ON CLASSIFICATION OF TORSION FREE POLYHEDRA

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This paper is a continuation of the study of stable homotopy types of polyhedra by H.-J. Baues and the author. Namely, we consider $(k-1)$-connected polyhedra of dimension $n+k$ in the *stable range*, i.e. with $k \ge n+1$. Such polyhedra have been described for $n \le 3$ [5] and it is known that for $n \ge 4$ their classification is a *wild* problem, i.e. contains classification of representations of all finitely generated algebras over a field [4]. If we only consider polyhedra with torsion free homologies, the situation differs. For $n \le 5$ such polyhedra have been described in [2, 3]; in this case there are only finitely many homotopy types of them. It was also known (and easy) that the classification of torsion free polyhedra is wild for $n \ge 11$ [12]. In this paper we give a description of torsion free polyhedra for $n = 6$ and show that for $n > 6$ this problem is wild.

As in our previous papers, we use the technique of *matrix problems*, mainly of *bimodule categories*, in the version elaborated in [12, Section 4]. In Section 2 we specialize it for our purpose. Section 1 contains the necessary definitions and preliminaries, and Section 3 presents polyhedra of small dimensions that play an essential role in our calculations. In Section 4 we calculate the local homotopy types of torsion free polyhedra for $n = 6$, and in Section 5 glue them into *congruence classes* in the sense of Freyd [13]. Recall that the congruence classes of local polyhedra form a basis of the corresponding Grothendieck group [13]. Finitely, in Section 6 we show that for $n > 6$ the classification problem of torsion free polyhedra is wild.

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1. Preliminaries

In this paper *space* means a *punctured space*; we denote by $*_X$ (or by $*$ if there can be no ambiguity) the marked point of the space X. B^n and S^{n-1} denote respectively the n-dimensional ball and the $(n-1)$-dimensional sphere. As usually, we denote by $X \vee Y$ the *wedge* (or one point union) of X and Y, i.e. the factor space of $X \sqcup Y$ by the relation $*_X = *_Y$, identify it with $*_X \times Y \cup X \times *_Y \subset X \times Y$, and set $X \wedge Y = X \times Y / X \vee Y$. In particular, we denote by $X[1] = S^1 \wedge X$ the *suspension* of X and by $X[n]$ its n times iterated suspension. The word "*polyhedron*" is used as a synonym of "*finite CW-complex.*" One can also consider wedges of several spaces $\bigvee_{i=1}^s X_i$; if all of them are copies of a fixed space X, we denote such a wedge by sX.

Recall several facts about the stable homotopy category (cf. [7]). We denote by $\mathrm{Hot}(X, Y)$ the set of homotopy classes of continuous maps $X \to Y$ and by CW the *homotopy category* of polyhedra, i.e. the category whose objects are polyhedra and morphisms are homotopy classes of continuous maps. The suspension functor defines a natural map $\mathrm{Hot}(X,Y) \to \mathrm{Hot}(X[1],Y[1])$. Moreover, the Whitehead theorem [15, Theorem 10.28 and Corollary 10.29] shows that the suspension functor *reflects isomorphisms* of simply connected polyhedra. It means that if $f \in \mathrm{Hot}(X,Y)$, where X and Y are simply connected, f is an isomorphism (i.e. a homotopy equivalence) if and only if so is $f[1]$. We set $\mathrm{Hos}(X,Y) = \varinjlim_n \mathrm{Hot}(X[n],Y[n])$. If $\alpha \in \mathrm{Hot}(X[n],Y[n])$, $\beta \in \mathrm{Hot}(Y[m],Z[m])$, one can consider the class $\beta[n] \circ \alpha[m] \in \mathrm{Hot}(X[m+n],Z[n+m])$ whose stabilization is, by definition, the product $\beta\alpha$ of the classes of α and β in $\mathrm{Hos}(X,Z)$. Thus we obtain the *stable homotopy category* of polyhedra CWS. Actually, if we only deal with *finite CW-complexes*, we need not go too far, since the following result holds [7, Theorem 1.21].

Proposition 1.1. *If* $\dim X \le d$ *and* Y *is* $(n-1)$*-connected, where* $d < 2n-1$*, then the map* $\mathrm{Hot}(X,Y) \to \mathrm{Hot}(X[1],Y[1])$ *is bijective. If* $d = 2n-1$*, this map is surjective. In particular, the map* $\mathrm{Hot}(X[m],Y[m]) \to \mathrm{Hos}(X,Y)$ *is bijective if* $m > d-2n+1$ *and surjective if* $m = d-2n+1$*.*

Here $(n-1)$-*connected* means, as usually, that the k-th homotopy group $\pi_k(X)$ is trivial for $k \le n-1$. Thus for *all* polyhedra of dimension at most d the map $\mathrm{Hot}(X[m],Y[m]) \to \mathrm{Hos}(X,Y)$ is bijective if $m \ge d$ and surjective if $m = d-1$. Note also that the natural functor $\mathsf{CW} \to \mathsf{CWS}$ reflects isomorphisms of simply connected polyhedra.

It is convenient to extend CWS adding formal negative shifts $X[-n]$ ($n \in \mathbb{N}$) of polyhedra and setting $\mathrm{Hos}(X[-n],Y[-m]) = \mathrm{Hos}(X[m],Y[n])$. In particular, we denote $S^{-n} = S^0[-n]$. Actually, we consider this extended category and denote it by CWS too. (It is the category $\mathcal{S}$ of [7]).

Since we are only interested in stable homotopy classification, we identify, in what follows, polyhedra and continuous maps with their images in CWS. We denote by CWF the full subcategory of CWS consisting of all spaces X with torsion free homology groups $\mathrm{H}_i(X) = \mathrm{H}_i(X,\mathbb{Z})$ for all i.

The stable category CWS is a *triangulated category* [14]. Namely, since any suspension $X[n]$ is an *H-cogroup* [15, Chapter 2], commutative if $n \ge 2$, the category CWS is an additive category. The suspension plays role of the *translation functor*, and the *distinguished triangles* are the *cone sequences* $X \xrightarrow{f} Y \to Cf \to X[1]$ (and isomorphic ones), where $Cf = CX \cup_f Y$ is the *cone of the map* f, i.e the factor space $CX \sqcup Y$ by the relation $(x,0) \sim f(x)$; $CX = X \times I/X \times 1$ is the cone over the space X. Note that cone sequences coincide with *cofibration sequences* in the category CWS [15, Proposition 8.30]. Recall that a cofibration sequence is a such one

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{f[1]} Y[1] \tag{1}$$

that for every polyhedron P the induced sequences

$$\begin{gathered}\mathrm{Hos}(P,X) \xrightarrow{f_*} \mathrm{Hos}(P,Y) \xrightarrow{g_*} \mathrm{Hos}(P,Z) \xrightarrow{h_*} \mathrm{Hos}(P,X[1]) \xrightarrow{f[1]_*} \mathrm{Hos}(P,Y[1]),\\ \mathrm{Hos}(Y[1],P) \xrightarrow{f[1]^*} \mathrm{Hos}(X[1],P) \xrightarrow{h^*} \mathrm{Hos}(Z,P) \xrightarrow{g^*} \mathrm{Hos}(Y,P) \xrightarrow{f^*} \mathrm{Hos}(X,P)\end{gathered} \tag{2}$$

are exact. In particular, we have an exact sequence of *stable homotopy groups*

$$\pi_k^S(X) \xrightarrow{f_*} \pi_k^S(Y) \xrightarrow{g_*} \pi_k^S(Z) \xrightarrow{h_*} \pi_{k-1}^S(X) \xrightarrow{f[1]_*} \pi_{k-1}^S(Y), \tag{3}$$

where $\pi_k^S(X) = \varinjlim_m \pi_{k+m}(X[m]) = \operatorname{Hos}(S^k, X)$. Certainly, one can prolong the sequences (2) and (3) into infinite exact sequences just taking further suspensions.

It is also known [7, Theorem 4.8] that the category CWS is *fully additive*, i.e. every idempotent $e \in \operatorname{Hos}(X, X)$ splits. In our case it means that there is a decomposition $X[m] \simeq Y \vee Z$ for some m, such that e comes from the map $\varepsilon : Y \vee Z \to Y \vee Z$ with $\varepsilon(y) = y$ for $y \in Y$ and $\varepsilon(z) = *_{Y\vee Z}$ for $z \in Z$. We call a polyhedron X *indecomposable* if $X \simeq Y \vee Z$ implies that either Y or Z are contractible (i.e. isomorphic in CW to the 1-point space).

Every CW-complex is obtained by *attaching cells*. Namely, if X^n is the n-th skeleton of X, then there is a wedge of balls $B = mB^{n+1}$ and a map $f : mS^n \to X^n$ such that X^{n+1} is isomorphic to the cone of f, i.e. to the space $B \cup_f X^n$. It gives cofibration sequences like (1) and exact sequences like (2) and (3).

We denote by CW_k^n the full subcategory of CW formed by $(k-1)$-connected $(n+k)$-dimensional polyhedra and by CWF_k^n the full subcategory of CW_k^n formed by the polyhedra X with torsion free homology groups $\mathrm{H}_i(X)$ for all i. Proposition 1.1 also implies the following fact.

Proposition 1.2. *The suspension functor induces equivalences* $\mathsf{CW}_k^n \xrightarrow{\sim} \mathsf{CW}_{k+1}^n$ *for all* $k > n+1$. *Moreover, if* $k = n+1$, *the suspension functor* $\mathsf{CW}_k^n \to \mathsf{CW}_{k+1}^n$ *is a* full representation equivalence, *i.e. it is full, dense and reflects isomorphisms.*

(*Dense* means that every object from CW_{k+1}^n is isomorphic (i.e. homotopy equivalent) to $X[1]$ for some $X \in \mathsf{CW}_k^n$.)

Therefore, setting $\mathsf{CW}^n = \mathsf{CW}_{n+2}^n \simeq \mathsf{CW}_k^n$ for $k > n+1$, we can consider it as a full subcategory of CWS. The same is valid for $\mathsf{CWF}^n = \mathsf{CWF}_{n+2}^n$. Moreover, all objects of these categories actually are of the form $X[1]$ for some $X \in \mathsf{CW}_{n+1}^n$. It leads to the following notion [1].

Definition 1.3. An *atom* is an indecomposable polyhedron $X \in \mathsf{CW}_{n+1}^n$ not belonging to the image of CW_n^n. A *suspended atom* is a polyhedron $X[m]$, where X is an atom. An atom (suspended atom) is called *torsion free* if it belongs to CWF. We denote by A^d (AF^d) the set of isomorphism classes of atoms (respectively, torsion free atoms) of dimension n.

Thus any polyhedron X from CW_k^n (CWF_k^n) with $k \geq n+1$ is homotopy equivalent to a wedge of spheres and suspended atoms (respectively, torsion free suspended atoms), in particular, spheres and suspended atoms are the only indecomposable objects in these categories. Note that, by definition, all atoms are of odd dimension, and the sphere $S^1 \simeq S^0[1]$ is not an atom.

The atoms of dimension d are known for $d \leq 9$ [5] (see also [12]). If $d > 9$, the classification of atoms of dimension d becomes a *wild problem* [4, 12]. It means that it contains the classification of pairs of linear maps in finite dimensional vector spaces over a field. It is known (see, e.g., [11]) that the last problem contains the classification of finite dimensional representations of *any* finitely generated algebra, so it is hardly believable that such a problem has more or less clear solution. A description of torsion free atoms of dimension $d \le 11$ is given in [2, 3] (see also [12]). Our aim is to give a description of torsion free atoms of dimension $d = 13$ and to prove that such a classification for $d > 13$ is a wild problem.

Recall the results of Freyd [13, 7] on localizations and Grothendieck group $\mathrm{G} = \mathrm{G}(\mathsf{CWS})$ of polyhedra. By definition, the latter is the abelian group generated by symbols $[X]$, where X runs through isomorphism classes (in CWS) of polyhedra, subject to the relations $[X \vee Y] = [X] + [Y]$. A polyhedron X is called *p-local*, where p is a prime number, if there is a commutative diagram (in CW)

$$\begin{tikzcd} X \arrow[rr, "p^s \cdot 1_X"] \arrow[dr] & & X, \\ & W \arrow[ur] & \end{tikzcd}$$

where W is a wedge of spheres and $s > 0$. We denote by A_p^d (AF_p^d) the subset of p-local atoms in A^d (AF^d). Two polyhedra, X and Y, are called *congruent* if there is a polyhedron Z such that $X \vee Z \simeq Y \vee Z$ (actually, Z can always be chosen as a wedge of spheres [7, Theorem 4.26]). Then we write $X \equiv Y$. Obviously, it implies that $[X] = [Y]$. Let $\tilde{\mathsf{A}}_p$ be the set of congruence classes of p-local atoms.

Theorem 1.4 (**Freyd's Theorem**). *The group* G *is free; a set of its free generators is*

$$\{\, [S^n] \mid n \in \mathbb{Z} \,\} \cup \Big(\bigcup_p \{\, [X[n]] \mid X \in \tilde{\mathsf{A}}_p,\ n \in \mathbb{Z} \,\} \Big).$$

More precisely, for any polyhedron X there is a wedge of spheres W such that $X \vee W \simeq \bigvee_{i=1}^s X_i$, where all X_i are p-local suspended atoms (perhaps for different p); moreover, if $X \vee W' \simeq \bigvee_{i=1}^r Y_i$ is another such decomposition, then $W \simeq W'$, $r = s$ and $X_i \equiv Y_{\sigma i}$ for some permutation σ of indices [7, Theorems 4.40 and 4.44].

Certainly, if X is torsion free, so are all summands X_i.

For any additive category $\mathcal{C}$ and any ring $\mathbf{R}$ we denote by $\mathbf{R} \otimes \mathcal{C}$ the category with the same set of objects, but with the sets of morphisms $\mathbf{R} \otimes \mathcal{C}(X, Y)$. We are especially interested in cases when $\mathbf{R} = \mathbb{Q}$ and $\mathbf{R} = \mathbb{Z}(p) = \{\, m/n \mid m, n \in \mathbb{Z},\ p \nmid n \,\} \subset \mathbb{Q}$, where p is a prime number. Then we denote $\mathcal{C}(p) = \mathbb{Z}(p) \otimes \mathcal{C}$ and $\mathcal{C}(\infty) = \mathbb{Q} \otimes \mathcal{C}$; $L(p)$ denotes the natural functor $\mathcal{C} \to \mathcal{C}(p)$ (p prime or ∞). If $X \in \mathsf{CWS}$ is not a wedge of spheres, we denote by $\mathbf{P}(X)$ the set of all primes dividing the order of one of the groups $\pi_k^S(X)$ with $k \le \dim X$; if X is a wedge of spheres, we set $\mathbf{P}(X) = \{\infty\}$. Then the following result holds [7, Theorem 4.41].

Theorem 1.5. *For any two polyhedra $X, Y \in \mathsf{CWS}$ the following properties are equivalent:*

1. $X \equiv Y$;
2. $L(p)X \simeq L(p)Y$ *for all prime p;*
3. $L(p)X \simeq L(p)Y$ *for all $p \in \mathbf{P}(X \vee Y)$.*

Note that, for any $X \in \mathsf{CWS}$, the additive group of the ring $\mathrm{Hos}(X, X)$ is finitely generated and its torsion part $\mathrm{T}(X, X)$ is a nilpotent ideal. Moreover, if X is a

p-local atom, the ring $\mathbb{Z}(p) \otimes \mathrm{Hos}(X, X)$ is local [7, Theorem 4.47]. It implies that the category $\mathsf{CWS}(p)$ is also fully additive. Moreover, decompositions in CWS can be constructed locally.

lemma 1.6. *Suppose that, for every $p \in \mathbf{P}(X)$, there is a decomposition $L(p)X \simeq Y_p \oplus Z_p$ such that $L(\infty)Y_p \simeq L(\infty)Y_q$ for all $p, q \in \mathbf{P}(X)$. Then $X \simeq Y \oplus Z$, where $L(p)Y \simeq Y_p$ and $L(p)Z \simeq Z_p$ for all p.*

Proof. Any decomposition of X comes from an equality $1 = e + f$, where $e, f \in \mathrm{Hos}(X, X)$ are orthogonal idempotents. Since $\mathrm{T}(X, X)$ is nilpotent, we may replace $\mathrm{Hos}(X, X)$ by the ring $\mathbf{R} = \mathrm{Hos}(X, X)/\mathrm{T}(X, X)$, which has free additive group and no nilpotent ideals. Therefore, decompositions of X are in one-to-one correspondence with decompositions of the regular $\mathbf{R}$-module. But for such decompositions the statement of the lemma is well known [8, Theorem 31.12]. $\square$

2. Technique

We use the technique developed in [12], which reduces the classification of polyhedra to a "matrix problem," more precisely, to the classification of elements of a bimodule.

Definition 2.1. Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories, $\mathcal{U}$ be an $\mathcal{A}$-$\mathcal{B}$-bimodule, i.e. a biadditive functor $\mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathsf{Ab}$, the category of abelian groups. We define the *bimodule category* $\mathbf{El}(\mathcal{U})$ (or the category of elements of the bimodule $\mathcal{U}$) as follows:

- Its set of objects is the disjoint union $\bigcup_{A,B} \mathcal{U}(A, B)$, where A runs through the objects of $\mathcal{A}$, B runs through the objects of $\mathcal{B}$.
- A morphism $\alpha \to \beta$, where $\alpha \in \mathcal{U}(A, B)$, $\beta \in \mathcal{U}(A', B')$ is a pair of morphisms $f : A \to A'$, $g : B \to B'$ such that $g\alpha = \beta f \in \mathcal{U}(A, B')$. (We write $g\alpha$ instead of $\mathcal{U}(1, g)\alpha$ and βf instead of $\mathcal{U}(f, 1)\beta$.)

Obviously, $\mathbf{El}(\mathcal{U})$ is also an additive category; it is fully additive if so are $\mathcal{A}$ and $\mathcal{B}$. The background of our calculations is the following.

Theorem 2.2. *Let $n < m \le 2n$. Denote by $\mathbf{B}$ the full subcategory of $\mathsf{CWF}^n = \mathsf{CWF}^n_{n+1}$ consisting of all complexes of dimension at most m and by $\mathbf{A}$ the full subcategory of CWF^n_{n+1} consisting of all $(m-1)$-connected complexes of dimension at most $2n$. Let $\Gamma(A, B)$, where $A \in \mathbf{A}$, $b \in \mathbf{B}$, denotes the subgroup of $\mathrm{Hos}(A, B)$ consisting of such maps $f : A \to B$ that $\mathrm{H}_m(f) = 0$. We consider Γ as $\mathbf{A}$-$\mathbf{B}$-bimodule. Denote by $\mathcal{I}$ the ideal of the category CWF^n consisting of all morphisms that factors both through $\mathbf{B}$ and through $\mathbf{A}[1]$ and by $\mathcal{J}$ the ideal in the category $\mathbf{El}(\Gamma)$ consisting of all morphisms $(\alpha, \beta) : f \to f'$ such that β factors through f' and α factors through f. Then $\mathsf{CWF}^n/\mathcal{I} \simeq \mathbf{El}(\Gamma)/\mathcal{J}$. Moreover, $\mathcal{I}^2 = \mathcal{J}^2 = 0$, hence both projections $\mathsf{CWF}^n \to \mathsf{CWF}^n/\mathcal{I}$ and $\mathbf{El}(\Gamma) \to \mathbf{El}(\Gamma)/\mathcal{J}$ are representation equivalences. In particular, there is a natural one-to-one correspondence between isomorphism classes of objects from CWF^n and $\mathbf{El}(\Gamma)$.*

Proof. Let $X \in \mathsf{CWF}^n$, B be the m-th skeleton of X. Then $X/B \in \mathsf{CW}^{2n-m}_{m+1}$ and $m > 2n - m + 1$, so $X/B \simeq A[1]$, where $A \in \mathsf{CW}^{2n-m}_m$. Moreover, there is a cofibration sequence

$$A \xrightarrow{f} B \xrightarrow{g} X \xrightarrow{h} A[1] \xrightarrow{f[1]} B[1], \tag{4}$$

which induces an exact sequence

$$\mathrm{H}_k(A) \xrightarrow{\mathrm{H}_k(f)} \mathrm{H}_k(B) \to \mathrm{H}_k(X) \to \mathrm{H}_{k-1}(A) \xrightarrow{\mathrm{H}_{k-1}(f)} \mathrm{H}_{k-1}(B)$$

for every k. If $k < m$, $\mathrm{H}_k(A) = \mathrm{H}_{k-1}(A) = 0$, thus $\mathrm{H}_k(X) \simeq \mathrm{H}_k(B)$, so the latter is torsion free. Since $\mathrm{H}_m(B)$ is also torsion free, $B \in \mathbf{B}$. If $k > m$, $\mathrm{H}_k(B) = 0$, hence $\mathrm{H}_k(X) \simeq \mathrm{Ker}\,\mathrm{H}_{k-1}(f)$. The latter subgroup contains the torsion part of $\mathrm{H}_{k-1}(A)$, since $\mathrm{H}_{k-1}(B)$ is torsion free. Thus this torsion part equals zero and $A \in \mathbf{A}$. At last, $\mathrm{H}_m(X) \simeq \mathrm{Cok}\,\mathrm{H}_m(f)$, thus $\mathrm{Cok}\,\mathrm{H}_m(f)$ is torsion free. If $\mathrm{H}_m(f) \neq 0$, there is a common direct summand of $\mathrm{H}_m(A)$ and $\mathrm{H}_m(B)$ isomorphic to $\mathbb{Z}$. It comes from some maps $\phi : S^m \to A$ and $\psi : B \to S^m$ such that $\psi f \phi = \mathrm{Id}$. Thus $A \simeq S^m \vee A'$, $B \simeq S^m \vee B'$ and with respect to these decompositions $f = \begin{pmatrix} 1 & 0 \\ 0 & f' \end{pmatrix}$ for some $f' : A' \to B'$. Hence we get a cofibration sequence

$$A' \xrightarrow{f'} B' \to X \to A[1]' \xrightarrow{f[1]'} B[1]'.$$

If still $\mathrm{H}_m(f') \neq 0$, we can repeat this procedure. At last we get a cofibration sequence like (4) but with $\mathrm{H}_m(f) = 0$. Hence, every object $X \in \mathsf{CWF}^n$ is isomorphic to the cone of a map $f : A \to B$ with $A \in \mathbf{A}$, $B \in \mathbf{B}$ and $\mathrm{H}_m(f) = 0$, so $f \in \Gamma(A, B)$.

Note that $\mathrm{Hos}(B, A[1]) = 0$ for all $A \in \mathbf{A}$, $B \in \mathbf{B}$. Suppose that a polyhedron X' is included into a cofibration sequence

$$A' \xrightarrow{f'} B' \xrightarrow{g'} X' \xrightarrow{h'} A[1]' \xrightarrow{f[1]'} B[1]',$$

where $A' \in \mathbf{A}$, $B' \in \mathbf{B}$, $f' \in \Gamma(A', B')$, and $\gamma : X \to X'$. Then $h'\gamma g = 0$, hence $\gamma g = g'\beta$ for some $\beta : B \to B'$. By the properties of cofibration sequences, the pair (β, γ) can be included into a commutative diagram

$$\begin{array}{ccccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & X & \xrightarrow{h} & A[1] & \xrightarrow{f[1]} & B[1] \\ \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \downarrow\alpha[1] & & \downarrow\beta[1] \\ A' & \xrightarrow{f'} & B & \xrightarrow{g'} & X & \xrightarrow{h'} & A[1]' & \xrightarrow{f[1]'} & B[1]'. \end{array} \tag{5}$$

In particular, $(\alpha, \beta) : f \to f'$ in $\mathbf{El}(\Gamma)$. Let $(\alpha', \beta') : f \to f'$ is another pair that can be included into a diagram of the form (5) with the same γ. Then $g'(\beta - \beta') = 0$ and $h(\alpha[1] - \alpha'[1]) = 0$, hence $\beta - \beta'$ factors through f' and $\alpha[1] - \alpha'[1]$ factors through $f[1]$. Thus $(\alpha - \alpha', \beta - \beta') \in \mathcal{J}$. On the other hand, if $(\alpha, \beta) \in \mathcal{J}$, then $g'\beta = (\alpha[1])h = 0$, so the diagram (5) with $\gamma = 0$ commutes.

In the same way, if $(\alpha, \beta) : f \to f'$, there is a map $\gamma : X \to X'$ such that the diagram (5) is commutative. If γ can be replaced by γ', then $(\gamma - \gamma')g = 0$, so $\gamma - \gamma'$ factors through h, and $h'(\gamma - \gamma') = 0$, so $\gamma - \gamma'$ factors through g'. Thus $\gamma - \gamma' \in \mathcal{I}$. On the other hand, if $\gamma \in \mathcal{I}$, then $h'\gamma = \gamma g = 0$, so the diagram (5) with $\alpha = \beta = 0$ commutes too. Therefore we really get an equivalence $\mathsf{CWF}^n/\mathcal{I} \simeq \mathbf{El}(\Gamma)/\mathcal{J}$.

The equality $\mathcal{I}^2 = 0$ follows from the fact that $\mathrm{Hos}(B, A[1]) = 0$ for all $A \in \mathbf{A}$, $B \in \mathbf{B}$. On the other hand, if $f : A \to B$ ($A \in \mathbf{A}$, $B \in \mathbf{B}$) and $\mathrm{H}^m(f) = 0$, then $f(A^m) \subseteq B^{m-1}$. For any morphisms $\xi : B' \to A$ and $\eta : B \to A'$, $\mathrm{Im}\,\xi \subseteq A^m$ and $\eta | B^{m-1} = 0$, wherefrom $\eta f \xi = 0$. It implies immediately that $\mathcal{J}^2 = 0$. $\square$

If we are interested in p-local atoms, especially in their congruence classes, we have to replace in Theorem 2.2 the categories CWF^n and $\mathbf{El}(\Gamma)$ by their localizations $\mathsf{CWF}^n(p)$ and $\mathbf{El}(\Gamma)(p)$. The latter coincide in fact with the bimodule category $\mathbf{El}(\Gamma(p))$, where $\Gamma(p)$ is the $\mathbf{A}(p)$-$\mathbf{B}(p)$-bimodule $\mathbb{Z}(p) \otimes \Gamma$.

3. Small dimensions

Recall the results for small values of d. If $n = 1$, $m = 2$, $\mathsf{ind}\,\mathbf{A} = \mathsf{ind}\,\mathbf{B} = \{ S^2 \}$ and there are no nonzero morphisms f with $\mathrm{H}^2(f) = 0$. Hence, there are no torsion free atoms of dimension 3. For $n = 2$, $m = 4$, $\mathsf{ind}\,\mathbf{B} = \{ S^3, S^4 \}$, $\mathsf{ind}\,\mathbf{A} = \{ S^4 \}$ and there is a unique indecomposable element in $\mathbf{El}(\Gamma)$, namely the (suspended) Hopf map $\eta : S^4 \to S^3$, which gives rise to a new atom C^5 of dimension 5 included into a cofibration sequence

$$S^4 \xrightarrow{\eta} S^3 \to C^5 \to S^5 \xrightarrow{\eta[1]} S^4.$$

It induces a commutative diagram of Hos-groups with exact rows and columns

$$\begin{tikzcd}
\mathbb{Z} \arrow[d] & 0 \arrow[l] \arrow[d] & \mathrm{Hos}(C^5,S^4) \arrow[l] \arrow[d] & \mathbb{Z}/2 \arrow[l] \arrow[d] & \mathbb{Z} \arrow[l] \arrow[d] \\
\mathbb{Z}/2 \arrow[d] & \mathbb{Z} \arrow[l] \arrow[d] & \mathrm{Hos}(C^5,S^3) \arrow[l] \arrow[d] & \mathbb{Z}/2 \arrow[l] \arrow[d] & \mathbb{Z}/2 \arrow[l] \arrow[d] \\
\mathrm{Hos}(S^4,C^5) \arrow[d] & \mathrm{Hos}(S^3,C^5) \arrow[l] \arrow[d] & \mathrm{Hos}(C^5,C^5) \arrow[l] \arrow[d] & \mathrm{Hos}(S^5,C^5) \arrow[l] \arrow[d] & \mathrm{Hos}(S^4,C^5) \arrow[l] \arrow[d] \\
0 \arrow[d] & 0 \arrow[l] \arrow[d] & \mathrm{Hos}(C^5,S^5) \arrow[l] \arrow[d] & \mathbb{Z} \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] \\
\mathbb{Z} & 0 \arrow[l] & \mathrm{Hos}(C^5,S^4) \arrow[l] & \mathbb{Z}/2 \arrow[l] & \mathbb{Z}\,, \arrow[l]
\end{tikzcd}$$

where all maps $\mathbb{Z} \to \mathbb{Z}/2$ are surjective and all maps $\mathbb{Z}/2 \to \mathbb{Z}/2$ are bijective. It gives that $\mathrm{Hos}(C^5, S^4) = \mathrm{Hos}(S^4, C^5) = 0$, and one can naturally identify $\mathrm{Hos}(S^3, C^5) = \mathrm{Hos}(C^5, S^5) = \mathbb{Z}$, $\mathrm{Hos}(C^3, S^3) = \mathrm{Hos}(S^5, C^5) = 2\mathbb{Z}$; then all maps $\mathbb{Z} \to \mathbb{Z}$ are identities and $2\mathbb{Z} \to \mathbb{Z}$ are natural inclusions. Therefore $\mathrm{Hos}(C^5, C^5)$ can be identified with the subring in $\mathbb{Z} \times \mathbb{Z}$ consisting of all pairs (a, b) with $a \equiv b \bmod 2$. We denote the suspended atoms $C^5[k]$ by C^{5+k}.

If $n = 3$, $m = 6$, $\mathsf{ind}\,\mathbf{A} = \{ S^6 \}$, $\mathsf{ind}\,\mathbf{B} = \{ S^4, S^5, S^6, C^6 \}$. Moreover, nonzero maps $S^6 \to S^6$ and $S^6 \to C^6$ induce nonzero maps of the 6th homologies, thus do not belong to Γ. Therefore, the only atom C_2^7 of dimension 7 comes from the iterated Hopf map $\eta^2 : S^6 \to S^4$. Again we denote $C_2^7[m]$ by C_2^{7+m}. The cofibration sequence

$$S^6 \xrightarrow{\eta^2} S^4 \to C^5 \to S^7 \xrightarrow{\eta^2[1]} S^5$$

induces a commutative diagram with exact rows and columns

$$\begin{tikzcd}
\mathbb{Z} \arrow[d] & 0 \arrow[l] \arrow[d] & \mathrm{Hos}(C_2^7,S^6) \arrow[l] \arrow[d] & \mathbb{Z}/2 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] \\
\mathbb{Z}/2 \arrow[d] & \mathbb{Z} \arrow[l] \arrow[d] & \mathrm{Hos}(C_2^7,S^4) \arrow[l] \arrow[d] & \mathbb{Z}/24 \arrow[l] \arrow[d] & \mathbb{Z}/2 \arrow[l] \arrow[d] \\
\mathrm{Hos}(S^6,C_2^7) \arrow[d] & \mathrm{Hos}(S^4,C_2^7) \arrow[l] \arrow[d] & \mathrm{Hos}(C_2^7,C_2^7) \arrow[l] \arrow[d] & \mathrm{Hos}(S^7,C_2^7) \arrow[l] \arrow[d] & \mathrm{Hos}(S^5,C_2^7) \arrow[l] \arrow[d] \\
0 \arrow[d] & 0 \arrow[l] \arrow[d] & \mathrm{Hos}(C_2^7,S^7) \arrow[l] \arrow[d] & \mathbb{Z} \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] \\
\mathbb{Z}/2 & \mathbb{Z}/2 \arrow[l] & \mathrm{Hos}(C_2^7,S^5) \arrow[l] & \mathbb{Z}/2 \arrow[l] & \mathbb{Z}\,, \arrow[l]
\end{tikzcd}$$

where all maps to $\mathbb{Z}/2$ are surjective and all maps from $\mathbb{Z}/2$ are injective. It implies that $\mathrm{Hos}(C_2^7,S^6) \simeq \mathrm{Hos}(S^5,C_2^7) \simeq \mathbb{Z}/2$, $\mathrm{Hos}(S^4,C_2^7) \simeq \mathrm{Hos}(C_2^7,S^7) \simeq \mathbb{Z}$, $\mathrm{Hos}(C_2^7,S^4) \simeq \mathrm{Hos}(S^7,C_2^7) \simeq 2\mathbb{Z} \oplus \mathbb{Z}/12$, and $\mathrm{Hos}(C_2^7,C_2^7)$ can be identified with the set of triangular matrices of the form

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \quad \text{with} \quad a,c \in \mathbb{Z},\ b \in \mathbb{Z}/12,\ a \equiv c \bmod 2.$$

The cases $n = 4$ and $n = 5$ are considered in [2, 3], where it was shown that there are only finitely many atoms of dimensions 9 and 11. Since we need not information about these atoms, we do not reproduce here the calculations. In [12] they are given using the same technique as in the present paper.

4. Atoms of dimension 13: localization

Let now $n = 6$, $m = 10$. Then $\mathsf{ind}\,\mathbf{A} = \{\, S^k\ (10 \le k \le 12), C^{12} \,\}$, $\mathsf{ind}\,\mathbf{B} = \{\, S^k\ (7 \le k \le 10), C^9, C^{10}, C_2^{10} \,\}$. Tables 1–3 on page 9 contain the values of $\Gamma(A,B)$, $\mathrm{Hos}(A',A)$ and $\mathrm{Hos}(B,B')$ for $A,A' \in \mathsf{ind}\,\mathbf{A}$, $B,B' \in \mathsf{ind}\,\mathbf{B}$. In these tables we present the values for the spaces C^k and C_2^k with respect to the defining cofibration sequence, as we did for $\mathrm{Hos}(C_2^7,C_2^7)$ above. Thus we show after colon the dimensions of spheres surrounding the corresponding space in such a cofibration sequence. The marks $^=$ at $\mathbb{Z}$ in the cells belonging to the same space shows that the corresponding entries are congruent modulo 2. The marks * show that we identify the elements of period 2 of the mentioned groups. For instance, in fact, $\Gamma(C^{12},C_2^{10}) \simeq \mathbb{Z}/24$, but it is convenient to consider it as $(\mathbb{Z}/24 \oplus \mathbb{Z}/2)/\langle\,(12,1)\,\rangle$. Under these notations, the action of Hos-groups on Γ-groups is just given by the matrix multiplication of the tables. All calculations for these tables are quite analogous to the calculations for C^5 and C_2^7 above, so we omit them.

Therefore, objects from $\mathbf{El}(\Gamma)$ can be considered as 10×5 block matrices $\gamma = (\gamma_{ij})$, where the entries of γ_{ij} are from the (ij)-th cell of Table 1. Morphisms $\gamma \to \gamma'$ are given by block matrices $\alpha = (\alpha_{ij})_{5\times 5}$ and $\beta = (\beta_{ij})_{10\times 10}$, where α_{ij} has entries from the (ij)-th cell of Table 2, β_{ij} has entries from the (ij)-th cell of Table 3, their sizes are compatible with those of γ_{ij} and γ'_{ij}, and $\beta\gamma = \gamma'\alpha$. Such a morphism is invertible if and only if all diagonal blocks of α and β are square, and both $\det\alpha$

TABLE 1. $\Gamma(A, B)$

	S^{10}	S^{11}	S^{12}	C^{12} : 10	12
S^7	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/24$	0
C_2^{10} : 7	$Z/12$	0	0	$\mathbb{Z}/24^*$	0
10	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2^*$
C^9 : 7	$\mathbb{Z}/12$	0	0	$\mathbb{Z}/24^*$	0
9	0	0	$\mathbb{Z}/24$	0	$\mathbb{Z}/24^*$
S^8	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	0
C^{10} : 8	0	$\mathbb{Z}/12$	0	0	0
10	0	0	0	0	0
S^9	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	$\mathbb{Z}/12$
S^{10}	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	0

TABLE 2. $\mathrm{Hos}(A, A')$

	S^{10}	S^{11}	S^{12}	C^{12} : 10	12
S^{10}	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$2\mathbb{Z}$	0
S^{11}	0	$\mathbb{Z}$	$\mathbb{Z}/2$	0	0
S^{12}	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$
C^{12} : 10	$\mathbb{Z}$	0	0	$\mathbb{Z}^=$	0
12	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}^=$

TABLE 3. $\mathrm{Hos}(B, B')$

	S^7	C_2^{10} : 7	10	C^9 : 7	9	S^8	C^{10} : 8	10	S^9	S^{10}
S^7	$\mathbb{Z}$	$2\mathbb{Z}$	$\mathbb{Z}/12$	$2\mathbb{Z}$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/24$
C_2^{10} : 7	$\mathbb{Z}$	$\mathbb{Z}^=$	$\mathbb{Z}/12$	$2\mathbb{Z}$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/12$	0	$\mathbb{Z}/12$
10	0	0	$\mathbb{Z}^=$	0	0	0	0	$\mathbb{Z}$	0	$2\mathbb{Z}$
C^9 : 7	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/12$	$\mathbb{Z}^=$	0	0	0	$\mathbb{Z}/12$	0	$\mathbb{Z}/12$
9	0	0	0	0	$\mathbb{Z}^=$	0	0	0	$2\mathbb{Z}$	0
S^8	0	0	0	0	0	$\mathbb{Z}$	$2\mathbb{Z}$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$
C^{10} : 8	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}^=$	0	0	0
10	0	0	$2\mathbb{Z}$	0	0	0	0	$\mathbb{Z}^=$	0	$2\mathbb{Z}$
S^9	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$
S^{10}	0	0	$\mathbb{Z}$	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$

and $\det\beta$ equal ± 1. One can easily see that only entries from $\mathbb{Z}$ or $2\mathbb{Z}$ give nonzero input to these determinants, so they belong indeed to $\mathbb{Z}$.

To solve this matrix problem, first localize it at 2, i.e. replace CWF^6 by $\mathsf{CWF}^6(2)$ and $\mathbf{El}(\Gamma)$ by $\mathbf{El}(\Gamma(2))$. It means that in all tables we must replace $\mathbb{Z}$ by $\mathbb{Z}(2)$, $\mathbb{Z}/24$ by $\mathbb{Z}/8$, and $\mathbb{Z}/12$ by $\mathbb{Z}/4$. Then consider it modulo 2, thus replace all nonzero entries in Table 1 by $\mathbb{Z}/2$. It gives a matrix problem, which is an example of a *bunch of chains* (cf. [6] or [10, Appendix]; we use the notations and terminology of the latter paper). Namely, there are 4 pairs of chains:

$$\begin{aligned}
&\mathfrak{E}_1 = \{\, e_1 < e_2 < e_4 \,\},\ \mathfrak{F}_1 = \{\, f_4 < f_1 \,\};\\
&\mathfrak{E}_2 = \{\, e_5 < e_9 \,\},\ \mathfrak{F}_2 = \{\, f_3 < f_5 \,\};\\
&\mathfrak{E}_3 = \{\, e_6 < e_7 \,\},\ \mathfrak{F}_3 = \{\, f_2 \,\};\\
&\mathfrak{E}_4 = \{\, e_3 < e_{10} < e'_9 < e'_6 \,\},\ \mathfrak{F}_3 = \{\, f'_1 < f'_2 < f'_3 \,\}
\end{aligned}$$

with the equivalence relation $\sim$ such that the only nontrivial equivalences are:

$$\begin{aligned}
&e_i \sim e'_i \ (i = 6, 9),\ e_2 \sim e_3,\ e_4 \sim e_5,\\
&f_j \sim f'_j \ (j = 1, 2, 3), \ \text{ and } \ f_4 \sim f_5.
\end{aligned}$$

We set $x - y$ if $x \in \mathfrak{E}_i$, $y \in \mathfrak{F}_i$ or vice versa, $\mathfrak{X}_i = \mathfrak{E}_i \cup \mathfrak{F}_i$ and $\mathfrak{X} = \bigcup_{i=1}^4 \mathfrak{X}_i$. Recall that, according to [6, 10], indecomposable objects for this bunch of chains are described by *full words*

$$w = x_1 r_2 x_2 r_2 \dots x_{n-1} r_n x_n,$$

where $x_k \in \mathfrak{X}$, $r_k \in \{\ \sim, - \}$ and $x_{k-1} r_k x_k$ according to the above definitions of $\sim$ and $-$; moreover, if $r_2 = -$ $(r_n = -)$, then $x_1 \not\sim y$ (respectively $x_n \not\sim y$) for any $y \in \mathfrak{X}$, $y \neq x_1$ $(y \neq x_n)$. Namely, each full word w defines an indecomposable *string object* $u_s(w)$. We call w a *cycle* if $r_2 = r_m =\sim$ and $x_n - x_1$; this cycle is called *aperiodic* if $w \neq v - v - \dots - v$ for any shorter word v. If w is an aperiodic cycle and $\pi(t) \neq t^d$ is a power of an irreducible polynomial from $\mathbb{Z}/2[t]$, there is also an indecomposable *band object* $u_b(w, \pi(t))$, and every indecomposable object is isomorphic either to a string or to a band one. Moreover, all isomorphisms between such objects come either from a cyclic shift of a cycle or from replacing a word w by the reverse word w^* ([10, Theorem A.12]). For every pair of mutually reverse words we choose one of them and denote by $\mathfrak{W}$ the set of all these representatives. We call x_1 (or x_n) an *exceptional* end of the word $w = x_1 r_2 \dots r_n x_n$ if $r_2 =\sim$ and $x_1 \notin \mathfrak{X}_4$ (respectively $r_n =\sim$ and $x_n \notin \mathfrak{X}_4$). Note that it can happen that both ends of w are exceptional. We also consider e_i $(i = 1, 7)$ as the unique exceptional end of the word $w = e_i$.

Suppose that we know the reduction $\bar\gamma$ of γ modulo 2: $\bar\gamma = \bigoplus_l r_l \bar\gamma_l$, where $\bar\gamma_l$ are pairwise non-isomorphic string and band objects. We lift each $\bar\gamma_l$ to a (fixed) object γ_l of $\mathbf{El}(\Gamma)$ and set $\hat\gamma = \bigoplus_l r_l \gamma_l$. An obvious way to do it is just to consider the elements of matrices defining $\bar\gamma$ (which are 0 and 1) as elements from $\mathbb{Z}/8$ or $\mathbb{Z}/4$, if necessary. Then $\gamma = \hat\gamma + 2\gamma'$, and the entries of γ' are at most from $\mathbb{Z}/4$. Again denote by $\bar\gamma'$ the reduction of γ' modulo 2.

Recall that, according to [10, Definitions A.10,A.11], string and band objects are actually block matrices, where all blocks are invertible (1×1 for string objects) and correspond to the entries $x_k - x_{k+1}$ in the word w. Note that if x_k and x_{k+1} belong to $\mathfrak{X}_4$, then neither x_{k-1} nor x_{k+2} (if they exist) belong to $\mathfrak{X}_4$ (if w is a cycle, we set $x_{n+k} = x_k$). Therefore, every new horizontal and vertical stripe in a band object contains an invertible block corresponding to an entry $x - y$ with $x, y \notin \mathfrak{X}_4$ (we call such a block a *ν-block*). It easily implies that if $\bar{\gamma}_l$ is a band object, $\hat{\gamma}_l$ splits out of the whole γ. If $\bar{\gamma}_l$ is a string object corresponding to a full word w, again all new stripes corresponding to $x_k - x_{k+1}$ contain an invertible ν-block, except the case when this stripe correspond to an exceptional end. So if w has no exceptional ends, $\hat{\gamma}_l$ also splits out of γ.

Denote by $\mathfrak{X}'$ the set of pairs (w, x), where $w \in \mathfrak{W}$ and x is an exceptional end of w. We call x the *type* of such a pair. Set also

$$\mathfrak{E}'_i = \left\{ (w, x) \in \mathfrak{X}' \mid x \in \mathfrak{E}_i \right\};$$
$$\mathfrak{F}'_j = \left\{ (w, x) \in \mathfrak{X}' \mid x \in \mathfrak{F}_j \right\}$$

and define orderings on each $\mathfrak{E}'_i$ and $\mathfrak{F}'_j$ setting $(w, x) < (w', x')$ if there is a morphism $u_w \to u_{w'}$ such that its component corresponding to the stripes x of w and x' of w' is nonzero. One can check that all these orderings are *linear* (it follows immediately from the reduction algorithm of [6]). We also set $(w, x) \sim (w, x')$ if x, x' are two exceptional ends of the same word. Thus we obtain a new bunch of chains. It obviously describes the admissible transformations of $\bar{\gamma}'$. Thus we can again use the description of indecomposable objects from [6, 10] and suppose that $\bar{\gamma}'$ is decomposed into a direct sum of string and band objects. Then rewrite γ' as $\hat{\gamma}' + 2\gamma''$, where $\hat{\gamma}'$ is a lift of this canonical form of $\bar{\gamma}'$ and γ'' only has entries corresponding to those blocks of the original problem, which had the entries from $\mathbb{Z}/24$. Moreover, we can repeat the same considerations as above to split out all new bands and only keep the strings having exceptional ends. This time the list of such strings is very short, since we can use transformations adding $\mathbb{Z}/2$-entries multiplied by 4 to $\mathbb{Z}/8$-entries. Namely, the only nonzero entries in γ'' can occur at the places corresponding to the horizontal stripes in the words $a_1 = e_1,\ a_2 = e_3 \sim e_2,\ a_5 = e_4 \sim e_5,\ a_6 = e'_6 \sim e_6$ or $a_9 = e'_9 \sim e_9$ and the vertical stripes in the words $b_1 = f_1 \sim f'_1,\ b_2 = f_2 \sim f'_2,\ b_4 = f_4 \sim f_5,\ b_{(3,0)} = f_3 \sim f'_3$ and other pairs $b_{(3,k)}$ of type f_3. Note the latter vertical stripes can only be combined with the horizontal stripe corresponding to $e_4 \sim e_5$ and it is impossible that one chain have two exceptional end of type f_3. This time we get one new pair $\mathfrak{X}''$ of chains:

$$\mathfrak{E}'' = \left\{ a_1 < a_2 < a_5 < a_6 < a_9 \right\}, \quad \mathfrak{F}'' = \left\{ b_{(3,k)} < b_4 < b_1 < b_2 \right\}$$

(under some linear order on the set $\left\{ b_{(3,k)} \right\}$) and the relation $\sim$ is empty. Therefore, it only has 1-dimensional representations corresponding to the words $a_i - b_j$ with $i = 1,\ j = 1, 4$, or $i = 2,\ j = 4$, or $i = 5,\ j \in \left\{ 4, (3, k) \right\}$, or $i = 6,\ j = 2$, or $i = 9,\ j = (3, 0)$.

It completes the 2-local description of $\mathbf{El}(\Gamma)$, hence of CWF^6, which can be presented as follows.

Definition 4.1. (1) Define symmetric relations $\overset{c}{\text{—}}$ ($c \in \{1,2,4\}$) on $\mathfrak{X}$ by the following exhaustive rule:

- $e_i \overset{c}{\text{—}} f_j$ if and only if $e_i - f_j$ and the (ij)-th entry in Table 1 is $\mathbb{Z}/m$, where $r \mid m$.

(2) Define an $\mathfrak{X}(2)$-*word* as a sequence

$$w = x_1 r_2 x_2 r_2 \dots x_{n-1} r_n x_n,$$

where $x_k \in \mathfrak{X}$, $r_k \in \left\{ \sim, \overset{c}{\text{—}} \mid r = 1,2,4 \right\}$ such that

(a) for each $1 < k \le n$, $x_{k-1} r_k x_k$ in $\mathfrak{X}$;
(b) if $r_k = \sim$, then $r_{k+1} = \overset{c}{\text{—}}$ for some c and vice versa;
(c) if $r_2 \ne \sim$ ($r_n \ne \sim$), then $x_1 \not\sim y$ (respectively, $x_n \not\sim y$) for any $y \in \mathfrak{X}$;
(d) if $r_k = \overset{2}{\text{—}}$, then either $2 < k < n$, or $k = 2$, $x_1 = e_1$, or $k = n$, $x_n = e_1$;
(e) $\overset{4}{\text{—}}$ only can occur in the following words or their reverse:

$$e_4 \sim e_5 \overset{4}{\text{—}} f_3 \sim \dots \quad \text{(any length)},$$
$$e_1 \overset{4}{\text{—}} f_1 \sim f_1', \quad e_1 \overset{4}{\text{—}} f_4 \sim f_5, \quad e_3 \sim e_2 \overset{4}{\text{—}} f_4 \sim f_5,$$
$$e_6' \sim e_6 \overset{4}{\text{—}} f_2 \sim f_2', \quad e_9' \sim e_9 \overset{4}{\text{—}} f_3 \sim f_3'.$$

(3) Define an $\mathfrak{X}(2)$-*cycle* as a pair (w, r_1), where w is an $\mathfrak{X}(2)$-word with $r_2 = r_n = \sim$, which contain no entry $\overset{4}{\text{—}}$, $r_1 \in \{\overset{1}{\text{—}}, \overset{2}{\text{—}}\}$ and $x_n r_1 x_1$ in $\mathfrak{X}$. For such a cycle we set $r_{qn+k} = r_k$ for any q and $1 \le k \le n$.

(4) A cycle (w, r_1) is called *periodic* if it is of the form $v r_1 v r_1 \dots r_1 v$ for a shorter cycle v.

(5) If (w, r_1) is an $\mathfrak{X}(2)$-cycle, $w = x_1 r_2 x_2 r_2 \dots x_{n-1} r_n x_n$, its k-*th shift* is, be definition, the cycle $(w(k), r_{2k+1})$, where $w(k) = x_{2k+1} r_{2k+2} x_{2k+2} \dots r_{2k} x_{2k}$.

Theorem 4.2. (1) *Every $\mathfrak{X}(2)$-word w defines an indecomposable object $P(w)$, called* string object, *in the category* $\mathbf{El}(\Gamma(2))$.

(2) *Two string objects, $P(w)$ and $P(w')$ are isomorphic if and only if either $w' = w$ or $w' = w^*$ (the reverse word).*

(3) *Let $\pi(t) \ne t$ is an irreducible polynomial over the field $\mathbb{Z}/2$ with the leading coefficient 1. Every quadruple $(w, r_1, \pi(t), m)$, where (w, r_1) is a non-periodic $\mathfrak{X}(2)$-cycle and m is a positive integer, defines an indecomposable object $P(w, r_1, \pi, m) \in \mathbf{El}(\Gamma(2))$, called* band object.

(4) *Two band objects, $P(w, r_1, \pi(t), m)$ and $P(w', r_1', \pi'(t), m')$, are congruent if and only if $m = m'$ and one of the following possibilities holds:*

(a) *$\pi'(t) = \pi(t)$ and either $w' \equiv w(k)$ with k even, $r_1' = r_{2k+1}$, or $w' \equiv w^*(k)$ with k odd, $r_1' = r_{n-2k}$;*
(b) *$\pi'(t) = t^d \pi(1/t)$, where $d = \deg \pi$, and either $w' \equiv w(k)$ with k odd, $r_1' = r_{2k+1}$ or $w' \equiv w^*(k)$ with k even, $r_1' = r_{n-2k}$.*

(5) *Every indecomposable object in $\mathbf{El}(\Gamma(2))$ is isomorphic to either a string or a band object.*

Thus congruence classes of indecomposable 2-local polyhedra in CWF^6 correspond to string and band objects of $\mathbf{El}(\Gamma(2))$. We denote these classes by the same symbols $P(w)$ and $P(w, r_1, \pi(t), m)$ and call such polyhedra, respectively, *string* and *band* polyhedra. Note that a string polyhedron from $P(w)$ (respectively, band polyhedron from $P(w, r_1, \pi(t), m))$ belongs to AF^{13} if and only if the word w contains at least one of the symbols e_1, e_2, e_4 (corresponding to the sphere S^7) and at least one of the symbols f_3, f_5 (corresponding to S^{12}).

If we consider the localizations $\mathsf{CWF}^6(3)$ and $\mathbf{El}(\Gamma(3))$, we get:

$$\begin{aligned}
&C^{12} \simeq S^{10} \oplus S^{12} \quad \text{in } \mathbf{A}(3);\\
&C^{10} \simeq S^{8} \oplus S^{10} \quad \text{in } \mathbf{B}(3);\\
&C^{9} \simeq S^{7} \oplus S^{9} \quad \text{in } \mathbf{B}(3);\\
&C^{10}_2 \simeq S^{7} \oplus S^{10} \quad \text{in } \mathbf{B}(3);\\
&\Gamma(3)(S^{10}, B) = 0 \quad \text{for all } B \in \mathbf{B}(3).
\end{aligned}$$

Therefore, the only indecomposable objects in $\mathsf{CWF}^6(3)$, except the images of spheres, correspond to 1×1 matrices (1) in $\Gamma(3)(S^{10}, S^7)$, in $\Gamma(3)(S^{11}, S^8)$ and in $\Gamma(3)(S^{12}, S^9)$. The first of them is an atom from AF^{11}, the other two are its suspensions. Hence there are no 3-local atoms of dimension 13. Thus we have accomplished a local description of atoms in CWS^6, in particular, of the Grothendieck group of this category.

It will be convenient to introduce a new equivalence relation $\approx$ and a new symmetric relation $\overset{*}{—}$ on $\mathfrak{X}$ by the following exhaustive rules:

$$\begin{aligned}
&e_1 \approx e_2 \approx e_4; \quad e_6 \approx e_7; \quad e_5 \approx e_9;\\
&f_1 \approx f_4; \quad f_3 \approx f_5;\\
&e_i \overset{*}{—} f_j \quad \text{if } e_i \approx e_1 \text{ and } f_j \approx f_1,\\
&\qquad\qquad \text{or } e_i \approx e_6 \text{ and } j = 2,\\
&\qquad\qquad \text{or } e_i \approx e_5 \text{ and } f_j \approx f_3.
\end{aligned}$$

Then we consider the 3-local polyhedra from CWF^6 as corresponding to the words $e_1 \overset{*}{—} f_1,\ e_6 \overset{*}{—} f_2,\ e_5 \overset{*}{—} f_3$ or their reverse.

5. Atoms of dimension 13: globalization

The description of indecomposable objects in $\mathsf{CWF}^6(2)$ and $\mathsf{CWF}^6(3)$, together with Theorem 1.5 and Lemma 1.6 implies a classification of the congruence classes of indecomposable polyhedra in CWF^6.

Definition 5.1. (1) We define new relations $\overset{c*}{—}$ ($c = 1, 2, 4$) on $\mathfrak{X}$: $x \overset{c*}{—} y$ if and only if both $x \overset{c}{—} y$ and $x \overset{*}{—} y$. Set $\mathfrak{R} = \{1, 2, 4, *, 1*, 2*, 4*\}$ and $\mathfrak{R}^* = \{*, 1*, 2*, 4*\}$.

(2) We define *global words* as sequences

$$w = x_1 r_2 x_2 r_3 \dots x_{n-1} r_n x_n,$$

where $x_k \in \mathfrak{X},\ r_k \in \mathfrak{R} \cup \{\sim\}$ and the following conditions hold:

(a) $x_k r_k x_{k+1}$ in $\mathfrak{X}$ for $1 \le k < n$;

(b) if $r_k \in \mathfrak{R}$, then $r_{k\pm1} =\sim$ and vice versa;

(c) if $r_2 \ne\sim$ ($r_n \ne\sim$), then $x_1 \not\sim y$ (respectively, $x_n \not\sim y$) for any $y \in \mathfrak{X}$;

(d) if $r_k = \overset{2}{\text{---}}$ or $r_k = \overset{2*}{\text{---}}$, then either $2 < k < n$, or $k = 2$, $x_1 = e_1$, or $k = n$, $x_n = e_1$;

(e) if $\mathsf{r} = \overset{4}{\text{---}}$ or $\mathsf{r} = \overset{4*}{\text{---}}$, then r only can occur in the following words or their reverse:

$$
\begin{aligned}
&e_4 \sim e_5 \;\mathsf{r}\; f_3 \sim \dots \quad \text{(any length)},\\
&e_1 \;\mathsf{r}\; f_1 \sim f_1', \quad e_1 \;\mathsf{r}\; f_4 \sim f_5, \quad e_3 \sim e_2 \;\mathsf{r}\; f_4 \sim f_5,\\
&e_6' \sim e_6 \;\mathsf{r}\; f_2 \sim f_2', \quad e_9' \sim e_9 \;\mathsf{r}\; f_3 \sim f_3',\\
&\dots \overset{*}{\text{---}} e_4 \sim e_5 \;\mathsf{r}\; f_3 \sim \dots \quad \text{(any length)},\\
&e_3 \sim e_2 \;\mathsf{r}\; f_4 \sim f_5 \overset{*}{\text{---}} \dots, \quad e_1 \;\mathsf{r}\; f_4 \sim f_5 \overset{*}{\text{---}} \dots \quad \text{(any length)}.
\end{aligned}
$$

(f) if w contains a subword $e_i \overset{*}{\text{---}} f_j$ or its reverse, it does not contain any subword $e_k \overset{c}{\text{---}} f_l$ or its reverse, where $c \in \{1,2,4\}$, $e_i \approx e_k$ and $f_j \approx f_l$.

(3) We define a *global cycle* as a pair (w, r_1), where w is a global word with $r_2 = r_n = \sim$, which contain no entry $\overset{4}{\text{---}}$ or $\overset{4*}{\text{---}}$, $r_1 \in \mathfrak{R} \setminus \{\overset{4}{\text{---}}, \overset{4*}{\text{---}}\}$ and $x_n r_1 x_1$ in $\mathfrak{X}$. For such a cycle we set $r_{qn+k} = r_k$ for any q and $1 \le k \le n$.

(4) A cycle (w, r_1) is called *periodic* if it is of the form $v r_1 v r_1 \dots r_1 v$ for a shorter cycle v.

(5) If (w, r_1) is a global cycle, $w = x_1 r_2 x_2 r_2 \dots x_{n-1} r_n x_n$, its *$k$-th shift* is, be definition, the cycle $(w(k), r_{2k+1})$, where $w(k) = x_{2k+1} r_{2k+2} x_{2k+2} \dots r_{2k} x_{2k}$.

(6) We call two global words w, w' *elementary congruent* if $w = x_1 r_2 x_2 r_3 \dots x_{n-1} r_n x_n$, $w' = x_1 r_2' x_2 r_3' \dots x_{n-1} r_n' x_n$ (with the same x_k) and there are two indices k, l such that

$$
\begin{aligned}
&r_k = \overset{c*}{\text{---}}, \quad r_l = \overset{d}{\text{---}} \quad \text{for some} \quad c, d \in \{1,2,4\},\\
&r_k' = \overset{c}{\text{---}}, \quad r_l' = \overset{d*}{\text{---}},\\
&x_k \approx x_l \quad \text{or} \quad x_k \approx x_{l-1}.
\end{aligned}
$$

(7) We call two global words w, w' *congruent* and write $w \equiv w'$ if there is a sequence of words $w = w_1, w_2, \dots, w_m = w'$ such that w_k and w_{k+1} are elementary congruent for $1 \le k < m$.

Theorem 5.2. (1) *Every global word w defines an indecomposable polyhedron $P(w) \in \mathsf{CWF}^6$, called* string polyhedron.

(2) *Two string polyhedra, $P(w)$ and $P(w')$, are congruent if and only if either $w' \equiv w$ or $w' \equiv w^*$ (the reverse word).*

(3) *Let $\pi(t) \ne t$ be an irreducible polynomial over the field $\mathbb{Z}/2$ with the leading coefficient* 1. *Every quadruple $(w, r_1, \pi(t), m)$, where (w, r_1) is a non-periodic global cycle and m is a positive integer, defines an indecomposable polyhedron $P(w, r_1, \pi, m) \in \mathsf{CWF}^6$, called* band polyhedron.

(4) *Two band polyhedra, $P(w, r_1, \pi(t), m)$ and $P(w', r_1', \pi'(t), m')$, are congruent if and only if $m = m'$ and one of the following possibilities holds:*

(a) *$\pi'(t) = \pi(t)$ and either $w' \equiv w(k)$ with k even, $r_1' = r_{2k+1}$, or $w' \equiv w^*(k)$ with k odd, $r_1' = r_{n-2k}$;*

(b) *$\pi'(t) = t^d \pi(1/t)$, where $d = \deg \pi$, and either $w' \equiv w(k)$ with k odd, $r_1' = r_{2k+1}$ or $w' \equiv w^*(k)$ with k even, $r_1' = r_{n-2k}$.*

(5) *Every indecomposable polyhedron in* CWF^6 *is congruent to either a string or a band polyhedron.*
(6) *A string polyhedron* $P(w)$ *(a band polyhedron* $P(w, r_1, \pi(t), m)$*) is an atom if and only if the word* w *contains at least one of the letters* e_1, e_2, e_4 *and at least one of the letters* f_3, f_5.

It is convenient to present the string and band polyhedra from CWF^6 by their *attachment diagrams*, as in [1–5,12]. In these diagrams the cells of dimension d are presented by vertices at the d-th level, while their attachments are presented by edges. Note that in our case all homotopy groups of spheres involved into the calculations are cyclic. If the corresponding homotopy group is not of order 2, and θ is its generator, we supply the edge with a coefficient a, which means that actually this attachments is by the map $a\theta$. If X arises from a cofibration sequence

$$A \xrightarrow{f} B \xrightarrow{g} X \xrightarrow{h} A[1] \xrightarrow{f[1]} B[1] \tag{6}$$

with $A \in \mathbf{A},\ B \in \mathbf{B},\ \mathrm{H}_{10}(f) = 0$, the cells on the levels from 7 to 10 describe the object $B \simeq X^{10}$; the cells on the levels from 11 to 13 describe the object $A[1] \simeq X/B$.

Let $w = x_1 r_2 x_2 r_3 \dots x_{n-1} r_n x_n$ be a global word and $P(w)$ be the corresponding string polyhedron. Then the cofibration sequence (6) and the attachment diagram for $X = P(w)$ can be reconstructed as follows.

(1) The indecomposable summands of A correspond to the following subwords (or their reverse):

$$\begin{aligned} S^{10} \quad &\text{to} \quad f_1 \sim f_1', \\ S^{11} \quad &\text{to} \quad f_2 \sim f_2', \\ S^{12} \quad &\text{to} \quad f_3 \sim f_3', \\ C^{12} \quad &\text{to} \quad f_4 \sim f_5. \end{aligned}$$

(2) The indecomposable summands of B correspond to the following subwords (or their reverse):

$$\begin{aligned} S^{7} \quad &\text{to} \quad e_1, \\ C^{10}_2 \quad &\text{to} \quad e_2 \sim e_3, \\ C^{9} \quad &\text{to} \quad e_4 \sim e_5, \\ S^{8} \quad &\text{to} \quad e_6 \sim e_6', \\ C^{10} \quad &\text{to} \quad e_7, \\ S^{9} \quad &\text{to} \quad e_9 \sim e_9', \\ S^{10} \quad &\text{to} \quad e_{10}. \end{aligned}$$

(3) The attachments correspond to the subwords $e_i\ \mathsf{r}\ f_j$ and their reverse, where $\mathsf{r} \in \mathfrak{R}$. Namely, an attachment starts at the f-end of the corresponding subword and terminates at its e-end. The coefficient a of this attachment

equals

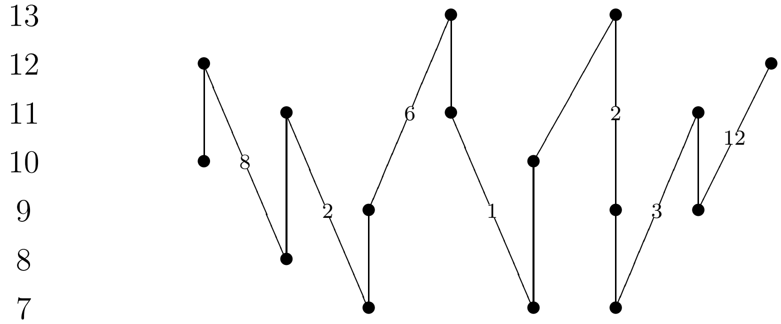
$$\begin{array}{rlll} 3c & \text{if} \quad \mathsf{r}=\overset{c}{-\!-}, & (c\in\{1,2,4\}), & \\ c & \text{if} \quad \mathsf{r}=\overset{c*}{-\!-}, & (c\in\{1,2,4\}), & \\ 8 & \text{if} \quad \mathsf{r}=\overset{*}{-\!-}, & \text{and} & e_i \overset{4}{-\!-} f_j, \\ 4 & \text{if} \quad \mathsf{r}=\overset{*}{-\!-}, & \text{and not} & e_i \overset{4}{-\!-} f_j. \end{array}$$

Recall that the congruence relation $\equiv$ for words allows to change these coefficients, namely to divide one of them by 3 and simultaneously multiply by 3 another at an "equivalent" place.

For instance, let

$$\begin{aligned} w = e_{10} \overset{1}{-\!-} f_2' \sim f_2 \overset{*}{-\!-} e_6 \sim e_6' \overset{1}{-\!-} f_1' \sim f_1 \overset{2*}{-\!-} e_4 \sim \\ \sim e_5 \overset{2}{-\!-} f_5 \sim f_4 \overset{1*}{-\!-} e_2 \sim e_3 \overset{1}{-\!-} f_3' \sim f_3 \overset{2*}{-\!-} e_5 \sim \\ \sim e_4 \overset{1}{-\!-} f_1 \sim f_1' \overset{1}{-\!-} e_9' \sim e_9 \overset{4}{-\!-} f_3 \sim f_3'. \end{aligned}$$

Then the attachment diagram for $P(w)$ is:

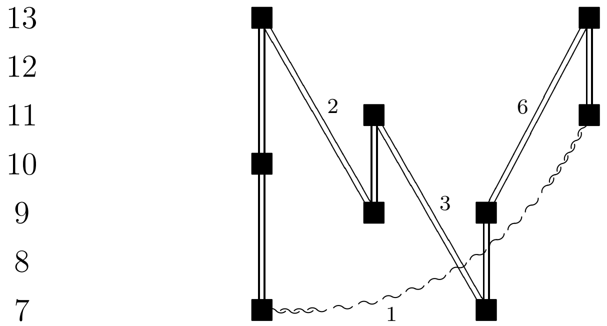


Let now $P(w, r_1, \pi(t), m)$ be a band polyhedron, where (w, r_1) is a non-periodic global cycle and $\deg \pi = d$. Then we must use the same procedure as before, but take dm copies of each cell. All attachments are natural: we attach the k-th copy from the upper level to the k-th copy from the lower one. Additionally, we must attach the copies of the last cell to the copies of the first one. This attachment must be twisted using the Frobenius matrix with the characteristic polynomial $\pi^m(t)$; the coefficient a of this attachment is defined as above by the relation r_1. Certainly, the Frobenius matrix can be replaced by any conjugate one (for instance, by the Jordan matrix, if the polynomial $\pi(t)$ is linear).

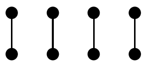
For instance, consider the polyhedron $P(w, \overset{1*}{-\!-}, t^2+t+1, 2)$, where

$$w = e_2 \sim e_3 \overset{1}{-\!-} f_3' \sim f_3 \overset{2*}{-\!-} e_9 \sim e_9' \overset{1}{-\!-} f_1' \sim f_1 \overset{1}{-\!-} e_4 \sim e_5 \overset{2}{-\!-} f_5 \sim f_4.$$

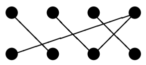
Its attachment diagram can be symbolically presented as follows:



To obtain the real attachment diagram, every square must be replaced by 4 copies of the corresponding cell. The double lines must be replaced by the attachments like



At last, the wavy line must be replaced by the attachments



since the Frobenius matrix with the characteristic polynomial $(t^2+t+1)^2$ is

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

6. Wildness of CWF^7

Recall that a classification problem is called *wild* if it contains the classification of pairs of linear mappings in a vector space over a field $\mathbf{k}$. Then it also contains the classification of finite dimensional representations of an *arbitrary* finitely generated $\mathbf{k}$-algebra (see, e.g., [11]).

To prove the wildness of CWF^7, we use Theorem 2.2 for $n=7, m=11$. Then $\mathbf{A}$ contains the polyhedron C_2^{14}, which arises from the cofibration sequence

$$S^{13} \xrightarrow{\eta^2} S^{11} \to C_2^{14} \to S^{14} \xrightarrow{\eta^2[1]} S^{12}, \tag{7}$$

while $\mathbf{B}$ contains the spheres S^k $(8 \le k \le 11)$. From the cofibration sequence (7) we get the following exact sequences:

$$\begin{gathered} 0 \leftarrow \mathbb{Z}/24 \leftarrow \mathrm{Hos}(C_2^{14}, S^8) \leftarrow \mathbb{Z}/2 \leftarrow 0\,, \\ \mathbb{Z}/2 \leftarrow \mathbb{Z} \leftarrow \mathrm{Hos}(C_2^{14}, S^{11}) \leftarrow \mathbb{Z}/24 \leftarrow \mathbb{Z}/2\,. \end{gathered}$$

Thus $\Gamma(C_2^{14},S^{11})\simeq\mathbb{Z}/12$. Consider now the commutative diagram

$$\begin{tikzcd}
S^{13} \arrow[r, "\eta"] \arrow[d, "\mathrm{Id}"'] & S^{12} \arrow[r] \arrow[d, "\eta"'] & C^{14} \arrow[r] \arrow[d] & S^{14} \arrow[r, "\eta"] \arrow[d, "\mathrm{Id}"'] & S^{13} \arrow[d, "{\eta[1]}"] \\
S^{13} \arrow[r, "\eta^2"'] & S^{11} \arrow[r] & C_2^{14} \arrow[r] & S^{14} \arrow[r, "\eta^2"'] & S^{12},
\end{tikzcd}$$

where both rows are cofibration sequences. It gives the commutative diagram with exact rows

$$\begin{tikzcd}
0 & 0 \arrow[l] & \mathrm{Hos}(C^{14},S_8) \arrow[l] & \mathbb{Z}/2 \arrow[l] & 0 \arrow[l] \\
0 \arrow[u] & \mathbb{Z}/24 \arrow[l] \arrow[u] & \mathrm{Hos}(C_2^{14},S^8) \arrow[l] \arrow[u] & \mathbb{Z}/2 \arrow[l] \arrow[u, "\mathrm{Id}"] & 0\,, \arrow[l] \arrow[u]
\end{tikzcd}$$

which shows that the lower row splits, i.e. $\Gamma(C_2^{14},S^8)=\mathrm{Hos}(C_2^{14},S^8)\simeq\mathbb{Z}/24\oplus\mathbb{Z}/2$. Consider the group $\Gamma(r,s,t)=\Gamma(rC_2^{14},sS^8\vee tS^{11})$. An element $\mu\in\Gamma(r,s,t)$ can be presented by a triangular matrix

$$\mu=\begin{pmatrix}\mu_1 & \mu_2\\ 0 & \mu_3\end{pmatrix},$$

where $\mu_1\in\mathrm{Mat}(s\times r,\mathbb{Z}/24)$ and $\mu_2\in\mathrm{Mat}(s\times r,\mathbb{Z}/2)$ correspond respectively to the direct summands $sr\mathbb{Z}/24$ and $sr\mathbb{Z}/2$ of $\Gamma(rC_2^{14},sS^8)$, while $\mu_3\in\mathrm{Mat}(t\times r,\mathbb{Z}/12)$ corresponds to the group $\Gamma(rC_2^{14},tS^{11})\simeq tr\mathbb{Z}/12$. If $\mu'\in\Gamma(r',s',t')$, morphisms $(\alpha,\beta):\mu\to\mu'$ can be presented by triangular matrices

$$\alpha=\begin{pmatrix}\alpha_1 & \alpha_2\\ 0 & \alpha_3\end{pmatrix},\quad \beta=\begin{pmatrix}\beta_1 & \beta_2\\ 0 & \beta_3\end{pmatrix},$$

where $\alpha_1,\alpha_3,\beta_1,\beta_3$ have coefficients from $\mathbb{Z}$, α_2 from $\mathbb{Z}/24$ and β_2 from $\mathbb{Z}/12$, $\beta_1\equiv\beta_3 \bmod 2$, $\alpha\mu=\mu'\beta$, and the subdivisions of α and β are compatible with those of μ and μ'. Note that actually only the residues of α_1,β_1 modulo 24, α_3,β_3 modulo 12 and of α_2,β_2 modulo 2 matters for the equality $\alpha\mu=\mu'\beta$.

Suppose now that $r=2r_1+r_2,\ s=2r_2,\ A,B\in\mathrm{Mat}(t\times r_1,\mathbb{Z}/2),\ C\in\mathrm{Mat}(r_1\times r_2,\mathbb{Z}/2)$, and consider the matrix $\mu=\mu(A,B,C)$ with components

$$\mu_1=\begin{pmatrix}6I_{r_1} & 0 & 0\\ 0 & 12I_{r_2} & 0\end{pmatrix},\quad \mu_2=\begin{pmatrix}0 & I_{r_1} & 0\\ 0 & 0 & C\end{pmatrix},\quad \mu_3=\begin{pmatrix}6A & 6B & 0\end{pmatrix}.$$

If $(\alpha,\beta):\mu(A,B,C)\to\mu(A',B',C')$, set $\alpha_1=(\xi_{ij})$ $(i,j=1,2)$, $\beta_1=(\eta_{ij})$, $\beta_3=(\zeta_{ij})$, $(1\le i,j\le 3)$, where theses subdivision are compatible with those of μ_k. Then one easily checks that the following congruences modulo 2 hold: $\xi_{21}\equiv\zeta_{21}\equiv 0$, $\eta_{ij}\equiv 0$ if $i<j$, $\xi_{11}\equiv\eta_{11}\equiv\zeta_{22}$ and $\xi_{22}\equiv\eta_{22}$. Since also $\zeta_{ij}\equiv\eta_{ij}$ for all i,j, we get

$$\xi_{22}C\equiv C'\zeta_{33},\quad \alpha_3A\equiv A'\xi_{22},\quad \alpha_3B\equiv B'\xi_{22}.$$

Therefore, if we consider the triples (A,B,C) as representations of the quiver

(8) $$\begin{tikzcd}
\bullet \arrow[r, "C"] & \bullet \arrow[r, bend left, "A"] \arrow[r, bend right, "B"'] & \bullet
\end{tikzcd}$$

over the field $\mathbb{Z}/2$, the elements $\mu(A,B,C)$ and $\mu(A',B',C')$ are isomorphic in $\mathbf{El}(\Gamma)$ if and only if the representations (A,B,C) and (A',B',C') of the quiver (8) are isomorphic. Since this quiver is wild [9], so is the category $\mathbf{El}(\Gamma)$, hence the category CWF^7 as well.

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