case of (4.1) can be rewritten as follows

$$\beta_n(a,k) = \sum_{i=0}^n \frac{\theta(k/a;p;q)_{n-i}\theta(k;p;q)_{n+i}}{\theta(q;p;q)_{n-i}\theta(aq;p;q)_{n+i}}\,\alpha_i(a,k). \tag{4.7}$$

In order to see this it is sufficient to apply the auxiliary formulae

$$\theta(z;p;q)_{n+i} = \theta(z;p;q)_n \theta(zq^n;p;q)_i \tag{4.8}$$

and

$$\theta(z;p;q)_{n-i} = (-1)^i z^{-i} q^{i(i+1)/2-ni} \frac{\theta(z;p;q)_n}{\theta(q^{1-n}/z;p;q)_i}. \tag{4.9}$$

The key new definition of the present paper is the following straightforward elliptic generalization of the Andrews' definition of WP-Bailey pairs [A2].

Definition. Two arbitrary sequences α_n and β_n, $n \in \mathbb{N}$, are said to form *an elliptic Bailey pair* if they satisfy the relation (4.7).

Suppose that we have found a regular way of generating different elliptic Bailey pairs out of a given one, e.g. from pair (4.5), (4.6). Then, (4.7) would result in various summation or transformation formulae for theta hypergeometric series, as it was the case with (4.1). The main result of the present paper is contained in the following theorem.

Theorem 3. *Whenever two sequences α_n and β_n form an elliptic Bailey pair, the new sequences defined as*

$$\alpha'_n(a,k) = \frac{\theta(\rho_1,\rho_2;p;q)_n}{\theta(aq/\rho_1, aq/\rho_2;p;q)_n} \left(\frac{aq}{\rho_1\rho_2}\right)^n \alpha_n\left(a, \frac{k\rho_1\rho_2}{aq}\right) \tag{4.10}$$

and

$$\begin{aligned}\beta'_n(a,k) = \frac{\theta(k\rho_1/a, k\rho_2/a;p;q)_n}{\theta(aq/\rho_1, aq/\rho_2;p;q)_n} \sum_{j=0}^{n} \frac{\theta(k\rho_1\rho_2 q^{2j-1}/a;p)\theta(\rho_1,\rho_2;p;q)_j}{\theta(k\rho_1\rho_2/aq;p)\theta(k\rho_2/a, k\rho_1/a;p;q)_j} \\ \times \frac{\theta(aq/\rho_1\rho_2;p;q)_{n-j}\theta(k;p;q)_{n+j}}{\theta(q;p;q)_{n-j}\theta(k\rho_1\rho_2/a;p;q)_{n+j}} \left(\frac{aq}{\rho_1\rho_2}\right)^j \beta_j\left(a, \frac{k\rho_1\rho_2}{aq}\right),\end{aligned} \tag{4.11}$$

where ρ_1, ρ_2 are two new arbitrary parameters, form an elliptic Bailey pair as well.

Proof. The proof consists in a long and cumbersome but fairly simple chain of algebraic computations involving a change of the order of summations and application of the Frenkel-Turaev ${}_{10}E_9$-sum. Indeed, substitute relation (4.7) into (4.11) and get

$$\beta'_n(a,k) = \frac{\theta(k\rho_1/a, k\rho_2/a;p;q)_n}{\theta(aq/\rho_1, aq/\rho_2;p;q)_n} \sum_{j=0}^{n} \frac{\theta(k\rho_1\rho_2 q^{2i-1}/a;p)\theta(\rho_1,\rho_2;p;q)_j}{\theta(k\rho_1\rho_2/aq;p)\theta(k\rho_2/a, k\rho_1/a;p;q)_j}$$
$$\times \frac{\theta(aq/\rho_1\rho_2;p;q)_{n-j}\theta(k;p;q)_{n+j}}{\theta(q;p;q)_{n-j}\theta(k\rho_1\rho_2/a;p;q)_{n+j}} \left(\frac{aq}{\rho_1\rho_2}\right)^j$$
$$\times \sum_{i=0}^{j} \frac{\theta(k\rho_1\rho_2/a^2q;p;q)_{j-i}\theta(k\rho_1\rho_2/aq;p;q)_{j+i}}{\theta(q;p;q)_{j-i}\theta(aq;p;q)_{j+i}} \alpha_i\left(a, \frac{k\rho_1\rho_2}{aq}\right)$$
$$= \frac{\theta(k\rho_1/a, k\rho_2/a;p;q)_n}{\theta(aq/\rho_1, aq/\rho_2;p;q)_n} \sum_{i=0}^{n}\sum_{j=0}^{n-i} \frac{\theta(k\rho_1\rho_2 q^{2i+2j-1}/a;p)\theta(\rho_1,\rho_2;p;q)_{i+j}}{\theta(k\rho_1\rho_2/aq;p)\theta(k\rho_2/a, k\rho_1/a;p;q)_{i+j}}$$
$$\times \frac{\theta(aq/\rho_1\rho_2;p;q)_{n-i-j}\theta(k;p;q)_{n+i+j}}{\theta(q;p;q)_{n-i-j}\theta(k\rho_1\rho_2/a;p;q)_{n+i+j}} \left(\frac{aq}{\rho_1\rho_2}\right)^{i+j}$$
$$\times \frac{\theta(k\rho_1\rho_2/a^2q;p;q)_j\theta(k\rho_1\rho_2/aq;p;q)_{j+2i}}{\theta(q;p;q)_j\theta(aq;p;q)_{j+2i}} \alpha_i\left(a, \frac{k\rho_1\rho_2}{aq}\right)$$
$$= \frac{\theta(k\rho_1/a, k\rho_2/a;p;q)_n}{\theta(aq/\rho_1, aq/\rho_2;p;q)_n} \sum_{i=0}^{n} \frac{\theta(k\rho_1\rho_2 q^{2i-1}/a;p)\theta(\rho_1,\rho_2;p;q)_i}{\theta(k\rho_1\rho_2/aq;p)\theta(k\rho_2/a, k\rho_1/a;p;q)_i}$$
$$\times \frac{\theta(aq/\rho_1\rho_2;p;q)_{n-i}\theta(k;p;q)_{n+i}\theta(k\rho_1\rho_2/kq;p;q)_{2i}}{\theta(q;p;q)_{n-i}\theta(k\rho_1\rho_2/a;p;q)_{n+i}\theta(aq;p;q)_{2i}}$$
$$\times \left(\frac{aq}{\rho_1\rho_2}\right)^i \alpha_i\left(a, \frac{k\rho_1\rho_2}{aq}\right)$$
$$\times {}_{10}V_9\left(\frac{k\rho_1\rho_2 q^{2i-1}}{a}; \rho_1 q^i, \rho_2 q^i, kq^{n+1}, q^{i-n}, \frac{k\rho_1\rho_2}{a^2q}; q,p;1\right). \quad (4.12)$$

By the Frenkel-Turaev summation formula, the latter ${}_{10}V_9$-series has a closed form

$${}_{10}V_9(\dots) = \frac{\theta(k\rho_1\rho_2 q^{2i}/a, k/a, \rho_1 q^{-n}/a, \rho_2 q^{-n}/a;p;q)_{n-i}}{\theta(q^{-n-i}/a, k\rho_1 q^i/a, k\rho_2 q^i/a, \rho_1\rho_2 q^{i-n}/a;p;q)_{n-i}}. \quad (4.13)$$

Substituting this expression into (4.12) and making various simplifications with the help of formulae (4.8), (4.9), one finds that

$$\beta'_n(a,k) = \sum_{i=0}^{n} \frac{\theta(k/a;p;q)_{n-i}\theta(k;p;q)_{n+i}\theta(\rho_1,\rho_2;p;q)_i}{\theta(q;p;q)_{n-i}\theta(aq;p;q)_{n+i}\theta(aq/\rho_1, aq/\rho_2;p;q)_i}$$
$$\times \left(\frac{aq}{\rho_1\rho_2}\right)^i \alpha_i\left(a, \frac{k\rho_1\rho_2}{aq}\right)$$
$$= \sum_{i=0}^{n} \frac{\theta(k/a;p;q)_{n-i}\theta(k;p;q)_{n+i}}{\theta(q;p;q)_{n-i}\theta(aq;p;q)_{n+i}} \alpha'_i(a,k),$$

as demanded in the formulation of the present theorem. □

Since we have already one Bailey pair (4.5) and (4.6), the Bailey chain machinery allows us to produce an infinite family of identities for elliptic hypergeometric series. As a first step along this chain, let us substitute expressions (4.5), (4.6) into formulae (4.10), (4.11). This yields an elliptic analogue of the q-Bailey pair considered in

[Si, A2]:

$$\alpha'(a,k) = \frac{\theta(aq^{2n};p)\theta(a,a/c,\rho_1,\rho_2;p;q)_n}{\theta(a;p)\theta(q,qc,aq/\rho_1,aq/\rho_2;p;q)_n}\left(\frac{k}{a}\right)^n, \tag{4.14}$$

$$\beta'(a,k) = \frac{\theta(k\rho_1/a,k\rho_2/a,k,k/c;p;q)_n}{\theta(aq/\rho_1,aq/\rho_2,q,qc;p;q)_n}, \tag{4.15}$$

where we denoted $c = k\rho_1\rho_2/aq$. Substitution of these sequences into relation (4.7) results in the Frenkel-Turaev sum (2.20). However, this does not mean that we have proved in such a way this remarkable summation formula because it was used already in the proof of the main theorem above, see (4.13). The next step along the Bailey chain brings a nice non-trivial result.

Theorem 4. *The elliptic Bailey identity for terminating ${}_{12}V_{11}$-series (2.19) holds true.*

Proof. Let us replace in (4.10) and (4.11) the parameters ρ_1,ρ_2 by *new variables* ρ_3,ρ_4 and substitute into these relations expressions (4.14) and (4.15). This yields

$$\alpha''_n(a,k) = \frac{\theta(aq^{2n};p)}{\theta(a;p)}\frac{\theta(a,a/\tilde{c},\rho_1,\ldots,\rho_4;p;q)_n}{\theta(q,q\tilde{c},aq/\rho_1,\ldots,aq/\rho_4;p;q)_n}\left(\frac{k}{a}\right)^n, \tag{4.16}$$

$$\begin{aligned}\beta''_n(a,k) = {} & \frac{\theta(k,k\rho_3/a,k\rho_4/a,aq/\rho_3\rho_4;p;q)_n}{\theta(q,aq/\rho_3,aq/\rho_4,k\rho_3\rho_4/a;p;q)_n} \\ & \times {}_{12}V_{11}(\tilde{k};\tilde{k}\rho_1/a,\tilde{k}\rho_2/a,\tilde{k}/\tilde{c},\rho_3,\rho_4,q^{-n},kq^n;q,p;1),\end{aligned} \tag{4.17}$$

where $\tilde{k} = k\rho_3\rho_4/aq$ and $\tilde{c} = k\rho_1\rho_2\rho_3\rho_4/a^2q^2$. Now we substitute the sequence $\alpha''_n(a,k)$ into the original relation (4.7) and get

$$\beta''_n(a,k) = \frac{\theta(k,k/a;p;q)_n}{\theta(q,aq;p;q)_n}{}_{12}V_{11}(a;\rho_1,\ldots,\rho_4,kq^n,q^{-n},a/\tilde{c};q,p;1). \tag{4.18}$$

Comparing (4.17) and (4.18), we get Frenkel-Turaev's elliptic generalization of the Bailey transformation for terminating ${}_{10}\Phi_9$-series (2.19) with $t_0 = a, t_1 = \rho_1, t_2 = \rho_2, t_3 = a/\tilde{c}, t_4 = \rho_3, t_5 = \rho_4, t_6 = q^{-n}$, and $t_7 = kq^n$. □

If we continue moving in this direction we will get more and more complicated identities involving multiple sums of particular combinations of theta functions. In order to get single sums, one can give the parameters special values so that the intermediate sums take closed form expressions.

5. A new theta hypergeometric series identity

In a fairly old paper [Br], Bressoud has constructed some examples of what Andrews coined as well-poised Bailey pairs. It appears that one of the original Bressoud's pairs, namely,

$$\alpha_n^{Br}(a,k) = \frac{1-aq^{2n}}{1-a}\frac{(a,qa^2/k^2;q)_n}{(q,k^2/a;q)_n}\left(\frac{k^2}{qa^2}\right)^n,$$

$$\beta_n^{Br}(a,k) = \frac{(k^2,q^2a^2/k^2;q^2)_n(-k^2/a;q)_{2n}}{(q^2,k^4/a^2;q^2)_n(-aq;q)_{2n}}\left(\frac{k^2}{qa^2}\right)^n,$$

admits an elliptic generalization. Consider the following sequences

$$\alpha_n(a,k)=\frac{\theta(aq^{2n};p)}{\theta(a;p)}\frac{\theta(a,qa^2/k^2;p;q)_n}{\theta(q,k^2/a;p;q)_n}\left(\frac{k^2}{qa^2}\right)^n, \tag{5.1}$$

$$\beta_n(a,k)=\frac{\theta(k^2,q^2a^2/k^2;p^2;q^2)_n\theta(-k^2/a;p;q)_{2n}}{\theta(q^2,k^4/a^2;p^2;q^2)_n\theta(-aq;p;q)_{2n}}\left(\frac{k^2}{qa^2}\right)^n. \tag{5.2}$$

Let us show that they satisfy relation (4.7) after squaring all parameters entering it (for $p=0$ this was done in [AB2]).

Plugging of (5.1) into (4.7) with q,p,a,k replaced by q^2,p^2,a^2,k^2 yields

$$\begin{aligned}\beta_n(a,k)&=\sum_{j=0}^n\frac{\theta(aq^{2j};p)\theta(k^2;p^2;q^2)_{n+j}\theta(k^2/a^2;p^2;q^2)_{n-j}}{\theta(a;p)\theta(a^2q^2;p^2;q^2)_{n+j}\theta(q^2;p^2;q^2)_{n-j}}\\&\quad\times\frac{\theta(a,qa^2/k^2;p;q)_j}{\theta(q,k^2/a;p;q)_j}\left(\frac{k^2}{qa^2}\right)^j\\&=\frac{\theta(k^2/a^2,k^2;p^2;q^2)_n}{\theta(q^2,a^2q^2;p^2;q^2)_n}{}_{10}V_9(a;kq^n,-kq^n,q^{-n},-q^{-n},qa^2/k^2;q,p;1),\end{aligned}$$

where we have used the factorization

$$\theta(z^2;p^2;q^2)_n=\theta(z;p;q)_n\theta(-z;p;q)_n.$$

Applying the Frenkel-Turaev sum to the latter ${}_{10}V_9$-series, we get

$$\beta_n(a,k)=\frac{\theta(k^2,q^2a^2/k^2,k^2/a^2;p^2;q^2)_n\theta(-aq^{1-2n}/k^2;p;q)_n}{\theta(q^2,a^2q^{2-2n}/k^2;p^2;q^2)_n\theta(-aq;p;q)_{2n}\theta(aq^{1-n}/k^2;p;q)_n}.$$

Further simplification of this expression with the help of the inversion relation $\theta(z;p)=-z\theta(z^{-1};p)$ yields expression (5.2), as desired. Thus we have proved that (5.1) and (5.2) form indeed an elliptic Bailey pair.

Substitution of (5.1), (5.2) into the relations (4.10), (4.11), where q,p,a,k,ρ_1,ρ_2 should be squared, yields a new Bailey pair:

$$\alpha'_n(a,k)=\frac{\theta(aq^{2n};p)\theta(\rho_1^2,\rho_2^2;p^2;q^2)_n\theta(a,qa^2/k^2;p;q)_n}{\theta(a;p)\theta(a^2q^2/\rho_1^2,a^2q^2/\rho_2^2;p^2;q^2)_n\theta(q,k^2/a;p;q)_n}\left(\frac{k}{\rho_1\rho_2}\right)^{2n} \tag{5.3}$$

$$\begin{aligned}\beta'_n(a,k)&=\frac{\theta(k^2\rho_1^2/a^2,k^2\rho_2^2/a^2,k^2,k^2/c^2;p^2;q^2)_n}{\theta(a^2q^2/\rho_1^2,a^2q^2/\rho_2^2,q^2c^2,q^2;p^2;q^2)_n}\sum_{j=0}^n\frac{\theta(c^2q^{4j};p^2)}{\theta(c^2;p^2)}\\&\quad\times\frac{\theta(\rho_1^2,\rho_2^2,k^2q^{2n},q^{-2n},c^2,a^2q^2/c^2;p^2;q^2)_j}{\theta(k^2\rho_2^2/a^2,k^2\rho_1^2/a^2,c^2q^{2-2n}/k^2,c^2q^{2+2n},q^2,c^4/a^2;p^2;q^2)_j}\\&\quad\times\frac{\theta(-c^2/a;p;q)_{2j}}{\theta(-aq;p;q)_{2j}}\left(\frac{qc^2}{a^2}\right)^j\\&=\frac{\theta(k^2\rho_1^2/a^2,k^2\rho_2^2/a^2,k^2,k^2/c^2;p^2;q^2)_n}{\theta(a^2q^2/\rho_1^2,a^2q^2/\rho_2^2,q^2c^2,q^2;p^2;q^2)_n}\,{}_{14}V_{13}(c^2;\rho_1^2,\rho_2^2,q^{-2n},\\&\quad k^2q^{2n},a^2q^2/c^2,-c^2/a,-c^2q/a,-c^2/ap,-c^2qp/a;q^2,p^2;1),\end{aligned} \tag{5.4}$$

where in the last equality we have applied the factorization

$$\frac{\theta(-c^2/a;p;q)_{2j}}{\theta(-aq;p;q)_{2j}}=\left(\frac{a^2q}{c^2}\right)^j\frac{\theta(-c^2/a,-c^2q/a,-c^2/ap,-c^2qp/a;p^2;q^2)_j}{\theta(-aq^2,-aq,-aq^2p,-aq/p;p^2;q^2)_j}. \tag{5.5}$$

Substituting now sequence (5.3) into relation (4.7) (with the squared parameters), it is not difficult to see that

$$\beta_n'(a,k)=\frac{\theta(k^2/a^2,k^2;p^2;q^2)_n}{\theta(q^2,a^2q^2;p^2;q^2)_n}\,{}_{14}V_{13}(a;qa^2/c^2,\\ \rho_1,-\rho_1,\rho_2,-\rho_2,kq^n,-kq^n,q^{-n},-q^{-n};q,p;1). \qquad (5.6)$$

Comparing expressions (5.4) and (5.6), one gets a new identity for elliptic hypergeometric series. Thus we have proved the following statement.

Theorem 5. *The following relation between two terminating very-well-poised balanced theta hypergeometric series of the unit argument holds true:*

$$\begin{aligned}&{}_{14}V_{13}(c^2;\rho_1^2,\rho_2^2,k^2q^{2n},a^2q^2/c^2,q^{-2n},\\ &\qquad -c^2/a,-c^2q/a,-c^2/ap,-c^2qp/a;q^2,p^2;1)\\ &=\frac{\theta(k^2/a^2,q^2c^2,a^2q^2/\rho_1^2,a^2q^2/\rho_2^2;p^2;q^2)_n}{\theta(a^2q^2,k^2/c^2,k^2\rho_1^2/a^2,k^2\rho_2^2/a^2;p^2;q^2)_n}\\ &\qquad\times{}_{14}V_{13}(a;\rho_1,-\rho_1,\rho_2,-\rho_2,kq^n,-kq^n,q^{-n},-q^{-n},qa^2/c^2;q,p;1), \quad (5.7)\end{aligned}$$

where $c=k\rho_1\rho_2/aq$.

Corollary 6. *Let us consider the limit* $p\to 0$. *Factorization (5.5) takes now the form*

$$\frac{(-c^2/a;q)_{2j}}{(-aq;q)_{2j}}=\frac{(-c^2/a,-c^2q/a;q^2)_j}{(-aq^2,-aq;q^2)_j}.$$

As a result, relation (5.7) reduces to

$$\begin{aligned}&{}_{10}W_9(c^2;\rho_1^2,\rho_2^2,k^2q^{2n},a^2q^2/c^2,q^{-2n},-c^2/a,-c^2q/a;q^2;qc^2/a^2)\\ &=\frac{(k^2/a^2,q^2c^2,a^2q^2/\rho_1^2,a^2q^2/\rho_2^2;q^2)_n}{(a^2q^2,k^2/c^2,k^2\rho_1^2/a^2,k^2\rho_2^2/a^2;q^2)_n}\\ &\qquad\times{}_{12}W_{11}(a;\rho_1,-\rho_1,\rho_2,-\rho_2,kq^n,-kq^n,q^{-n},-q^{-n},qa^2/c^2;q;q), \quad (5.8)\end{aligned}$$

which is the Andrews-Berkovich transformation formula for two very-well-poised basic hypergeometric series (see (4.9) in [AB2]*).*

A few remarks to this result are in order. First, both ${}_{14}V_{13}$ series in (5.7) are balanced, i.e. we deal with elliptic hypergeometric series. In sharp distinction, in formula (5.8) the ${}_{10}W_9$ series is not balanced according to the definition of balancing from [GR], whereas the series ${}_{12}W_{11}$ is balanced. This mismatch shows that our definition of balancing, which descends from the condition of ellipticity, is more flexible than the standard one. Second, both ${}_{14}V_{13}$ series have the unit argument, i.e. the $z=1$ constraint still accompanies the very-well-poisedness and balancing conditions in order to create some non-trivial identities (within the q-series definitions [GR] this corresponds to the choice $z=q$). Third, relation (5.7) may be interpreted as a rule for simultaneous change of the bases $q\to q^2$, $p\to p^2$ for particular ${}_{r+1}V_r$-series (cf. [BIS, Wa2]). Finally, let us denote $a=q^{2\alpha},k=q^{\kappa},\rho_1=q^{u_1},\rho_2=q^{u_2}$. Since both terminating ${}_{14}V_{13}$-series are totally elliptic, i.e. they are elliptic functions of α,κ,u_1,u_2 variables, relation (5.7) is nothing else than an infinite sequence of identities for elliptic functions parametrized by the integer $n\in\mathbb{N}$.

The results of the present paper and [A2, AB2] seem to open doors to elliptic generalization of the multivariable Bailey chain machinery. Some relevant constructions and identities for that were discussed in [DG, ML1, ML2, BS, Wa2, DS1, DS3, Ro]. It would be quite interesting to understand also whether the elliptic Bailey pairs are related to supersymmetric theories [BMS, AB1].

Let us stress once more that the proposed elliptic Bailey chain (4.7), (4.10), (4.11) does not admit confluence limits, i.e. it is not possible to scale parameters either to zero or to infinity without setting $p = 0$, and this fact limits severely the number of identities that can be derived at the elliptic level. In particular, existence of elliptic analogues of the trinomial Bailey pairs [AB1, Wa1] is not clear. Moreover, it appears that "an alternative Bailey lemma" of [A2, AB2] does not admit the direct elliptic extension because of the absence of an elliptic analogue of the required q-Pfaff-Saalschütz ${}_3\Phi_2$ summation formula. Therefore, the attractive Bailey lattice [AAB] or Bailey tree [A2] type constructions are not available yet at the elliptic level.

Appendix A. The elliptic beta integral and Frenkel-Turaev sum

A century ago there was a great deal of activity devoted to generalizations of the Euler's gamma function $\Gamma(u)$ which centered around Barnes and Jackson. Barnes defined the multiple gamma function through some r-ple infinite products [Ba3], the double gamma function being [Ba1]

$$\Gamma_2^{-1}(u;\omega) = e^{\gamma_{22}u+\gamma_{21}\frac{u^2}{2}}\, u \prod_{n,m=0}^{\infty}{}' \left(1+\frac{u}{\Omega}\right) e^{-\frac{u}{\Omega}+\frac{u^2}{2\Omega^2}}, \tag{A.1}$$

where γ_{21}, γ_{22} are some constants analogous to the Euler's constant and $\Omega = n\omega_1 + m\omega_2$ with the ratio ω_1/ω_2 not equal to a real negative number. The prime in the product sign means that the $n = m = 0$ point is skipped. The Γ_2-function satisfies two equations,

$$\frac{\Gamma_2(u+\omega_1;\omega)}{\Gamma_2(u;\omega)} = \frac{1}{\Gamma_1(u;\omega_2)}, \qquad \frac{\Gamma_2(u+\omega_2;\omega)}{\Gamma_2(u;\omega)} = \frac{1}{\Gamma_1(u;\omega_1)},$$

where $\Gamma_1(u;\omega) = \rho(\omega)\omega^{u/\omega}\Gamma(u/\omega)$ for some constant $\rho(\omega)$. Jackson proposed the q-gamma function [J],

$$\Gamma_q(u) = \frac{(q;q)_\infty(1-q)^{1-u}}{(q^u;q)_\infty}, \tag{A.2}$$

which satisfies the equation $\Gamma_q(u+1)/\Gamma_q(u) = (1-q^u)/(1-q)$, and reduces in the limit $q \to 1$ to $\Gamma(u)$.

Due to the relation $\Gamma(u)\Gamma(1-u) = \pi/\sin \pi u$, one may interpret $\Gamma^{-1}(u)$ as a "half-sine" function (in the sense that it has only half of sine's zeros). Since the product $(z;q)_\infty(qz^{-1};q)_\infty$ defines for $z = q^u$ the divisor of Jacobi's $\theta_1(u)$-function (1.4), $(q^u;q)_\infty$ contains only "half" of the zeros of $\theta_1(u)$ and $\Gamma_q^{-1}(u)$ may be taken as "a half" of the $\theta_1(u)$-function. For $\mathrm{Im}(\omega_1/\omega_2) > 0$ one has [Ba1, Ba2]

$$\Gamma_2^{-1}(u;\omega_1,\omega_2)\Gamma_2^{-1}(u-\omega_2;\omega_1,-\omega_2) = \kappa(\omega)\, (e^{2\pi i u/\omega_2}; e^{2\pi i\omega_1/\omega_2})_\infty$$

for some coefficient $\kappa(\omega)$ independent of u, and, due to this property, Barnes was interpreting (A.1) as "a quarter" of $\theta_1(u)$. A different combination of the double gamma functions $\Gamma_2(\omega_1+\omega_2-u;\omega)/\Gamma_2(u;\omega)$ is known nowadays as the double sine function [M].

Following Barnes, Jackson introduced also a double q-gamma function and an elliptic gamma function as a particular ratio of two double q-gamma functions [J].

In the modern time the elliptic gamma function was reintroduced in [Ru] and some of its properties were investigated in [FV]. For $|q|, |p| < 1$ let us represent the Jackson's double infinite product as follows:

$$(z;q,p)_\infty = \prod_{j,k=0}^{\infty} (1 - zq^j p^k). \tag{A.3}$$

It satisfies the equations

$$\frac{(z;q,p)_\infty}{(qz;q,p)_\infty} = (z;p)_\infty, \qquad \frac{(z;q,p)_\infty}{(pz;q,p)_\infty} = (z;q)_\infty.$$

¿From them one easily finds that $(pz^{-1};p)_\infty = (qp(qz)^{-1};q,p)_\infty/(qpz^{-1};q,p)_\infty$ and a similar relation with q and p interchanged. As a result, the elliptic gamma function

$$\Gamma(z;q,p) = \prod_{j,k=0}^{\infty} \frac{1 - z^{-1}q^{j+1}p^{k+1}}{1 - zq^j p^k} \tag{A.4}$$

satisfies the difference equations

$$\Gamma(qz;q,p) = \theta(z;p)\Gamma(z;q,p), \qquad \Gamma(pz;q,p) = \theta(z;q)\Gamma(z;q,p).$$

In similarity with the plain and q-hypergeometric situations, the elliptic shifted factorials $\theta(z;p;q)_s$ for arbitrary complex s are defined as ratios of the elliptic gamma functions

$$\theta(z;p;q)_s = \frac{\Gamma(zq^s;q,p)}{\Gamma(z;q,p)}.$$

The shorhand notation $\Gamma(t_1, \ldots, t_k; q,p) \equiv \prod_{j=1}^k \Gamma(t_j;q,p)$ is assumed below.

The following Askey-Wilson type integral representing an elliptic generalization of the Nassrallah-Rahman q-beta integral has been calculated in [S1, S2]. Denote by $\mathbb{T}$ the positively oriented unit circle. Assume that five complex parameters $t_m, m = 0, \ldots, 4$, satisfy the inequalitites $|t_m| < 1,\ |pq| < |A|$, where $A \equiv \prod_{r=0}^4 t_r$. Define the elliptic beta integral as a contour integral,

$$\mathcal{N}_E(\mathbf{t}) = \frac{1}{2\pi i}\int_{\mathbb{T}} \Delta_E(z,\mathbf{t})\frac{dz}{z}, \tag{A.5}$$

where

$$\Delta_E(z,\mathbf{t}) = \frac{\prod_{m=0}^4 \Gamma(zt_m, z^{-1}t_m;q,p)}{\Gamma(z^2, z^{-2}, zA, z^{-1}A;q,p)}. \tag{A.6}$$

Then the following equality holds true:

$$\mathcal{N}_E(\mathbf{t}) = \frac{2\prod_{0\le m<s\le 4}\Gamma(t_m t_s;q,p)}{(q;q)_\infty (p;p)_\infty \prod_{m=0}^4 \Gamma(At_m^{-1};q,p)}. \tag{A.7}$$

For $p = 0$, integral (A.5) is reduced to the Nassrallah-Rahman integral [NR, R]:

$$\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{(z^2, z^{-2}, zA, z^{-1}A;q)_\infty}{\prod_{m=0}^4 (zt_m, z^{-1}t_m;q)_\infty}\frac{dz}{z} = \frac{2\prod_{m=0}^4 (t_m^{-1}A;q)_\infty}{(q;q)_\infty \prod_{0\le m<s\le 4}(t_m t_s;q)_\infty}, \tag{A.8}$$

where $|t_r|<1$. When one of the parameters, say t_4, goes to zero, integral (A.8) is reduced to the celebrated Askey-Wilson integral [AW, GR],

$$\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{(z^2,z^{-2};q)_\infty}{\prod_{m=0}^3(zt_m,z^{-1}t_m;q)_\infty}\frac{dz}{z} = \frac{2(t_0t_1t_2t_3;q)_\infty}{(q;q)_\infty\prod_{0\le m<s\le 3}(t_mt_s;q)_\infty}. \tag{A.9}$$

Integrals (A.9) and (A.8) may be viewed as multiparameter q-extensions of the Euler's beta integral [AAR],

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \qquad \mathrm{Re}\ \alpha,\ \mathrm{Re}\ \beta>0, \tag{A.10}$$

where $\Gamma(z)$ is the standard gamma function. Because the function on the r.h.s. of (A.10) is called the beta function, the function $\mathcal{N}_E(\mathbf{t})$ in (A.7) may be called the elliptic beta function. Note that its form cannot be simplified by taking parameters t_r to zero or infinity, i.e. its complexity structure is ineradicable for $0<|q|,|p|<1$.

Suppose that $|t_s|<1$, $s=1,\dots,4$, $|pt_0|<1<|t_0|$, $|pq|<|A|$, and that the arguments of all t_s and p,q are linearly independent over $\mathbb{Z}$. Denote by C a contour separating sequences of poles of $\Delta_E(z,\mathbf{t})$ converging to zero, which are located at the points $z=t_rq^jp^k$, $A^{-1}q^{j+1}p^{k+1}$, $j,k\in\mathbb{N}$, from the ones diverging to infinity along the points $z=t_r^{-1}q^{-j}p^{-k}$, $Aq^{-j-1}p^{-k-1}$, $j,k\in\mathbb{N}$. Then the following residue formula takes place (see [DS1]):

$$\frac{1}{2\pi i}\int_C \Delta_E(z,\mathbf{t})\frac{dz}{z} = \frac{1}{2\pi i}\int_{\mathbb{T}}\Delta_E(z,\mathbf{t})\frac{dz}{z} + 2\kappa(\mathbf{t})\sum_{\substack{n\ge 0\\ |t_0q^n|>1}}\nu_n(\mathbf{t}), \tag{A.11}$$

where

$$\begin{aligned}\kappa(\mathbf{t}) &= \frac{\prod_{m=1}^4\Gamma(t_mt_0,t_mt_0^{-1};p,q)}{(q;q)_\infty(p;p)_\infty\Gamma(t_0^{-2},t_0^{-1}A,t_0A;p,q)},\\ \nu_n(\mathbf{t}) &= q^n\frac{\theta(t_0^2q^{2n};p)}{\theta(t_0^2;p)}\prod_{m=0}^5\frac{\theta(t_mt_0;p;q)_n}{\theta(qt_m^{-1}t_0;p;q)_n},\end{aligned}$$

and $\prod_{m=0}^5 t_m=q$. Similar residue analysis for simpler beta integrals can be found, e.g. in [AW, W1], and, actually, (A.11) is a subcase of much more complicated formulae found by van Diejen and the author via residue calculus for elliptic Selberg integrals [DS1, DS3].

Let us consider the limit $t_0t_4\to q^{-N}$, $N\in\mathbb{N}$. The expression on the l.h.s. of (A.11), given by formula (A.7), and the factor $\kappa(\mathbf{t})$ in front of the residues sum on the r.h.s. blow up, whereas the integral over the unit circle $\mathbb{T}$ remains finite. Dividing all the terms by $2\kappa(\mathbf{t})$ and taking the limit, one arrives at the following identity

$$\sum_{n=0}^N q^n\frac{\theta(t_0^2q^{2n};p)}{\theta(t_0^2;p)}\prod_{r=0}^5\frac{\theta(t_0t_r;p;q)_n}{\theta(qt_0t_r^{-1};p;q)_n} = \frac{\theta(qt_0^2;p;q)_N\prod_{1\le r<s\le 3}\theta(q/t_rt_s;p;q)_N}{\theta(q/t_0t_1t_2t_3;p;q)_N\prod_{r=1}^3\theta(qt_0/t_r;p;q)_N}, \tag{A.12}$$

which is nothing else than the Frenkel-Turaev ${}_{10}V_9$-series sum (2.20) in different notations. Thus, the proof of the elliptic beta integral (A.5) provides simultaneously a proof of (A.12). The described method is completely different from the one used by Frenkel and Turaev in [FT] and it has a purely analytical nature immanent to the special functions theory. Note that the integral (A.5) itself is essentially more complicated object than the above summation formula and it cannot be deduced from (A.12).

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